# On measure solutions of backward stochastic differential equations

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**Abstract**

We consider backward stochastic differential equations (BSDE) with nonlinear generators typically of quadratic growth in the control variable. A measure solution of such a BSDE will be understood as a probability measure under which the generator is seen as vanishing, so that the classical solution can be reconstructed by a combination of the operations of conditioning and using martingale representations. We show that classical solutions entail the existence of measure solutions. To go the other way, we prove a priori inequalities providing bounds on exponential moments of the control processes. Then we give some algorithms based for instance on approximations of singular generators by smoother ones, which construct measure solutions from first principles, in particular without reference to classical solutions. This way we provide an elegant and efficient method to at least recover classical existence Theorems for BSDE.

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# Introduction

The generally accepted natural framework for the most efficient formulation of pricing and hedging contingent claims on complete financial markets, for instance in the classical Merton-Scholes problem, is given by martingale theory, more precisely by the elegant notion of martingale measures. Martingale measures represent a view of the world in which price dynamics do not have inherent trends. From the perspective of this world, pricing a claim amounts to taking expectations, while hedging boils down to pure conditioning and using martingale representation.

At first glance, hedging a claim is, however, a problem calling upon stochastic control: it consists in choosing strategies to steer the portfolio into a terminal random endowment the portfolio holder has to ensure. Solving stochastic backward equations (BSDE) is a technique tailor-made for this purpose. This powerful tool has been introduced to stochastic control theory by Bismut [1]. Its mathematical treatment in terms of stochastic analysis was initiated by Pardoux and Peng [16], and its particular significance for the field of utility maximization in financial stochastics clarified in El Karoui, Peng and Quenez [8]. To fix ideas, we restrict our attention to a Wiener space probabilistic environment. In this framework, a BSDE with terminal variable $\xi$ at time horizon $T$ and generator $f$ is solved by a pair of processes $(Y, Z)$ on the interval $[0, T]$ satisfying

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s, Y_s, Z_s) ds, \quad t \in [0, T]. \tag{1}$$

In the case of vanishing generator, the solution just requires an application of the martingale representation theorem in the Wiener filtration, and $Z$ will be given as the stochastic integrand in the representation, to which we will refer as *control process* in the sequel. The classical approach of existence and uniqueness for BSDE involves a priori inequalities as a basic ingredient, by which unique solutions are constructed via fixed point arguments, just as in the case of forward stochastic differential equations.

In this paper we are looking for a notion in the context of BSDE that plays the role of the martingale measure in the context of hedging claims. Our main interest is directed to BSDE of the type (1) with generators that are non-Lipschitzian, and depend on the control variable $z$ quadratically, typically $f(s, y, z) = z^2\, b(s, z)$, $s \in [0, T]$, $z \in \mathbb{R}$, with a bounded function $b$. These generators were given a through treatment in Kobylanski [13], Briand, Hu [3], and Lepeltier, San Martin [14]. While [13] and [14] consider existence and uniqueness questions for bounded terminal variables $\xi$, [3] goes to the limit of possible terminal variables by considering $\xi$ for which $\exp(\gamma|\xi|)$ has finite expectation for some $\gamma > 2||b||_\infty$. All these papers employ different methods of approach following the classical pattern of arguments mentioned above. In contrast to this, we shall investigate an alternative notion of solution of BSDE, the generators of which fulfill similar conditions. In analogy with martingale measures in hedging which effectively eliminate drifts in price dynamics, we shall look for probability measures under which the generator of a given BSDE is seen as vanishing. Given such a measure $\mathbb{Q}$ which we call *measure solution* of the BSDE and supposing that $\mathbb{Q} \sim \mathbb{P}$, the processes $Y$ and $Z$ are the results of projection and representation respectively, i.e.

$Y = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_\cdot) = Y_0 + \int_0^\cdot Z_s d\widetilde{W}_s$, where $\widetilde{W}$ is a Wiener process under $\mathbb{Q}$. To summarize the findings of the paper, we basically show that existence Theorems obtained in the papers quoted are recovered in a more elegant and concise way in terms of measure solutions. On the other hand, the notion of measure solution being closer in spirit to weak solutions or solutions of martingale problems, we obtain new existence results for some generators, without touching uniqueness questions in general. Of course, determining a measure $\mathbb{Q}$ under which the generator vanishes amounts to doing a Girsanov change of probability that eliminates it. We therefore have to look at the BSDE in the form

$$Y_t = \xi - \int_t^T Z_s \left[ dW_s - \frac{f(s, Y_s, Z_s)}{Z_s} ds \right], \quad t \in [0, T], \tag{2}$$

define $g(s, y, z) = \dfrac{f(s, y, z)}{z}$, and study the measure

$$\mathbb{Q} = \exp\left( M - \frac{1}{2}\langle M \rangle \right) \cdot \mathbb{P}$$

for the martingale $M = \int_0^\cdot g(s, Y_s, Z_s) dW_s$. One of the fundamental problems that took some effort to solve consists in showing that $\mathbb{Q}$ is a probability measure. Here one has to dig essentially deeper than Novikov's or Kazamaki's criteria allow. We successfully employed a criterion which is based on the explosion properties of the quadratic variation $\langle M \rangle$, which we learnt from a conversation with M. Yor, and has been latent in the literature for a while, see Liptser, Shiryaev [15], or the more recent paper by Wong, Heyde [18]. This criterion allows a simple treatment of the problem of existence of measure solutions in the case of bounded terminal variable, and a still elegant and efficient one in the borderline case of exponentially integrable terminal variable considered by Briand, Hu [3].

Here is an outline of the presentation of our material. Throughout we consider BSDE possessing generators with quadratic nonlinearity in $z$. In a first section we start with strong (classical) solutions $(Y, Z)$ of our BSDE, to show existence of measure solutions. In case terminal variables are bounded, this is a relatively easy task. Things become essentially more complex, as soon as one passes to exponentially integrable terminal variables. We consider several scenarios in which we can show by exhibiting examples that in case of non-uniqueness there can be solutions not corresponding to measure solutions. In the second section, we prove a priori inequalities for the control process $Z$ provided a (measure) solution exists. In our setting this means we establish bounds for higher, eventually exponential moments of the square norms of $Z$. For given sequences of measure solutions corresponding to approximations of either the generator by a sequence of more regular ones, or the terminal variable by a sequence of bounded ones, we show that the a priori bounds are uniform. This crucial step then allows us to extract strongly convergent subsequences of control processes, and therefrom obtain a limiting probability measure for the corresponding sequence of approximate measure solutions, which turns out to be a good candidate for a measure solution of (1). This idea is basic for the algorithms presented in section 3, in which we construct measure solutions for BSDE without reference to classical ones. In the first subsection, we construct measure solutions for Lipschitzian generators, under less restrictive conditions than in El Karoui, Huang [7]. In the second subsection, we assume boundedness of the terminal variable, and approximate non-Lipschitzian generators by a sequence of Lipschitzian ones to obtain a measure solution version of Kobylanski's [13] or Lepeltier, San Martin's [14] results.

# 1 Strong solutions generate measure solutions

In this section we concentrate on the case in which strong solutions of our genuine BSDE exist. We show that measure solutions $\mathbb{Q}$ with $\mathbb{Q}$ equivalent to $\mathbb{P}$ exist. We consider the following class of generators. Let

$$f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$$

satisfy the **Assumption (H1)**:

- (i): $f(s,z) = f(\cdot,s,z)$ is adapted for any $z \in \mathbb{R}$,
- (ii): $f$ is continuous in $z$,
- (iii): $f(s,z) = z^2\, b(s,z)$,

with some bounded continuous function $b$ bounded by $|\alpha| > 0$. At places, we will use the more restrictive hypothesis

- (iii)': $f(s,z) = \alpha(z^2 + z\, b(s,z))$,

with some constant $\alpha \neq 0$. Consider the BSDE

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s,Z_s)ds, \quad t \in [0,T]. \tag{3}$$

Let us now define our concept of measure solution. Let

$$g(s,z) = \frac{f(s,z)}{z}, \quad s \in [0,T], z \in \mathbb{R}.$$

According to (iii) resp. (iii)', we have

$$g(s,z) = z\, b(s,z) \quad \text{resp.} \quad \alpha\,(z + b(s,z)), \quad s \in [0,T], z \in \mathbb{R}.$$

Note that $g$ is continuous.

**Definition 1.1** *A triple* $(Y,Z,\mathbb{Q})$ *is called* measure solution *of the BSDE (3), if* $\mathbb{Q}$ *is a probability measure on* $(\Omega,\mathcal{F})$*,* $(Y,Z)$ *a pair of* $(\mathcal{F}_t)$*–adapted stochastic processes such that the following conditions are satisfied:*

$$\begin{aligned} \widetilde{W} &= W - \int_0^\cdot g(s,Z_s)ds \quad \textit{is a} \quad \mathbb{Q}- \textit{ Brownian motion,} \\ \xi &\in L^1(\Omega,\mathcal{F},\mathbb{Q}), \\ Y_t &= \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_t) = \xi - \int_t^T Z_s d\widetilde{W}_s, \quad t \in [0,T]. \end{aligned}$$

## 1.1 Bounded terminal variable

In this subsection, we shall assume that the terminal variable $\xi$ is bounded, and of course that (3) possesses a strong solution $(Y, Z)$. In accordance with results of the literature (see for example Kobylanski [13]), it is natural to assume that $Y$ is bounded, by the same constant as $\xi$. Note that due to

$$|g(s,z)| \leq c|z|, \quad \text{s} \in [0,T], z \in \mathbb{R}$$

for some $c > 0$ and the definition of strong solutions we have

$$\int_0^T g^2(s, Z_s)ds < \infty \quad \mathbb{P} - a.s.. \tag{4}$$

We shall prove that under this condition, also a measure solution exists. For this purpose, denote by

$$M = \int_0^{\cdot} g(s, Z_s)dW_s. \tag{5}$$

It is clear that all we have to establish is that the measure

$$\mathbb{Q} = V_T \cdot \mathbb{P},$$

with

$$V = \exp\left(M - \frac{1}{2}\langle M\rangle\right)$$

leads to a probability measure equivalent to $\mathbb{P}$. This will be done by investigating possible explosions of the quadratic variation process $\langle M\rangle$. For $n \in \mathbb{N}$, let

$$\tau_n = \inf\{t \geq 0 : \langle M\rangle_t \geq n\}. \tag{6}$$

Let

$$Q^n = V_T|_{\mathcal{F}_{\tau_n}} \cdot \mathbb{P}$$

be the measure change locally on $\mathcal{F}_{\tau_n}$. We know that $Q^n$ is a probability measure equivalent to $\mathbb{P}$. Note first that the Radon-Nikodym density of $Q^n$ with respect to $\mathbb{P}$ on $\mathcal{F}_{\tau_n}$ is given by

$$V_{\tau_n} = \exp\left(M_{\tau_n} - \frac{1}{2}\langle M\rangle_{\tau_n}\right).$$

Moreover, the drifted process

$$\widetilde{W}^n = W - \int_0^{\tau_n \wedge \cdot} g(s, Z_s)ds$$

is a $Q^n-$ Brownian motion, in particular locally up to time $\tau_n$. In order to show that $\mathbb{Q}$ is a probability measure, we have to show

$$Q^n(\tau_n < T) \to 0 \quad (n \to \infty) \tag{7}$$

(see Heyde, Wong [18], or Bühler [2], or Liptser, Shiryaev [15]). Recall that by the very definition of the measure change,

$$Y_{\tau_n \wedge \cdot} = Y_0 + \int_0^{\tau_n \wedge \cdot} Z_s d\widetilde{W}_s^n$$

is a martingale under $Q^n$, up to time $\tau_n$, which is bounded by the same constant $c_1$ as $\xi$, due to the boundedness of the latter.

Hence we obtain for any $n \in \mathbb{N}$, starting with an application of Chebyshev-Markov's inequality, and, due to (iii), another constant $c_2$ independent of $n$

$$\begin{aligned} Q^n(\tau_n < T) &\leq \frac{1}{n} \mathbb{E}^{Q^n} \left( \int_0^{\tau_n} g(s, Z_s)^2 ds \right) = \frac{1}{n} \mathbb{E}^n \left( \int_0^{\tau_n} g(s, Z_s)^2 ds \right) \\ &\leq c_2 \frac{1}{n} \mathbb{E}^n \left( \int_0^{\tau_n} (Z_s)^2 ds \right) \\ &= c_2 \frac{1}{n} \mathbb{E}^n \left( \left| \int_0^{\tau_n} Z_s d\tilde{W}_s^n \right|^2 \right) = c_2 \frac{1}{n} \mathbb{E}^n \left( |Y_{\tau_n} - Y_0|^2 \right) \\ &\leq \frac{1}{n} c_1 c_2. \end{aligned}$$

Hence, clearly we obtain (7), and have established our main result.

**Theorem 1.1** *Assume that $\xi$ is bounded, and that $f$ satisfies Assumption (H1). Then there is a measure solution of (3) such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$.*

**Proof:** In Kobylanski [13] or Lepeltier and San Martin [14], it is proved that under our conditions (i)-(iii) there exists a strong solution $(Y, Z)$ satisfying

$$\int_0^T Z_s^2 ds < \infty \quad \mathbb{P} - a.s..$$

But due to (iii) we have

$$|g(s, z)| \leq c|z|$$

for all $s \in [0, T], z \in \mathbb{R}$. Hence (4) follows.

By (7), we may apply the result in Wong, Heyde [18], or Bühler [2], according to which

$$\mathbb{Q} = \exp \left( M_T - \frac{1}{2} \langle M \rangle_T \right) \cdot \mathbb{P}$$

is a probability measure equivalent to $\mathbb{P}$. Under $\mathbb{Q}$, by definition,

$$W^{\mathbb{Q}} = W - \int_0^{\cdot} g(s, Z_s)\, ds$$

is a Brownian motion, and our BSDE may be written

$$Y_t = \xi - \int_t^T Z_s dW_s^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}(\xi | \mathcal{F}_t)$$

for $t \in [0, T]$. This shows that $(Y, Z, \mathbb{Q})$ is a measure solution.□

## 1.2 Exponentially integrable lower bounded terminal variable

In the following subsections, we shall discuss terminal variables that are not bounded. As is known from literature, see for example Briand, Hu [3], [4], this case is by far more complex. For example, it is here that even if the generators are smooth, solutions stop to be unique. We shall exhibit examples below which complement the result shown in Briand, Hu [4], according to which uniqueness is granted in case the generator of the BSDE possesses additional convexity properties, and the terminal variable possesses exponential moments of all orders. This fact underlines that also variations in the generator affect questions of existence and uniqueness of solutions a lot. For this reason, and also to keep better oriented on a windy track with many bifurcations, we shall choose a simpler generator, and work under (iii)' with $b = 0$, i.e. our generator is given by

$$f(s, z) = \alpha z^2.$$

We shall further assume without loss of generality that $\alpha > 0$. This can always be obtained in our BSDE by changing the signs of $\xi$, and the solution pair $(Y, Z)$. Nonetheless, it turns out that positive and negative terminal variables need a separate treatment. We start in this subsection by assuming that $\xi$ be bounded below. Note that by a linear shift of $Y$ we may assume that $\xi \geq 0$. We shall further work under exponential integrability assumptions in the spirit of Briand, Hu [3]. According to this paper, exponential integrability of the terminal variable of the form

$$\mathbb{E}(\exp(\gamma|\xi|)) < \infty \tag{8}$$

for some $\gamma > 2\alpha$ is sufficient for the existence of a solution. Let us first exhibit an example to show that one cannot go essentially beyond this condition without losing solvability.

**Example:**

Let $T = 1$, and let $\alpha = \frac{1}{2}$. Let us first consider

$$\xi = \frac{W_1^2}{2}.$$

It is immediately clear from the fact that $W_1$ possesses the standard normal density, that $\mathbb{E}\exp(2\alpha|\xi|) = \infty$, hence of course also for $\gamma > 2\alpha$ (8) is not satisfied. To find a solution $(Y, Z)$ of (3) on any interval $[t, 1]$ with $t > 0$ define

$$Z_s = \frac{W_s}{s}, \quad s > 0,$$

and set for completeness $Z_0 = 0$. Let $t > 0$ and use the product formula for Itô integrals to deduce

$$\begin{aligned} \int_t^1 Z_s dW_s &= \frac{1}{2}\frac{W_s^2}{s}\Big|_t^1 + \frac{1}{2}\int_t^1 \frac{W_s^2}{s^2} ds \qquad (9) \\ &= \xi - \frac{1}{2}\frac{W_t^2}{t} + \frac{1}{2}\int_t^1 Z_s^2 ds. \end{aligned}$$

This means that, if we set for convenience again $Y_0 = 0$, the pair of processes $(Y_s, Z_s) = (\frac{1}{2}\frac{W_s^2}{s}, \frac{W_s}{s})$, $s \in [0,1]$, solves the BSDE (3) on $[t,1]$ for any $t > 0$. Of course, the definition of $Y_0$ is totally inconsistent with the BSDE. Worse than that, $Z$ is not square integrable on $[0,1]$, as is well known from the path behavior of Brownian motion. Hence $(Y,Z)$ is not a solution of (3). To put it more strictly, there is no strong solution of (3) on $[0,1]$, since, due to local Lipschitz conditions, any such solution would have to coincide with $(Y,Z)$ on any interval $[t,1]$ with $t > 0$.

According to Jeulin, Yor [9], transformations of this type are related to a phenomenon they call *appauvrissement de filtrations*. In fact, if $\frac{1}{2}$ is replaced with a parameter $\lambda$, they show that the natural filtration of the transformed process gets poorer than the one of the Wiener process, iff $\lambda > \frac{1}{2}$. Hence in the case we are interested in the Wiener filtration is preserved.

Let us now reduce the factor of $W_1^2$ in the definition of $\xi$ a bit, to show that solutions exist in this setting. For $k \in \mathbb{N}$, let

$$\xi_k = \frac{W_1^2}{2(1+1/k)},$$

and consider the BSDE (3) with the generator $f$ chosen above, and terminal condition $\xi_k$. In this setting, we clearly have

$$\mathbb{E}\exp(\gamma\xi_k) < \infty \quad \text{for} \quad 2\alpha \le \gamma < 2\alpha(1+1/k).$$

This shows that the condition of Briand, Hu [3] is satisfied. It is not hard to construct the solutions of the corresponding BSDEs explicitly, in the same way as above. In fact, for $k \in \mathbb{N}$ we may define $f_k(t) = \frac{1}{k} + t, t \in [0,1]$, and set

$$Z_t^k = \frac{W_t}{f_k(t)}, \quad t \in [0,1].$$

We may then repeat the product formula for Itô integrals argument used above to obtain for $t \geq 0$

$$\begin{aligned} \int_t^1 Z_s^k dW_s &= \frac{1}{2}\frac{W_s^2}{f_k(s)}\Big|_t^1 + \frac{1}{2}\int_t^1 \frac{W_s^2 f_k'(s)}{f_k(s)^2} ds \\ &= \frac{1}{2}\frac{W_1^2}{f_k(1)} - \frac{1}{2}\frac{W_t^2}{f_k(t)} + \frac{1}{2}\int_t^1 (Z_s^k)^2 ds. \end{aligned} \tag{10}$$

Hence we set

$$Y_t^k = \frac{1}{2}\frac{W_t^2}{f_k(t)}, \quad t \in [0,1],$$

to identify the pair of processes $(Y^k, Z^k)$ as a solution of the BSDE

$$Y_t^k = \xi_k - \int_t^1 Z_s^k dW_s + \frac{1}{2}\int_t^1 (Z_s^k)^2 ds, \quad t \in [0,1]. \tag{11}$$

We do not know at this moment whether (3) possesses more solutions.□

### 1.2.1 Measure solution property of particular solution

Under the assumption of exponential integrability, we will now derive measure solutions from given strong solutions. Leaving the difficult question of uniqueness apart for a moment, we remark that with our simple generator, we obtain an explicit solution given by the formula

$$Y_t = \frac{1}{2\alpha} \ln M_t - \frac{1}{2\alpha} \ln M_0, \quad Z_t = \frac{1}{2\alpha} \frac{H_t}{M_t}, \tag{12}$$

where

$$M_t = \mathbb{E}(\exp(2\alpha\xi)|\mathcal{F}_t) = M_0 + \int_0^t H_s dW_s, \quad t \in [0, T].$$

In the sequel, we shall work with these explicit solutions. In the following Lemma, we prove integrability properties for the square norm of $Z$ which will be crucial for stating the martingale property of $M$ and other related processes later.

**Lemma 1.1** *For any $p \geq 1$ we have*

$$\mathbb{E}\left(\left[\int_0^T Z_s^2 ds\right]^p\right) < \infty.$$

*In particular, $\int_0^\cdot Z_s dW_s$ is a uniformly integrable martingale.*

**Proof:** Let $t \in [0, T]$. By Itô's formula, applied to $N$

$$\frac{1}{2\alpha}[\ln M_t - \ln M_0] = \frac{1}{2\alpha}\left[\int_0^t \frac{H_s}{M_s} dW_s - \frac{1}{2}\int_0^t \left(\frac{H_s}{M_s}\right)^2 ds\right] = \int_0^t Z_s dW_s - \alpha \int_0^t Z_s^2 ds.$$

Hence

$$\alpha \int_0^t Z_s^2 ds = -\frac{1}{2\alpha}[\ln M_t - \ln M_0] + \int_0^t Z_s dW_s. \tag{13}$$

By concavity of the ln and Jensen's inequality

$$\ln M_t = \ln \mathbb{E}(\exp(2\alpha\xi)|\mathcal{F}_t) \geq \mathbb{E}(2\alpha\xi|\mathcal{F}_t).$$

Using this in (13), we obtain

$$\alpha \int_0^t Z_s^2 ds \leq -\mathbb{E}(\xi|\mathcal{F}_t) + \frac{1}{2\alpha} \ln M_0 + \int_0^t Z_s dW_s.$$

Taking $p-$norms in this inequality and using the inequality of Burkholder, Davis and Gundy for the stochastic integral, we obtain with universal constants $c_1, c_2, c_3$

$$\begin{aligned} \mathbb{E}\left(\left[\int_0^t Z_s^2 ds\right]^p\right) &\leq c_1 \left[\mathbb{E}\left(|\mathbb{E}(\xi|\mathcal{F}_t)|^p\right) + |\ln M_0|^p + \mathbb{E}\left(\left[\int_0^t Z_s^2 ds\right]^{\frac{p}{2}}\right)\right] \\ &\leq c_2 \left[\mathbb{E}(|\xi|^p) + |\ln M_0|^p + \mathbb{E}\left(\left[\int_0^t Z_s^2 ds\right]^{\frac{p}{2}}\right)\right]. \end{aligned}$$

By a standard argument this entails

$$\mathbb{E}\left(\left[\int_0^t Z_s^2 ds\right]^p\right) \le c_3[\mathbb{E}(|\xi|^p) + |\ln M_0|^p + 1],$$

and finishes the proof. □

We shall now prove that $(Y, Z)$ gives rise to a measure solution.

**Theorem 1.2** *Assume that that $f$ satisfies $f(s,z) = \alpha z^2$, $z \in \mathbb{R}, s \in [0,T]$, and that $\xi$ is bounded below and satisfies (8). Then there is a measure solution of (3) such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$.*

**Proof:** Let

$$S = \int_0^\cdot Z_s \, dW_s.$$

Due to Lemma 1.1, we know that $S$ is a uniformly integrable martingale. We may write

$$\begin{aligned} \alpha S - \frac{1}{2}\alpha^2 \langle S \rangle &= \alpha \left[\int_0^\cdot Z_s dW_s - \alpha \int_0^\cdot Z_s^2 ds\right] + \int_0^\cdot (\alpha^2 Z_s^2 - \frac{1}{2}\alpha^2 Z_s^2) \, ds \qquad (14) \\ &= \alpha(Y - Y_0) + \frac{1}{2}\alpha^2 \int_0^\cdot Z_s^2 ds. \end{aligned}$$

Now let the stopping times $\tau_n$ be defined as in the previous subsection. For any $n \in \mathbb{N}$ we have

$$\mathbb{E}\exp\left(\alpha S_{\tau_n} - \frac{1}{2}\alpha^2 \langle S \rangle_{\tau_n}\right) = 1,$$

and consequently Fatou's lemma implies

$$\mathbb{E}\exp\left(\alpha[\xi - Y_0] + \frac{1}{2}\alpha^2 \int_0^T Z_s^2 ds\right) \le \liminf_{n\to\infty} \mathbb{E}\exp\left(\alpha S_{\tau_n} - \frac{1}{2}\alpha^2 \langle S \rangle_{\tau_n}\right) = 1. \qquad (15)$$

Using this and the positivity of the terminal variable $\xi$, we can now obtain the exponential integrability property

$$\mathbb{E}\exp\left[\frac{1}{2}\alpha(\xi - Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2 \, ds\right] < \infty. \qquad (16)$$

We shall now use (14) together with (15) to prove the exponential integrability of $\frac{1}{2}\alpha S_T$. In fact, we have

$$\frac{1}{2}\alpha S_T = \frac{1}{2}\alpha(\xi - Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2 \, ds.$$

Hence we obtain

$$\mathbb{E}\exp\left(\frac{1}{2}\alpha S_T\right) < \infty, \qquad (17)$$

and together with the uniform integrability of the martingale $S$, proved in Lemma 1.1, this enables us to apply the criterion of Kazamaki (see Revuz, Yor [17], p. 332). Hence we have proved the existence of a measure solution to our BSDE (3). □

As a by-product of our main result, we obtain the exponential integrability of the quadratic variation of $S$.

**Corollary 1.1** *Under the conditions of Theorem 1.2 we have*

$$\mathbb{E}\exp\left(\frac{1}{2}\alpha^2\int_0^T Z_s^2\,ds\right) < \infty.$$

**Proof:** This follows immediately from (16) and the lower boundedness of $\xi$. □

### 1.2.2 A quadratic BSDE with two solutions

Let us now come back to the question of uniqueness of solutions, and their measure solution property. Briand, Hu [3] prove the existence of solutions $(Y, Z)$ in the usual sense, given that (8) is satisfied. In a setting with more general generators the nonlinear $z$-part being bounded by $\alpha z^2$, they provide pathwise upper and lower bounds for $Y$, given by the known explicit solution for this generator $(\frac{1}{2\alpha}\log\mathbb{E}(\exp(2\alpha\xi)|\mathcal{F}_t)_{t\in[0,T]}$ used above, and its negative counterpart $(-\frac{1}{2\alpha}\log\mathbb{E}(\exp(-2\alpha\xi)|\mathcal{F}_t)_{t\in[0,T]}$. In a more recent paper, Briand, Hu [4] also provide a uniqueness result for the same setting, which is satisfied under the stronger integrability hypothesis

$$\mathbb{E}(\exp(\gamma|\xi|)) < \infty \tag{18}$$

for all $\gamma > 0$ and a convexity assumption concerning the generator. Let us start our discussion of uniqueness and the measure solution property by giving some examples.

For $b > 0$, let $\tau_b = \inf\{t \geq 0 : W_t \leq bt - 1\}$. We first consider a BSDE with random time horizon $\tau_b$. Let the generator be further specified by $\alpha = \frac{1}{2}$. Let $\xi = 2a(b-a)\tau_b - 2a$, where $a > 0$. It will become clear along the way why this choice of terminal variable is made. In the first place, it is motivated by the striking simplicity of the solutions we shall construct. We shall in fact give two explicit solutions of the BSDE

$$Y_{t\wedge\tau_b} = \xi - \int_t^{\tau_b} Z_s dW_s + \int_t^{\tau_b} \frac{1}{2}Z_s^2 ds. \tag{19}$$

Appropriate choices of $a$ and $b$ allow for terminal variables that are bounded below as well as bounded above. The fact that the time horizon is random is not crucial. Indeed, by using a time change, any solution of Equation (19) can be transformed into a solution of a BSDE with the same generator and with time horizon 1. To this end consider the time change $\rho(t) = \frac{t}{1+t}$, $t \in [0, \infty]$, and observe that the inverse of $\rho$ is given by $\rho^{-1}(t) = \frac{t}{1-t}$, $t \in [0, 1]$. Let $h(t) = \frac{1}{1-t}$ for all $t \in [0, 1]$. Then the process defined by

$$\tilde{W}_t = \int_0^t h^{-1}(s)d(W_{\rho^{-1}(s)}), \qquad t \in [0, 1], \tag{20}$$

is a Brownian motion on $[0, 1]$. Note that $W_t = \int_0^{\rho(t)} h(s) d\tilde{W}_s$ (and this is how we have to define $W$, if $\tilde{W}$ is given). Moreover, the stopping time

$$\hat{\tau}_b = \inf \left\{ t \geq 0 : \int_0^t h(s) d\tilde{W}_s \leq \frac{t}{1-t} - 1 \right\}$$

is equal to $\rho(\tau_b)$. We can now define a time changed analogue of Equation (19) with time horizon 1.

**Lemma 1.2** *Let $(Y_t, Z_t)$ be a solution of the BSDE (19), and let $\hat{\xi} = 2a(b-a)\frac{\hat{\tau}_b}{1-\hat{\tau}_b} - 2a$. Then $(y_t, z_t) = (Y_{\rho^{-1}(t)}, h(t) Z_{\rho^{-1}(t)})$ is a solution of the BSDE*

$$y_t = \hat{\xi} - \int_t^1 z_s d\tilde{W}_s + \int_t^1 \frac{1}{2} z_s^2 ds. \tag{21}$$

**Proof:** Since stochastic integration and continuous time changes can be interchanged (see Proposition 1.5, Chapter V in [17]) we have

$$\begin{aligned} y_t &= Y_{\rho^{-1}(t)} = \int_0^{\rho^{-1}(t)} Z_s dW_s - \frac{1}{2} \int_0^{\rho^{-1}(t)} Z_s^2 ds \\ &= \int_0^t Z_{\rho^{-1}(s)} dW_{\rho^{-1}(s)} - \frac{1}{2} \int_0^t Z_{\rho^{-1}(s)}^2 d\rho^{-1}(s) \\ &= \int_0^t Z_{\rho^{-1}(s)} h(s) d\tilde{W}_s - \frac{1}{2} \int_0^t Z_{\rho^{-1}(s)}^2 h^2(s) ds, \end{aligned}$$

and hence the result. $\square$

Let us first assess exponential integrability properties of $\xi$. For this, let $\gamma > 0$ be arbitrary. Then

$$\mathbb{E} e^{\gamma|\xi|} = \mathbb{E} e^{\gamma|2a(b-a)\tau_b - 2a|} \leq e^{2a\gamma} \mathbb{E} e^{\gamma 2a|b-a|\tau_b}$$

Define the auxiliary stopping time

$$\sigma_b = \inf\{t \geq 0 : W_t \leq t - b\}.$$

It is well known and proved by the scaling properties of Brownian motion that the laws of $\tau_b$ and $\frac{\sigma_b}{b^2}$ are identical. See Revuz, Yor [17]. Moreover, the Laplace transform of $\sigma_b$ is equally well known. According to Revuz, Yor [17] we therefore have for $\lambda > 0$

$$E(\exp(-\lambda\tau_b)) = E(\exp(-\frac{\lambda}{b^2}\sigma_b) = \exp(-b[\sqrt{1 + \frac{2\lambda}{b^2}} - 1]). \tag{22}$$

Moreover, it is seen by analytic continuation arguments that this formula is even valid for $\lambda \geq -\frac{b^2}{2}$. Now choose $\lambda = -2a|b-a|\gamma$. Then the inequality

$$-2a|b-a|\gamma \geq -\frac{1}{2} b^2$$

amounts to

$$\gamma \leq \frac{b^2}{4a|b-a|}. \tag{23}$$

This in turn means that we have exponential integrability of orders bounded by $\frac{b^2}{4a|b-a|}$ , in particular we may reach arbitrarily high orders by choosing $a$ and $b$ sufficiently close. But no combination of $a$ and $b$ allows exponential integrability of all orders. In the light of Briand, Hu [4] this means that the entire field of pairs of positive $a$ and $b$ promises multiple solutions, and this is precisely what we will exhibit.

**The first solution**

It is clear from the definition that the pair $(Y_t, Z_t)$, defined by $Y_t = 2aW_{t\wedge\tau_b} - 2a^2(\tau_b \wedge t)$ and $Z = 2a1_{[0,\tau_b]}$, is a solution of (19). To answer the question whether this defines a measure solution, we have to investigate

$$\mathbb{E}\exp\left[\int_0^{\tau_b} \frac{1}{2} Z_s dW_s - \frac{1}{8}\int_0^{\tau_b} Z_s^2 ds\right] = \mathbb{E}\exp\left[aW_{\tau_b} - \frac{a^2}{2}\tau_b\right] = \mathbb{E}(\exp(a(b-\frac{a}{2})\tau_b - a)).$$

Due to (22) we have

$$\mathbb{E}(\exp(a(b-\frac{a}{2})\tau_b - a)) = \exp(-b[\sqrt{1-\frac{2}{b^2}a(b-\frac{a}{2})} - 1] - a) = \exp(-b[|1-\frac{a}{b}| - 1] - a),$$

and the latter equals 1 in case $b \geq a$ and $\exp(2(b-a)) < 1$ in case $a > b$. This simply means that our first solution is a measure solution of (21) provided $b \geq a$, and it fails to be one in case $a > b$. We will show that this first solution does not necessarily correspond to the particular solution discussed in the beginning of the section.

**The second solution**

We show now that the BSDE (19) with the same terminal variable as above possesses a second solution. By Lemma 1.2 there exists a second solution of (21) as well. Once this is shown, for any possible degree $\gamma$ of exponential integrability we will have exhibited a negative random variable satisfying $\mathbb{E}(\exp(\gamma|\xi|)) < \infty$ for which (19) possesses at least two solutions. This in turn will underline that Briand, Hu's [4] uniqueness result, valid under (18) cannot be improved by much. Note that the solution we will exhibit is again of the explicit form (12) encountered earlier. Let $M_t = \mathbb{E}[e^{\xi}|\mathcal{F}_t]$ for all $t \geq 0$. Due to the martingale representation property there exists a process $H$ such that $M_t = M_0 + \int_0^t H_s dW_s$. We know that $(\ln M_{\tau_b\wedge t}, \frac{H_{\tau_b\wedge t}}{M_{\tau_b\wedge t}})$ is a solution of (19). We will show below that

$$\ln M_{\tau_b\wedge t} = 2b - 4a + 2(b-a)W_{\tau_b\wedge t} - 2(b-a)^2(\tau_b \wedge t), \quad \text{if} \quad 2a > b, \tag{24}$$

$$\ln M_{\tau_b} = 2aW_{\tau_b\wedge t} - 2a^2\tau_b \wedge t, \quad \text{if} \quad 2a \leq b. \tag{25}$$

This implies that the solution $(\ln M_{\tau_b\wedge t}, \frac{H_{\tau_b\wedge t}}{M_{\tau_b\wedge t}})$ is different from the solution $(2aW_{\tau_b\wedge t} - 2a^2(\tau_b \wedge t), 2a)$ obtained above in case $2a > b$. Hence by Lemma 1.2 we obtain a second solution of (21) in this case.

First note that

$$
\begin{aligned}
M_t &= e^{-2a}\mathbb{E}[e^{2a(b-a)\tau_b}|\mathcal{F}_t] \\
&= e^{-2a}\left(e^{2a(b-a)\tau_b}1_{\{\tau_b\le t\}} + e^{2a(b-a)t}\mathbb{E}[e^{2a(b-a)[\tau_b-t]}|\mathcal{F}_t]1_{\{\tau_b>t\}}\right)
\end{aligned}
\tag{26}
$$

Let $\sigma_b(x,t) = \inf\{s \ge 0 : W_{s+t} - W_t \le b(s+t) - 1 - x\}$ and observe that on the set $\{\tau_b > t\}$ we have $\tau_b - t = \sigma_b(W_t, t)$. Therefore, by using again our knowledge on the Laplace transforms of $\sigma(x,t)$ (see [17]), we get

$$
\begin{aligned}
\mathbb{E}[e^{2a(b-a)[\tau_b-t]}|\mathcal{F}_t]1_{\{\tau_b>t\}} &= \mathbb{E}[e^{2a(b-a)\sigma_b(x,t)}]\Big|_{x=W_t}1_{\{\tau_b>t\}} \\
&= e^{-b(1+W_t-bt)[\sqrt{1-\frac{4a(1-a)}{b^2}}-1]}1_{\{\tau_b>t\}} \\
&= e^{-b(1+W_t-bt)[|1-\frac{2a}{b}|-1]}1_{\{\tau_b>t\}}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
M_t &= e^{-2a}\left(e^{2a(b-a)\tau_b}1_{\{\tau_b\le t\}} + e^{2a(b-a)t}e^{-b(W_t+1-bt)[|1-\frac{2a}{b}|-1]}1_{\{\tau_b>t\}}\right) \\
&= e^{2a((1-a)(\tau_b\wedge t)-1)}1_{\{\tau_b\le t\}} + e^{-2(b-a)(W_t+1-bt)}1_{\{\tau_b>t\}}.
\end{aligned}
$$

Hence in case $2a > b$

$$
\begin{aligned}
\ln M_{\tau_b\wedge t} &= 2a((b-a)(\tau_b\wedge t)-1) - 2(a-b)(W_{\tau_b\wedge t} + 1 - (\tau_b\wedge t)) \\
&= -4a + 2b + [-2b + 4a - 2a^2](\tau_b\wedge t) - 2(a-b)W_{\tau_b\wedge t} \\
&= 2b - 4a + 2(b-a)W_{\tau_b\wedge t} - 2[b-a]^2(\tau_b\wedge t).
\end{aligned}
$$

This confirms the first equation (24). Let finally $2a \le b$. Then we have

$$
\begin{aligned}
M_t &= e^{-2a}\left(e^{2a(b-a)\tau_b}1_{\{\tau_b\le t\}} + e^{2a(b-a)t}e^{2a(W_t+1-bt)}1_{\{\tau_b>t\}}\right) \\
&= e^{2a((b-a)(\tau_b\wedge t)+2a(W_{\tau_b\wedge t}+1-b\tau_b\wedge t)} \\
&= e^{2aW_{\tau_b\wedge t}-2a^2\tau_b\wedge t}.
\end{aligned}
$$

Hence in this case

$$
\ln M_{\tau_b\wedge t} = 2aW_{\tau_b\wedge t} - 2a^2\tau_b\wedge t.
$$

Note that in case $2a \le b$ we recover the solution already obtained as the first solution.

Let us finally show that this second solution is in fact a measure solution for any possible combination of parameters.

**Lemma 1.3** $(\ln M_{\tau_b\wedge t}, \frac{H_{\tau_b\wedge t}}{M_{\tau_b\wedge t}})$ *is a measure solution of (19), hence provides a measure solution of (21).*

**Proof:** For the first solution in case $a \le b$, which is identical to the one considered in case $2a \le b$, we have already established the measure solution property. Let us therefore consider the case $2a > b$. Note that for all $t$, $M_{t\wedge\tau_b} = e^{2b-4a} + \int_0^{t\wedge\tau_b} H_s dW_s$. Itô's formula applied to $e^{2(b-a)W_{\tau_b\wedge t}-2[b-a]^2(\tau_b\wedge t)}$ yields

$$H_{s\wedge\tau_b} = 2(b-a)e^{2(b-a)W_{\tau_b\wedge t}-2[b-a]^2(\tau_b\wedge t)}.$$

As a consequence, we have

$$Z_{s\wedge\tau_b} = \frac{H_{s\wedge\tau_b}}{M_{s\wedge\tau_b}} = 2(b-a)1_{[0,\tau_b]}(s),$$

and therefore

$$\begin{aligned}\mathcal{E}\Big(\frac{1}{2}\int ZdW\Big)_{\tau_b} &= e^{(b-a)W_{\tau_b}-\frac{1}{2}(b-a)^2\tau_b} \\ &= e^{(b-a)(b\tau_b-1)-\frac{1}{2}(b-a)^2\tau_b} \\ &= e^{(a-b)}e^{\frac{1}{2}(b-a)(b+a)\tau_b}.\end{aligned}$$

Again the explicit representation of the Laplace transform in (22) gives

$$\mathbb{E}\mathcal{E}\Big(\frac{1}{2}\int ZdW\Big)_{\tau_b} = e^{(a-b)}\mathbb{E}e^{-\frac{1}{2}(b-a)(b+a)\tau_b} = e^{(a-b)}e^{-b(\sqrt{1-(1-\frac{a^2}{b^2})}-1)} = 1.$$

This implies the claimed result that our second solution $(\ln M_{\tau_b\wedge t}, \frac{H_{\tau_b\wedge t}}{M_{b\wedge t}})$ is a measure solution of (19). □

**Remarks:**

1. We can summarize the findings of our investigations of the examples by stating that there are three basic scenarios: a) for $b \geq 2a$ we obtained one solution which is a measure solution at the same time; b) in the range $2a > b \geq a$ we found two different solutions both of which are measure solutions; c) if $a > b$ we finally encountered two solutions one of which is a measure solution, while the other one is not.

2. Note that our examples exhibiting solutions of (19) that are not measure solutions are all for negative terminal variables $\xi$. Positive terminal variables arise in scenarios a) or b), and therefore only produce multiple measure solutions.

**A continuum of solutions**

Let us now combine the first and second solutions to obtain a continuum of solutions of our BSDE (19). To do this, we have to consider a still somewhat more general class of stopping times. For $c \in \mathbb{R}$, let

$$\rho_c = \inf\{t \geq 0 : W_t \leq t - c\}.$$

We investigate the terminal variables

$$\xi = 2a(a-1)\rho_c + d$$

with further constants $a \neq 0, d \in \mathbb{R}$. Note first that the integrability properties of $\xi$ are the same as those obtained before for $b = 1$. According to the preceding paragraphs, our BSDE (19) possesses the following two solutions

$$Z^1 = 2a1_{[0,\rho_c]}, \quad Y^1 = d_1 + 2aW_{\rho_c \wedge \cdot} - 2a^2 \rho_c \wedge \cdot, \tag{27}$$

$$Z^2 = 2(1-a)1_{[0,\rho_c]}, \quad Y^2 = d_2 + 2(1-a)W_{\rho_c \wedge \cdot} - 2(1-a)^2 \rho_c \wedge \cdot, \tag{28}$$

with $d_1 = -2ac$ resp. $d_2 = -2(a-1)c$. Let us now take $c = 1$ and combine the two solutions to obtain a continuum of new ones. To do this, we start with the equation

$$\rho_1 = \rho_c + \rho_{1-c} \circ \theta_{\rho_c},$$

where $theta_t$ is the shift on Wiener space defined by

$$\theta_t(\omega) = W_{t+\cdot}(\omega) - W_t(\omega),$$

and $c \in ]0, 1[$. It describes the first time to reach the line with slope 1 that cuts the vertical at level $-1$, by decomposition with the intermediate time to reach the line with slope 1 cutting the vertical at $-c$. We mix the two solutions on the two resulting stochastic intervals, more precisely we put for $c \in ]0, 1[, l \in \mathbb{R}$

$$\begin{aligned} Z^c &= 2a1_{[0,\rho_c]} + 2(1-a)1_{[\rho_c,\rho_1]}, \qquad (29) \\ Y^c &= l + 2aW_{\rho_c \wedge \cdot} - 2a^2 \rho_c \wedge \cdot + 2(1-a)[W_{\rho_1 \wedge \cdot} - W_{\rho_c \wedge \cdot}] - 2(1-a)^2[\rho_1 \wedge \cdot - \rho_c \wedge \cdot]. \end{aligned}$$

Since we have

$$\begin{aligned} Y^c_{\rho_1} &= l + 2aW_{\rho_c} - 2a^2 \rho_c + 2(1-a)[W_{\rho_1} - W_{\rho_c}] - 2(1-a)^2[\rho_1 - \rho_c] \\ &= l + 2a(1-a)\rho_1 - 2ac - 2(1-a)(1-c), \end{aligned}$$

we have to set

$$l - 2ac - 2(1-a)(1-c) = d$$

in order to obtain a solution of (19) with $c = 1$. According to the treatment of the first and second solution, the constructed mixture is a measure solution if and only if both components of the mixture are. This is the case for $2a(1-a) > 0$, whereas for $2a(1-a) < 0$ we obtain a continuum of solutions that are no measure solutions.

**Remarks:**

1. This time, we may summarize our results by saying that there are two scenarios: a) for $2a(1-a) > 0$ there is a continuum of measure solutions of (19), while for $2a(1-a) < 0$ a continuum of non measure solutions is obtained.

2. Note that the initial conditions of our solutions continuum vary in a convex way between $-2a$ and $-2(1-a)$ as $c$ varies in $]0, 1[$, spanning the whole interval.

We shall now point out that the measure solution property of the second solution in case $a > b$ exhibited in the example above is not a coincidence. In fact, it will turn out that also for negative exponentially integrable $\xi$, solutions given by (12) provide measure solutions. To prove this, we will reverse the sign of $\xi$ by looking at our BSDE from the perspective of an equivalent measure.

## 1.3 Exponentially integrable upper bounded terminal variable

Sticking with the positivity of $\alpha$ in the generator

$$f(s,z) = \alpha z^2, \quad s \in [0,T], z \in \mathbb{R}$$

we shall now consider terminal variables $\xi$ that fulfill the exponential integrability condition (8), but are bounded above by a constant. Again, by a constant shift of the solution component $Y$, we can assume that the upper bound is 0, i.e. $\xi \leq 0$. So fix a non-positive terminal variable $\xi$ satisfying (8) for some $\gamma > 2\alpha$, and denote by $(Y,Z)$ the pair of processes given by the explicit representation of (12) solving our BSDE according to Briand, Hu [3]. With respect to the following probability measure, $\xi$ will effectively change its sign, so that we can hook up to the previous discussion. Recall $S = \int_0^\cdot Z_s dW_s$.

**Lemma 1.4** *Let $V = \exp(2\alpha S - 2\alpha^2\langle S\rangle)$. Then $V$ is a martingale of class (D), and consequently*

$$R = V_T \cdot \mathbb{P}$$

*is a probability measure equivalent to $\mathbb{P}$. Moreover,*

$$W^R = W - 2\alpha \int_0^\cdot Z_s ds$$

*is a Brownian motion under $R$.*

**Proof:** By (3), we may write

$$2\alpha[Y - Y_0] = 2\alpha S - 2\alpha^2\langle S\rangle,$$

hence also

$$2\alpha[\xi - Y_0] = 2\alpha S_T - 2\alpha^2\langle S\rangle_T.$$

According to Briand, Hu [3], Theorem 2, there exists $\delta > 2\alpha$ such that

$$\mathbb{E}(\sup_{t\in[0,T]} \exp(\delta|Y_t|)) < \infty, \tag{30}$$

and therefore $\beta > 1$ with the property

$$\mathbb{E}(\sup_{t\in[0,T]} V_t^\beta) < \infty. \tag{31}$$

This clearly implies that $V$ is a martingale of class (D), and consequently $R$ is a probability measure. Finally, Girsanov's theorem implies that $W^R$ is a Brownian motion under $R$.□

Now consider our BSDE under the perspective of the measure $R$ with respect to the Brownian motion $W^R$. We may write

$$Y = \xi - \int_\cdot^T Z_s dW_s + \alpha \int_\cdot^T Z_s^2 ds = \xi - \int_\cdot^T Z_s dW_s^R - \alpha \int_\cdot^T Z_s^2 ds. \tag{32}$$

But this just means that by switching signs in $(Y, Z)$, we may return, under the new measure $R$, to our old BSDE with $\xi$ replaced with $-\xi$. So our measure change puts us back into the framework of the previous subsection, and we may resume our arguments there by setting

$$S^R = -\int_0^\cdot Z_s dW_s^R.$$

We need an analogue of Lemma 1.1 to guarantee that $R$ is a uniformly integrable martingale.

**Lemma 1.5** *For any $p \geq 1$ we have*

$$\mathbb{E}^R\left(\left[\int_0^T Z_s^2 ds\right]^p\right) < \infty.$$

*In particular, $S^R$ is a uniformly integrable martingale under $R$.*

**Proof:** By definition of $R$, we have for any $p > 1$

$$\mathbb{E}^R\left(\left[\int_0^T Z_s^2 ds\right]^p\right) = \mathbb{E}\left(\exp(2\alpha[\xi - Y_0])\left[\int_0^T Z_s^2 ds\right]^p\right).$$

Now since $\xi \leq 0$, the density $\exp(2\alpha[\xi - Y_0])$ is bounded above. Therefore the asserted moment finiteness follows from Lemma 1.1. □

We are in a position to prove the main result of this subsection.

**Theorem 1.3** *Assume that that $f$ satisfies $f(s, z) = \alpha z^2$, $z \in \mathbb{R}, s \in [0, T]$, and that $\xi$ is bounded above and satisfies (8). Then there is a measure solution of (3) such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$.*

**Proof:** We may assume $\xi \leq 0$. Let us first show, in analogy to the proof of Theorem 1.2, that

$$V^R = \exp(\alpha S^R - \frac{1}{2}\alpha^2 \langle S^R \rangle)$$

is a uniformly integrable martingale under $R$, using Kazamaki's criterion. For this purpose, let

$$\tau_n^R = \inf\{t \geq 0 : \langle S^R \rangle_t \geq n\} \wedge T, \quad n \in \mathbb{N}.$$

Then, due to $\langle S \rangle = \langle S^R \rangle$, we deduce for all $n \in \mathbb{N}$ that $\tau_n = \tau_n^R$. Since $\tau_n^R \to T$ as $n \to \infty$, even with $\tau_n^R = T$ for all but finitely many $n$, Fatou's lemma allows to deduce

$$\mathbb{E}^R(V_T) \leq \liminf_{n \to \infty} \mathbb{E}^R(V^R_{\tau_n^R}) \leq 1. \tag{33}$$

Moreover, by the form of the BSDE translated to $W^R$ under $R$,

$$\begin{aligned}
\alpha S^R - \frac{1}{2}\alpha^2 \langle S^R \rangle &= \alpha[-\int_0^\cdot Z_s dW_s^R - \frac{1}{2}\alpha \int_0^\cdot Z_s^2 ds] \\
&= \alpha[-\int_0^\cdot Z_s dW_s^R - \alpha \int_0^\cdot Z_s^2 ds] + \frac{1}{2}\alpha^2 \int_0^\cdot Z_s^2 ds \\
&= \alpha[-Y + Y_0] + \frac{1}{2}\alpha^2 \int_0^\cdot Z_s^2 ds.
\end{aligned}$$

Using the negativity of $\xi$ and the identity just derived, we get the integrability property

$$\mathbb{E}^R \exp\left[\frac{1}{2}\alpha(-\xi + Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2\, ds\right] < \infty. \tag{34}$$

Using this and the positivity of the terminal variable $\xi$, we can now obtain the exponential integrability property

$$\mathbb{E} \exp\left[\frac{1}{2}\alpha(\xi - Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2\, ds\right] < \infty. \tag{35}$$

Again, we may now use (34) together with (33) to prove the exponential integrability of $\frac{1}{2}\alpha S_T^R$. In fact, from the BSDE viewed with $W^R$ under $R$ we have

$$\frac{1}{2}\alpha S_T^R = \frac{1}{2}\alpha(-\xi + Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2\, ds.$$

Hence we obtain

$$\mathbb{E}^R \exp\left(\frac{1}{2}\alpha S_T^R\right) < \infty. \tag{36}$$

Now appeal to the uniform integrability of the martingale $S^R$ under $R$, proved in Lemma 1.5, to see that the criterion of Kazamaki (see Revuz, Yor [17], p. 332) may be applied. Hence $V_R$ is a uniformly integrable martingale under $R$.

We have to show that this implies uniform integrability of

$$V = \exp(\alpha S - \frac{1}{2}\alpha^2 \langle S \rangle)$$

under $\mathbb{P}$. To see this, note that

$$\begin{aligned} \exp(\alpha S - \frac{1}{2}\alpha^2\langle S\rangle) &= \exp(2\alpha S - 2\alpha^2\langle S\rangle) \cdot \exp(-\alpha S + \frac{3}{2}\alpha^2\langle S\rangle) \\ &= \exp(2\alpha S - 2\alpha^2\langle S\rangle) \cdot \exp(\alpha S^R - \frac{1}{2}\alpha^2\langle S^R\rangle). \end{aligned}$$

Hence for $n \in \mathbb{N}$

$$\mathbb{E}(V_{\tau_n} 1_{\{\tau_n < T\}}) = \mathbb{E}^R(V^R_{\tau_n^R} 1_{\{\tau_n^R < T\}}), \tag{37}$$

and the latter expression tends to 0 as $n \to \infty$ by the first part of the proof. Hence the uniform integrability of $V$ under $\mathbb{P}$ follows from the criterion by Liptser, Shiryaev [15] already used earlier. This completes the proof. □

**Remark:**

The results of the preceding two subsections clearly call for similar ones for our BSDE with exponentially integrable terminal variable that are not bounded. Due to the nonlinearity of the generator of the BSDE, it seems impossible to derive such properties by combining the results of Theorems 1.2 and 1.3.

## 2 Some a priori estimates

Of course, we are more interested in going the other direction than in the preceding section. In other words, we would like to use our concept of solution to derive measure solutions, and eventually to obtain strong solutions thereof, without knowing in advance anything about strong solvability. The present section will provide some preliminary results for this purpose. Similarly to the classical theory of BSDE, we will discuss some a priori estimates. They will consist in (exponential) integrability properties of the square norms of the control process $Z$, if a solution pair $(Y, Z)$ is specified. So in all that follows in this section, we shall suppose that a solution $(Y, Z)$ is given.

Let us first assume that (iii) is satisfied such that with a constant $c$ we have $|b(s, z)| \leq c$ for all $s \in [0, T], z \in \mathbb{R}$. For bounded terminal variable $\xi$ we then get a bound for the square norm of $Z$ in terms of exponential moments of $\xi$. In fact, this bound can be extended to capture the BMO norm of the martingale generated by the stochastic integrand $Z$. The arguments are adopted from Kobylanski [13]. See also El Karoui, Barrieu [6].

**Theorem 2.1** *Suppose that the generator $f$ satisfies (iii) with a bound $c = |\alpha|$ for $b$. Assume $\xi$ is bounded, and let $L$ be its $L^\infty$–norm. Let $\mathbb{Q}$ be a probability measure such that $(Y, Z, \mathbb{Q})$ is a measure solution. Then $Y$ is bounded in $L^\infty(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}([0, T]), \mathbb{P} \otimes \lambda)$ and there is a constant $k$ only depending on $f$ such that we have*

$$\mathbb{E}\left(\int_t^T (Z_s)^2 ds \Big| \mathcal{F}_t\right) \leq k\, e^{2cL} < \infty, \tag{38}$$

*for all $t \in [0, T]$. In particular, $\int_0^\cdot Z_s dW_s$ is a $\mathbb{P}$-BMO martingale.*

**Proof:** See [12], chapter 2, for the definition of a BMO martingale. First of all, by choice of $\mathbb{Q}$, $Y_t = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_t)$. Hence the process $|Y|$ is bounded by $L$.

To resume the main argument of the proof of Proposition 2.1. in Kobylanski [13], we apply Itô's formula with the smooth strictly increasing real valued function $\Phi$ defined on $[-L, L]$ by

$$\Phi(x) = e^{2cx} - 2ce^{-2cL}x,$$

to obtain for $t \in [0, T]$

$$\begin{aligned} \Phi(\xi) - \Phi(Y_t) &= \int_t^T \Phi'(Y_s) Z_s dW_s \\ &- \int_t^T \Phi'(Y_s) f(s, Z_s) ds + \frac{1}{2}\int_t^T \Phi''(Y_s)(Z_s)^2 ds. \end{aligned}$$

Note that $\Phi' \geq 0$ on $[-L, L]$. Putting $M = \int_0^\cdot \Phi'(Y_s) Z_s dW_s$ and rearranging, we obtain the following inequality by using the growth hypotheses assumed for $f$

$$\begin{aligned} \Phi(Y_t) &= M_T - M_t + \Phi(\xi) + \int_t^T [\Phi'(Y_s) f(s, Z_s) - \frac{1}{2}\Phi''(Y_s)\,(Z_s)^2]\, ds \\ &\leq M_T - M_t + \Phi(\xi) + \int_t^T [c\Phi'(Y_s) - \frac{1}{2}\Phi''(Y_s)]\; (Z_s)^2\, ds. \end{aligned} \tag{39}$$

We may choose $\Phi$ non-negative, and such that

$$c\Phi' - \frac{1}{2}\Phi'' = -2c^2 e^{-2cL} = -K.$$

Moreover $M$ is a martingale, because since $Y$ is bounded, so is $\Phi'(Y)$, and because from the definition of a measure solution, we know that $Z$ belongs to $L^2(\Omega \times [0,T], \mathbb{P} \otimes \lambda)$. Taking conditional expectations on both sides of (39) yields the bound

$$K\mathbb{E}\left(\int_t^T (Z_s^n)^2 ds \Big| \mathcal{F}_t\right) \leq \mathbb{E}(\exp(2c\xi)|\mathcal{F}_t) \leq \exp(2cL) < \infty,$$

by hypothesis. This establishes the desired boundedness with $k = 1/K$. □

Let us next replace the assumption (iii) with (iii)', and stick with the terminology of the preceding section. We start with describing $M - \frac{1}{2}\langle M \rangle$ in an alternative way, involving $\mathbb{Q}$-martingales, even more generally only $Q_n$-martingales for all $n \in \mathbb{N}$. We just assume that $Z$ is given as an a.s. square integrable process. Observe first that by Itô's formula we have for $t \in [0,T]$

$$Y_t - Y_0 = \int_0^\cdot Z_s dW_s - \int_0^\cdot f(s, Z_s) ds. \tag{40}$$

To give our decomposition, let

$$\widetilde{W} = W - \int_0^\cdot g(s, Z_s) ds.$$

We take the already introduced by (5) and (6) notations. $\widetilde{W}$ is a $Q_n$-martingale up to time $\tau_n$ for all $n \in \mathbb{N}$. Let now

$$N = \int_0^\cdot [g(s, Z_s) - \alpha Z_s] \, dW_s, \text{ and } \widetilde{N} = \int_0^\cdot [g(s, Z_s) - \alpha Z_s] \, d\widetilde{W}_s, \tag{41}$$

and denote

$$B = \frac{1}{2} \int_0^\cdot g^2(s, Z_s) ds. \tag{42}$$

Then we may write

$$M_T - \frac{1}{2}\langle M \rangle_T = \tilde{N}_T + \alpha[\xi - Y_0] + B_T = H. \tag{43}$$

Now for any $n \in \mathbb{N}$

$$\mathbb{E} \exp\left(M_{\tau_n} - \frac{1}{2}\langle M \rangle_{\tau_n}\right) = \mathbb{E}^{Q^n}(1) = 1,$$

and consequently by Fatou's lemma we have

$$\mathbb{E} \exp\left(\tilde{N}_T + \alpha[\xi - Y_0] + B_T\right) \leq \liminf_{n \to \infty} \mathbb{E} \exp\left(M_{\tau_n} - \frac{1}{2}\langle M \rangle_{\tau_n}\right) = 1. \tag{44}$$

Here we have also used the fact that due to (4) $\tau_n \to T$ as $n \to \infty$. This property implies the following exponential integrability property.

**Theorem 2.2** *Let $f$ satisfy (i), (ii), and (iii)', and assume that for some $\gamma > |\alpha|$ we have $\mathbb{E}\exp(\gamma|\xi|) < \infty$. Then there exists $\beta > 0$ only depending on $\alpha$ and $\gamma$ such that (45) holds:*

$$\mathbb{E}\exp\left(\beta\int_0^T Z_s^2\, ds\right) < \infty. \tag{45}$$

**Proof:** From (41) and (42), we have

$$\widetilde{N} + \alpha[Y - Y_0] + B = N + \alpha[Y - Y_0] + B - \alpha\int_0^\cdot b(s, Z_s)\, g(s, Z_s)\, ds.$$

Hence for $\rho < 1$

$$\begin{aligned}
&\mathbb{E}\exp\left[\rho\left(B_T - \alpha\int_0^T b(s, Z_s)\, g(s, Z_s)\, ds\right)\right] \\
&\le \left[\mathbb{E}\exp\left(\widetilde{N}_T + \alpha[\xi - Y_0] + B_T\right)\right]^\rho \times \left[\mathbb{E}\exp\left(-\frac{\rho}{1-\rho}(N_T + \alpha[\xi - Y_0])\right)\right]^{1-\rho} \\
&\le \left[\mathbb{E}\exp\left(-\frac{\rho}{1-\rho}(N_T + \alpha[\xi - Y_0])\right)\right]^{1-\rho},
\end{aligned}$$

using (44). It remains to further estimate the second factor on the right hand side. For this purpose, let us choose $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we may estimate, using Hölder's inequality

$$\begin{aligned}
&\mathbb{E}\exp\left(-\frac{\rho}{1-\rho}(N_T + \alpha[\xi - Y_0])\right) \\
&\le \left[\mathbb{E}\exp\left(-\frac{\rho}{1-\rho}q\, N_T\right)\right]^{\frac{1}{q}} \left[\mathbb{E}\exp\left(-\frac{\rho}{1-\rho}p\cdot\alpha(\xi - Y_0)\right)\right]^{\frac{1}{p}}.
\end{aligned}$$

Now observe that $\frac{\rho}{1-\rho}p$ may be chosen as close to 0 as we wish, at the expense of choosing $\rho$ close to 0. Hence for $\rho$ small enough we deduce that

$$\mathbb{E}\exp\left[\rho\left(B_T - \alpha\int_0^T b(s, Z_s)\, g(s, Z_s)\, ds\right)\right] \le c_1,$$

where the finite constant $c_1$ as well as $\rho$ only depend on $\alpha$ and $\gamma$, i.e. are universal.

Now once again by the boundedness of $b$, we obtain with another constant $c_2$

$$[B_T - \alpha\int_0^T b(s, Z_s)\, g(s, Z_s)\, ds] \ge B_T - c_2\sqrt{B_T}.$$

We choose $\delta < \rho < 1$ and a constant $c_3$ big enough to ensure

$$\delta B_T \le \rho[B_T - c_2\sqrt{B_T}] + c_3.$$

Hence we obtain

$$\mathbb{E}\exp(\delta\, B_T) \le e^{c_3} c_1. \tag{46}$$

Finally note that (iii)' implies that we may estimate

$$|\alpha||z| \leq c_4 + |g(s,z)|$$

for any $z \in \mathbb{R}, s \in [0,T]$, with a suitable constant $c_4$. So the asserted inequality follows from (46) with

$$\beta < \frac{\alpha^2}{2}\left(1 - \frac{1}{1+\frac{\gamma}{\alpha}}\right),$$

and the proof is finished. □

Here of course the degree of exponential boundedness of the process $Z$ depends on the integrability exponent for the terminal variable. If $\xi$ is bounded, we can take $\gamma$ as large as we want, and we obtain the following bound $\beta < \frac{\alpha^2}{2}$.

# 3 The existence of measure solutions

We shall now construct measure solutions from first principles. In particular, we shall not assume any knowledge about strong solutions. As in the two first sections, we shall mainly be interested in the case of quadratically bounded generators. Our construction consists in modifying some parameters of the system, such as the terminal variable, or the Lipschitz continuity properties of the coefficient functionals. These modifications will be done to approximate a system for which no solution is known by simpler systems for which this is the case.

## 3.1 The Lipschitz case

In order to obtain a self-contained theory that is not using any knowledge on classical solutions, we first construct measure solutions in the setting for which they have been studied mostly: for generators that possess Lipschitz properties and increase at most linearly. We shall even allow the control process $Z$ to be multi-dimensional. More formally, in this subsection we consider the following class of generators. Let

$$f : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$$

satisfy the **Assumption (H2)**:

- (1) $f(s,z) = f(\cdot,s,z)$ is adapted for any $z \in \mathbb{R}^d$;
- (2) $\mathbb{E}\left(\int_0^T |f(s,0)|^\gamma ds\right) < \infty$;
- (3) the set $\{s \in [0,T],\ f(s,.) \text{ is not continuous}\}$ is of Lebesgue measure zero;
- (4) $|f(s,z) - f(s,0)| \leq \phi_s |z|$ for all $s \in [0,T],\ z \in \mathbb{R}^d$;

with some constants $K > 0$, $\gamma \geq 1$ and some non negative process $\phi$. We suppose that $\xi \in L^\gamma$, $\gamma \geq 1$, and we shall assume in the following that $f(s, 0) = 0$ for all $s \in [0, T]$. This can be done without loss of generality, since we may replace $\xi$ with the $\gamma$-integrable random variable

$$\tilde{\xi} = \xi + \int_0^T f(s, 0) ds.$$

Now for $z \in \mathbb{R}^d$, for all $1 \leq j \leq d$, let $z^{(j)}$ denote the $d$-dimensional vector whose first $j$ components are equal to those of $z$, and whose last $d - i$ components are equal to 0. With this notation, we define a function $g^j : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ by the requirement that for all $s \in [0, T]$, $z \in \mathbb{R}^d$:

$$\begin{aligned} g^j(s, z) &= \frac{f(s, z^{(j)}) - f(s, z^{(j-1)})}{z^j}, \text{ if } z^j \neq 0, \\ &= 0, \text{ if } z^j = 0. \end{aligned}$$

Here $z^i$ denotes the $i$ component of the vector $z$. Therefore we have defined the function $g$ with values in $\mathbb{R}^d$ and $g$ is bounded by the process $\phi$.

The process $\phi$ verifies either

$$\exists \kappa > 1, \ \mathbb{E}\left[\exp\left(\frac{\kappa}{2}\int_0^T \phi_r^2 dr\right)\right] < +\infty; \tag{47}$$

or

$$\text{the martingale } \left(L_t = \int_0^t \phi_r dW_r\right)_{t \in [0,T]} \text{ is BMO.} \tag{48}$$

We denote by $\|L\|$ the $BMO_2$-norm of $L$. From Theorem 2.2 in [12], (48) implies (47), with $1/\kappa = 2\|L\|^2$. Remark that (47) is a stronger Novikov condition. From these assumptions (see [12], theorem 2.3), we know that for $0 \leq t \leq T$,

$$\mathcal{E}(\phi W)_t = \exp\left(\int_0^t \phi_r dW_r - \frac{1}{2}\int_0^t \phi_r^2 dr\right)$$

is a uniformly integrable martingale.

**Remark 3.1** *If $\phi$ is a constant, then (47) is satisfied for all $\kappa > 1$. If $f$ is a Lipschitz function:*

$$|f(t, y, z) - f(t, y, z')| \leq K|z - z'|,$$

*then $\phi$ is the constant $K$.*

Our solution algorithm for (3) is based on a recursively defined change of measure. Let $Q^0 = \mathbb{P}$, and $W^0 = W$, the coordinate process which is a Wiener process under $Q^0$. Set

$$Y^1 = \mathbb{E}(\xi|\mathcal{F}_\cdot) = \mathbb{E}(\xi) + \int_0^\cdot Z_s^1 dW_s^0,$$

and

$$Q^1 = \exp\left(\int_0^T g(s, Z_s^1) dW_s - \frac{1}{2}\int_0^T g(s, Z_s^1)^2 ds\right) \cdot \mathbb{P} = R_T^1 \cdot \mathbb{P}.$$

Then

$$W^1 = W - \int_0^{\cdot} g(s, Z_s^1) ds$$

is a Wiener process under $Q^1$. Indeed under (47), the Novikov condition is satisfied, and under (48), the martingale

$$M_t^1 = \int_0^t g(s, Z_s^1) dW_s$$

is BMO. Now since $(Q^1, Q^0)$ is a Girsanov pair, it is well known that the predictable representation property is inherited from the Brownian motion $W^0$ to the Brownian motion $W^1$. See for example Revuz, Yor [17], p. 335. Hence there exists a pair $(Y^2, Z^2)$ of processes such that for all $t \in [0, T]$

$$Y_t^2 = \mathbb{E}^{Q^1}(\xi|\mathcal{F}_t) = \mathbb{E}^{Q^1}(\xi) + \int_0^t Z_s^2 dW_s^1.$$

Assume that $Q^n$ is recursively defined, along with the Brownian motion

$$W^n = W - \int_0^{\cdot} g(s, Z_s^n) ds$$

under $Q^n$. Then Revuz, Yor [17] may be applied to obtain two processes $(Y^{n+1}, Z^{n+1})$ such that

$$Y^{n+1} = \mathbb{E}^{Q^n}(\xi|\mathcal{F}_{\cdot}) = \mathbb{E}^n(\xi|\mathcal{F}_{\cdot}) = \mathbb{E}^n(\xi) + \int_0^{\cdot} Z_s^{n+1} dW_s^n.$$

Now set

$$Q^{n+1} = \exp\left[\int_0^T g(s, Z_s^{n+1}) dW_s - \int_0^T g(s, Z_s^{n+1})^2 ds\right] \cdot \mathbb{P} = R_T^{n+1} \cdot \mathbb{P}$$

to complete the recursion step. Then from our assumptions on $\phi$, and from the boundedness of $g$, the sequence of probability measures $(Q^n)_{n\in\mathbb{N}}$ is well defined and consists of measures equivalent with $P$. It is not hard to show tightness for this sequence.

**Theorem 3.1** *The sequence $(Q^n)_{n\in\mathbb{N}}$ is tight.*

**Proof:** In this proof, $\mathbb{E}^n$ denotes the expectation under $Q^n$. For $0 \le s \le t \le T$, $n \in \mathbb{N}$, we have, recalling that $W$ is the coordinate process on the canonical space:

$$\begin{aligned}
\mathbb{E}^n\left(|W_t - W_s|^4\right) &\le \mathbb{E}^n\left(|W_t^n - W_s^n + \int_s^t g(u, Z_u^n) du|^4\right) \\
&\le C\left[\mathbb{E}^n\left(|W_t^n - W_s^n|^4\right) + \mathbb{E}^n\left(|\int_s^t g(u, Z_u^n) du|^4\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq C|t-s|^2 + C|t-s|^2 \mathbb{E}^n \left( \int_s^t g(u, Z_u^n)^2 du \right)^2 \\
&\leq C|t-s|^2 + C|t-s|^2 \mathbb{E}^n \left( \int_s^t \phi_u^2 du \right)^2 \\
&\leq C|t-s|^2 + C|t-s|^2 \mathbb{E} \left[ \left( \int_s^t \phi_u^2 du \right)^2 R_T^n \right] \\
&\leq C|t-s|^2 \left\{ 1 + \left[ \mathbb{E} \left( \int_s^t \phi_u^2 du \right)^{2p} \right]^{1/p} [\mathbb{E}(R_T^n)^q]^{1/q} \right\}, \quad (49)
\end{aligned}
$$

from the Hölder inequality with $p > 1$ and $p^{-1} + q^{-1} = 1$.

Suppose that $\phi$ satisfies the assumption (47). From the Novikov condition applied to the martingale

$$
M_t^n = \int_0^t g(u, Z_u^n) dW_u,
$$

we know that $\mathcal{E}(M^n)$ is a uniformly integrable martingale under $\mathbb{P}$. Moreover

$$
\mathbb{E}\left[\exp\left(\frac{\sqrt{C}}{2} M_T^n\right)\right] \leq \mathbb{E}\left[\exp\left(\frac{C}{2}\langle M^n\rangle_T\right)\right]^{1/2} \leq \mathbb{E}\left[\exp\left(\frac{C}{2}\langle L\rangle_T\right)\right]^{1/2} < +\infty.
$$

From Theorem 1.5 in [12], we deduce that if $p$ is s.t.

$$
\frac{\sqrt{p}}{\sqrt{p}-1} = \sqrt{C} = \sqrt{\kappa},
$$

then

$$
\mathbb{E}\left[\mathcal{E}\left(g(., Z^n)W\right)_T^p\right] = \mathbb{E}(R_T^n)^p \leq C. \quad (50)
$$

Now if $\phi$ verifies the assumption (48), the martingale $M^n$ is also BMO, and the BMO-norm of $M^n$ is smaller than the BMO-norm of $L$. Therefore from Theorem 3.1 in [12] (or more precisely from the proof of this result), we deduce that there exists $q > 1$ and $C$ s.t.

$$
\mathbb{E}\left[\mathcal{E}\left(g(., Y^n, Z^n)W\right)_T^q\right] = \mathbb{E}R_n^q \leq C. \quad (51)
$$

The constant $q$ must satisfy the following inequality

$$
\|L\| < \theta(q) = \left\{ 1 + \frac{1}{q^2} \ln \frac{2q-1}{2(q-1)} \right\}^{\frac{1}{2}} - 1.
$$

The function $\theta : ]1, +\infty[ \to \mathbb{R}_+^*$ is a continuous decreasing function with $\theta(1) = \infty$ and $\theta(\infty) = 0$. Moreover, from the John-Nirenberg inequality (see [12], Theorem 2.2):

$$
\mathbb{E}\left[\exp\left(\frac{1}{4\|L\|_{BMO_2}^2} \int_0^T \phi_u^2 du\right)\right] \leq 2 \Longrightarrow \mathbb{E}\left(\int_s^t \phi_u^2 du\right)^{2p} < +\infty.
$$

Finally from (49)

$$
\mathbb{E}^n\left(|W_t - W_s|^4\right) \leq C|t-s|^2.
$$

Hence by a well known criterion (see for example Kallenberg [10], p. 261), tightness follows. □

In a second step, we shall now establish the boundedness in $L^2$ of the control sequence $(Z^n)_{n\in\mathbb{N}}$ obtained by the algorithm.

**Theorem 3.2** *There exists a function $\Psi$ such that, if $\gamma > \Psi(\kappa)$ for (47) or $\gamma > \Psi(\|L\|)$ for (48), then there exists $p > 1$ such that $\mathbb{E}\left[\left(\int_0^T (Z_s^n)^2 ds\right)^{\frac{p}{2}}\right]$ is a bounded sequence.*

**Proof:** Denote for $n \in \mathbb{N}$

$$R^n = R_T^n = \exp\left(\int_0^T g(s, Z_s^n) dW_s - \frac{1}{2}\int_0^T g(s, Z_s^n)^2 ds\right).$$

Then for $p > 1$ and $\varepsilon > 0$

$$\begin{aligned}
\mathbb{E}\left(\int_0^T (Z_s^n)^2 ds\right)^{\frac{p}{2}} &= \mathbb{E}\left[\left(\int_0^T (Z_s^n)^2 ds\right)^{\frac{p}{2}} (R^{n-1})^{\frac{1}{1+\varepsilon}} (R^{n-1})^{-\frac{1}{1+\varepsilon}}\right] \\
&\le \left[\mathbb{E}\left(\int_0^T (Z_s^n)^2 ds\right)^{\frac{p(1+\varepsilon)}{2}} R^{n-1}\right]^{\frac{1}{1+\varepsilon}} \left[\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}} \\
&= \left[\mathbb{E}^{n-1}\left(\int_0^T (Z_s^n)^2 ds\right)^{\frac{p(1+\varepsilon)}{2}}\right]^{\frac{1}{1+\varepsilon}} \left[\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}}.
\end{aligned}$$

With Burkholder-Davis-Gundy's inequality we obtain:

$$\mathbb{E}^{n-1}\left(\int_0^T (Z_s^n)^2 ds\right)^{\frac{p(1+\varepsilon)}{2}} \le C\mathbb{E}^{n-1}\left(|\xi|^{p(1+\varepsilon)}\right).$$

Thus for some $\eta > 0$

$$\begin{aligned}
\mathbb{E}\left(\int_0^T (Z_s^n)^2 ds\right)^{\frac{p}{2}} &\le C\left[\mathbb{E}\left(|\xi|^{p(1+\varepsilon)} R^{n-1}\right)\right]^{\frac{1}{1+\varepsilon}} \left[\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}} \\
&\le C\left\{\mathbb{E}\,|\xi|^{p(1+\varepsilon)(1+\eta)}\right\}^{\frac{1}{(1+\varepsilon)(1+\eta)}} \left\{\mathbb{E}(R^{n-1})^{\frac{1+\eta}{\eta}}\right\}^{\frac{\eta}{(1+\varepsilon)(1+\eta)}} \left\{\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}}\right\}^{\frac{\varepsilon}{1+\varepsilon}} \quad (52)
\end{aligned}$$

From the conditions (47) or (48), we can prove that there exists $\eta > 0$ and $\varepsilon > 0$ s.t.

$$\sup_{n\in\mathbb{N}} \left\{\mathbb{E}(R^{n-1})^{\frac{1+\eta}{\eta}}\right\}^{\frac{\eta}{(1+\varepsilon)(1+\eta)}} \left\{\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}}\right\}^{\frac{\varepsilon}{1+\varepsilon}} < +\infty.$$

First assume that (47) holds. Then

$$
\begin{aligned}
(R^{n-1})^{-\frac{1}{\varepsilon}} &= \exp\left[-\frac{1}{\varepsilon}\int_0^T g(s,Z_s^{n-1})dW_s + \frac{1}{2\varepsilon}\int_0^T g(s,Z_s^{n-1})^2 ds\right] \\
&= \exp\left[\int_0^T \left(-\frac{g(s,Z_s^{n-1})}{\varepsilon}\right) dW_s - \frac{1}{2}\int_0^T \left(\frac{g(s,Z_s^{n-1})}{\varepsilon}\right)^2 ds\right] \\
&\quad \times \exp\left[\frac{1}{2\varepsilon^2}(1+\varepsilon)\int_0^T g(s,Z_s^{n-1})^2 ds\right] \qquad (53)
\end{aligned}
$$

Now if

$$
\Gamma^{n-1,\varepsilon} = -\int_0^T \frac{g(u,Z_u^{n-1})}{\varepsilon} dW_u,
$$

we have for $C>1$

$$
\mathbb{E}\left[\exp\left(\frac{\sqrt{C}}{2}\Gamma^{n-1,\varepsilon}\right)\right] \leq \mathbb{E}\left[\exp\left(\frac{C}{2}\langle \Gamma^{n-1,\varepsilon}\rangle\right)\right]^{1/2} \leq \mathbb{E}\left[\exp\left(\frac{C}{2\varepsilon^2}\langle L\rangle_T\right)\right]^{1/2} < +\infty,
$$

when $C/\varepsilon^2 = \kappa$. Thus

$$
\mathbb{E}\left[\exp\left[\int_0^T \left(-\frac{g(s,Z_s^{n-1})}{\varepsilon}\right) dW_s - \frac{1}{2}\int_0^T \left(\frac{g(s,Z_s^{n-1})}{\varepsilon}\right)^2 ds\right]\right]^q < +\infty
$$

when $1/q + 1/p = 1$ and

$$
\frac{\sqrt{p}}{\sqrt{p}-1} = C = \varepsilon\sqrt{\kappa} \Longleftrightarrow p = \frac{\kappa\varepsilon^2}{(\varepsilon\sqrt{\kappa}-1)^2}.
$$

And we have

$$
\mathbb{E}\exp\left[\frac{p}{2\varepsilon^2}(1+\varepsilon)\int_0^T g(s,Z_s^{n-1})^2 ds\right] \leq \mathbb{E}\exp\left[\frac{p(1+\varepsilon)}{2\varepsilon^2}\int_0^T \phi_s^2 ds\right] < +\infty,
$$

if

$$
\frac{p(1+\varepsilon)}{\varepsilon^2} \leq \kappa \Longleftrightarrow \varepsilon \geq \frac{1+2\sqrt{\kappa}}{\kappa}.
$$

From (53) and with Hölder inequality we deduce that $\mathbb{E}R_{n-1}^{-\frac{1}{\varepsilon}} \leq C$. With (50) we already know that there exists $\eta$ s.t. $\mathbb{E}R_{n-1}^{\frac{1+\eta}{\eta}} \leq C$. We have to take $\sqrt{1+\eta} = \dfrac{\sqrt{\kappa}}{\sqrt{\kappa}-1}$.

Assume that (48) holds. Then we already know (51): there exists $\eta > 0$ such that $\mathbb{E}R_{n-1}^{\frac{1+\eta}{\eta}} \leq C$, if $\eta$ satisfies

$$
\|L\| < \theta\left(\frac{1+\eta}{\eta}\right).
$$

We use theorem 2.4 in [12] in order to prove that $\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}} \leq C$. We must choose $\varepsilon$ s.t.

$$
\|L\| \leq \sqrt{2}\left(\sqrt{1+\varepsilon}-1\right).
$$

The two constants $\eta$ and $\varepsilon$ depend on the constant $\kappa$ in (47) or the BMO-norm $\|L\|$ in (48). Coming back to (52) we deduce that:

$$\mathbb{E}\left(\int_0^T (Z_s^n)^2\, ds\right)^{\frac{p}{2}} \leq C\left\{\mathbb{E}\,|\xi|^{p(1+\varepsilon)(1+\eta)}\right\}^{\frac{1}{(1+\varepsilon)(1+\eta)}}.$$

Since $\xi$ belongs to $L^\gamma$, if $\gamma > (1+\varepsilon)(1+\eta)$, the desired boundedness follows for some $p > 1$ such that $\gamma \geq p(1+\varepsilon)(1+\eta)$. $\square$

We have an explicit formula for the function $\Psi$. For (47):

$$\Psi(\kappa) = \Psi_{47}(\kappa) = 1 + 4\frac{\sqrt{\kappa}}{(\sqrt{\kappa}-1)^2},$$

and for (48):

$$\Psi(\|L\|) = \Psi_{48}(\|L\|) = \left(1 + \frac{\|L\|}{2}\right) \times \frac{\theta^{-1}(\|L\|)}{\theta^{-1}(\|L\|)-1}.$$

We can check that $\Psi_{48} : ]0, +\infty[ \to ]1, +\infty[$ is an increasing function such that $\Psi(0) = 1$ and $\Psi(\infty) = \infty$.

**Remark 3.2** *If $\phi$ is a constant, recall that (47) holds for all $\kappa > 1$. In this case, we just have to suppose that $\gamma > 1$.*

**Lemma 3.1** *There exists a subsequence of $Z^n$ (still denoted $Z^n$) which converges $\mathbb{P}\otimes\lambda$-a.e. to some process $Z$.*

**Lemma 3.2** *The sequence $R_T^n$ converges also $\mathbb{P}$-a.s. to*

$$R_T = \exp\left(\int_0^T g(s, Z_s)dW_s - \frac{1}{2}\int_0^T (g(s,Z_s))^2 ds\right).$$

**Proof:** We may w.l.o.g. assume that $g(s, .)$ is continuous for all $s \in [0, T]$. The rest follows from Lemma 3.1. $\square$

Equipped with these results, we are now in a position to state our existence Theorem.

**Theorem 3.3** *Assumption (H1) holds. There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and an adapted process $Z$ such that $\mathbb{E}([\int_0^T |Z_s|^2 ds]^{\frac{1}{2}}) < \infty$ such that, setting*

$$R_T = \exp\left(\int_0^T g(s, Z_s)dW_s - \frac{1}{2}\int_0^T g(s,Z_s)^2 ds\right), \quad W^{\mathbb{Q}} = W - \int_0^\cdot g(s, Z_s)ds,$$

*we have*

$$\mathbb{Q} = R_T \cdot \mathbb{P},$$

*and such that the pair $(Y, Z)$ defined by*

$$Y = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_\cdot) = \mathbb{E}^{\mathbb{Q}}(\xi) + \int_0^\cdot Z_s dW_s^{\mathbb{Q}}$$

*solves the BSDE (3).*

**Proof:** Using Theorem 3.1, choose a probability measure $\mathbb{Q}$ and another subsequence of the corresponding subsequence of $(Q_n)_{n\in\mathbb{N}}$ which converges weakly to $\mathbb{Q}$. We denote this subsequence again by $(Q_n)_{n\in\mathbb{N}}$ and the corresponding subsequence of controls by $(Z^n)_{n\in\mathbb{N}}$. We have:

$$\mathbb{Q} = R_T \cdot \mathbb{P}.$$

Moreover for all $n \in \mathbb{N}$,

$$Y_t^n = \mathbb{E}^{n-1}(\xi) + \int_0^t Z_s^n dW_s^n = \mathbb{E}^{n-1}(\xi) + \int_0^t Z_s^n dW_s - \int_0^t Z_s^n g(s, Z_s^{n-1}) ds.$$

The only thing we have to prove, is that the sequence $Y_0^n = \mathbb{E}^n(\xi)$ also converges. But $Y_0^n = \mathbb{E}^n(\xi) = \mathbb{E}(\xi R^n)$, and $\xi$ belongs to $L^\gamma$, $R^n$ also belongs to some $L^p$ space with $1/p + 1/\gamma = 1$ if and only if

$$\gamma \geq \frac{\kappa}{(\sqrt{\kappa} - 1)^2}.$$

But it is true since $\gamma \geq \Psi(\kappa)$. Taking a subquence if necessary, we deduce that $Y_0^n$ converges to $\mathbb{E}^{\mathbb{Q}}(\xi)$.

Hence we obtain

$$Y_t = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_\cdot) = \mathbb{E}^{\mathbb{Q}}(\xi) + \int_0^t Z_s dW_s^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion. Finally $(Y, Z)$ solves the BSDE (3). □

Here the generator $f$ does not depend on $Y$. But some straightforward modifications in the previous arguments show that we can also consider $f$ and $g$ depending on $y$ if we assume that there exists some constant $K$ s.t. for all $(s, y) \in [0, T] \times \mathbb{R}$, $|f(s, y, 0| \leq K$. The proof of Theorem 3.1 remains unchanged, and in Theorem 3.2, we add that $\mathbb{E}\left(\int_0^T (Y_s^n)^2 ds\right)^{p/2}$ is bounded.

## 3.2 Quadratic non-linearity: bounded terminal variable

We shall now pass to the case of non-Lipschitz generators. This will be done in several steps, consisting in a hierarchy of approximations by solutions known from the previous steps. Let us first assume that the terminal variable $\xi$ is bounded. In this case we can base our arguments on the first a priori inequality in the preceding section. The generators for which we thus are able to derive existence of a solution are just continuous in the control variable. This corresponds to the setting considered in Lepeltier and San Martin [14]. Our method of construction of solutions is based on the notion of measure solution and therefore different from the ones used in this paper and Kobylanski [13]. It is of independent interest, and seen in the subsequent section to provide an extension to a setting not treated so far, given by an exponential integrability condition weaker than the one discussed in the first section, see Briand, Hu [3].

Let us assume in this subsection that the generator of our BSDE satisfies Assumption (H1), and let $\xi$ be bounded and $\mathcal{F}_T$–measurable. Our solution algorithm for (3) is based on a recursively defined change of measure, on the basis of solutions of the corresponding BSDE with an approximating sequence of generators linearly bounded in $z$. For $n \in \mathbb{N}$ let

$$f_n : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$$

be chosen with the following properties

- $(i)$ $f_n(s,z) = f_n(\cdot,s,z)$ is adapted for any $z \in \mathbb{R}$,
- $(ii)$ $f_n$ is continuous in $z$,
- $(iii)$ there is a constant $c_n > 0$ such that $|f_n(s,z)| \le c_n|z|$ for all $s \in [0,T], z \in \mathbb{R}$,
- $(iv)$ $|f_n| \le |f|$, and $f_n \to f$ uniformly on compact sets in $\mathbb{R} \times [0,T]$ as $n \to \infty$.

For $n \in \mathbb{N}$, consider the BSDE associated with generator $f_n$

$$Y_t^n = \xi - \int_t^T Z_s^n dW_s + \int_t^T f_n(s, Z_s^n) ds, \quad t \in [0,T]. \tag{54}$$

Let

$$g_n(s,z) = \frac{f_n(s,z)}{z}, \quad z \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Note that $f_n$ satisfies Assumption (H2). By our previous section, there exist a probability measure

$$Q^n = \exp\left(\int_0^T g_n(s, Z_s^n) dW_s - \frac{1}{2}\int_0^T g_n(s, Z_s^n)^2 ds\right) \cdot \mathbb{P},$$

a Wiener process under $Q^n$

$$W^n = W - \int_0^{\cdot} g_n(s, Z_s^n) ds$$

and we have

$$Y^n = \mathbb{E}^{Q^n}(\xi|\mathcal{F}_\cdot) = \int_0^{\cdot} Z_s^n dW_s^n.$$

We now aim at constructing a limit point of the sequence $(Q^n)_{n\in\mathbb{N}}$, which will in fact represent a measure solution of (3). For this purpose it will be necessary to prove boundedness of $(Z^n)_{n\in\mathbb{N}}$ in $L^2$, and tightness of the sequence $(Q^n)_{n\in\mathbb{N}}$. Let us start with the first goal, for which we shall employ an a priori inequality from the preceding section.

**Theorem 3.4** *Let $\xi$ be bounded, and assume that $f$ satisfies Assumption (H1), with a bound $c$ for the function $b$. Then there exists a constant $k$ such that for any $t \in [0,T]$ we have*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left(\int_t^T (Z_s^n)^2 ds \Big| \mathcal{F}_t\right) \le k\, \mathbb{E}(\exp(2c\xi)) < \infty. \tag{55}$$

**Proof:** We shall apply Theorem 2.1 to each measure in the sequence $(Q^n)_{n\in\mathbb{N}}$. This is feasible, due to condition (iv) for $f_n$. Note that the constant $k$ figuring in the Theorem only depends on the constant $c$ from (iii), which due to (iv) is the same for all $f_n$. Hence we immediately obtain (55) from (38), and complete the proof.□

Let us next discuss tightness of the sequence $(Q_n)_{n\in\mathbb{N}}$.

**Theorem 3.5** *Assume that $\xi$ is bounded. Then $(Q^n)_{n\in\mathbb{N}}$ is tight.*

**Proof:** We begin as in the proof of Theorem 3.1. For $s,t\in[0,T], s\le t, n\in\mathbb{N}$ we have, using the inequality of Cauchy-Schwarz for the last inequality

$$\begin{aligned}\mathbb{E}^{Q^n}|W_t-W_s|^4 &= \mathbb{E}^{Q^n}\left(\left|W_t^n-W_s^n+\int_s^t g(u,Z_u^n)du\right|^4\right)\\ &\le 2^4\left[\mathbb{E}^{Q^n}|W_t^n-W_s^n|^4+\mathbb{E}^{Q^n}\left|\int_s^t g(u,Z_u^n)du\right|^4\right]\\ &\le c_1\left[|t-s|^2+|t-s|^2\mathbb{E}^{Q^n}\left(\int_s^t (Z_s^n)^2ds\right)^2\right],\end{aligned}$$

with a constant $c_1$ independent of $n$. We now use the inequality of Burkholder-Davis-Gundy to deduce

$$\sup_{n\in\mathbb{N}}\mathbb{E}^{Q^n}\left[\int_s^t (Z_s^n)^2ds\right]^2\le c_2\sup_{n\in\mathbb{N}}\mathbb{E}^{Q^n}(\xi^4)\le c_3$$

with some positive constants $c_2,c_3$. The last inequality is due to the boundedness of $\xi$. Hence the well known tightness criterion (see for example Kallenberg [10], p. 261) applies. □

We are prepared to state our main result.

**Theorem 3.6** *Suppose that $\xi$ is bounded. There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and an adapted process $Z$ such that $\mathbb{E}([\int_0^T Z_s^2ds]^{\frac12})<\infty$ such that, setting*

$$R=\exp(\int_0^T g(s,Z_s)dW_s-\frac12\int_0^T g(s,Z_s)^2ds),\quad W^{\mathbb{Q}}=W-\int_0^{\cdot} g(s,Z_s)ds,$$

*we have*

$$\mathbb{Q}=R\cdot\mathbb{P},$$

*and such that the pair $(Y,Z)$ defined by*

$$Y=\mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_{\cdot})=\int_0^{\cdot} Z_sdW_s^{\mathbb{Q}}$$

*solves the BSDE (3).*

**Proof:** Using Theorem 3.4, we may state that the sequence $(Z^n)_{n\in\mathbb{N}}$ is bounded in $L^2$, hence uniformly integrable in $L^1$, and therefore weakly compact in $L^1$ (see the characterization of weakly compact sets in $L^1$ in Dunford-Schwartz [5]). We may consequently extract a weakly convergent sequence which we also call $(Z^n)_{n\in\mathbb{N}}$ for ease of notation, with weak limit $Z$, so that the $L^2$–norms converge as well. Hence the Theorem of Radon-Riesz applies and shows that $(Z^n)_{n\in\mathbb{N}}$ even converges to $Z$ in the strong $L^2$ sense. By eventually extracting another subsequence, we may finally assume that $(Z^n)_{n\in\mathbb{N}}$ converges to $Z$ $\mathbb{P}\otimes\lambda$–a.e. on $\Omega\times[0,T]$.

As a second ingredient, we use Theorem 3.5, to choose a corresponding subsequence of $(Q^n)_{n\in\mathbb{N}}$ which converges weakly to the probability measure $\mathbb{Q}$. As before, we denote this subsequence again by $(Q^n)_{n\in\mathbb{N}}$. Now denote

$$R_n = \exp\left(\int_0^T g_n(s,Z_s^n)dW_s - \frac{1}{2}\int_0^T g_n(s,Z_s^n)^2 ds\right),$$

$$R = \exp\left(\int_0^T g(s,Z_s)dW_s - \frac{1}{2}\int_0^T g(s,Z_s)^2 ds\right).$$

By continuity of $g$ and (iv), we conclude that $(R_n)_{n\in\mathbb{N}}$ converges $\mathbb{P}$–a.s. to the random variable $R$. Since due to Theorem 3.4 the BMO norm of $\int_0^\cdot Z_s^n dW_s$ is bounded w.r.t. $n$, $\int_0^\cdot Z_s dW_s$ belongs to BMO. Moreover $|g(s,z)|\leq z$, thus $R$ is a uniformly integrable martingale. Hence

$$\mathbb{Q} = R\cdot\mathbb{P}$$

on the one hand, and, by weak convergence and uniform integrability of the Radon-Nikodym densities which follows from the boundedness of the $|f_n|$ by $|f|$ we have on the other hand

$$Y = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_\cdot) = \int_0^\cdot Z_s dW_s^{\mathbb{Q}}.$$

Hence as before $(Y,Z)$ solves the BSDE (3). □

**Remark:**

We shall refrain from presenting results about the existence of measure solutions from first principles in the case of only exponentially integrable terminal variables in this paper. In subsection 1.3 we showed by giving examples that this terrain is rather rough, and requires some additional effort, for example to single out possible limit points of a tight sequence of (unique) measure solutions associated with bounded truncations of unbounded terminal variables, among the possibly non-unique measure solutions associated with the latter. The problem will be addressed in upcoming work.

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