

# On measure solutions of backward stochastic differential equations

Stefan Ankirchner

Institut für Mathematik  
Humboldt-Universität zu Berlin  
Unter den Linden 6  
10099 Berlin  
Germany

Peter Imkeller

Institut für Mathematik  
Humboldt-Universität zu Berlin  
Unter den Linden 6  
10099 Berlin  
Germany

Alexandre Popier

CMAP  
Ecole Polytechnique  
91128 Palaiseau Cédex  
France

February 5, 2008

## Abstract

We consider backward stochastic differential equations (BSDE) with nonlinear generators typically of quadratic growth in the control variable. A measure solution of such a BSDE will be understood as a probability measure under which the generator is seen as vanishing, so that the classical solution can be reconstructed by a combination of the operations of conditioning and using martingale representations. In case the terminal condition is bounded and the generator fulfills the usual continuity and boundedness conditions, we show that measure solutions with equivalent measures just reinterpret classical ones. In case of terminal conditions that have only exponentially bounded moments, we discuss a series of examples which show that in case of non-uniqueness classical solutions that fail to be measure solutions can coexist with different measure solutions.

**2000 AMS subject classifications:** primary 60H20, 60H07; secondary 60G44, 93E20, 60H30.

**Key words and phrases:** backward stochastic differential equation; stochastic control; hedging of contingent claim; martingale measure; martingale representation; Girsanov's theorem; weak solution; measure solution; Brownian motion.

# Introduction

The generally accepted natural framework for the most efficient formulation of pricing and hedging contingent claims on complete financial markets, for instance in the classical Merton-Scholes problem, is given by martingale theory, more precisely by the elegant notion of martingale measures. Martingale measures represent a view of the world in which price dynamics do not have inherent trends. From the perspective of this world, pricing a claim amounts to taking expectations, while hedging boils down to pure conditioning and using martingale representation.

At first glance, hedging a claim is, however, a problem calling upon stochastic control: it consists in choosing strategies to steer the portfolio into a terminal random endowment the portfolio holder has to ensure. Solving stochastic backward equations (BSDE) is a technique tailor-made for this purpose. This powerful tool has been introduced to stochastic control theory by Bismut [1]. Its mathematical treatment in terms of stochastic analysis was initiated by Pardoux and Peng [15], and its particular significance for the field of utility maximization in financial stochastics clarified in El Karoui, Peng and Quenez [7]. To fix ideas, we restrict our attention to a Wiener space probabilistic environment. In this framework, a BSDE with terminal variable  $\xi$  at time horizon  $T$  and generator  $f$  is solved by a pair of processes  $(Y, Z)$  on the interval  $[0, T]$  satisfying

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s, Y_s, Z_s) ds, \quad t \in [0, T]. \quad (1)$$

In the case of vanishing generator, the solution just requires an application of the martingale representation theorem in the Wiener filtration, and  $Z$  will be given as the stochastic integrand in the representation, to which we will refer as *control process* in the sequel. The classical approach of existence and uniqueness for BSDE involves a priori inequalities as a basic ingredient, by which unique solutions are constructed via fixed point arguments, just as in the case of forward stochastic differential equations.

In this paper we are looking for a notion in the context of BSDE that plays the role of the martingale measure in the context of hedging claims. Our main interest is directed to BSDE of the type (1) with generators that are non-Lipschitzian, and depend on the control variable  $z$  quadratically, typically  $f(s, y, z) = z^2 b(s, z)$ ,  $s \in [0, T]$ ,  $z \in \mathbb{R}$ , with a bounded function  $b$ . These generators were given a thorough treatment in Kobylanski [12], Briand, Hu [2], and Lepeltier, San Martin [13]. While [12] and [13] consider existence and uniqueness questions for bounded terminal variables  $\xi$ , [2] goes to the limit of possible terminal variables by considering  $\xi$  for which  $\exp(\gamma|\xi|)$  has finite expectation for some  $\gamma > 2\|b\|_\infty$ . All these papers employ different methods of approach following the classical pattern of arguments mentioned above. In contrast to this, we shall investigate an alternative notion of solution of BSDE, the generators of which fulfill similar conditions. In analogy with martingale measures in hedging which effectively eliminate drifts in price dynamics, we shall look for probability measures under which the generator of a given BSDE is seen as vanishing. Given such a measure  $\mathbb{Q}$  which we call *measure solution* of the BSDE and supposing that  $\mathbb{Q} \sim \mathbb{P}$ , the processes  $Y$  and  $Z$  are the results

of projection and representation respectively, i.e.  $Y = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_t) = Y_0 + \int_0^t Z_s d\widetilde{W}_s$ , where  $\widetilde{W}$  is a Wiener process under  $\mathbb{Q}$ . The first main finding of the paper roughly states that provided the terminal variable  $\xi$  is bounded, all classical solutions can be interpreted as measure solutions. More precisely, we show that if the generator satisfies the usual continuity and quadratic boundedness conditions, classical solutions  $(Y, Z)$  exist if and only if measure solutions with  $\mathbb{Q} \sim \mathbb{P}$  exist. So existence Theorems obtained in the papers quoted are recovered in a more elegant and concise way in terms of measure solutions. We do not touch uniqueness questions in general. Of course, determining a measure  $\mathbb{Q}$  under which the generator vanishes amounts to doing a Girsanov change of probability that eliminates it. We therefore have to look at the BSDE in the form

$$Y_t = \xi - \int_t^T Z_s \left[ dW_s - \frac{f(s, Y_s, Z_s)}{Z_s} ds \right], \quad t \in [0, T], \quad (2)$$

define  $g(s, y, z) = \frac{f(s, y, z)}{z}$ , and study the measure

$$\mathbb{Q} = \exp \left( M - \frac{1}{2} \langle M \rangle \right) \cdot \mathbb{P}$$

for the martingale  $M = \int_0^\cdot g(s, Y_s, Z_s) dW_s$ . One of the fundamental problems that took some effort to solve consists in showing that  $\mathbb{Q}$  is a probability measure. Here one has to dig essentially deeper than Novikov's or Kazamaki's criteria allow. We successfully employed a criterion which is based on the explosion properties of the quadratic variation  $\langle M \rangle$ , which we learnt from a conversation with M. Yor, and has been latent in the literature for a while, see Liptser, Shiryaev [14], or the more recent paper by Wong, Heyde [17]. This criterion allows a simple treatment of the problem of existence of measure solutions in the case of bounded terminal variable, and a still elegant and efficient one in the borderline case of exponentially integrable terminal variable considered by Briand, Hu [2]. If  $\xi$  is only exponentially bounded, things turn essentially more complex immediately. Specializing to a very simple generator, we find a wealth of different situations looking confusing at first sight. Just to quote three basic scenarios exhibited in a series of examples of different types: in the first type we obtain one solution which is a measure solution at the same time; in the second one we find two different solutions both of which are measure solutions; in the third one we encounter two solutions one of which is a measure solution, while the other one is not. We even combine these basic examples to develop a scenario in which there exists a continuum of measure solutions, and another one in which a continuum of non-measure solutions is given.

Here is an outline of the presentation of our material. Throughout we consider BSDE possessing generators with quadratic nonlinearity in  $z$ . In a first section we discuss the case of bounded terminal variable  $\xi$ , and show that if the generator satisfies continuity and quadratic boundedness conditions, classical solutions  $(Y, Z)$  exist if and only if measure solutions with  $\mathbb{Q} \sim \mathbb{P}$  exist. Things become essentially more complex in the second section, where we pass to exponentially integrable terminal variables.

Taking the simple generator  $f(s, z) = \alpha z^2, s \in [0, t], z \in \mathbb{R}$ , with some  $\alpha \in \mathbb{R}$ , that a wealth of different scenarios arises in which in case of non-uniqueness in particular solutions can be measure solutions, while different ones fail to have this property.

## 1 Measure solutions: Definition and first examples

In this section we first recall some basic definitions concerning BSDEs. We then introduce and exemplify the notion of a measure solution by looking at a special class of BSDEs.

Throughout let  $T$  be a non-negative real,  $(\Omega, \mathcal{F}, \mathbb{P})$  a probability space, and  $(W_t)_{0 \leq t \leq T}$  a one-dimensional Brownian motion, whose natural filtration, augmented by  $\mathcal{N}$ , is denoted by  $(\mathcal{F}_t)_{0 \leq t \leq T}$ , where

$$\mathcal{N} = \{A \subset \Omega, \exists G \in \mathcal{F}, A \subset G \text{ and } \mathbb{P}(G) = 0\}.$$

Let  $\xi$  be an  $\mathcal{F}_T$ -measurable random variable, and let  $f : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function such that for all  $z \in \mathbb{R}$  the mapping  $f(\cdot, \cdot, z)$  is predictable. A classical solution of the BSDE with *terminal condition*  $\xi$  and *generator*  $f$  is defined to be a pair of predictable processes  $(Y, Z)$  such that almost surely we have  $\int_0^T Z_s^2 ds < \infty$ ,  $\int_0^T |f(s, Y_s, Z_s)| ds < \infty$ , and for all  $t \in [0, T]$ ,

$$Y_t = \xi - \int_t^T Z_s dW_s + \int_t^T f(s, Z_s) ds. \quad (3)$$

The solution processes  $(Y, Z)$  are often shown to satisfy some integrability properties and to belong to the following function spaces. For  $p \geq 1$  let  $\mathcal{H}^p$  denote the set of all  $\mathbb{R}$ -valued predictable processes  $\zeta$  such that  $E \int_0^1 |\zeta_t|^p dt < \infty$ , and by  $\mathcal{S}^\infty$  we denote the set of all essentially bounded  $\mathbb{R}$ -valued predictable processes.

If  $\xi$  is square integrable and  $f$  satisfies a Lipschitz condition, then it is known that there exists a unique pair  $(Y, Z) \in \mathcal{H}^2 \otimes \mathcal{H}^2$  solving (3). Recall that the solution process  $Y_t$  has a nice representation as a conditional expectation with respect to a new probability measure if  $f$  is a linear function of the form

$$f(s, z) = b_s z, \quad (4)$$

where  $b$  is a predictable and bounded process. More precisely, if  $D_t = \exp(\int_0^t b_s dW_s - \frac{1}{2} \int_0^t b_s^2 ds)$ , and  $Q$  is the probability measure with density  $Q = D_T \cdot P$ , then

$$Y_t = \mathbb{E}^Q[\xi | \mathcal{F}_t]. \quad (5)$$

In the following we will discuss whether  $Y$  still can be written as a conditional expectation of  $\xi$  if  $f$  does not have a representation as in (4) with  $b$  bounded, but satisfies only a quadratic growth condition in  $z$ . We aim at finding sufficient conditions guaranteeing that the process  $Y_t$  of a classical solution of a quadratic BSDE has a representation as a conditional expectation of  $\xi$  with respect to a new probability measure. For this purpose we consider the class of generators  $f : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , satisfying for some constant  $c \in \mathbb{R}_+$ ,

**Assumption (H1):**

- (i)  $f(s, z) = f(\cdot, s, z)$  is adapted for any  $z \in \mathbb{R}$ ,
- (ii)  $g(s, z) = \frac{f(s, z)}{z}$ ,  $z \in \mathbb{R}$ , is continuous in  $z$ , for all  $s \in [0, T]$ ,
- (iii)  $|f(s, z)| \leq c(1 + z^2)$  for any  $s \in [0, T]$ ,  $z \in \mathbb{R}$ .

Let  $\xi$  be an  $\mathcal{F}_T$ -measurable random variable. We introduce for BSDEs with generators satisfying (H1) our concept of measure solutions.

**Definition 1.1** *A triplet  $(Y, Z, \mathbb{Q})$  is called measure solution of the BSDE (3), if  $\mathbb{Q}$  is a probability measure on  $(\Omega, \mathcal{F})$ ,  $(Y, Z)$  a pair of  $(\mathcal{F}_t)$ -predictable stochastic processes such that  $\int_0^T Z_s^2 ds < \infty$ ,  $\mathbb{Q}$ -a.s. and the following conditions are satisfied:*

$$\begin{aligned} \widetilde{W} &= W - \int_0^\cdot g(s, Z_s) ds \quad \text{is a } \mathbb{Q} - \text{Brownian motion,} \\ \xi &\in L^1(\Omega, \mathcal{F}, \mathbb{Q}), \\ Y_t &= \mathbb{E}^{\mathbb{Q}}(\xi | \mathcal{F}_t) = \xi - \int_t^T Z_s d\widetilde{W}_s, \quad t \in [0, T]. \end{aligned}$$

It is known from the literature that if the terminal condition  $\xi$  is bounded and the generator  $f$  satisfies Assumption (H1), then the BSDE (3) has a classical solution  $(Y, Z)$  (see for example Kobylanski [12]). We show that in this case there exists a probability measure  $\mathbb{Q}$ , equivalent to  $\mathbb{P}$ , such that  $(Y, Z, \mathbb{Q})$  is a measure solution.

**Theorem 1.1** *Assume that  $\xi$  is bounded, and that  $f$  satisfies Assumption (H1). Then for every classical solution  $(Y, Z)$  there exists a probability measure  $\mathbb{Q}$ , equivalent to  $\mathbb{P}$ , such that  $(Y, Z, \mathbb{Q})$  is a measure solution of (3).*

**Proof:** Let  $(Y, Z)$  be a classical solution of (3). The very definition entails

$$\int_0^T Z_s^2 ds < \infty, \quad \mathbb{P} - a.s.$$

Note that due to (ii) and (iii)

$$|g(s, z)|^2 \leq C(1 + z^2), \quad s \in [0, T], z \in \mathbb{R}^d,$$

for some  $C > 0$ , and hence we have

$$\int_0^T g^2(s, Z_s) ds < \infty, \quad \mathbb{P} - a.s. \tag{6}$$

We shall prove that under this condition also a measure solution exists. For this purpose, we define

$$M = \int_0^\cdot g(s, Z_s) dW_s. \tag{7}$$

It is clear that all we have to establish is that the measure

$$\mathbb{Q} = V_T \cdot \mathbb{P},$$

with

$$V = \exp\left(M - \frac{1}{2}\langle M \rangle\right)$$

leads to a probability measure equivalent to  $\mathbb{P}$ . This will be done by investigating possible explosions of the quadratic variation process  $\langle M \rangle$ . For  $n \in \mathbb{N}$ , let

$$\tau_n = T \wedge \inf\{t \geq 0 : \langle M \rangle_t \geq n\}. \quad (8)$$

Let

$$Q^n = V_T \cdot \mathbb{P}|_{\mathcal{F}_{\tau_n}}$$

be the measure change defined locally on  $\mathcal{F}_{\tau_n}$ . We know that  $Q^n$  is a probability measure equivalent to  $\mathbb{P}$ , and the Radon-Nikodym density of  $Q^n$  with respect to  $\mathbb{P}$  on  $\mathcal{F}_{\tau_n}$  is given by

$$V_{\tau_n} = \exp\left(M_{\tau_n} - \frac{1}{2}\langle M \rangle_{\tau_n}\right).$$

Moreover, the drifted process

$$\widetilde{W}^n = W - \int_0^{\tau_n \wedge \cdot} g(s, Z_s) ds$$

is a  $Q^n$ -Brownian motion, in particular locally up to time  $\tau_n$ . In order to show that  $\mathbb{Q}$  is a probability measure, it is sufficient to verify

$$Q^n(\tau_n < T) \rightarrow 0 \quad (n \rightarrow \infty). \quad (9)$$

Namely, (9) implies

$$\lim_n \mathbb{E}(V_T 1_{\{\tau_n = T\}}) = \lim_n [\mathbb{E}(V_{\tau_n}) - \mathbb{E}(V_{\tau_n} 1_{\{\tau_n < T\}})] = 1 - \lim_n Q^n(\tau_n < T) = 1. \quad (10)$$

On the other hand, dominated convergence yields  $E(V_T) = \lim_n \mathbb{E}(V_T 1_{\{\tau_n = T\}})$ , and hence that  $\mathbb{Q}$  is a probability measure. We remark that the criterion (9) can be found in [14], and appears also as Lemma 1.5 in [11].

Recall that by the very definition of the measure change,

$$Y_{\tau_n \wedge \cdot} = Y_0 + \int_0^{\tau_n \wedge \cdot} Z_s d\widetilde{W}_s^n$$

is a martingale under  $Q^n$ , up to time  $\tau_n$ , which is bounded by a constant  $c_1$ , due to the boundedness of  $\xi$  (see Theorem 2.3 in [12]).

Hence we obtain for any  $n \in \mathbb{N}$ , starting with an application of Chebyshev-Markov's inequality, and, due to (iii), another constant  $c_2$  independent of  $n$ , such that

$$\begin{aligned} \mathbb{Q}^n(\tau_n < T) &\leq \frac{1}{n} \mathbb{E}^{\mathbb{Q}^n} \left( \int_0^{\tau_n} g(s, Z_s)^2 ds \right) = \frac{1}{n} \mathbb{E}^n \left( \int_0^{\tau_n} g(s, Z_s)^2 ds \right) \\ &\leq \frac{1}{n} c_2 \left( 1 + \mathbb{E}^n \int_0^{\tau_n} Z_s^2 ds \right) \\ &= \frac{1}{n} c_2 \left( 1 + \mathbb{E}^n \left| \int_0^{\tau_n} Z_s d\tilde{W}_s^n \right|^2 \right) = \frac{1}{n} c_2 (1 + \mathbb{E}^n |Y_{\tau_n} - Y_0|^2) \\ &\leq \frac{1}{n} c_2 (1 + c_1). \end{aligned}$$

Thus we have shown (9), and hence that  $\mathbb{Q}$  is a probability measure. Under  $\mathbb{Q}$ , by definition,

$$W^{\mathbb{Q}} = W - \int_0^\cdot g(s, Z_s) ds$$

is a Brownian motion, and our BSDE may be written as

$$Y_t = \xi - \int_t^T Z_s dW_s^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}(\xi | \mathcal{F}_t)$$

for  $t \in [0, T]$ . This shows that  $(Y, Z, \mathbb{Q})$  is a measure solution.  $\square$

It is straightforward to see that every measure solution gives rise to a classical solution. Consequently, under the assumptions of Theorem 1.1, measure solutions exist if and only if classical solutions exist. More precisely, we obtain the following.

**Corollary 1.1** *Assume that  $\xi$  is bounded, and that  $f$  satisfies Assumption (H1). Then  $(Y, Z)$  is a classical solution if and only if there exists a probability measure  $\mathbb{Q}$ , equivalent to  $\mathbb{P}$ , such that  $(Y, Z, \mathbb{Q})$  is a measure solution of (3).*

We remark that the previous results can be extended to the case where  $W$  is a  $d$ -dimensional Brownian motion. Let  $f : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a generator for which there exists a constant  $c \in \mathbb{R}_+$  such that

$$|f(s, z)| \leq c(1 + |z|^2), \quad s \in [0, T], z \in \mathbb{R}^d, \quad (11)$$

and assume that  $g : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a function that is continuous in  $z$  and satisfies

$$\langle z, g(s, z) \rangle = f(s, z), \quad \text{for all } z \in \mathbb{R}^d \text{ and } s \in [0, T]. \quad (12)$$

If  $\xi$  is bounded and  $\mathcal{F}_T$ -measurable, then one can show with similar arguments as used in the preceding proof that, starting from a classical solution  $(Y, Z)$ , there exists a probability measure  $\mathbb{Q}$  such that  $W - \int_0^\cdot g(s, Z_s)$  is a  $\mathbb{Q}$ -Brownian motion, and  $Y_t = E^{\mathbb{Q}}(\xi | \mathcal{F}_t)$ .

Notice that the relation (12) may be satisfied by more than one continuous  $g$ , and consequently there may exist more than one measure solution in the multidimensional

case. For example, let  $d = 2$ ,  $f(s, z) = z_1 z_2$ , and observe that  $|f(z)| \leq \frac{1}{2}|z|^2$ . For any  $a \in (0, \infty)$  let  $g_a(z) = (az_1, \frac{1}{a}z_2)$ . Then, we have  $\langle z, g_a(s, z) \rangle = f(s, z)$ , and thus there exist more than one measure solution for a BSDE with generator  $f$  and a bounded terminal condition  $\xi$ .

In the following sections we shall discuss quadratic BSDEs with terminal conditions that are not bounded. As is known from literature, see for example Briand, Hu [2], [3], this case is by far more complex. For example, it is here that even if the generators are smooth, solutions stop to be unique. We shall exhibit examples below which complement the result shown in Briand, Hu [3], according to which uniqueness is granted in case the generator of the BSDE possesses additional convexity properties, and the terminal variable possesses exponential moments of all orders. This fact underlines that also variations in the generator affect questions of existence and uniqueness of solutions a lot. For this reason, and also to keep better oriented on a windy track with many bifurcations, in the next section we shall choose a simpler generator, and assume that our generator is given by

$$f(s, z) = \alpha z^2.$$

## 2 Measure and non-measure solutions of quadratic BSDEs with unbounded terminal condition

In this section we will study in more detail the BSDEs with generator of the form

$$f(z) = \alpha z^2.$$

We shall further assume without loss of generality that  $\alpha > 0$ . This can always be obtained in our BSDE by changing the signs of  $\xi$ , and the solution pair  $(Y, Z)$ .

Nonetheless, it turns out that positive and negative terminal variables need a separate treatment. We will first show (see Subsection 2.1) the existence of measure solutions for terminal conditions  $\xi$  bounded from below. Note that by a linear shift of  $Y$  we may assume that  $\xi \geq 0$ . We shall further work under exponential integrability assumptions in the spirit of Briand, Hu [2]. According to this paper, exponential integrability of the terminal variable of the form

$$\mathbb{E}(\exp(\gamma|\xi|)) < \infty \tag{13}$$

for some  $\gamma > 2\alpha$  is sufficient for the existence of a solution. Let us first exhibit an example to show that one cannot go essentially beyond this condition without losing solvability.

**Example:**

Let  $T = 1$ , and let  $\alpha = \frac{1}{2}$ . Let us first consider

$$\xi = \frac{W_1^2}{2}.$$

It is immediately clear from the fact that  $W_1$  possesses the standard normal density, that  $\mathbb{E} \exp(2\alpha|\xi|) = \infty$ , hence of course also for  $\gamma > 2\alpha$  (13) is not satisfied. To find a



solution  $(Y, Z)$  of (3) on any interval  $[t, 1]$  with  $t > 0$  define

$$Z_s = \frac{W_s}{s}, \quad s > 0,$$

and set for completeness  $Z_0 = 0$ . Let  $t > 0$  and use the product formula for Itô integrals to deduce

$$\begin{aligned} \int_t^1 Z_s dW_s &= \frac{1}{2} \frac{W_s^2}{s} \Big|_t^1 + \frac{1}{2} \int_t^1 \frac{W_s^2}{s^2} ds \\ &= \xi - \frac{1}{2} \frac{W_t^2}{t} + \frac{1}{2} \int_t^1 Z_s^2 ds. \end{aligned} \quad (14)$$

This means that, if we set for convenience again  $Y_0 = 0$ , the pair of processes  $(Y_s, Z_s) = (\frac{1}{2} \frac{W_s^2}{s}, \frac{W_s}{s})$ ,  $s \in [0, 1]$ , solves the BSDE (3) on  $[t, 1]$  for any  $t > 0$ . Of course, the definition of  $Y_0$  is totally inconsistent with the BSDE. Worse than that,  $Z$  is not square integrable on  $[0, 1]$ , as is well known from the path behavior of Brownian motion. Hence  $(Y, Z)$  is not a solution of (3). To put it more strictly, there is no classical solution of (3) on  $[0, 1]$ , since, due to local Lipschitz conditions, any such solution would have to coincide with  $(Y, Z)$  on any interval  $[t, 1]$  with  $t > 0$ .

According to Jeulin, Yor [8], transformations of this type are related to a phenomenon they call *appauvrissement de filtrations*. In fact, if  $\frac{1}{2}$  is replaced with a parameter  $\lambda$ , they show that the natural filtration of the transformed process gets poorer than the one of the Wiener process, iff  $\lambda > \frac{1}{2}$ . Hence in the case we are interested in the Wiener filtration is preserved.

Let us now reduce the factor of  $W_1^2$  in the definition of  $\xi$  a bit, to show that solutions exist in this setting. For  $k \in \mathbb{N}$ , let

$$\xi_k = \frac{W_1^2}{2(1 + 1/k)},$$

and consider the BSDE (3) with the generator  $f$  chosen above, and terminal condition  $\xi_k$ . In this setting, we clearly have

$$\mathbb{E} \exp(\gamma \xi_k) < \infty \quad \text{for} \quad 2\alpha \leq \gamma < 2\alpha(1 + 1/k).$$

This shows that the condition of Briand, Hu [2] is satisfied. It is not hard to construct the solutions of the corresponding BSDEs explicitly, in the same way as above. In fact, for  $k \in \mathbb{N}$  we may define  $f_k(t) = \frac{1}{k} + t$ ,  $t \in [0, 1]$ , and set

$$Z_t^k = \frac{W_t}{f_k(t)}, \quad t \in [0, 1].$$

We may then repeat the product formula for Itô integrals argument used above to obtain for  $t \geq 0$

$$\begin{aligned} \int_t^1 Z_s^k dW_s &= \frac{1}{2} \frac{W_s^2}{f_k(s)} \Big|_t^1 + \frac{1}{2} \int_t^1 \frac{W_s^2 f_k'(s)}{f_k(s)^2} ds \\ &= \frac{1}{2} \frac{W_1^2}{f_k(1)} - \frac{1}{2} \frac{W_t^2}{f_k(t)} + \frac{1}{2} \int_t^1 (Z_s^k)^2 ds. \end{aligned} \quad (15)$$

Hence we set

$$Y_t^k = \frac{1}{2} \frac{W_t^2}{f_k(t)}, \quad t \in [0, 1],$$

to identify the pair of processes  $(Y^k, Z^k)$  as a solution of the BSDE

$$Y_t^k = \xi_k - \int_t^1 Z_s^k dW_s + \frac{1}{2} \int_t^1 (Z_s^k)^2 ds, \quad t \in [0, 1]. \quad (16)$$

We do not know at this moment whether (3) possesses more solutions.  $\square$

## 2.1 Exponentially integrable lower bounded terminal variable

Under the exponential integrability assumption  $\mathbb{E}(\exp(2\alpha\xi)) < \infty$ , we will now derive measure solutions from given classical solutions. Leaving the difficult question of uniqueness apart for a moment, we remark that with our simple generator, we obtain an explicit solution given by the formula

$$Y_t = \frac{1}{2\alpha} \ln M_t - \frac{1}{2\alpha} \ln M_0, \quad Z_t = \frac{1}{2\alpha} \frac{H_t}{M_t}, \quad (17)$$

where

$$M_t = \mathbb{E}(\exp(2\alpha\xi) | \mathcal{F}_t) = M_0 + \int_0^t H_s dW_s, \quad t \in [0, T].$$

In the sequel, we shall work with this explicit solution. In the following lemma, we prove integrability properties for the square norm of  $Z$  which will be crucial for stating the martingale property of  $M$  and other related processes later.

**Lemma 2.1** *For any  $p \geq 1$  we have*

$$\mathbb{E} \left( \left[ \int_0^T Z_s^2 ds \right]^p \right) < \infty.$$

*In particular,  $\int_0^\cdot Z_s dW_s$  is a uniformly integrable martingale.*

**Proof:** Let  $t \in [0, T]$ . By Itô's formula, applied to  $N$

$$\frac{1}{2\alpha} [\ln M_t - \ln M_0] = \frac{1}{2\alpha} \left[ \int_0^t \frac{H_s}{M_s} dW_s - \frac{1}{2} \int_0^t \left( \frac{H_s}{M_s} \right)^2 ds \right] = \int_0^t Z_s dW_s - \alpha \int_0^t Z_s^2 ds.$$

Hence

$$\alpha \int_0^t Z_s^2 ds = -\frac{1}{2\alpha} [\ln M_t - \ln M_0] + \int_0^t Z_s dW_s. \quad (18)$$

By concavity of the  $\ln$  and Jensen's inequality

$$\ln M_t = \ln \mathbb{E}(\exp(2\alpha\xi) | \mathcal{F}_t) \geq \mathbb{E}(2\alpha\xi | \mathcal{F}_t).$$

Using this in (18), we obtain

$$\alpha \int_0^t Z_s^2 ds \leq -\mathbb{E}(\xi | \mathcal{F}_t) + \frac{1}{2\alpha} \ln M_0 + \int_0^t Z_s dW_s.$$

Taking  $p$ -norms in this inequality and using the inequality of Burkholder, Davis and Gundy for the stochastic integral, we obtain with universal constants  $c_1, c_2, c_3$

$$\begin{aligned} \mathbb{E} \left( \left[ \int_0^t Z_s^2 ds \right]^p \right) &\leq c_1 \left[ \mathbb{E}(|\mathbb{E}(\xi | \mathcal{F}_t)|^p) + |\ln M_0|^p + \mathbb{E} \left( \left[ \int_0^t Z_s^2 ds \right]^{\frac{p}{2}} \right) \right] \\ &\leq c_2 \left[ \mathbb{E}(|\xi|^p) + |\ln M_0|^p + \mathbb{E} \left( \left[ \int_0^t Z_s^2 ds \right]^{\frac{p}{2}} \right) \right]. \end{aligned}$$

By a standard argument this entails

$$\mathbb{E} \left( \left[ \int_0^t Z_s^2 ds \right]^p \right) \leq c_3 [\mathbb{E}(|\xi|^p) + |\ln M_0|^p + 1],$$

and finishes the proof.  $\square$

We shall now prove that  $(Y, Z)$  gives rise to a measure solution.

**Theorem 2.1** *Assume that  $(Y, Z)$  are defined as in (17). Then there exists a probability measure  $\mathbb{Q}$ , equivalent to  $\mathbb{P}$ , such that  $(Y, Z, \mathbb{Q})$  is a measure solution of (3).*

**Proof:** Let

$$S = \int_0^\cdot Z_s dW_s.$$

Due to Lemma 2.1, we know that  $S$  is a uniformly integrable martingale. We may write

$$\begin{aligned} \alpha S - \frac{1}{2} \alpha^2 \langle S \rangle &= \alpha \left[ \int_0^\cdot Z_s dW_s - \alpha \int_0^\cdot Z_s^2 ds \right] + \int_0^\cdot (\alpha^2 Z_s^2 - \frac{1}{2} \alpha^2 Z_s^2) ds \quad (19) \\ &= \alpha(Y - Y_0) + \frac{1}{2} \alpha^2 \int_0^\cdot Z_s^2 ds. \end{aligned}$$

Now define stopping times  $\tau_n = T \wedge \inf\{t \geq 0 : \langle S \rangle_t \geq n\}$ . For any  $n \in \mathbb{N}$  we have

$$\mathbb{E} \exp \left( \alpha S_{\tau_n} - \frac{1}{2} \alpha^2 \langle S \rangle_{\tau_n} \right) = 1,$$

and consequently Fatou's lemma implies

$$\mathbb{E} \exp \left( \alpha[\xi - Y_0] + \frac{1}{2} \alpha^2 \int_0^T Z_s^2 ds \right) \leq \liminf_{n \rightarrow \infty} \mathbb{E} \exp \left( \alpha S_{\tau_n} - \frac{1}{2} \alpha^2 \langle S \rangle_{\tau_n} \right) = 1. \quad (20)$$

Using this and the positivity of the terminal variable  $\xi$ , we can now obtain the exponential integrability property

$$\mathbb{E} \exp \left[ \frac{1}{2} \alpha (\xi - Y_0) + \frac{1}{2} \alpha^2 \int_0^T Z_s^2 ds \right] < \infty. \quad (21)$$

We shall now use (19) together with (20) to prove the exponential integrability of  $\frac{1}{2} \alpha S_T$ . In fact, we have

$$\frac{1}{2} \alpha S_T = \frac{1}{2} \alpha (\xi - Y_0) + \frac{1}{2} \alpha^2 \int_0^T Z_s^2 ds.$$

Hence we obtain

$$\mathbb{E} \exp \left( \frac{1}{2} \alpha S_T \right) < \infty, \quad (22)$$

and together with the uniform integrability of the martingale  $S$ , proved in Lemma 2.1, this enables us to apply the criterion of Kazamaki (see Revuz, Yor [16], p. 332). Hence we have proved the existence of a measure solution to our BSDE (3).  $\square$

As a by-product of our main result, we obtain the exponential integrability of the quadratic variation of  $S$ .

**Corollary 2.1** *Under the conditions of Theorem 2.1 we have*

$$\mathbb{E} \exp \left( \frac{1}{2} \alpha^2 \int_0^T Z_s^2 ds \right) < \infty.$$

**Proof:** This follows immediately from (21) and the lower boundedness of  $\xi$ .  $\square$

## 2.2 A quadratic BSDE with two solutions

Let us now come back to the question of uniqueness of solutions, and their measure solution property. Briand, Hu [2] prove the existence of solutions  $(Y, Z)$  in the usual sense, given that (13) is satisfied. In a setting with more general generators the nonlinear  $z$ -part being bounded by  $\alpha z^2$ , they provide pathwise upper and lower bounds for  $Y$ , given by the known explicit solution for this generator  $(\frac{1}{2\alpha} \log \mathbb{E}(\exp(2\alpha\xi)|\mathcal{F}_t)_{t \in [0, T]}$  used above, and its negative counterpart  $(-\frac{1}{2\alpha} \log \mathbb{E}(\exp(-2\alpha\xi)|\mathcal{F}_t)_{t \in [0, T]}$ . In a more recent paper, Briand, Hu [3] also provide a uniqueness result for the same setting, which is satisfied under the stronger integrability hypothesis

$$\mathbb{E}(\exp(\gamma|\xi|)) < \infty \quad (23)$$

for all  $\gamma > 0$  and a convexity assumption concerning the generator. Let us start our discussion of uniqueness and the measure solution property by giving some examples.

For  $b > 0$ , let  $\tau_b = \inf\{t \geq 0 : W_t \leq bt - 1\}$ . We first consider a BSDE with random time horizon  $\tau_b$ . Let the generator be further specified by  $\alpha = \frac{1}{2}$ . Let  $\xi = 2a(b - a)\tau_b - 2a$ , where  $a > 0$ . It will become clear along the way why this choice of

terminal variable is made. In the first place, it is motivated by the striking simplicity of the solutions we shall construct. We shall in fact give two explicit solutions of the BSDE

$$Y_{t \wedge \tau_b} = \xi - \int_t^{\tau_b} Z_s dW_s + \int_t^{\tau_b} \frac{1}{2} Z_s^2 ds. \quad (24)$$

Appropriate choices of  $a$  and  $b$  allow for terminal variables that are bounded below as well as bounded above. The fact that the time horizon is random is not crucial. Indeed, by using a time change, any solution of Equation (24) can be transformed into a solution of a BSDE with the same generator and with time horizon 1. To this end consider the time change  $\rho(t) = \frac{t}{1+t}$ ,  $t \in [0, \infty]$ , and observe that the inverse of  $\rho$  is given by  $\rho^{-1}(t) = \frac{t}{1-t}$ ,  $t \in [0, 1]$ . Let  $h(t) = \frac{1}{1-t}$  for all  $t \in [0, 1]$ . Then the process defined by

$$\tilde{W}_t = \int_0^t h^{-1}(s) d(W_{\rho^{-1}(s)}), \quad t \in [0, 1], \quad (25)$$

is a Brownian motion on  $[0, 1]$ . Note that  $W_t = \int_0^{\rho(t)} h(s) d\tilde{W}_s$  (and this is how we have to define  $W$ , if  $\tilde{W}$  is given). Moreover, the stopping time

$$\hat{\tau}_b = \inf \left\{ t \geq 0 : \int_0^t h(s) d\tilde{W}_s \leq \frac{t}{1-t} - 1 \right\}$$

is equal to  $\rho(\tau_b)$ . We can now define a time changed analogue of Equation (24) with time horizon 1.

**Lemma 2.2** *Let  $(Y_t, Z_t)$  be a solution of the BSDE (24), and let  $\hat{\xi} = 2a(b-a)\frac{\hat{\tau}_b}{1-\hat{\tau}_b} - 2a$ . Then  $(y_t, z_t) = (Y_{\rho^{-1}(t)}, h(t)Z_{\rho^{-1}(t)})$  is a solution of the BSDE*

$$y_t = \hat{\xi} - \int_t^1 z_s d\tilde{W}_s + \int_t^1 \frac{1}{2} z_s^2 ds. \quad (26)$$

**Proof:** Since stochastic integration and continuous time changes can be interchanged (see Proposition 1.5, Chapter V in [16]), we have

$$\begin{aligned} y_t &= Y_{\rho^{-1}(t)} = \int_0^{\rho^{-1}(t)} Z_s dW_s - \frac{1}{2} \int_0^{\rho^{-1}(t)} Z_s^2 ds \\ &= \int_0^t Z_{\rho^{-1}(s)} dW_{\rho^{-1}(s)} - \frac{1}{2} \int_0^t Z_{\rho^{-1}(s)}^2 d\rho^{-1}(s) \\ &= \int_0^t Z_{\rho^{-1}(s)} h(s) d\tilde{W}_s - \frac{1}{2} \int_0^t Z_{\rho^{-1}(s)}^2 h^2(s) ds, \end{aligned}$$

and hence the result.  $\square$

Let us first assess exponential integrability properties of  $\xi$ . For this, let  $\gamma > 0$  be arbitrary. Then we have

$$\mathbb{E}e^{\gamma|\xi|} = \mathbb{E}e^{\gamma|2a(b-a)\tau_b - 2a|} \leq e^{2a\gamma} \mathbb{E}e^{\gamma 2a|b-a|\tau_b}.$$

Define the auxiliary stopping time

$$\sigma_b = \inf\{t \geq 0 : W_t \leq t - b\}.$$

It is well known and proved by the scaling properties of Brownian motion that the laws of  $\tau_b$  and  $\frac{\sigma_b}{b^2}$  are identical (see Revuz, Yor [16]). Moreover, the Laplace transform of  $\sigma_b$  is equally well known. According to Revuz, Yor [16] we therefore have for  $\lambda > 0$

$$E(\exp(-\lambda\tau_b)) = E(\exp(-\frac{\lambda}{b^2}\sigma_b)) = \exp(-b[\sqrt{1 + \frac{2\lambda}{b^2}} - 1]). \quad (27)$$

Moreover, it is seen by analytic continuation arguments that this formula is even valid for  $\lambda \geq -\frac{b^2}{2}$ . Now choose  $\lambda = -2a|b - a|\gamma$ . Then the inequality

$$-2a|b - a|\gamma \geq -\frac{1}{2}b^2$$

amounts to

$$\gamma \leq \frac{b^2}{4a|b - a|}. \quad (28)$$

This in turn means that we have exponential integrability of orders bounded by  $\frac{b^2}{4a|b-a|}$ , in particular we may reach arbitrarily high orders by choosing  $a$  and  $b$  sufficiently close. But no combination of  $a$  and  $b$  allows exponential integrability of all orders. In the light of Briand, Hu [3] this means that the entire field of pairs of positive  $a$  and  $b$  promises multiple solutions, and this is precisely what we will exhibit.

### The first solution

It is clear from the definition that the pair  $(Y_t, Z_t)$ , defined by  $Y_t = 2aW_{t \wedge \tau_b} - 2a^2(\tau_b \wedge t)$  and  $Z = 2a1_{[0, \tau_b]}$ , is a solution of (24). To answer the question whether this defines a measure solution, we have to investigate

$$\mathbb{E} \exp \left[ \int_0^{\tau_b} \frac{1}{2} Z_s dW_s - \frac{1}{8} \int_0^{\tau_b} Z_s^2 ds \right] = \mathbb{E} \exp \left[ aW_{\tau_b} - \frac{a^2}{2} \tau_b \right] = \mathbb{E}(\exp(a(b - \frac{a}{2})\tau_b - a)).$$

Due to (27) we have

$$\mathbb{E}(\exp(a(b - \frac{a}{2})\tau_b - a)) = \exp(-b[\sqrt{1 - \frac{2}{b^2}a(b - \frac{a}{2})} - 1] - a) = \exp(-b[|1 - \frac{a}{b}| - 1] - a),$$

and the latter equals 1 in case  $b \geq a$  and  $\exp(2(b - a)) < 1$  in case  $a > b$ . This simply means that our first solution is a measure solution of (26) provided  $b \geq a$ , and it fails to be one in case  $a > b$ . We will show that this first solution does not necessarily correspond to the particular solution discussed in the beginning of the section.

## The second solution

We show now that the BSDE (24) with the same terminal variable as above possesses a second solution. By Lemma 2.2 there exists a second solution of (26) as well. Once this is shown, for any possible degree  $\gamma$  of exponential integrability we will have exhibited a negative random variable satisfying  $\mathbb{E}(\exp(\gamma|\xi|)) < \infty$  for which (24) possesses at least two solutions. This in turn will underline that Briand, Hu's [3] uniqueness result, valid under (23) cannot be improved by much. Note that the solution we will exhibit is again of the explicit form (17) encountered earlier. Let  $M_t = \mathbb{E}[e^\xi | \mathcal{F}_t]$  for all  $t \geq 0$ . Due to the martingale representation property there exists a process  $H$  such that  $M_t = M_0 + \int_0^t H_s dW_s$ . We know that  $(\ln M_{\tau_b \wedge t}, \frac{H_{\tau_b \wedge t}}{M_{\tau_b \wedge t}})$  is a solution of (24). We will show below that

$$\ln M_{\tau_b \wedge t} = 2b - 4a + 2(b-a)W_{\tau_b \wedge t} - 2(b-a)^2(\tau_b \wedge t), \quad \text{if } 2a > b, \quad (29)$$

$$\ln M_{\tau_b} = 2aW_{\tau_b \wedge t} - 2a^2\tau_b \wedge t, \quad \text{if } 2a \leq b. \quad (30)$$

This implies that the solution  $(\ln M_{\tau_b \wedge t}, \frac{H_{\tau_b \wedge t}}{M_{\tau_b \wedge t}})$  is different from the solution  $(2aW_{\tau_b \wedge t} - 2a^2(\tau_b \wedge t), 2a)$  obtained above in case  $2a > b$ . Hence by Lemma 2.2 we obtain a second solution of (26) in this case.

First note that

$$\begin{aligned} M_t &= e^{-2a} \mathbb{E}[e^{2a(b-a)\tau_b} | \mathcal{F}_t] \\ &= e^{-2a} (e^{2a(b-a)\tau_b} 1_{\{\tau_b \leq t\}} + e^{2a(b-a)t} \mathbb{E}[e^{2a(b-a)[\tau_b - t]} | \mathcal{F}_t] 1_{\{\tau_b > t\}}) \end{aligned} \quad (31)$$

Let  $\sigma_b(x, t) = \inf\{s \geq 0 : W_{s+t} - W_t \leq b(s+t) - 1 - x\}$  and observe that on the set  $\{\tau_b > t\}$  we have  $\tau_b - t = \sigma_b(W_t, t)$ . Therefore, by using again our knowledge on the Laplace transforms of  $\sigma(x, t)$  (see [16]), we get

$$\begin{aligned} \mathbb{E}[e^{2a(b-a)[\tau_b - t]} | \mathcal{F}_t] 1_{\{\tau_b > t\}} &= \mathbb{E}[e^{2a(b-a)\sigma_b(x, t)}] \Big|_{x=W_t} 1_{\{\tau_b > t\}} \\ &= e^{-b(1+W_t-bt)[\sqrt{1-\frac{4a(1-a)}{b^2}}-1]} 1_{\{\tau_b > t\}} \\ &= e^{-b(1+W_t-bt)[|1-\frac{2a}{b}|-1]} 1_{\{\tau_b > t\}}. \end{aligned}$$

Consequently,

$$\begin{aligned} M_t &= e^{-2a} \left( e^{2a(b-a)\tau_b} 1_{\{\tau_b \leq t\}} + e^{2a(b-a)t} e^{-b(W_t+1-bt)[|1-\frac{2a}{b}|-1]} 1_{\{\tau_b > t\}} \right) \\ &= e^{2a((1-a)(\tau_b \wedge t) - 1)} 1_{\{\tau_b \leq t\}} + e^{-2(b-a)(W_t+1-bt)} 1_{\{\tau_b > t\}}. \end{aligned}$$

Hence in case  $2a > b$

$$\begin{aligned} \ln M_{\tau_b \wedge t} &= 2a((b-a)(\tau_b \wedge t) - 1) - 2(a-b)(W_{\tau_b \wedge t} + 1 - (\tau_b \wedge t)) \\ &= -4a + 2b + [-2b + 4a - 2a^2](\tau_b \wedge t) - 2(a-b)W_{\tau_b \wedge t} \\ &= 2b - 4a + 2(b-a)W_{\tau_b \wedge t} - 2[b-a]^2(\tau_b \wedge t). \end{aligned}$$

This confirms the first equation (29). Let finally  $2a \leq b$ . Then we have

$$\begin{aligned}
M_t &= e^{-2a} \left( e^{2a(b-a)\tau_b} 1_{\{\tau_b \leq t\}} + e^{2a(b-a)t} e^{2a(W_t + 1 - bt)} 1_{\{\tau_b > t\}} \right) \\
&= e^{2a((b-a)(\tau_b \wedge t) + 2a(W_{\tau_b \wedge t} + 1 - b\tau_b \wedge t))} \\
&= e^{2aW_{\tau_b \wedge t} - 2a^2\tau_b \wedge t}.
\end{aligned}$$

Hence in this case

$$\ln M_{\tau_b \wedge t} = 2aW_{\tau_b \wedge t} - 2a^2\tau_b \wedge t.$$

Note that in case  $2a \leq b$  we recover the solution already obtained as the first solution.

Let us finally show that this second solution is in fact a measure solution for any possible combination of parameters.

**Lemma 2.3** ( $\ln M_{\tau_b \wedge t}, \frac{H_{\tau_b \wedge t}}{M_{\tau_b \wedge t}}$ ) can be extended to a measure solution of (24), hence provides a measure solution of (26).

**Proof:** For the first solution in case  $a \leq b$ , which is identical to the one considered in case  $2a \leq b$ , we have already established the measure solution property. Let us therefore consider the case  $2a > b$ . Note that for all  $t$ ,  $M_{t \wedge \tau_b} = e^{2b-4a} + \int_0^{t \wedge \tau_b} H_s dW_s$ . Itô's formula applied to  $e^{2(b-a)W_{\tau_b \wedge t} - 2[b-a]^2(\tau_b \wedge t)}$  yields

$$H_{s \wedge \tau_b} = 2(b-a)e^{2(b-a)W_{\tau_b \wedge t} - 2[b-a]^2(\tau_b \wedge t)}.$$

As a consequence, we have

$$Z_{s \wedge \tau_b} = \frac{H_{s \wedge \tau_b}}{M_{s \wedge \tau_b}} = 2(b-a)1_{[0, \tau_b]}(s),$$

and therefore

$$\begin{aligned}
\mathcal{E}\left(\frac{1}{2} \int Z dW\right)_{\tau_b} &= e^{(b-a)W_{\tau_b} - \frac{1}{2}(b-a)^2\tau_b} \\
&= e^{(b-a)(b\tau_b - 1) - \frac{1}{2}(b-a)^2\tau_b} \\
&= e^{(a-b)} e^{\frac{1}{2}(b-a)(b+a)\tau_b}.
\end{aligned}$$

Again the explicit representation of the Laplace transform in (27) gives

$$\mathbb{E}\mathcal{E}\left(\frac{1}{2} \int Z dW\right)_{\tau_b} = e^{(a-b)} \mathbb{E}e^{-\frac{1}{2}(b-a)(b+a)\tau_b} = e^{(a-b)} e^{-b(\sqrt{1 - (1 - \frac{a^2}{b^2})} - 1)} = 1.$$

This implies the claimed result that our second solution ( $\ln M_{\tau_b \wedge t}, \frac{H_{\tau_b \wedge t}}{M_{\tau_b \wedge t}}$ ) is a measure solution of (24).  $\square$



**Remarks:**

1. We can summarize the findings of our investigations of the examples by stating that there are three basic scenarios: a) for  $b \geq 2a$  we obtained one solution which is a measure solution at the same time; b) in the range  $2a > b \geq a$  we found two different solutions both of which are measure solutions; c) if  $a > b$  we finally encountered two solutions one of which is a measure solution, while the other one is not.

2. Note that our examples exhibiting solutions of (24) that are not measure solutions are all for negative terminal variables  $\xi$ . Positive terminal variables arise in scenarios a) or b), and therefore only produce multiple measure solutions.

**A continuum of solutions**

Let us now combine the first and second solutions to obtain a continuum of solutions of our BSDE (24). To do this, we have to consider a still somewhat more general class of stopping times. For  $c \in \mathbb{R}$ , let

$$\rho_c = \inf\{t \geq 0 : W_t \leq t - c\}.$$

We investigate the terminal variables

$$\xi = 2a(a - 1)\rho_c + d$$

with further constants  $a \neq 0, d \in \mathbb{R}$ . Note first that the integrability properties of  $\xi$  are the same as those obtained before for  $b = 1$ . According to the preceding paragraphs, our BSDE (24) possesses the following two solutions

$$Z^1 = 2a1_{[0, \rho_c]}, \quad Y^1 = d_1 + 2aW_{\rho_c \wedge \cdot} - 2a^2\rho_c \wedge \cdot, \quad (32)$$

$$Z^2 = 2(1 - a)1_{[0, \rho_c]}, \quad Y^2 = d_2 + 2(1 - a)W_{\rho_c \wedge \cdot} - 2(1 - a)^2\rho_c \wedge \cdot, \quad (33)$$

with  $d_1 = -2ac$  resp.  $d_2 = -2(a - 1)c$ . Let us now take  $c = 1$  and combine the two solutions to obtain a continuum of new ones. To do this, we start with the equation

$$\rho_1 = \rho_c + \rho_{1-c} \circ \theta_{\rho_c},$$

where  $\theta_{\rho_c}$  is the shift on Wiener space defined by

$$\theta_t(\omega) = W_{t+\cdot}(\omega) - W_t(\omega),$$

and  $c \in ]0, 1[$ . It describes the first time to reach the line with slope 1 that cuts the vertical at level  $-1$ , by decomposition with the intermediate time to reach the line with slope 1 cutting the vertical at  $-c$ . We mix the two solutions on the two resulting stochastic intervals, more precisely we put for  $c \in ]0, 1[, l \in \mathbb{R}$

$$\begin{aligned} Z^c &= 2a1_{[0, \rho_c]} + 2(1 - a)1_{[\rho_c, \rho_1]}, \\ Y^c &= l + 2aW_{\rho_c \wedge \cdot} - 2a^2\rho_c \wedge \cdot + 2(1 - a)[W_{\rho_1 \wedge \cdot} - W_{\rho_c \wedge \cdot}] - 2(1 - a)^2[\rho_1 \wedge \cdot - \rho_c \wedge \cdot]. \end{aligned} \quad (34)$$

Since we have

$$\begin{aligned} Y_{\rho_1}^c &= l + 2aW_{\rho_c} - 2a^2\rho_c + 2(1-a)[W_{\rho_1} - W_{\rho_c}] - 2(1-a)^2[\rho_1 - \rho_c] \\ &= l + 2a(1-a)\rho_1 - 2ac - 2(1-a)(1-c), \end{aligned}$$

we have to set

$$l - 2ac - 2(1-a)(1-c) = d$$

in order to obtain a solution of (24) with  $c = 1$ . According to the treatment of the first and second solution, the constructed mixture is a measure solution if and only if both components of the mixture are. This is the case for  $2a(1-a) > 0$ , whereas for  $2a(1-a) < 0$  we obtain a continuum of solutions that are no measure solutions.

**Remarks:**

1. This time, we may summarize our results by saying that there are two scenarios: a) for  $2a(1-a) > 0$  there is a continuum of measure solutions of (24), while for  $2a(1-a) < 0$  a continuum of non measure solutions is obtained.

2. Note that the initial conditions of our solutions continuum vary in a convex way between  $-2a$  and  $-2(1-a)$  as  $c$  varies in  $]0, 1[$ , spanning the whole interval.

We shall now point out that the measure solution property of the second solution in case  $a > b$  exhibited in the example above is not a coincidence. In fact, it will turn out that also for negative exponentially integrable  $\xi$ , solutions given by (17) provide measure solutions. To prove this, we will reverse the sign of  $\xi$  by looking at our BSDE from the perspective of an equivalent measure.

### 2.3 Exponentially integrable upper bounded terminal variable

Sticking with the positivity of  $\alpha$  in the generator

$$f(s, z) = \alpha z^2, \quad s \in [0, T], z \in \mathbb{R}$$

we shall now consider terminal variables  $\xi$  that fulfill the exponential integrability condition (13), but are bounded above by a constant. Again, by a constant shift of the solution component  $Y$ , we can assume that the upper bound is 0, i.e.  $\xi \leq 0$ . So fix a non-positive terminal variable  $\xi$  satisfying (13) for some  $\gamma > 2\alpha$ , and denote by  $(Y, Z)$  the pair of processes given by the explicit representation of (17) solving our BSDE according to Briand, Hu [2]. With respect to the following probability measure,  $\xi$  will effectively change its sign, so that we can hook up to the previous discussion. Recall  $S = \int_0^\cdot Z_s dW_s$ .

**Lemma 2.4** *Let  $V = \exp(2\alpha S - 2\alpha^2 \langle S \rangle)$ . Then  $V$  is a martingale of class (D), and consequently*

$$R = V_T \cdot \mathbb{P}$$

*is a probability measure equivalent to  $\mathbb{P}$ . Moreover,*

$$W^R = W - 2\alpha \int_0^\cdot Z_s ds$$

*is a Brownian motion under  $R$ .*

**Proof:** By (3), we may write

$$2\alpha[Y - Y_0] = 2\alpha S - 2\alpha^2\langle S \rangle,$$

hence also

$$2\alpha[\xi - Y_0] = 2\alpha S_T - 2\alpha^2\langle S \rangle_T.$$

According to Briand, Hu [2], Theorem 2, there exists  $\delta > 2\alpha$  such that

$$\mathbb{E}(\sup_{t \in [0, T]} \exp(\delta|Y_t|)) < \infty, \quad (35)$$

and therefore  $\beta > 1$  with the property

$$\mathbb{E}(\sup_{t \in [0, T]} V_t^\beta) < \infty. \quad (36)$$

This clearly implies that  $V$  is a martingale of class (D), and consequently  $R$  is a probability measure. Finally, Girsanov's theorem implies that  $W^R$  is a Brownian motion under  $R$ .  $\square$

Now consider our BSDE under the perspective of the measure  $R$  with respect to the Brownian motion  $W^R$ . We may write

$$Y = \xi - \int_0^T Z_s dW_s + \alpha \int_0^T Z_s^2 ds = \xi - \int_0^T Z_s dW_s^R - \alpha \int_0^T Z_s^2 ds. \quad (37)$$

But this just means that by switching signs in  $(Y, Z)$ , we may return, under the new measure  $R$ , to our old BSDE with  $\xi$  replaced with  $-\xi$ . So our measure change puts us back into the framework of the previous subsection, and we may resume our arguments there by setting

$$S^R = - \int_0^\cdot Z_s dW_s^R.$$

We need an analogue of Lemma 2.1 to guarantee that  $R$  is a uniformly integrable martingale.

**Lemma 2.5** *For any  $p \geq 1$  we have*

$$\mathbb{E}^R \left( \left[ \int_0^T Z_s^2 ds \right]^p \right) < \infty.$$

*In particular,  $S^R$  is a uniformly integrable martingale under  $R$ .*

**Proof:** By definition of  $R$ , we have for any  $p > 1$

$$\mathbb{E}^R \left( \left[ \int_0^T Z_s^2 ds \right]^p \right) = \mathbb{E} \left( \exp(2\alpha[\xi - Y_0]) \left[ \int_0^T Z_s^2 ds \right]^p \right).$$

Now since  $\xi \leq 0$ , the density  $\exp(2\alpha[\xi - Y_0])$  is bounded above. Therefore the asserted moment finiteness follows from Lemma 2.1.  $\square$

We are in a position to prove the main result of this subsection.

**Theorem 2.2** Assume that  $f$  satisfies  $f(s, z) = \alpha z^2$ ,  $z \in \mathbb{R}, s \in [0, T]$ , and that  $\xi$  is bounded above and satisfies (13). Then there is a measure solution of (3) with a measure  $\mathbb{Q}$  that is equivalent to  $\mathbb{P}$ .

**Proof:** We may assume  $\xi \leq 0$ . Let us first show, in analogy to the proof of Theorem 2.1, that

$$V^R = \exp(\alpha S^R - \frac{1}{2}\alpha^2 \langle S^R \rangle)$$

is a uniformly integrable martingale under  $R$ , using Kazamaki's criterion. For this purpose, let

$$\tau_n^R = \inf\{t \geq 0 : \langle S^R \rangle_t \geq n\} \wedge T, \quad n \in \mathbb{N}.$$

Then, due to  $\langle S \rangle = \langle S^R \rangle$ , we deduce for all  $n \in \mathbb{N}$  that  $\tau_n = \tau_n^R$ . Since  $\tau_n^R \rightarrow T$  as  $n \rightarrow \infty$ , even with  $\tau_n^R = T$  for all but finitely many  $n$ , Fatou's lemma allows to deduce

$$\mathbb{E}^R(V_T) \leq \liminf_{n \rightarrow \infty} \mathbb{E}^R(V_{\tau_n^R}) \leq 1. \quad (38)$$

Moreover, by the form of the BSDE translated to  $W^R$  under  $R$ ,

$$\begin{aligned} \alpha S^R - \frac{1}{2}\alpha^2 \langle S^R \rangle &= \alpha \left[ - \int_0^\cdot Z_s dW_s^R - \frac{1}{2}\alpha \int_0^\cdot Z_s^2 ds \right] \\ &= \alpha \left[ - \int_0^\cdot Z_s dW_s^R - \alpha \int_0^\cdot Z_s^2 ds \right] + \frac{1}{2}\alpha^2 \int_0^\cdot Z_s^2 ds \\ &= \alpha[-Y + Y_0] + \frac{1}{2}\alpha^2 \int_0^\cdot Z_s^2 ds. \end{aligned}$$

Using the negativity of  $\xi$  and the identity just derived, we get the integrability property

$$\mathbb{E}^R \exp \left[ \frac{1}{2}\alpha(-\xi + Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2 ds \right] < \infty. \quad (39)$$

Using this and the positivity of the terminal variable  $\xi$ , we can now obtain the exponential integrability property

$$\mathbb{E} \exp \left[ \frac{1}{2}\alpha(\xi - Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2 ds \right] < \infty. \quad (40)$$

Again, we may now use (39) together with (38) to prove the exponential integrability of  $\frac{1}{2}\alpha S_T^R$ . In fact, from the BSDE viewed with  $W^R$  under  $R$  we have

$$\frac{1}{2}\alpha S_T^R = \frac{1}{2}\alpha(-\xi + Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2 ds.$$

Hence we obtain

$$\mathbb{E}^R \exp \left( \frac{1}{2}\alpha S_T^R \right) < \infty. \quad (41)$$

Now appeal to the uniform integrability of the martingale  $S^R$  under  $R$ , proved in Lemma 2.5, to see that the criterion of Kazamaki (see Revuz, Yor [16], p. 332) may be applied. Hence  $V_R$  is a uniformly integrable martingale under  $R$ .

We have to show that this implies uniform integrability of

$$V = \exp(\alpha S - \frac{1}{2}\alpha^2\langle S \rangle)$$

under  $\mathbb{P}$ . To see this, note that

$$\begin{aligned} \exp(\alpha S - \frac{1}{2}\alpha^2\langle S \rangle) &= \exp(2\alpha S - 2\alpha^2\langle S \rangle) \cdot \exp(-\alpha S + \frac{3}{2}\alpha^2\langle S \rangle) \\ &= \exp(2\alpha S - 2\alpha^2\langle S \rangle) \cdot \exp(\alpha S^R - \frac{1}{2}\alpha^2\langle S^R \rangle). \end{aligned}$$

Hence for  $n \in \mathbb{N}$

$$\mathbb{E}(V_{\tau_n} 1_{\{\tau_n < T\}}) = \mathbb{E}^R(V_{\tau_n^R}^R 1_{\{\tau_n^R < T\}}), \quad (42)$$

and the latter expression tends to 0 as  $n \rightarrow \infty$  by the first part of the proof. Hence the uniform integrability of  $V$  under  $\mathbb{P}$  follows from the explosion criterion (9) already used earlier. This completes the proof.  $\square$

**Remark:** The results of the preceding two subsections clearly call for similar ones for our BSDE with exponentially integrable terminal variable that are not bounded. Due to the nonlinearity of the generator of the BSDE, it seems impossible to derive such properties by combining the results of Theorems 2.1 and 2.2.

## References

- [1] Bismut, J.-M. “Théorie probabiliste du contrôle des diffusions.” *Mem. Amer. Math. Soc.* **4** (1976), no. 167.
- [2] P. Briand, Y. Hu *BSDE with quadratic growth and unbounded terminal value*. Probab. Theory Related Fields 136 (2006), 604-618.
- [3] P. Briand, Y. Hu *Quadratic BSDEs with convex generators and unbounded terminal conditions*. Preprint (2007). arXiv:math/0703423v1
- [4] N. Dunford, J. T. Schwartz. *Linear Operators. Part I: General theory*. Second printing. New York-London: Interscience Publishers 1964.
- [5] El Karoui, N., Barrieu, P. *Pricing, Hedging and Optimally Designing Derivatives via Minimization of Risk Measures*. To appear in: R. Carmona (ed.) Volume on Indifference Pricing. Princeton University Press: 2006.
- [6] El Karoui, N., Huang, S.-J., *A general result of existence and uniqueness of backward stochastic differential equations*, Backward stochastic differential equations (Paris, 1995–1996), Pitman Res. Notes Math. Ser., 364, 27–36, 1997.
- [7] N. El Karoui, S. Peng, M.-C. Quenez. *Backward stochastic differential equations in finance*. Math. Finance 7 (1997), 1–71.

- [8] T. Jeulin, M. Yor. *Filtration des ponts Browniens et équations différentielles stochastiques linéaires*. Séminaire de probabilités XXIV 1988/89, Lect. Notes Math. 1426, 227- 265 (1990).
- [9] O. Kallenberg. *Foundations of modern probability*. Springer: Berlin (1997).
- [10] I. Karatzas and S. E. Shreve, *Methods of mathematical finance*, Springer, New York, 1998; MR1640352 (2000e:91076)
- [11] Kazamaki, N. “ Continuous Exponential Martingales and BMO”. *Lecture Notes in Mathematics* **1579**, Springer, Berlin, 1994.
- [12] M. Kobylanski. *Backward stochastic differential equations and partial differential equations with quadratic growth*. Annals of Probability 28(2) (2000), 558-602.
- [13] J.-P. Lepeltier, J. San Martin *Existence for BSDE with superlinear-quadratic coefficient*. Stochastics and Stochastics Reports 63 (1998), 227-240.
- [14] R. Liptser, A. Shiryaev *Statistics of random processes. 1: General theory*. Applications of Mathematics. 5. Berlin: Springer 2001.
- [15] Pardoux, E.; Peng, S.G. “Adapted solution of a backward stochastic differential equation.” Systems Control Lett. **14** (1990), 55-61.
- [16] D. Revuz, M. Yor. *Continuous Martingales and Brownian Motion*. Third Edition. Springer: Berlin 1999.
- [17] B. Wong, C. Heyde. *On the martingale property of stochastic exponentials*. J. Appl. Probability 41 (2004), 654-664.