Equilibrium trading of climate and weather risk and numerical simulation in a Markovian framework *

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Abstract

We consider financial markets with agents exposed to external sources of risk caused for example by short term climate events such as the South Pacific sea surface temperature anomalies widely known under the name El Nino. Since such risks cannot be hedged through investments on the capital market alone, we face a typical example of an incomplete financial market. In order to make this risk tradable, we use a financial market model in which an additional insurance asset provides another possibility of investment besides the usual capital market. Given one of many possible market prices of risk each agent can maximize his individual exponential utility from his income obtained from trading in the capital market, the additional security, and his risk exposure function. Under the equilibrium market clearing condition for the insurance security the market price of risk is uniquely determined by a backward stochastic differential equation. We translate these stochastic equations via the Feynman-Kac formalism into semi-linear parabolic partial differential equations. Numerical schemes are available by which these semilinear pde can be simulated. We choose two simple qualitatively interesting models to describe sea surface temperature, and with an ENSO risk exposed fisher and farmer and a climate risk neutral bank three model agents with simple risk exposure functions. By simulating the expected appreciation price of risk trading, the optimal utility of the agents as a function of temperature, and their optimal investment into the risk trading security we obtain first insight into the dynamics of such a market in simple situations.

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Introduction

In this paper, we continue the treatment of control and dynamical hedging of risks exterior to usual financial markets, which we started in [15]. In the latter paper we designed a simple model for making market external risk tradable which is based on the key ideas of market completion in a partial equilibrium. We proved the existence of a unique partial equilibrium, in which all the agents were able to optimally trade their individual exposure to the external risk.

Here we illustrate these theoretical results by a detailed study of a particular risk source, created by the El Niño Southern Oscillation (ENSO). This risk source will suggest a particular market consisting of agents exposed to ENSO risk. We shall use numerical schemes to be detailed below, by means of which we simulate the performance of the market of ENSO risk trading in some simple toy situations. This provides first insight into the market dynamics and triggers further conceptual questions.

We first briefly explain this climate risk source. ENSO is understood as the randomly periodic event of an anomalous rise of the sea surface temperature of the Eastern Pacific just south of the equator near the American coast, striking in a random period every 3-8 years around Christmas. Some of the widely known local climatic changes it triggers are: an increased precipitation rate on the western hemisphere, while the eastern hemisphere may suffer from draughts. These local climate changes have severe economic consequences. Due to higher sea surface temperatures, catch rates for many species of fish in South American countries drop significantly during ENSO years. At the same time, for instance rice or cotton farming may be more profitable in usually dry areas of the same countries, thanks to the wetter weather. So, while certain parts of local economies suffer from ENSO, at the same time other parts may profit. This leads us to conclude that economically the external risk source given by ENSO may create different types of agents with complementary or at least negatively correlated interests. It is not hard to figure out that the numerous globally felt climatic and thus economical consequences generate a variety of agents exposed to the risk in very different, often complementary ways. See for example Gaol and Manurung [13] and Mizuno [26] for the dependence of catch numbers for big eye tuna in the South Java sea waters on the ENSO cycle.

The model we propose to transfer this risk among affected agents is discussed in detail in the companion paper [15]. In our climate risk example it takes the following form. We consider an economy with a finite number of agents $a \in I$, represented for instance by individual farmers or fishers, banks, or companies like insurance or even reinsurance companies. Their common feature is their exposure to the risks caused by ENSO. In order to describe this exposure in an analytically accessible way, we represent the climate (sea surface temperature) process K by simple low dimensional stochastic differential equations suggested by the climate physics literature. The most common used for the purpose of predictions of the event is based on an Ornstein-Uhlenbeck process in dimension 15, where the dimensionality comes from statistical data fitting (see Penland [27]). A 2-dimensional conceptual bi-stable model with intrinsic random periodicity is obtained from a deterministic nonlinear equation coupling thermocline depth and sea surface temperature perturbed by random noise representing trade wind coupling at sea level (see Fang, Barcilon, Wang [1]). Another way to obtain the random periodicity in a simple conceptual model is described by Battisti [4]. Here the delay coupling to the state the randomly perturbed system experienced before it sent Kelvin waves from the South American Pacific coast across the ocean, which were reflected at the Japanese coast, is responsible for an intrinsic periodicity. In our simulations we use two simple models consistent with these reduced models to describe K. In the simplest one, K is a one-dimensional Ornstein-Uhlenbeck process. The more realistic second one is given by a conceptual bi-stable diffusion model driven by a Brownian motion with a time-periodic potential function which has two minima the depths of which fluctuate periodically. The noise is implemented with intensities at which the solution trajectories show some random periodicity which can be measured by means of quality of periodic tuning notions (see [18], [17], [16]).

The agents composing our market are allowed to have three sources of income. Firstly, they can trade on a stock market represented by a stock price process X^S as small traders - a hypothesis made for simplicity, which needs some further elaboration in future work in view of the role big agents like re-insurers may play. Secondly, the individual exposure of each agent a to climate risk is mathematically described by some payoff functional $H^a(K, X^S)$ depending both on the climate process K and on the stock price process X^S. The risks represented by H^a cannot be hedged by the stocks. For this reason we of course face a typical incomplete market. We *complete the market* by constructing a special security X^E, through which climate risk becomes tradable and which therefore acts as the third source of income. Agents active on the market may buy or sell individual amounts of this climate index according to their random risk exposures. If a particular market price of risk θ is given, every agent is able to price his share of risk to be traded. He will then choose an investment strategy which optimizes the individual utility from his total income composed of his investment both into the usual capital market and into the climate index, and of his random risky income. There will be a unique market price of risk θ^* for which a *market equilibrium* is achieved, i.e. for which there is zero excess demand for the climate index. This pricing rule is determined by the intervention of one of the main tools of stochastic control theory in incomplete markets, backward stochastic differential equations (BSDE). For details of this model and references to literature concerning our key techniques of market completion, utility maximization, BSDE, and asset design see [15].

On the one side, typical toy agents we have in mind for our simulations will be just a pair composed of a fisher and a (rice) farmer (f or r) subject to the hazard of ENSO, and whose random income $H^f (H^r)$ depends uniquely on the climate process K. They are given by some cumulative functional H of the form

$$H = \int_0^T \phi(K_s) ds \quad \text{or} \quad H = \int_0^T \phi(s, K_s, X^S(s)) ds,$$

where ϕ is an individual bounded revenue function taking its maximum for example at some low temperature k_f close to the normal sea surface temperature in the case of the fisher, or at some higher temperature k_r in the case of the rice farmer. The functions may in turn be relatively small at the corresponding opposite values k_r resp. k_f. On the other side, there is a climate risk neutral agent such as a bank (b) whose income H^b is a function of the stock market evolution alone. Trading climate risk for these agents can be viewed in the following way. f wants to hedge fluctuations caused by the external factor and signs a contract with b to transfer part of this risk. b's interest in the contract could be based on the wish to diversify its portfolio. The main example of the global ENSO risk provides a number of further relevant risk functionals for different, eventually complementary groups of agents treated in the mathematical parts, but not in the simulations below. For example, the exposure to ENSO for a big agent such as a re-insurance company i will be a functional of the type

$$H^i = g(\tau, K_\tau), \quad \text{or} \quad H^i = g(\tau, K_\tau, X^S(\tau)),$$

if τ is the time ENSO strikes, which is realized by some entrance time for the process K.

This choice of income functionals for model traders used for the simulations will imply that the mathematical treatment is possible in the framework of forward-backward stochastic differential equations of Markovian character. Since the numerical analysis of BSDE is still in its infancy (see for example [6]) and no techniques at all are available if - as in our case - the equation is only locally Lipschitz, we first transfer our stochastic simulation problem into a problem of simulating non-linear PDE. In fact, via the generalized Feynman-Kac formalism, BSDE are associated with systems of linear or semi-linear parabolic partial differential equations. Their solutions exist in general in the viscosity sense, much as in Chaumont [7]. In the concrete situations we consider, they turn out to be classical. We will use newly developed numerical schemes for non-linear PDE from [7] to approximate and simulate them for the Ornstein-Uhlenbeck or bistable diffusion climate processes, and the risk functionals for fishers, farmers and bank just described. Notably, we shall be able to simulate the expected price of X^E which indicates the cumulative appreciation of trading the external risk by the affected agents, the temporal evolution of the optimal utility for the agents in dependence on the level of the temperature process K, and the shape of the optimal investments of the agents into X^E. This way we obtain first information on the dynamics of such a market which will, if not quantitatively, be of interest at least for qualitative issues.

The paper is organized as follows. In section 1 we give a more formal and detailed account of our market model, including in particular the formal links to the theory of semi-linear parabolic pde via generalized Feynman-Kac formulas, as well as proofs for existence, uniqueness and regularity for the pde expressing the Markovian optimal control problem derived from our utility maximization problems on completed markets under the equilibrium constraint. We shall also explain the concrete elementary models for temperature processes and risk functionals used in the simulations. Section 2 is devoted to exhibiting and explaining the numerical approximation schemes and convergence results. In section 3 we present our simulation results for the optimal allocation of risk given the particular temperature processes and risk exposure functionals for fisher, farmer and bank, and interpret the findings intuitively.

1 Analytical versions of model equations and concrete examples

In this section we describe formally the equations governing our model. We start on the stochastic side. In subsection 1.1 we shall recall our market model designed to make external risk tradable, and the main conclusions from the conceptual companion paper [15]. For its key ingredient, a nonlinear BSDE describing the equilibrium price of external risk, no methods of numerical simulation are known. To make our equations amenable to better known numerical techniques, we make use of a translation technique from BSDE to PDE theory. This crucial link between stochastic forward and backward differential equations on the one hand and nonlinear PDE, possibly with solutions in the viscosity sense, on the other hand is provided by a nonlinear extension of the Feynman-Kac formula and will be explained in subsection 1.2. All stochastic equations relevant for our purposes will be transferred into linea and semi-linear PDE by means of this link in section 1.3. For instance, the analytical description of the optimal investment policies of the different agents into the asset price process of the market X^S and the insurance asset X^E leads to an optimal control problem in terms of a nonlinear Hamilton-Jacobi-Bellman equation. In the final subsection 1.4 we discuss concrete examples of risk exposure functionals which depict some of the situations alluded to in the introduction. Existence and uniqueness questions for the different PDEs governing the analytical description in this section will be discussed in section 2.

1.1 An equilibrium model for trading market external risks

We consider a probability space $(\Omega, \mathcal{F}, I\!P)$ with two independent Brownian motions W_1 and W_2 which we take for simplicity one-dimensional, indexed by the finite time interval $[0, T]$, where $T > 0$ is a deterministic time horizon. Let $\mathcal{F} = \{\mathcal{F}_t\}_{0 \le t \le T}$ be the completion of the natural filtration of $W = (W_1, W_2)$ by the sets of measure 0.

We consider a simple financial market composed of 2 securities, consisting of one bond with null interest rate

$$X_{0,t} = 1, \quad \text{for all} \quad t \in [0, T],$$

and 1 stock. We assume that the stock price vector process X^S is given by a Markovian SDE, i.e. :

$$\begin{cases} dX_s^S = X_s^S \left(b^S(s, X_s^S) ds + \sigma^S(s, X_s^S) dW_{1,s} \right), & t \le s \le T, \\ X_t^S = x_1 \in I\!R. \end{cases}$$

The coefficients $b^S : [0,T] \times I\!R \mapsto I\!R$, $\sigma^S : [0,T] \times I\!R \mapsto I\!R \times I\!R$ are supposed to satisfy Lipschitz conditions in the state variables.

We also consider a 1-dimensional climate process, the dynamics of which is described by an SDE of the form

$$\begin{cases} dK_s = b_K(s, K_s)ds + \sigma_K(s, K_s)dW_{2,s}, & t \le s \le T, \\ K_t = k \in I\!R. \end{cases}$$

The coefficients $b_K : [0,T] \times I\!R \mapsto I\!R$ and $\sigma_K : [0,T] \times I\!R \mapsto I\!R \times I\!R$ are again Lipschitz functions of the state variables. Our model assumes independence of the uncertainties inherent in the stock market or the climate process respectively. This can in fact be weakened considerably. If the Gaussian noise processes driving X^S and K are correlated, say $E(W_{1,t}W_{2,t}) = \rho t$, $t \ge 0$, we split the climate noise into two independent components, by setting

$$X_t = \frac{1}{\sqrt{1-\rho}}[W_{2,t} - \sqrt{\rho}W_{1,t}], \quad t \ge 0,$$

to get

$$W_{2,t} = \sqrt{1-\rho}X_t + \sqrt{\rho}W_{1,t}, t \ge 0.$$

Hence essentially we obtain, with respect to another independent pair of Wiener processes (W_1, X), a stock price process X^S only depending on the first component W_1, and a climate process depending on both components (W_1, X). This is the setting treated in the general framework of [15]. It is shown in a manner analogous to the one sketched above that a unique market price of risk exists in partial equilibrium. A numerical simulation of the market dynamics as the one performed in this paper for the case of independent X^S and K, however, is still missing in the correlated case.

1.1.1 Market completion

Market completion is achieved by adding a security X^E whose market price of risk process θ^E parametrizes the completion and thus the valuation of risky claims. X^E will be determined by the solution of an SDE driven by the climate uncertainty process W_2 of the form

$$\begin{cases} dX_s^E = X_s^E \left(b_s^E ds + \sigma_s^E dW_{2,s}\right), & t \le s \le T, \\ X_t^E = x_2 \in I\!R^d. \end{cases}$$

We will note $\theta^S = \frac{b^S}{\sigma^S}$, $\theta^E = \frac{b^E}{\sigma^E}$, and $\theta = (\theta^S, \theta^E)$. The probability Q^θ, under which (X^S, X^E) is a martingale, is given by Girsanov's formula

$$\left.\frac{dQ^\theta}{dI\!P}\right|_{\mathcal{F}_s} = Z_s^\theta = \mathcal{E}\left(-\int_0^s \theta_t dW_t\right) = \exp\left(-\int_0^s \theta_t dW_t - \frac{1}{2}\int_0^s ||\theta_t||^2 \, dt\right), \quad s \in [0,T]. \tag{1}$$

1.1.2 Agents and income

Let I be the finite set of small agents active on the market. Each agent $a \in I$ is supposed to be endowed with an initial capital $v_0^a \geq 0$. He invests in the market including the insurance asset and uses an admissible trading strategy $\pi = (\pi_1, \pi_2)$.

Therefore his wealth process is given by

$$V_s^a = v_0^a + \int_0^s \pi_{1,s} \frac{dX_s^S}{X_s^S} + \pi_{2,s} \frac{dX_s^E}{X_s^E}, \ s \in [0,T] \tag{2}$$

("not investing" means investing in X_0, i.e. choosing the strategy $\pi = 0$).

At the end T of the trading interval, each agent a receives a stochastic income H^a, describing the profits the company he represents obtains, which can depend on the market and on the climate. For our purposes we assume that it has the form

$$H^a = g^a(\tau, X_{1,\tau}, K_\tau) + \int_0^\tau \varphi^a(t, X_t^S, K_t) dt,$$

where g^a and φ^a are real valued bounded smooth (C^∞) functions, with

$$\tau = \inf \left\{ s \in]t,T] \,\middle|\, (s, X_s^S, K_s) \notin \mathcal{O} \right\}$$

the entrance time of some critical set $\mathcal{O}$, an open subset of $]0,T[\times \mathbb{R} \times \mathbb{R}$.

1.1.3 Individual utility maximization

Each agent $a \in I$, by acting on its trading strategy π, wants to maximize the expected utility of the sum of the terminal wealth V_T^a and the income H^a. His preferences are described by an individual exponential utility function

$$U^a(x) = -\exp(-\alpha_a x), \ x \in \mathbb{R},$$

with individual risk aversion coefficient $\alpha_a > 0$. In mathematical terms, every agent wants to attain

$$J^a = \sup_{\pi \ admissible} \mathbb{E}\left[U^a\left(V_T^a + H^a\right)\right]. \tag{3}$$

Under simple assumptions (see [15]), this quantity can be computed via well known techniques based on duality and Legendre transforms for U. See Karatzas, Lehoczky, Shreve [19] or Kramkov, Schachermayer [23]. The formula valid in our setting is derived in [15], Theorem 2.2. We have

$$J^a = \mathbb{E}\left[-\frac{\lambda_a}{\alpha_a} Z_T^\theta\right] = -\frac{\lambda_a}{\alpha_a}, \tag{4}$$

since Z^θ is a $\mathbb{P}-$martingale, where λ_a is defined by

$$\log(\lambda_a) = \log(\alpha_a) - \alpha_a v_0^a + \mathbb{E}^\theta\left[-\log\left(Z_T^\theta\right) - \alpha_a H^a\right]. \tag{5}$$

1.1.4 Local Equilibrium Measure

The optimal income agent a obtains from trading in the two securities and his exposure to external risk therefore depends on the market price of external risk θ^E. To determine a unique price of risk θ^* under which the market reaches an equilibrium under which every agent obtains his maximal income we impose a partial market clearing condition. It states that the total investment $(\pi_2^a)_{a\in I}$ in the insurance asset satisfies the condition

$$\sum_{a\in I} \pi_2^a = 0 \ a.s. \tag{6}$$

The corresponding equilibrium price of risk process θ^{E*}, which determines completely the structure of the security X^E and a unique martingale measure $Q^{\theta^{E*}}$, is computed in [15], Theorems 3.3 and 3.5, as the solution of a nonlinear BSDE. We shall briefly recall how this can be seen. The structure result for the optimal utility of agent a reflected for example in (5) combines with (1) to produce for any $a \in I$ the following formula for the optimal income from trading in (X^S, X^E) including the income due to risk exposure

$$\begin{aligned} &-\frac{1}{\alpha_a}\log(\frac{1}{\alpha_a}\lambda_a Z_T^\theta) - H^a \qquad (7)\\ &= \ -\frac{1}{\alpha_a}\log\left(\frac{\lambda_a}{\alpha_a}\right) + \frac{1}{\alpha_a}\int_0^T (\theta_t^S dW_{1,t} + \theta_t^E dW_{2,t}) + \frac{1}{2\alpha_a}\int_0^T ((\theta_t^S)^2 + (\theta_t^E)^2) dt - H^a. \end{aligned}$$

To take into account the market clearing condition, we now calculate the total optimal income of all agents on the market due to their trading strategies (π_1^a, π_2^a). It amounts to the following quantity

$$\begin{aligned} &\sum_{a\in I}(B^a - H^a) \\ &= \sum_{a\in I} v_0^a + \int_0^T (\sum_{a\in I}\pi_{1,t}^a) dX_t^S + \int_0^T (\sum_{a\in I}\pi_{2,t}^a) dX_t^E \qquad (8)\\ &= \sum_{a\in I} v_0^a + \int_0^T (\sum_{a\in I}\pi_{1,t}^a)\, \sigma_{1,t} X_t^S\, (dW_{1,t} + \theta_t^S dt) \\ &\qquad + \int_0^T (\sum_{a\in I}\pi_{2,t}^a)\, \sigma_t^E (dW_{2,t} + \theta_t^E dt) \\ &= \sum_{a\in I} v_0^a + \int_0^T (\sum_{a\in I}\pi_{1,t}^a)\, \sigma_{1,t} X_t^S\, (dW_{1,t} + \theta_t^S dt). \end{aligned}$$

We next sum (7) in $a \in I$ and equate the result to (8). The equation thus obtained is interpreted as an equation for the unknown process θ^E with given parameter θ^S and given risk exposure functionals H^a. To explain this, we abbreviate

$$
\begin{aligned}
\overline{\alpha} &= \left(\sum_{\alpha\in I}\frac{1}{\alpha_a}\right)^{-1}, \\
\overline{H} &= \sum_{\alpha\in I} H^a + \frac{1}{2\overline{\alpha}}\int_0^\tau (\theta_t^S)^2 dt, \\
z_1 &= \theta^S - \overline{\alpha}\sigma^S \sum_{a\in I}\pi_1^a, \\
z_2 &= \theta^E.
\end{aligned}
$$

Observe that the equation obtained from taking the overall balance of (7) and (8) is a static equation valid only for time T, and given by

$$\overline{\alpha}\overline{H} = c + \int_0^T z_t dW_t + \int_0^T \theta_t^S z_{1,t} dt + \frac{1}{2}\int_0^T z_{2,t}^2 dt. \tag{9}$$

We extend it to a dynamic equation of processes, which controls the system into the final variable $\overline{\alpha}\overline{H}$. This is the job of the following nonlinear BSDE to be solved by a pair of processes $(h, (z_1, z_2))$. The role of the first component h is to define the momentary price of the claim $\overline{\alpha}\overline{H}$, while the control process z_2 provides the equilibrium price of climate risk θ^{E*}. The equation is given by

$$h_s = \overline{\alpha}\overline{H} - \int_s^T z_t dW_t - \int_s^T \theta_t^S z_{1,t} dt - \frac{1}{2}\int_s^T z_{2,t}^2 dt. \tag{10}$$

It is one of the main goals of this paper to compute and simulate the equilibrium market price of risk θ^{E*} in simple scenarios of exposure of few agents to climate risk. We shall assume in the sequel, and justify in the computations later, that θ^{E*} can be written as a regular function of the state of the system, i.e.

$$\theta_s^{E*} = \widetilde{\theta}^{E*}(s, X_s^S, K_s) \text{ with } \widetilde{\theta^E} \in C^2.$$

In the sequel we shall use the same symbol θ^E for both the random process and the real valued regular function of $(s, x_1, k) \in [t, T] \times \mathbb{R} \times \mathbb{R}$, since it will be clear from the context which object we are dealing with. Hereby for notational simplicity the "*" is suppressed. Under the conditions we impose on the coefficients of our equations, it will be seen that $\widetilde{\theta}^E$ is even C^∞. We emphasize that expectations are usually taken with respect to the partial equilibrium probability Q^{θ^E} and denoted by the symbol E^{θ^E}.

1.1.5 Local equilibrium under a family of linear pricing rules

The idea of market completion with a partial equilibrium just sketched can be reinterpreted in the framework of a family of pricing rules. At the same time, this interpretation allows an extension of our method to a more general situation. How the agents on our market get along with these pricing rules also gives an alternative description of how the market uses the fictitious asset featured in the concepts explained above.

The external risk might be too complicated to complete the market with only one additional security. In this case *completion* may be attained in a different way. The agents can trade the risk by buying and selling random payoffs among each other which they are able to choose freely. At the beginning of the trading interval the agents sign contracts that describe those payoffs. As before, they are allowed to freely trade at the stock market.

Random payoffs are priced using one *pricing rule* for all payoffs on the market. The value of a payoff which is replicable by a trading strategy must equal its initial capital. In particular, a pricing rule consistent with the stock price is linear on the agents' replicable payoffs. We therefore use pricing rules which are linear functions of the payoffs and which can be described as expectations under probability measures equivalent to the real world measure P. The condition to be consistent with the stock price in addition leads us to those equivalent probability measures for which the stock price process is a martingale. In the version of our model described above market completion was achieved by choosing a second security parameterized by a market prize of risk process θ^E which generates a martingale under a unique probability measure Q^{θ^E}. In this version it is replaced by a step procedure: at first, the agents are allowed to trade random payoffs which in the second step are priced with rules directly parameterizing martingale measures Q^{θ^E}.

The budget set of an agent consists of all random payoffs that are not more expensive than the sum of the initial capital and the value of his random income due to risk exposure. It depends of course on the pricing rule which formally replaces the introduction of the insurance asset in the model version above. To correspond to the latter, given an admissible pricing rule, every agent chooses in his budget set the payoff which maximizes his expected utility.

In this setting, the market clearing condition leading to the *partial equilibrium* reads as follows. The difference between the sum of the incomes of the agents due to external risk exposure and the sum of the preferred payoffs viewed with particular linear pricing rules must be replicable by a trading strategy based on the stock alone. So partial equilibrium is achieved through the construction of a linear pricing rule for which the difference between risk exposure and optimal payoff can be replicated on the stock market.

1.2 The PDE link: generalized Feynman-Kac formulas

Since methods for simulating (10) are not available at the moment, we shall transfer our stochastic control problem of determining the equilibrium price of risk and the associated stochastic differential equations into a problem facing linear and semi-linear parabolic PDE. The translation uses a well known generalization of the Feynman-Kac formalism. We will recall in this subsection the main results from this technique we have to appeal to in our particular situation.

Let $n, d \in \mathbb{N}$. Let $\mathcal{O}$ be an open subset of $]0, T[\times \mathbb{R}^n$. Let $t \in [0, T]$ be an arbitrary time, representing time of initial action. For a $d-$dimensional Brownian motion W, and $x \in \mathbb{R}^n$, we construct the process $X^{t,x}$ as the solution of the following SDE

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dW_s, \quad t \leq s \leq \tau \\ X_t^{t,x} = x \in \{y \,|(t,y) \in \mathcal{O}\}, \end{cases} \tag{11}$$

and we define $\tau = \inf \{s \in]t,T] \,|(s, X_s^{t,x}) \notin \mathcal{O}\}$, the first exit time of $X^{t,x}$ from the domain $\mathcal{O}$.

In this section, we will denote by L the infinitesimal generator associated with $X^{t,x}$, i.e. for a regular (C^2) function ϕ,

$$L\phi(s,x) = b(s,x)D\phi(s,x) + \frac{1}{2}\text{trace}\left[\sigma\sigma^*(s,x)D^2\phi(s,x)\right], \quad (s,x) \in \mathcal{O},$$

where $D\phi$ stands for the gradient and $D^2\phi$ the Hessian matrix of ϕ.

In all the following, we will suppose that the drift b and the diffusion matrix σ are C^∞ functions of the state variable with linear growth at infinity, and that L is uniformly elliptic.

We first recall the well-known classical Feynman-Kac formula for linear problems, which can be found in [21].

Theorem 1.1 *Suppose that the coefficients f and h are Lipschitz functions of the state variable with linear growth at infinity. Assume further that h is bounded, g is continuous with polynomial growth in the state variable. We define the function v on $\overline{\mathcal{O}}$ by the expectation $E_{t,x}$ with respect to the diffusion measure starting at time t in x*

$$v(t,x) = \mathbb{E}_{t,x}\left[\int_t^\tau f(s, X_s^{t,x})e^{-\int_s^\tau h(r,X_r^{t,x})dr}ds + g(\tau, X_\tau)e^{-\int_t^\tau h(s,X_s)ds}\right]. \tag{12}$$

Then v is a classical solution of the following backward linear system

$$\begin{cases} -\dfrac{\partial v}{\partial s} - Lv + f - hv = 0 \text{ in } \mathcal{O}, \\ u(s,x) = g(s,x) \text{ on } \partial\mathcal{O}. \end{cases} \tag{13}$$

There is a similar formula for forward PDEs with an initial condition instead of a terminal one.

Proof:
The simplest way to prove that the function v solves (13) is to prove that there exists a classical solution to the PDE (13). Once this is guaranteed, we just have to apply Itô's formula in a well known manner to read off the PDE. So it is enough to quote a classical existence and uniqueness result. It is valid under certain hypotheses for the coefficients which will be seen to be satisfied in all the applications we have in mind below. To complete the proof we therefore recall a result which can be found for example in [12] or [14].□

Theorem 1.2 *Under the assumptions of Theorem 1.1, system (13) has a unique classical solution.*

We now recall the nonlinear Feynman-Kac formula for BSDEs.

Theorem 1.3 *Suppose that $\sigma\sigma^*$ is uniformly elliptic, and $\mathcal{O} =]0,T[\times I\!R^n$. In addition to the family $(X^{t,x})_{(t,x)\in[0,T]\times I\!R^n}$ given by (11) consider two additional processes Y and Z defined by the following BSDE*

$$\begin{cases} -dY_s^{t,x} = F(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds - Z_s^{t,x}dW_s, \quad t \le s \le \tau \\ Y_\tau^{t,x} = g(\tau, X_\tau^{t,x}). \end{cases} \tag{14}$$

Assume that $F : [0,T] \times I\!R^n \times I\!R \times I\!R^n \to I\!R$ is C^∞ and $g \in C([0,T] \times C^1(I\!R^n))$. Then, for every $t \le s \le \tau$, we have

$$\begin{cases} Y_s^{t,x} = u(s, X_s^{t,x}) \\ Z_s^{t,x} = \sigma^* Du(s, X_s^{t,x}), \end{cases}$$

where u is the unique classical solution of the PDE

$$\begin{cases} -\dfrac{\partial u}{\partial s} - Lu - F(s,x,u,\sigma^*(t,x)Du) = 0 \text{ in } \mathcal{O}, \\ u(s,x) = g(s,x) \text{ on } \partial\mathcal{O}. \end{cases} \tag{15}$$

Proof:
Again, we shall invoke a classical existence, regularity and uniqueness result, in order to prove that the generally existing solution in the viscosity sense of (15) is in fact a unique regular classical solution. Once this is guaranteed, the proof of the existence may be completed by an appeal to Itô's formula (see [22], p. 581) in a well known manner. □

The theorem alluded to above which guarantees the existence, uniqueness and regularity of classical solutions for (15) is taken from Taylor [29].

Theorem 1.4 *Under the assumptions of theorem 1.3, system (15) has a unique classical solution $w \in C\left([0,T], C^1\left(I\!R^n\right)\right) \cap C^\infty\left([0,T[\times I\!R^n\right)$.*

Proof:
The proof is given in Taylor [29], in Proposition 15.1.1 on p.273. Note first that we may and do assume that the infinitesimal generator L is in divergence form, and thus self adjoint as a linear operator. This can be achieved by shuffling the drift part as well as an additional drift containing $D\sigma\sigma^*$ to the function F in Taylor's Proposition. This is possible due to the regularity assumptions on b and σ. With F and L thus modified, we next have to make sure that under the given assumptions the hypotheses of this Proposition are satisfied. For convenience, we recall these hypotheses. For any integer $r \ge 0$, they claim

$$e^{tL} : C^{r+1}([0,T] \times I\!R^n) \to C^{r+1}([0,T] \times I\!R^n) \quad \text{is a strongly continuous semigroup, for } t \ge 0, \tag{16}$$

$$\Phi : \begin{array}{ccc} C^{r+1}([0,T] \times I\!R^n) & \to & C^r([0,T] \times I\!R^n) \\ \varphi & \mapsto & F(\varphi, D\varphi) \end{array} \quad \text{is a locally Lipschitz map,} \tag{17}$$

and, for some $\gamma < 1$,

$$||e^{tL}||_{\mathcal{L}(C^r([0,T]\times I\!R^n),C^{r+1}([0,T]\times I\!R^n))} \leq Ct^{-\gamma}. \tag{18}$$

The condition on F is evidently satisfied. To verify the conditions on the semigroup of L, we refer to Davies [8]. Strong continuity is due to [8], Theorem 1.4.1, p.22. The smoothing property is related to [8], Theorem 5.2.1, p. 149, and the large time asymptotic property can be obtained from [8], Theorem 2.3.6, p. 73. □

1.3 The key PDE of our model

In the following subsection, we shall use the techniques of subsection 1.2 to derive the main parabolic linear or nonlinear PDE relevant for our model. We start with the individual utility maximization problem for the agents on the market according to subsubsection 1.1.3.

1.3.1 PDE for individual maximal utility

Fix $a \in I$. The expectation in the explicit representation of the maximal utility for agent a in (5) leads to a linear PDE. To see this, define for $t \in [0,T], x_1 \in I\!R, k \in I\!R$

$$\chi(t,x_1,k) = I\!E^{\theta^E}\left[-\log\left(Z_T^\theta\right) - \alpha_a H^a\right]$$

$$= I\!E^{\theta^E}\left[\int_t^T \left(\left|\left|-\frac{1}{2}\theta\left(s,X_s^S,K_s\right)\right|\right|^2 - \alpha_a\varphi^a\left(s,X_s^S,K_s\right)\right)ds - \alpha_a g^a\left(T,X_T^S,K_T\right)\right].$$

An appeal to the backward version of Theorem 1.1 translates the stochastic utility maximization formula into a linear backward PDE.

Corollary 1.1 *Let $\widetilde{L}$ be the infinitesimal generator of the diffusion (X^S,K) under Q^θ, determined for a regular function ϕ by*

$$\widetilde{L}\phi = \left(b_K - \theta^E\sigma_K\right)\frac{\partial\phi}{\partial k} + \frac{1}{2}trace\left\{\begin{pmatrix} x_1^2(\sigma^S)^2 & 0 \\ 0 & \sigma_K^2\end{pmatrix} D^2\phi\right\}.$$

Then χ is the unique classical solution of the following backward PDE

$$\begin{cases} -\dfrac{\partial\chi}{\partial t} - \widetilde{L}\chi - \dfrac{1}{2}||\theta||^2 - \alpha_a\varphi^a = & 0 \\ \chi(T,x_1,k) = & -\alpha_a g^a(x_1,k). \end{cases} \tag{19}$$

Proof:

The result follows from Theorem 1.1 in dimension $n=2$ with $b = \begin{pmatrix} 0 \\ b_K - \theta^E\sigma_K\end{pmatrix}$, $\sigma\sigma^* = \begin{pmatrix} x_1^2(\sigma^S)^2 & 0 \\ 0 & \sigma_K^2\end{pmatrix}$, $f = -\frac{1}{2}||\theta||^2 - \alpha_a\varphi^a$, $g = -\alpha_a g^a$ and $h = 0$. Obviously, f and h are Lipschitz continuous and possess linear growth at infinity, g is continuous and bounded. There is one small gap here, which can be easily overcome. The diffusion matrix $\sigma\sigma^*$ is not uniformly elliptic, due to the appearance of x_1^2 in the first diagonal entry. But since the generated diffusion does not visit the boundary $x_1 = 0$, we may argue by using a logarithmic coordinate change in x_1 at the beginning of the analysis (see the proof of Corollary 1.3). By this change, the diffusion matrix becomes constant in the first diagonal entry, and thus uniformly elliptic. The drift is modified, but stays Lipschitz with linear growth at infinity. The change of variable being a regular bijection of the domain, existence and uniqueness of solutions in the two coordinate systems are equivalent. □

If, as usual, the initial time of action is 0, we have

$$J^a = -\exp\left(-\alpha_a v_0^a + \chi(0, x_1, k)\right).$$

1.3.2 PDE for optimal portfolio

While Corollary 1.1 offers a convenient possibility of describing the optimal utility, an analytic access to the actual optimal portfolio strategies (π_1, π_2), the quantities of (X^S, X^E) to be invested, requires to dig a little deeper. We have to invoke the basic results of stochastic control theory (see for example [24] or [5]).

Fix as before $a \in I$ and suppose that the trading period begins at a time $t \in]0, T]$, each agent starting with an initial capital v_t^a (and $X_t^S = x_1$ and $K_t = k$). a wants to attain his value function

$$J^a(t, x_1, k, v_t^a) = \sup_{\pi \ admissible} I\!E_{t,x_1,k,v_t^a}\left[U^a\left(V_T^a + H^a\right)\right].$$

Here E_{t,x_1,k,v_t^a} denotes the expectation with respect to the two dimensional diffusion (X^S, K) starting at time t in (x_1, k), with initial capital v_t^a. By the same calculus we have

$$J^a(t, x_1, k, v_t^a) = -\exp\left(-\alpha_a v_t^a + \chi(t, x_1, k)\right), \tag{20}$$

where χ is the classical solution of the PDE of Corollary 1.1.

Let us rewrite the wealth process V^a defined in (2) in terms of the proportions $p = (p_1, p_2)$ to be invested in (X^S, X^E). Formally $\frac{\pi_i}{V^a} = p_i$, and we now have $p_0+p_1+p_2 = 1$. In these terms we may write

$$\frac{dV_s^a}{V_s^a} = p_{1,s}\left(b^S ds + \sigma^S dW_{1,s}\right) + p_{2,s}\left(\theta^E ds + dW_{2,s}\right),$$

so that the coefficients of this SDE controlled by p do not depend on X^E.

Equation (20) yields that the function J^a is C^2, as is the function χ. So, using theorem 1.3.1, p. 25, in [7] this implies that J^a solves the following Hamilton-Jacobi-Bellman (HJB) equation

$$\left\{\begin{array}{l} \dfrac{\partial J^a}{\partial t}(s, x_1, k, v) + \sup\limits_p \left\{\mathcal{L}^p J^a(s, x_1, k, v)\right\} = 0 \text{ for } (s, x_1, k) \in \mathcal{O} \\ J^a(s, x_1, k, v) = U^a(v + g^a(x_1, k)) \text{ for } (s, x_1, k) \text{ on } \partial\mathcal{O}, \end{array}\right. \tag{21}$$

where $\mathcal{O}$ is the open set from 1.1.2, $\mathcal{L}^p$ is the infinitesimal generator of the diffusion $s \mapsto (X_s^S, K_s, V_s^a,)$, i.e. the differential operator determined by its value for a regular function ϕ by

$$\mathcal{L}^p\phi(s, x_1, k, v) = \begin{pmatrix} x_1 b^S(s, x_1) \\ b_K(s, k) \\ v\left(p_1 b^S + p_2\theta^E\right) \end{pmatrix} \begin{pmatrix} \frac{\partial\phi}{\partial x_1} \\ \frac{\partial\phi}{\partial k} \\ \frac{\partial\phi}{\partial v} \end{pmatrix}$$

$$+\frac{1}{2}\text{trace}\left\{\begin{pmatrix} x_1^2(\sigma^S)^2 & 0 & vp_1x_1(\sigma^S)^2 \\ 0 & \sigma_K^2 & vp_2\sigma_K \\ vp_1x_1(\sigma^S)^2 & vp_2\sigma_K & v^2\left(p_1^2(\sigma^S)^2 + p_2^2\right) \end{pmatrix} \begin{pmatrix} \frac{\partial^2\phi}{\partial x_1^2} & \frac{\partial^2\phi}{\partial x_1\partial k} & \frac{\partial^2\phi}{\partial x_1\partial v} \\ \frac{\partial^2\phi}{\partial x_1\partial k} & \frac{\partial^2\phi}{\partial k^2} & \frac{\partial^2\phi}{\partial k\partial v} \\ \frac{\partial^2\phi}{\partial x_1\partial v} & \frac{\partial^2\phi}{\partial k\partial v} & \frac{\partial^2\phi}{\partial v^2} \end{pmatrix}\right\}. \tag{22}$$

If the optimal control process exists, it is given in *feedback form*, i.e. as a function of the state of the system by

$$p(s, X_s^S, K_s, V_s^a) = \arg\max_p \mathcal{L}^p J^a(s, X_s^S, K_s, V_s^a). \tag{23}$$

Formulas of this type have been derived for example in Fleming, Soner [11], p.170, in a general setting, and also in [5] and [24]. As soon as this process p is well-defined, it coincides with the optimal strategy. In our case, existence problems for p are covered by (19) which guarantees the existence of a classical solution of the HJB equation.

Using (20), we can express the optimal proportions p in terms of the function χ defined by system (5) of Corollary 1.1. This will allow us to justify the existence of the optimal control.

We then have to find p_1 which maximizes

$$p_1\left(vb^S\frac{\partial J^a}{\partial v} + vx_1\sigma^S\frac{\partial^2 J^a}{\partial v\partial x_1}\right) + \frac{1}{2}(p_1)^2\left(v^2(\sigma^S)^2\frac{\partial^2 J^a}{\partial v^2}\right)$$

and, independently, p_2 which maximizes

$$p_2\left(v\theta^E\frac{\partial J^a}{\partial v} + v\sigma_K\frac{\partial^2 J^a}{\partial v\partial k}\right) + \frac{1}{2}(p_2)^2\left(v^2\frac{\partial^2 J^a}{\partial v^2}\right).$$

This is seen by applying (22) to J^a, separating the p_1- and the p_2-terms from the resulting polynomial in (p_1, p_2) and separately maximizing these. By (20), we have $\frac{\partial J^a}{\partial v} = -\alpha_a J^a$, hence $\frac{\partial^2 J^a}{\partial v^2} = (\alpha_a)^2 J^a$ and $\frac{\partial^2 J^a}{\partial kv} = -\alpha_a\frac{\partial J^a}{\partial k}$. Therefore, to compute p_2, we have to maximize the expression

$$-p_2\alpha_a v\left(\theta^E J^a + \sigma_K\frac{\partial J^a}{\partial k}\right) + \frac{1}{2}(p_2)^2\left(v^2(\alpha_a)^2 J\right).$$

Now, again by (20), $\frac{\partial J^a}{\partial k} = \frac{\partial\chi}{\partial k}J^a$. Moreover, by definition of the utility functions, it is clear that $J^a \leq 0$. We are therefore led to the problem of minimizing

$$-p_2\alpha_a v\left(\theta^E + \sigma_K\frac{\partial\chi}{\partial k}\right) + \frac{1}{2}(p_2)^2 v^2(\alpha_a)^2.$$

The result is easily obtained by minimizing the given polynomial of degree 2 and, together with the analogous calculation for p_1 leads to the following formulas.

Corollary 1.2 *Let $a \in I$. Let χ be a solution of (5), define J^a by (20), and let X^E and therefore θ^E be given according to subsection 1.1. Then the solution (p_1^a, p_2^a) of the optimal control problem (23) at $(X_s^S, K_s, V_s^a) = (x_1, k, v)$ is given by*

$$p_1^a = \frac{b^S + x_1 \sigma^S \dfrac{\partial \chi}{\partial x_1}}{v(\sigma^S)^2 \alpha_a},$$

$$p_2^a = \frac{\theta^E + \sigma_K \dfrac{\partial \chi}{\partial k}}{v \alpha_a}.$$

Accordingly, the quantity

$$\pi_{2,s}^a = V_s^a p_{2,s}^a = \frac{1}{\alpha_a}\left(\theta^E(s, X_s^S, K_s) + \sigma_K(s, K_s)\frac{\partial \chi}{\partial k}(s, X_s^S K_s)\right)$$

is the optimal amount of money to be invested in X^E by agent a at time $s \in [t, T]$.

1.3.3 PDE for equilibrium price of risk

Next we shall obtain a semi-linear parabolic PDE for the equilibrium price of external risk, determined in the stochastic setting by (10). To simplify its derivation, let us further abbreviate

$$\begin{aligned} g &= \overline{\alpha} \sum_{\alpha \in I} g^a, \\ \varphi &= \overline{\alpha} \sum_{\alpha \in I} \varphi^a + \frac{1}{2}(\theta^S)^2, \\ R_s &= \int_0^s \varphi(t, X_t^S, K_t) dt. \end{aligned}$$

In these terms, we obtain $\overline{\alpha}\overline{H} = g(\tau, X_{1,\tau}, K_\tau) + R_\tau$ and we can rewrite the BSDE (10) as

$$-Y_s = h_s - R_s = g(\tau, X_{1,\tau}, K_\tau) - \int_s^T z_t dW_t - \int_s^T \theta_t^S z_{1,t} dt - \frac{1}{2}\int_s^T z_{2,t}^2 dt + \int_s^T \varphi_t dt. \quad (24)$$

Now using the nonlinear Feynman-Kac formula in its version of Theorem 1.3, we see that z and thus θ^E can be obtained by computing the function u, which is the classical solution of a backward nonlinear PDE, provided the coefficient and risk functions satisfy the following regularity hypotheses.

(H1) the system state domain $\mathcal{O}$ is given by the cylinder $]0, T[\times]0, \infty) \times I\!R$ (there is no stopping time, no Dirichlet condition).

(H2) the terminal income g is a $C^1(]0,\infty) \times I\!R)$ function and all the other coefficients $b^S, \sigma^S, b_K, \sigma_K, \varphi$ are $C^\infty([0,T]\times]0,\infty) \times I\!R)$ functions.

(H3) $(\sigma^S)^2, \sigma_K^2$ are bounded below by positive constants.

Corollary 1.3 *Assume that the domain and coefficient functions satisfy the hypotheses* **(H1), (H2), (H3).** *Let u be a classical solution of the nonlinear PDE*

$$\begin{cases} -\dfrac{\partial u}{\partial t} - b_K \dfrac{\partial u}{\partial K} - \dfrac{1}{2}\left(x_1^2(\sigma^S)^2 \dfrac{\partial^2 u}{\partial x_1^2} + \sigma_K^2 \dfrac{\partial^2 u}{\partial K^2}\right) + \dfrac{1}{2}\left(\sigma_K \dfrac{\partial u}{\partial K}\right)^2 - \varphi(t, x_1, k) = 0 \quad in\ \mathcal{O}, \\ u = -g \ on\ \partial\mathcal{O}. \end{cases} \tag{25}$$

Then by setting

$$\begin{aligned} Y_s &= R_s - h_s = u(s, X_s^S, K_s), \\ z_s &= \begin{pmatrix} X_s^S \sigma^S & 0 \\ 0 & \sigma_K \end{pmatrix} Du(s, X_s^S, K_s) \end{aligned} \tag{26}$$

we obtain the unique solution of BSDE (24).

Proof:

We shall prove that our system, under a regular change of variables, can be written in the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - Lu - F(t, x, u, \gamma^*(t,x)Du) = 0 \text{ in } \mathcal{O}, \\ u(t,x) = g(t,x) \text{ on } \partial\mathcal{O}, \end{cases} \tag{27}$$

with

$$Lu = \frac{1}{2}\left(\gamma_1^2 \frac{\partial^2 u}{\partial x^2} + \gamma_2^2 \frac{\partial^2 u}{\partial y^2}\right),$$

and coefficients γ_1, γ_2 whose squares are bounded below by positive constants. Let us begin formally. Suppose that $\widetilde{u}$ is a solution, in some sense, of (25). Consider a function $\widetilde{w}$ defined by

$$\widetilde{w}(t,x,y) = \widetilde{u}(T-t, e^x, y) \text{ on } [0,T] \times I\!R^2.$$

It is straightforward to see that $\widetilde{w}$ is associated with the system (27) with terminal condition

$$f(x,y) = -g(T, e^x, y), \tag{28}$$

coefficients

$$\gamma_1(x) = \sigma^S(e^x), \quad \gamma_2(y) = \sigma_K(y), \quad x, y \in I\!R,$$

and generator

$$F(t, (x,y), w, (w_x, w_y)) = -\frac{1}{2}\gamma_1^2 w_x + \gamma_2 w_y - \frac{1}{2}\gamma_2^2 w_y^2 + \varphi(T-t, e^x, y). \tag{29}$$

Due to **(H2)**, f and F are regular functions, and **(H3)** guarantees the uniform ellipticity of the operator L. Hence the assumptions of Theorem 1.4 hold. There exists a unique classical solution $w \in C\left([0,T], C^1\left(\mathbb{R}^2\right)\right) \cap C^\infty\left([0,T[\times \mathbb{R}^2\right)$ of the system (27),(28) and (29). Now we can define rigorously u by setting

$$u(t,x,k) = w\left(T-t, \log(x), k\right) \text{ for } (t,x,k) \in [0,T]\times]0,\infty) \times \mathbb{R}.$$

This function has clearly the announced regularity. Finally, using (26) and Itô's formula, it is easy to check that u solves (25) in the classical sense.□

Recall that θ^E is defined as a partial derivative of the function u in (30). The preceding result allows us to justify this definition, moreover we obviously have

$$\theta^E \in C\left([0,T], C\left(]0,\infty)^p \times \mathbb{R}^d\right)\right) \cap C^\infty\left([0,T[\times]0,\infty)^p \times \mathbb{R}^d\right).$$

In particular θ^E is a Lipschitz continuous function, so the process X^E is well-defined by (30).

Recalling the definitions of z above and of θ in subsection 1.1, we can use Corollary 1.3 to compute explicitly θ^E and thus the equilibrium insurance asset process X^E through the following formulas

$$\theta^E_s = \sigma_K \frac{\partial u}{\partial K}(s, X^S_s, K_s), \tag{30}$$

$$dX^E_s = X^E_s\left(\theta^E_s ds + dW_{2,s}\right). \tag{31}$$

1.3.4 PDE for moments of X^E

We finally derive a linear PDE enabling us to compute the moments of the insurance asset process X^E in case σ_K is invertible. Under this hypothesis, we can write

$$dW_{2,s} = \frac{dK_s - b_K(s,K_s)ds}{\sigma_K(s,K_s)}.$$

This leads us to an integral expression for X_2 in terms of the trajectories of the process K, given by

$$\log\left(X^E_t\right) = \log\left(x_2\right) + \int_0^t \left[\theta^E(s, X^S_s, K_s) - \frac{1}{2} - \frac{b_K(s,K_s)}{\sigma_K(s,K_s)}\right] ds + \int_0^t \frac{1}{\sigma_K(s,K_s)} dK_s.$$

If σ_K is even constant which is the case if for example K is an Ornstein-Uhlenbeck process, we have

$$\log\left(X^E_t\right) = \log\left(x_2\right) + \int_0^t \left[\theta^E(s, X^S_s, K_s) - \frac{1}{2} - \frac{b_K(s,K_s)}{\sigma_K}\right] ds + \frac{K_s - K_0}{\sigma_K}.$$

In this case the expectation of X^E_t with the initial conditions $X_{1,0} = x_1, X_{2,0} = x_2, K_0 = k$ may be expressed by the formula

$$\begin{aligned} \mathbb{E}_{x_1,x_2,k}[X^E_t] &= x_2 e^{-\frac{k}{\sigma_K}} \mathbb{E}_{x_1,k}\left[e^{\frac{K_s}{\sigma_K}} \exp\left(\int_0^t \left[\theta^E(s, X^S_s, K_s) - \frac{1}{2} - \frac{b_K(s,K_s)}{\sigma_K}\right] ds\right)\right] \\ &= x_2 e^{-\frac{k}{\sigma_K}} f(t, x_1, k). \end{aligned} \tag{32}$$

In this case again, we may translate its computation into analysis by associating with this expectation a PDE possessing a simple derivation from the forward linear Feynman-Kac formula in Theorem 1.1.

Corollary 1.4 *Suppose $\sigma_K \neq 0$ is constant. Define*

$$f(s, x_1, k) = \frac{1}{x_2} e^{\frac{k}{\sigma_K}} \mathbb{E}_{x_1,x_2,k}[X_t^E], \quad s \in [t, T], x_1, x_2, k \in \mathbb{R}.$$

Let L be the infinitesimal generator of the diffusion (X^S, K), i.e.

$$Lf = \begin{pmatrix} x_1 b^S \\ b_K \end{pmatrix} Df + \frac{1}{2} trace \left\{ \begin{pmatrix} x_1 \sigma^S & 0 \\ 0 & \sigma_K \end{pmatrix} D^2 f \right\}.$$

Then f is the solution of the forward linear PDE

$$\begin{cases} \frac{\partial f}{\partial t} - Lf - \left(\theta^E - \frac{1}{2} - \frac{b_K}{\sigma_K}\right) f & = 0 \\ f(0, x_1, k) & = \exp\left(\frac{k}{\sigma_K}\right). \end{cases} \tag{33}$$

Remark that the implicit dependence on x_2 in the definition of f above can indeed be suppressed, since X^E depends only in a multiplicative way on its initial condition x_2.

Proof:

The result is directly given by the forward form of theorem 1.1 with $b = \begin{pmatrix} x_1 b^S \\ b_K \end{pmatrix}$, $\sigma = \begin{pmatrix} x_1 \sigma^S & 0 \\ 0 & \sigma_K \end{pmatrix}$, $f = 0$, $g = \exp\left(\frac{k}{\sigma_K}\right)$, and $h = \theta^E - \frac{1}{2} - \frac{b_K}{\sigma_K}$. For obtaining uniform ellipticity of the diffusion part, a procedure as in the proof of Corollary 1.3, based on a logarithmic coordinate change in x_1, again applies. □

With the same technique, we can compute any moment of X^E. For all $n \in \mathbb{N}$,

$$\mathbb{E}_{x_1,x_2,k}[X_{2,t}^n] = x_2 \exp\left(-n \frac{k}{\sigma_K}\right) f_n(t, x_1, k),$$

where f_n is the solution of

$$\begin{cases} \frac{\partial f_n}{\partial t} - Lf_n - n\left(\theta^E - \frac{1}{2} - \frac{b_K}{\sigma_K}\right) f_n & = 0, \\ f_n(0, x_1, k) & = \exp\left(n \frac{k}{\sigma_K}\right). \end{cases}$$

1.4 Examples

We now specify some climate processes, stock price models, and risk exposure functionals we shall investigate in our numerical simulations in section 3.

1.4.1 Temperature process

The climate process affecting the agents on our market will model the local temperature (of air, of ocean water) evolution as a random function of time. It is therefore usually modelled as a one-dimensional stochastic process, also for the simplicity of qualitative numerical simulations. The reduced physical models they come from usually lead to finite dimensional stochastic equations and describe some nonlinear interaction between finitely many physical quantities including the local temperature. We shall base our simulations on two of these. The first one comes from a nonlinear two-dimensional stochastic differential equation coupling the thermocline depth in some area of the South Pacific with the sea surface temperature (see [1]). The system turns out to be an autonomous nonlinear stochastic oscillator which in some parameter regimes acts as a stochastically perturbed bistable differential equation with an intrinsically defined periodicity. For our purposes, we mimic it by taking a one-dimensional SDE driven by a Brownian motion. It describes the motion of a state variable travelling through a bi-stable potential landscape, with an explicit periodic dependence of the potential shape creating a non-autonomous stochastic system that retains the characteristics of the two-dimensional model. The second one comes from a 15-dimensional linear SDE of the Ornstein-Uhlenbeck type with a $15 \times 15-$matrix with non-trivial rotational part and entries determined by satellite measurements which is used in linear prediction models for ENSO (see [27]). It creates a diffusion with non-trivial rotation numbers implying random periodicity for the sea surface temperature contained in the model. For our qualitative problems we may describe the temperature curve as a simple mean-reverting linear sde with an additional deterministic periodic forcing. This leads to the following concrete examples.

1. **Ornstein-Uhlenbeck**. A simple model for a temperature process fluctuating around an average value $K_a \in I\!R$ is given by an Ornstein-Uhlenbeck process (centered in K_a), determined by

   $$dK_s = C(K_a - K_s)ds + \sigma_K dW_{2,s},$$

   where $C > 0$ is the strength of restoring force to K_a, and $\sigma_K > 0$ the volatility. We use this process in our simulations (cf. model A in section 3).

2. **Ornstein-Uhlenbeck with periodic term**. This is a rudimentary version of the temperature part of the model used for ENSO prediction. It is obtained by modifying the preceding example in adding a periodical perturbation

   $$dK_s = \left[C(K_a - K_s) + C' \sin\left(\frac{2\pi}{T_0}s\right)\right] ds + \sigma_K dW_{2,s},$$

   where $C' > 0$ is the amplitude and T_0 the period of the sinusoidal periodic term.

3. **Periodically forced bi-stable temperature**. This is a phenomenological version of the stochastic oscillator model for ENSO sketched above, where intrinsic periodicity is replaced by a non-autonomous periodic dependence of the bi-stable

function U. U is a double-well potential function, for example $U(k) = \frac{k^4}{4} - \frac{k^2}{2}, k \in \mathbb{R}$. The diffusion process K given by the SDE

$$dK_s = -U'(K_s)ds + Q.\sin\left(\frac{2\pi}{T_0}s\right)ds + \sqrt{\varepsilon}dW_{2,s}$$

models temperature in a bi-stable environment. For ε chosen appropriately, the trajectories of K are almost periodic. This phenomenon is investigated under the name *stochastic resonance.* See [17] for a review. We use this process in our simulations (cf. model B and C in section 3).

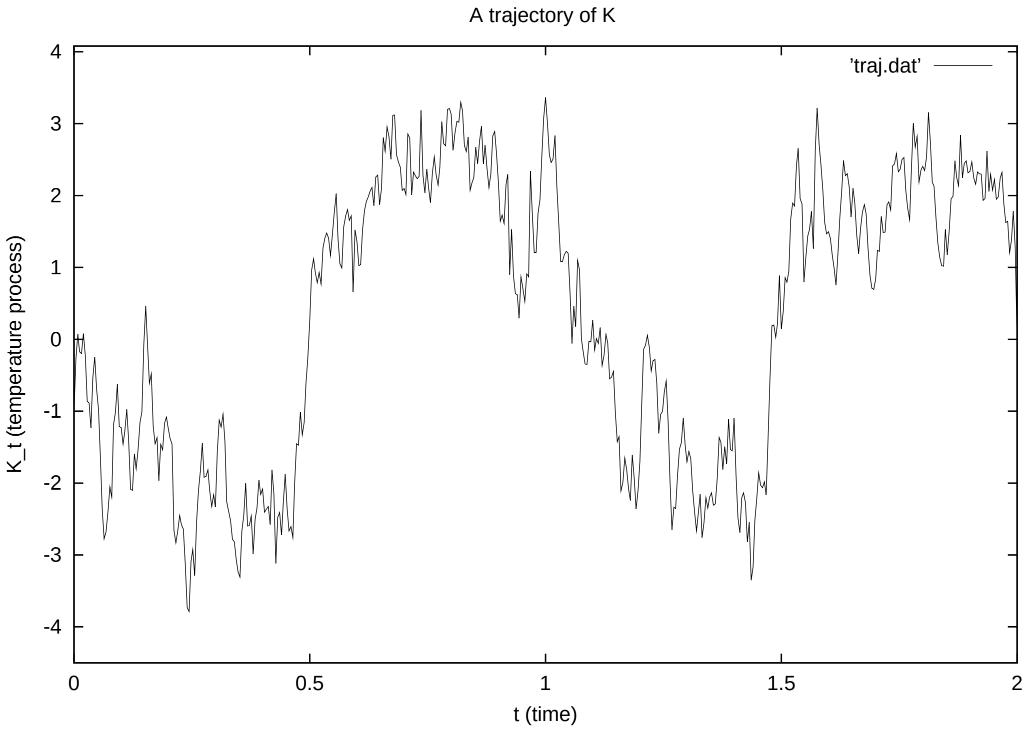

Figure 1: A sample path of the bi-stable temperature process K.

1.4.2 Asset price process

The stock price model is for simplicity taken to be a geometric Brownian motion.

1. **Black-Scholes**. The stock price X_t^S at time t is described by

$$dX_s^S = X_s^S \left(b^S ds + \sigma^S dW_{1,s}\right), \quad s \in [t, T], \tag{34}$$

where $b^S > 0$ is the rate and $\sigma^S > 0$ the volatility.

1.4.3 Risk exposure of the agents

Three typical qualitative risk exposures will be considered: the one of a fisher describing profits from fishing whose efficiency depends on the surface temperature of the ocean and is optimal at some fixed temperature value while it drops off as temperature deviates from this optimum. A rice farmer's risk exposure functional may be quite similar, his interests, however, complementary to the fisher's. Think of the sea surface temperature process possessing two meta-stable equilibria, a low and a high one. As explained earlier and corresponds with the ENSO scenarios, the fisher may have his temperature of optimal income near the lower equilibrium, while the farmer might profit more from higher precipitation rates at the higher temperature equilibrium. This in particular means that the fisher profits from temperature values under which the farmer suffers most, and vice versa. The exposure of a bank may not directly dependent on climate risk.

1. **Fisher**. Let $\tau = T$, the final time of the trading interval. Let K be a local sea surface temperature, and imagine a fishing company $f \in I$ that makes most profits if the temperature is near an optimal value k_1. We can describe the income H^f of this company on the period $[0, T]$ qualitatively by

$$H^f = \int_0^T \varphi^f(K_s) ds,$$

   where φ^f is a positive function taking its global maximum in k_1, for example

$$\varphi^f(k) = e^{-(k-k_1)^2}.$$

2. **Farmer**. The (rice) farmer or farming company may have an exposure of the same type as the fisher. The optimal income is just obtained at a different value k_r, which is higher than k_f, and may be given by the second meta-stable point of a bi-stable process K. The income of the farmer may therefore be described by

$$H^r = \int_0^T \varphi^r(K_s) ds,$$

   where φ^r is a positive function taking its global maximum in k_r, for example

$$\varphi^f(k) = e^{-(k-k_2)^2}.$$

   If we work with a bi-stable K, we see immediately that farmer and fisher have complementary interests, and therefore are likely to profit from trading the climate risk among each other.

3. **Bank**. As an additional agent, we can consider a bank b whose profits only come from its portfolio management from investment on the financial market, and which participates in the climate risk share only by investing in the insurance security X^E. So its exposure functional will be the trivial $H^b = 0$.

2 Numerical approximations results

As the main result of section 1, the stochastic equations relevant to our model have been translated into linear or non-linear PDE. The main equations we obtained this way are given by

- the backward linear PDE (19) describing the value function and providing the optimal strategy for any agent on the market.
- the forward linear PDE (33) computing the moments of X^E.
- the backward non-linear PDE (25) providing the coefficient θ^E which determines the insurance asset X^E.

In this section we design numerical schemes approximating the solutions of these parabolic PDEs and prove their convergence. We shall employ a method initiated by Barles and Souganidis [3] based on the well known stability results for viscosity solutions (see [10], [2] for a general presentation) to derive a basic convergence result which will be applicable to our schemes.

In subsection 2.1 we recall theorem 2.1 from [3], which deals with general fully non-linear second order PDEs, and proves that any monotone, stable and consistent scheme converges, provided that there exists a comparison result for the limiting equation.

In the next subsections, we shall explain the numerical approximation schemes we use for our simulations, starting in the linear case[1] in subsection 2.2, which can be used to compute the solution of PDEs (19) and (33).

In subsection 2.3, we consider a non-linear equation with quadratic terms, like PDE (25). For such an equation, the comparison result is known (see Kobylanski [22]) but we cannot use classical schemes, and we present a new algorithm, generalizing finite differences. The non-linearity of the original equation weakens the stability and monotonicity properties of such schemes, and we solve this problem showing that under a stronger Courant-Friedrichs-Levy (C.F.L.) condition, this scheme satisfies the conditions required by the convergence result.

2.1 Convergence

To state our convergence result in a fairly general framework, let $\mathcal{O}$ be an open subset of $]0,T[\times I\!\!R^n$, and let us consider a general possibly non-linear PDE of the second order written in the forward form

$$\begin{cases} \frac{\partial v}{\partial t} + G(t,x,v,Dv,D^2v) & = 0 \text{ in } \mathcal{O}, \\ v & = \Psi \text{ on } \partial\mathcal{O}. \end{cases} \tag{35}$$

Here G and Ψ are scalar functions, respectively continuous on $\mathcal{O} \times I\!\!R \times I\!\!R^n \times \mathcal{S}^n$ and $\partial\mathcal{O}$, and $\mathcal{S}$ denotes the set of symmetric $n \times n-$matrices. Let $\varepsilon > 0$. We consider time-explicit schemes of the form

[1] already presented as an example in [3]

$$\begin{cases} v_\varepsilon(t+\varepsilon,x) &= S(\varepsilon)v_\varepsilon(t,x) \quad \text{if} \quad (t,x)\in\mathcal{O}, \\ v_\varepsilon(t+\varepsilon,x) &= \Psi(t+\varepsilon,x) \quad \text{in any other case,} \end{cases} \tag{36}$$

where, for all $\varepsilon > 0$, $S(\varepsilon)$ is an operator defined on $L^\infty(\mathcal{O})$ with values in $L^\infty(\mathcal{O})$.

We assume that the following assumptions hold.

Monotonicity :
For any $\varepsilon > 0$, and any function $u, v \in L^\infty(\mathcal{O})$,

$$S(\varepsilon)u \leq S(\varepsilon)v \text{ if } u \leq v \text{ in } \mathcal{O}. \tag{37}$$

Let us note that this assumption can be relaxed (see [3] remark 2.1 p. 276), this inequality needs only to hold within up to $o(\varepsilon)$ terms.

Commutation with constants :
For any $\xi \in I\!R$,

$$S(\varepsilon)(u+\xi) = S(\varepsilon)u + \xi. \tag{38}$$

Stability :

$$\begin{array}{c}\text{There exists a sequence } (v_\varepsilon)_{\varepsilon>0} \text{ of solutions to the scheme (36)} \\ \text{which are locally uniformly bounded in } L^\infty(\overline{\mathcal{O}}).\end{array} \tag{39}$$

Consistency :
For any $(t,x) \in \mathcal{O}$ and any test function $\phi \in C_b^\infty(\overline{\mathcal{O}})$,

$$\lim_{\substack{\varepsilon \to 0 \\ (s,y)\to(t,x)}} \frac{\phi(s,y) - S(\varepsilon)\phi(s,y)}{\varepsilon} = G(t,x,\phi(t,x),D\phi(t,x),D^2\phi(t,x)). \tag{40}$$

We also assume that a **strong comparison result** holds for the equation (35) (see [2], [3]), i.e.

$$\begin{array}{c}\text{If } u \text{ is a bounded viscosity subsolution to (35)} \\ \text{and } v \text{ is a bounded viscosity supersolution to (35),} \\ \text{then } u \leq v \text{ on } \overline{\mathcal{O}}.\end{array} \tag{41}$$

Under these conditions, we have the following convergence result derived in [3], Theorem 2.1, p. 275, and also in [7], Theorem 2.4.5, page 81.

Theorem 2.1 *Under the assumptions (37), (38), (39), (40) and (41), the solution v_ε of the scheme (36) converges locally uniformly as $\varepsilon \to 0$ to the unique viscosity solution of PDE (35).*

We note that a unique classical solution of (35) coincides with the viscosity solution.

2.2 Approximation schemes for linear equations

Let us first treat a general backward parabolic linear PDE of the second order. Note that equations (5) and (33) are of this form:

$$\begin{cases} -\dfrac{\partial u}{\partial t} - b.Du - \dfrac{1}{2}\mathrm{trace}\left[\sigma\sigma^* D^2 u\right] & = 0 \quad \text{in} \quad \mathcal{O}, \\ u & = \Psi \quad \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{42}$$

We here assume that $b : \mathcal{O} \to I\!R^n$ and $\sigma : \mathcal{O} \to I\!R^{n\times n}$ are Lipschitz continuous, Ψ is continuous and also that $a = \sigma\sigma^*$ is a diagonal dominant matrix, i.e.

$$\text{for all } j,\ \sigma\sigma^*_{i,i} \geq \sum_{j\neq i} \left|\sigma\sigma^*_{i,j}\right|.$$

We use a time-explicit upwind finite differences scheme (see [7], section 2.4, page 65, and [25]). Let $\Delta t = \varepsilon > 0$ and $\Delta x = \Delta x(\varepsilon) > 0$ be the mesh size of a space-time grid. We denote by $\mathcal{V}_{\Delta x}$ the set of neighboring points of $x = 0$ on the space grid of mesh size Δx.

Let us describe our scheme in the particular case $\mathcal{O} =]0, T[\times I\!R^n$.

Scheme 2.1 *Given $\Delta t > 0$ and $\Delta x > 0$, we construct a function $\overline{u}$ such that*

$$\overline{u}(T, x) = \Psi(x),$$

and

$$\overline{u}(t - \Delta t, x) = \sum_{h\in\mathcal{V}_{\Delta x}} p(x, h)\overline{u}(t, x + h) = S(\Delta t, \Delta x)_{\overline{u}(t,.)}(x), \tag{43}$$

with :

$$\begin{aligned} p(x, 0) &= 1 - \frac{\Delta t}{\Delta x}\sum_{i=1}^{d} |b_i|(x) - \frac{\Delta t}{(\Delta x)^2}\sum_{i=1}^{d}\left(a_{ii} - \sum_{j\neq i}|a_{ij}|\right)(x), \\ p(x, \pm e_i\Delta x) &= \frac{\Delta t}{(\Delta x)^2}(b_i)^{\pm}(x) + \frac{1}{2}\frac{\Delta t}{(\Delta x)^2}\left(a_{ii} - \sum_{j\neq i}|a_{ij}|\right)(x), \\ p(x, (e_i \pm e_j)\Delta x) &= p(x, -(e_i \pm e_j)\Delta x) = \frac{1}{2}\frac{\Delta t}{(\Delta x)^2}(a_{ij})^{\pm}(x), \\ p(x, h) &= 0 \quad \textit{in any other case.} \end{aligned}$$

Consistency, monotonicity and commutation with constants of the scheme are straightforward. Under the C.F.L. condition

$$\Delta t \leq (\Delta x)^2\left\{\sum_{i=1}^{d}\left(\Delta x|b_i| + a_{ii} - \sum_{j\neq i}|a_{ij}|\right)\right\}^{-1},$$

the scheme is stable, because $p(x,h) \geq 0$ and $\sum_{h \in \mathcal{V}_{\Delta x}} p(x,h) = 1$, and S is a contraction. In fact, we have

$$|\overline{u}(t-\Delta t, x)| \leq ||\overline{u}(t,.)||_\infty \leq ... \leq ||\Psi||_\infty.$$

Moreover, the classical uniqueness result of theorem 1.2 implies that a strong comparison result holds for equations (5) and (33) (see also theorem 3.3 p.18 in [10]).

Then, for both these equations, theorem 2.1 proves that $\overline{u}$ converges locally uniformly to the unique continuous viscosity solution, thus the unique classical solution, as Δt and Δx converge to 0.

2.3 Approximation schemes for non-linear equations with quadratic terms

We now consider a more complicated equation, the following general semilinear PDE with quadratic terms which generalizes (25):

$$\begin{cases} -\dfrac{\partial u}{\partial t} - b.Du - \dfrac{1}{2}\text{trace}\left[\sigma\sigma^* D^2 u\right] + ||MDu||^2 & = 0 \quad \text{in} \quad \mathcal{O}, \\ u & = \Psi \quad \text{on} \quad \partial\mathcal{O}, \end{cases} \tag{44}$$

where M is a $n \times n-$matrix.

This kind of equation has been studied in the viscosity solutions framework in Kobylanski [22]. In particular, if we assume that the coefficients b, σ and M are Lipschitz functions of the state variable with linear growth at infinity, theorem 3.3.2 p. 582 in [22] states that a strong comparison result holds for (44).

Since $||MDu||^2 = MDu \cdot MDu = \text{trace}\,[MDuDu^*M^*] = \text{trace}\,[M^*MDuDu^*] = (M^*MDu).Du$, we can rewrite (44) as

$$\begin{cases} -\dfrac{\partial u}{\partial t} - (b - M^*MDu).Du - \dfrac{1}{2}\text{trace}\left[\sigma\sigma^* D^2 u\right] & = 0 \quad \text{in} \quad \mathcal{O}, \\ u & = \Psi \quad \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{45}$$

This gives us a simple idea for defining an approximating scheme which we again describe in the case $\mathcal{O} =]0,T[\times I\!R^n$.

Scheme 2.2 *Given $\Delta t > 0$ and $\Delta x > 0$, we construct a function $\overline{u}$ such that*

$$\overline{u}(T,x) = \Psi(x),$$

and

$$\overline{u}(t-\Delta t, x) = \sum_{h \in \mathcal{V}_{\Delta x}} \widetilde{p}(x,h)\overline{u}(t, x+h) = \widetilde{S}(\Delta t, \Delta x)_{\overline{u}(t,.)}(x). \tag{46}$$

This time the transition coefficients $\widetilde{p}$ depend on $\overline{u}$ in the following way

$$\begin{array}{rcl}\widetilde{p}(x,0) &=& p(x,0) - \dfrac{\Delta t}{(\Delta x)^2}\displaystyle\sum_{i=1}^{d} \left| \left(M^* M \delta^{\Delta x}\overline{u}(t,x)\right)_i \right|, \\ \widetilde{p}(x,\pm e_i \Delta x) &=& p(x,\pm e_i\Delta x) + \dfrac{\Delta t}{(\Delta x)^2}\left(\left(M^* M \delta^{\Delta x}\overline{u}(t,x)\right)_i\right)^{\pm}, \\ \widetilde{p}(x,(e_i \pm e_j)\Delta x) &=& \widetilde{p}(x,-(e_i \pm e_j)\Delta x) = p(x,(e_i \pm e_j)\Delta x), \\ \widetilde{p}(x,h) &=& 0 \quad \textit{in any other case,}\end{array}$$

where

$$\delta^{\Delta x}\overline{u}(t,x) = \begin{pmatrix} \overline{u}(t,x+e_1\Delta x) - \overline{u}(t,x) \\ \overline{u}(t,x+e_2\Delta x) - \overline{u}(t,x) \\ \dots \\ \overline{u}(t,x+e_d\Delta x) - \overline{u}(t,x) \end{pmatrix}.$$

It is straightforward to check the consistency of this scheme with (44), and the property of commutation with constants. Under the C.F.L. condition

$$\Delta t \leq (\Delta x)^2 \left\{ \sum_{i=1}^{d} \left(\Delta x |b_i| + 2||\Psi||_\infty ||M||_\infty^2 + a_{ii} - \sum_{j\neq i} |a_{ij}| \right) \right\}^{-1}, \tag{47}$$

the scheme is also stable : we have $\widetilde{p}(x,h) \in [0,1]$ for all (x,h) and $\displaystyle\sum_h \widetilde{p}(x,h) = 1$ for all x. So $\widetilde{S}$ is a contraction and thus $||\overline{u}||_\infty \leq ||\Psi||_\infty$.

To check the monotonicity condition, let us further assume that

$$\Delta t \leq (\Delta x)^3. \tag{48}$$

For any functions u and v such that $u \leq v$, we obtain the following chain of inequalities, denoting $\hat{p} = \widetilde{p} - p$ and adding the subscript p_u to indicate that the transition matrix belongs to u etc:

$$\begin{array}{rcl}\widetilde{S}(\Delta t,\Delta x)(u) - \widetilde{S}(\Delta t,\Delta x)(v) &=& \displaystyle\sum_{h\in\mathcal{V}_{\Delta x}} \left(\widetilde{p}_u(x,h)u(t,x+h) - \widetilde{p}_v(x,h)v(t,x+h)\right) \\ &=& \displaystyle\sum_{h\in\mathcal{V}_{\Delta x}} p(x,h)\left(u(t,x+h) - v(t,x+h)\right) \\ && + \displaystyle\sum_{h\in\mathcal{V}_{\Delta x}} \left(\hat{p}_u(x,h)u(t,x+h) - \hat{p}_v(x,h)v(t,x+h)\right) \\ &\leq& \displaystyle\sum_{h\in\mathcal{V}_{\Delta x}} \left(\hat{p}_u(x,h)u(t,x+h) - \hat{p}_v(x,h)v(t,x+h)\right) \\ &\leq& \dfrac{\Delta t}{(\Delta x)^2} C \leq \Delta x C.\end{array}$$

Here, thanks to stability, the constant C depends only on bounds on Ψ, M, b, σ and on the Lispschitz constants of b and σ.

Moreover, according to [22], we have a strong comparison result for the PDE (44).

Hence, Theorem 2.1 allows us to conclude that $\overline{u}$ converges locally uniformly to the unique continuous viscosity solution u of (44), as Δt and Δx converge to 0.

3 Simulations and their interpretations

We now choose different types of simple toy agents and different temperature models (as given in section 1.6.3). We concentrate on simulating the expectation of the additional security X^E (in subsection 3.1), the maximal expected utility J^a for each agent (in subsection 3.2) and the optimal strategy of investment in X^E (in subsection 3.3). We use the following concrete models :

Model A
The time horizon is chosen to be $T = 2$. We use an Ornstein-Uhlenbeck process to describe the climate process K, with the following coefficients :

$$dK_s = -K_s + \frac{1}{2} dW_{2,s}, \quad s \in [0, T].$$

Here we consider only two model agents, a fisher and a bank as described in section 1.6.1. The fisher's random income function is

$$H^f = \int_0^T \varphi^f(K_s) ds,$$

with $\varphi^f(k) = 5 \exp\left(-10k^2\right)$, for all $k \in I\!R$. This means that the optimal temperature for the fisher is normalized to be 0. The bank has no risky income, i.e. $H^b = 0$. We assume that each agent uses the risk aversion coefficient $\alpha^f = \alpha^b = 1$.

Model B
The temperature is now modelled by a periodically forced bi-stable temperature process with coefficients

$$dK_s = -8(K_s^3 - K_s) - \sin(2\pi s) + 4.5 dW_{2,s}, \quad s \in [0, T].$$

See Figure 1 for a sample path of this process. Again, we choose $T = 2$ for the time horizon, i.e. 2 periods of the temperature process. This process is close to the high temperature value $k_r = 2.5$ for $t \in [0; 0.5] \cup [1; 1.5]$ and symmetrically close to the low value $k_f = -2.5$ for $t \in [0.5; 1] \cup [1.5; 2]$. Again we consider only two agents, a fisher and a farmer with respective income

$$H^f = \int_0^T 5 \exp\left(-10(K_s - k_f)^2\right) ds \text{ and } H^r = \int_0^T 5 \exp\left(-10(K_s - k_r)^2\right) ds,$$

where the optimal temperature is $k_f = -2.5$ for the fisher and $k_r = 2.5$ for the farmer, which coincide with the bistable states of the temperature process. We again assume that each agent uses the risk aversion coefficient $\alpha^f = \alpha^r = 1$.

Model C
This model uses the same characteristics as model B except for the time horizon, which is now chosen to be $T = 3/2$, i.e. 3 half-periods for the temperature process K. This gives an advantage to the farmer, since the temperature spends 1 unit of time i.e. 2/3 of the trading interval near the meta-stable state favorable for the farmer, and only 0.5 units of time near its low meta-equilibrium favorable for the fisher.

In all the models, the share price is a geometrical Brownian motion given by (34) with very strong coefficients $b^S = 1$ and $\sigma^S = 1$.

3.1 Expectation of X^E

Here we exhibit the expectation of X_t^E at the same time $t = 1.5$ for each model, as a function of the initial condition (x_1, k) at time $t = 0$. X^E is starting from 1 at time $t = 0$.

3.1.1 Model A

We observe that $\mathbb{E}[X^E]$ has a minimum if the temperature starts from the value 0 which is optimal for the fisher. Indeed, in this case, the fisher's income is maximal, since the temperature will only slightly oscillate around 0. So there is no need to transfer risk from the fisher to the bank : the expectation of X^E (starting from $k = 0$) almost stays at the initial value 1. This can serve as an indication that we could interpret the size of X^E as an appreciation rate for the trading of climate risk among the affected agents.

If, on the other hand, the initial temperature is far from 0, the fact that the expectation of X^E grows with time indicates that the fisher has an interest to invest in X^E. In this case the growth of X^E compensates the smaller income of the fisher.

3.1.2 Model B

The dependence on K seems reversed in this model as compared to model A. We now see that the expectation of X^E is maximal when starting from $k = 0$, i.e. in the middle between the optimal temperatures. At this temperature obviously both agents like to trade risk, since on the scale between -2.5 and 2.5 it corresponds to the worst situation for the totality of the affected agents. This is why X^E is expected to be higher.

3.1.3 Model C

This case is very similar to the preceding one. We just observe that the maximum of the expectation has been translated to lower temperatures, to account for the difference of exposition of the agents.

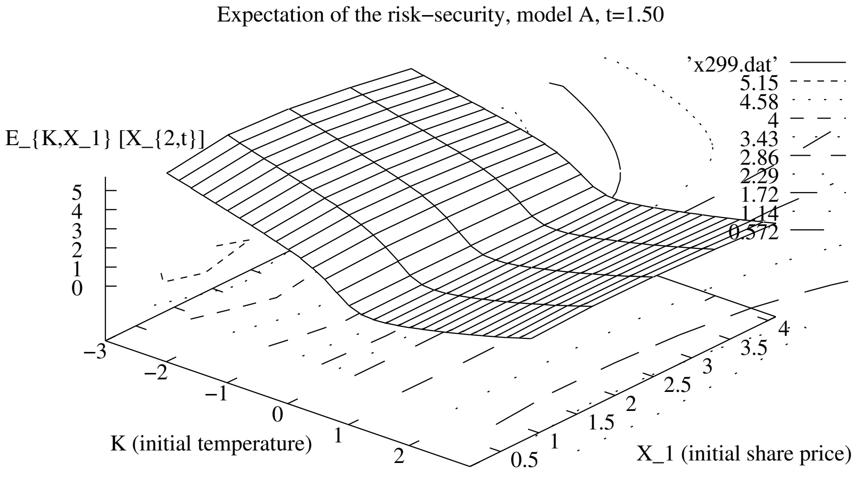

Figure 2: The expectation of X_t^E (model A).

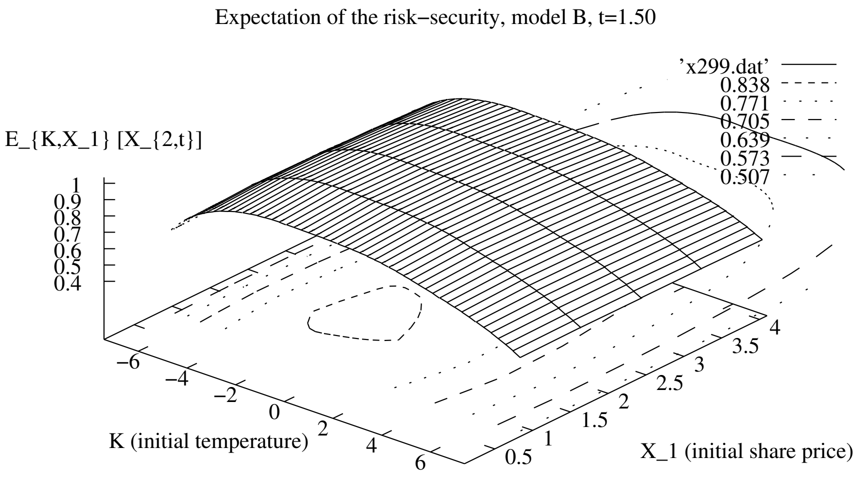

Figure 3: The expectation of X_t^E (model B).

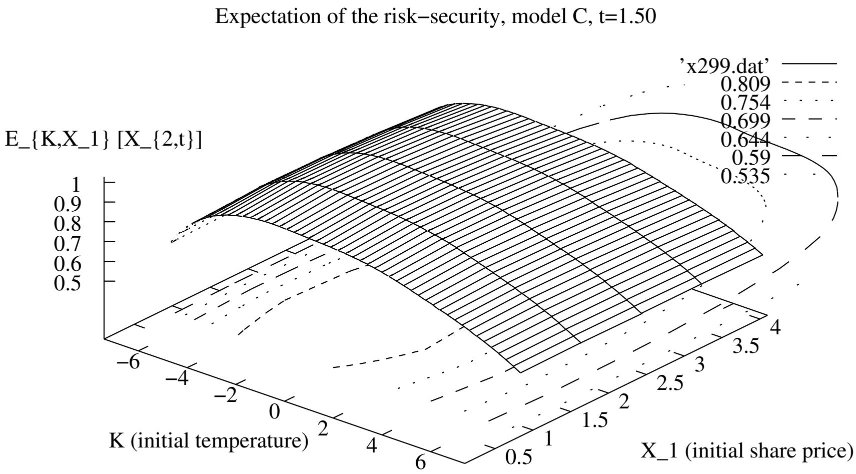

Figure 4: The expectation of X_t^E (model C).

3.2 Optimal value J^a

We now turn to numerical simulations of the underlying optimal control problem. First we will show the value J^a, the optimal utility, for both agents involved, at different times $t \in [0,1]$, as a function of the current value of the temperature k at time t. Due to simplicity of our model for the share price X^S, and since the climate affected agents are chosen to have incomes not depending on X^S, J^a does not depend on X_t^S.

In a real situation, the wealth process of an agent with initial capital $v_0^a = 1$ should increase with time. Here we assume that the initial capital of each agent is normed by $v_t^a = 1$ at time t. This is why the expected terminal value J^a we simulated is decreasing with time. Indeed we have $J_T^a = 1$. This is not a limitation, since J^a depends in a multiplicative way on the initial value of the wealth process of agent a. Also, in our simulations we are more interested in exhibiting the dependence of J^a on k at different times.

3.2.1 Model A

Let us recall that if the fisher does not invest in X^E, his only benefits will be given by H^f. The dependence of this random income on K shows a narrow peak around the optimal temperature 0. The fisher benefits a lot when the temperature is near 0 and almost nothing not very far from there. We clearly observe that investing in X^E reduces the fisher's risk exposure. The optimal utility curve exhibited by the simulations at different times has a very wide maximal zone around 0.

The bank's optimal utility curve as a function of temperature shows the following features. The bank's situation is best if the temperature is in a neighborhood of 0, but not too close to 0. Indeed, if it is very close to 0, it is not interesting for the fisher to invest in X^E: there is no risk to transfer. If temperature changes a little, both agents clearly have an interest in the exchange of X^E. If the temperature is

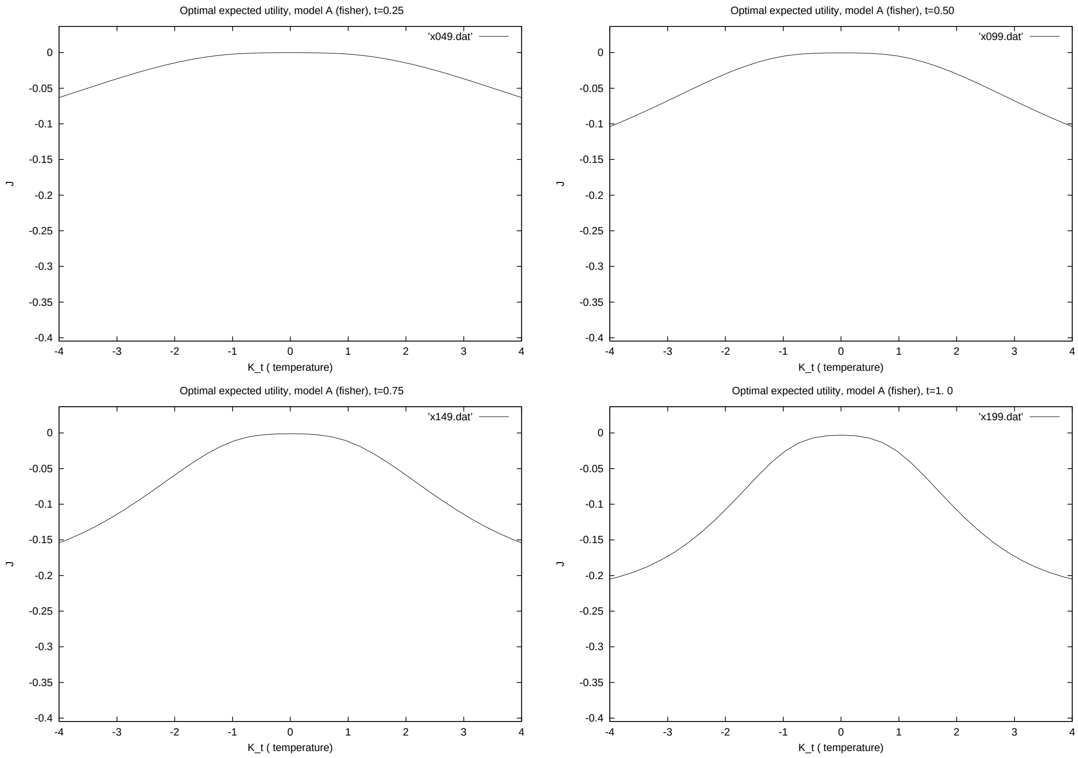

Figure 5: The maximal expected utility J for the fisher (model A).

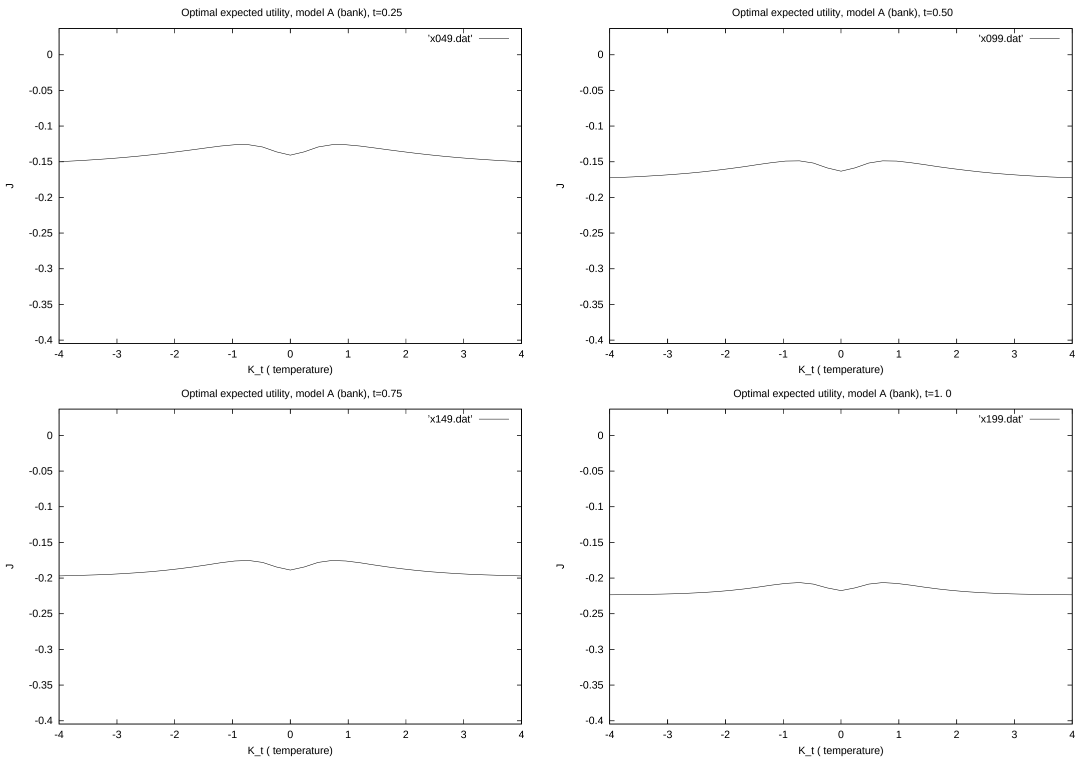

Figure 6: The maximal expected utility J for the bank (model A).

too far from 0, then of course the situation is bad for both agents: the fisher has not much money to invest. The latter situation is, however, very unlikely to happen. The Ornstein-Uhlenbeck process used here reaches ± 2 before time 1 only with a very small probability.

3.2.2 Models B and C

We just show diagrams from the farmer's point of view for theses models, since there is symmetry in the exposure of the agents. The optimal expected utility for the farmer seems a very flat curve, which is maximal around the optimal temperature. This may indicate that trading on the risk asset brought security to the agents. There is no real qualitative difference in the shape of the curves between model B and model C. We just observe that model C reflects, of course, a better situation for the farmer.

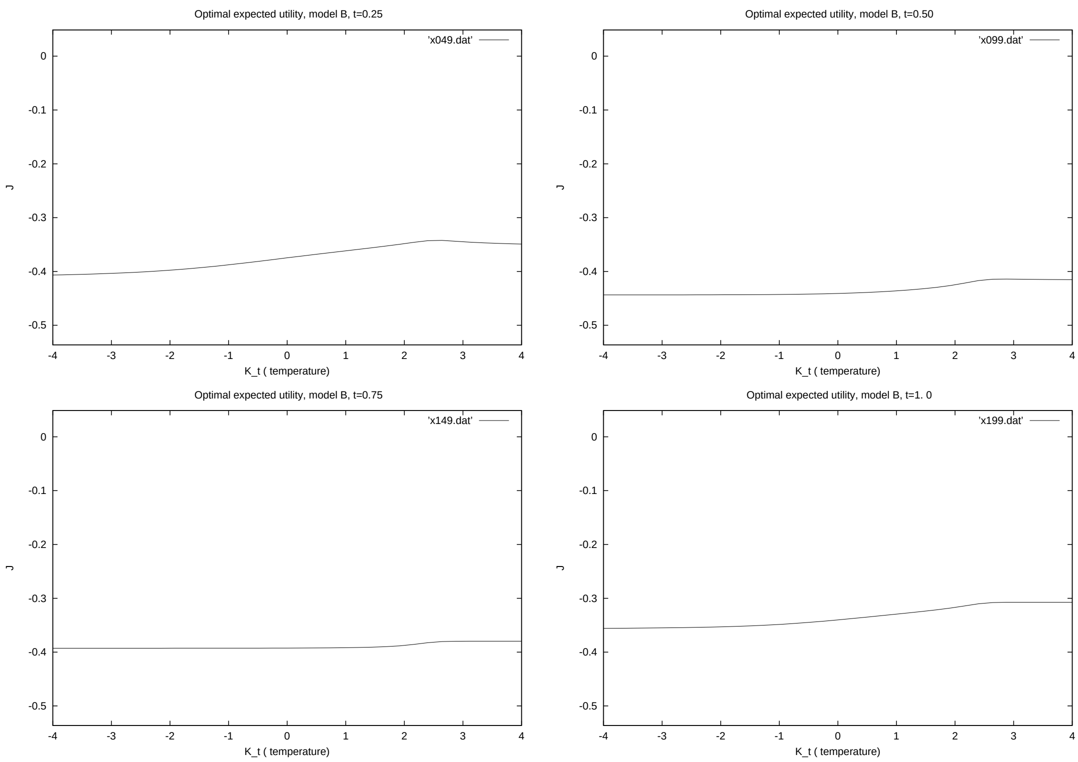

Figure 7: The maximal expected utility J for the farmer (model B).

3.3 Optimal strategies

We finally describe the optimal amount of money to be invested in X^E by each agent during the trading interval, i.e. the strategy of investment which allow the agents to attain maximal expected utility J^a.

Since only two agents are active on the market, the local equilibrium condition (6) implies that at each time t the entire quantity of X^E sold by one agent is bought by

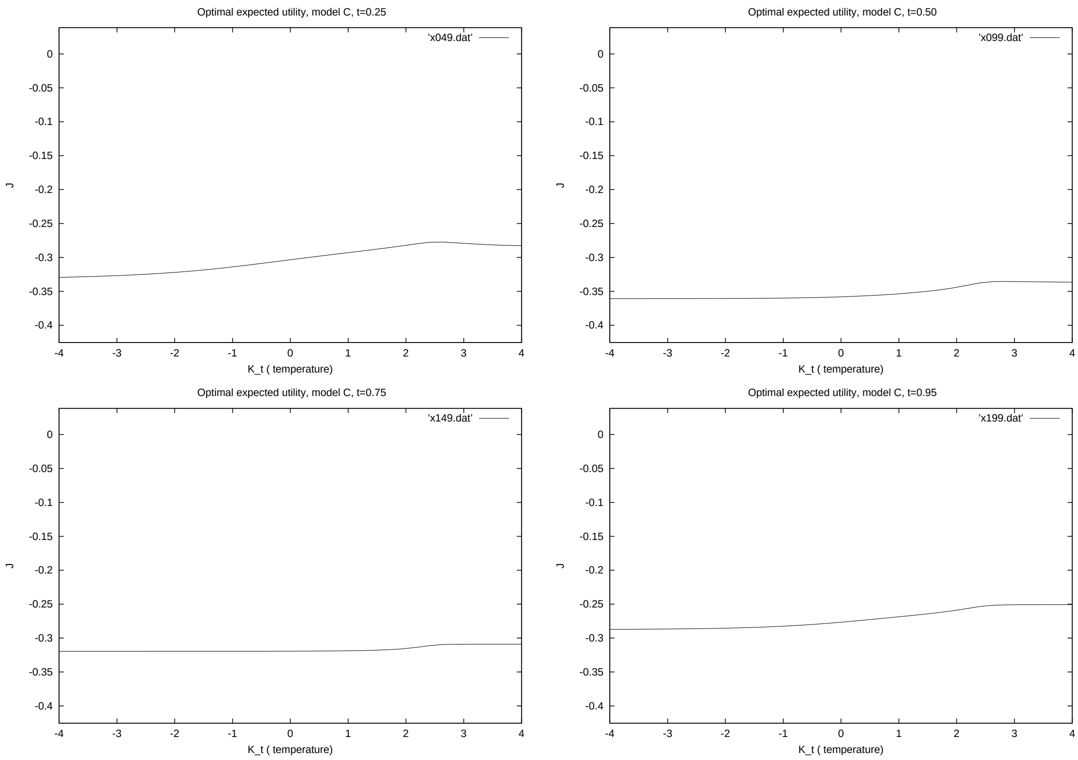

Figure 8: The maximal expected utility J for the farmer (model C).

the other, i.e.

$$\pi^f_{2,t} = -\pi^b_{2,t} \text{ in model A, or } \pi^r_{2,t} = -\pi^f_{2,t} \text{ in models B and C.}$$

Therefore it will be enough to show diagrams of the strategy of one agent (fisher in model A and farmer in models B and C). Since we are able to approximate numerically the strategies of both agents, we remark that the local equilibrium condition may be used to check the accuracy of our schemes.

We show the optimal strategies as functions of t (on the period [0,1]) and the current temperature K_t. As in the preceding subsection, in our simple example this strategy does not depend on X^S_t. The diagrams also display the optimal amount of money to be exchanged between the agents, from the selected agent's point of view.

3.3.1 Model A

Here we only show the fisher's optimal strategy π^f_2. At the optimal temperature for fishing $K_t = 0$, the fisher makes his maximal profit, and we observe that there is no exchange of risk trading money. As soon as the temperature grows a little, the fisher has to buy a certain quantity of X^E from the bank. This exchange will bring security to the fisher and profits to the bank.

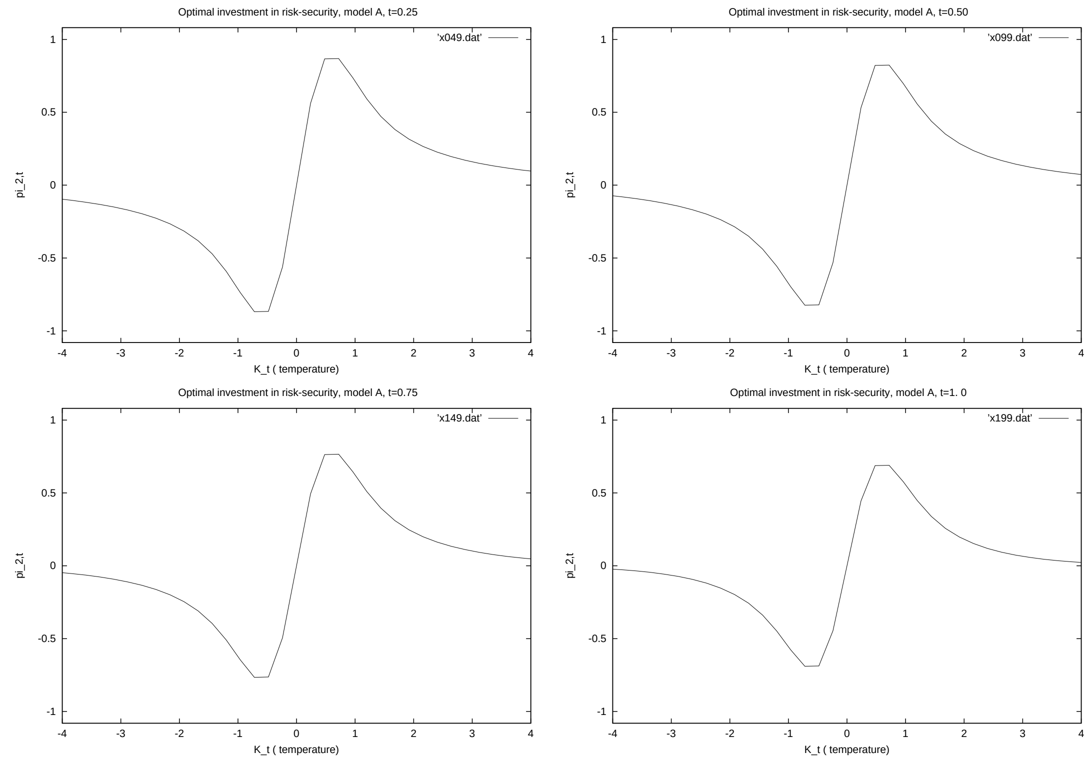

Figure 9: The optimal strategy for the fisher (model A).

3.3.2 Models B and C

We only show the farmer's optimal strategy π_2^r. We can first notice, by taking into account the estimates for the expectation of X^E in subsection 3.1, that the appreciation of risk trading is very low compared to model A.

On theses diagrams, we see that the farmer invests in X^E when the temperature is high in the first half period [0,0.5], i.e. an interval that favors him, and sells X^E (to the fisher) when the temperature is low, for $t \in [0.5, 1]$, i.e. when he needs money. This reflects the intuition that the agents have an interest to share their risks by exchanging money this way.

Again, the qualitative difference between models B and C is not big. We just observe that the farmer invests a little more than the fisher.

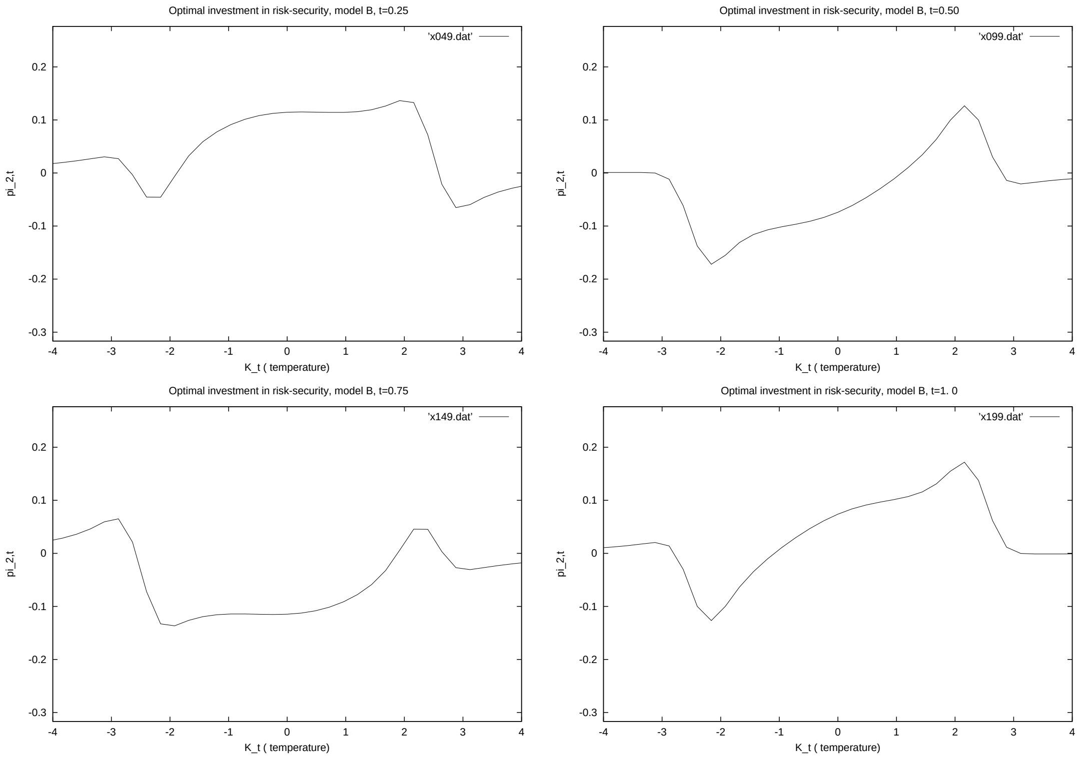

Figure 10: The optimal strategy for the farmer (model B).

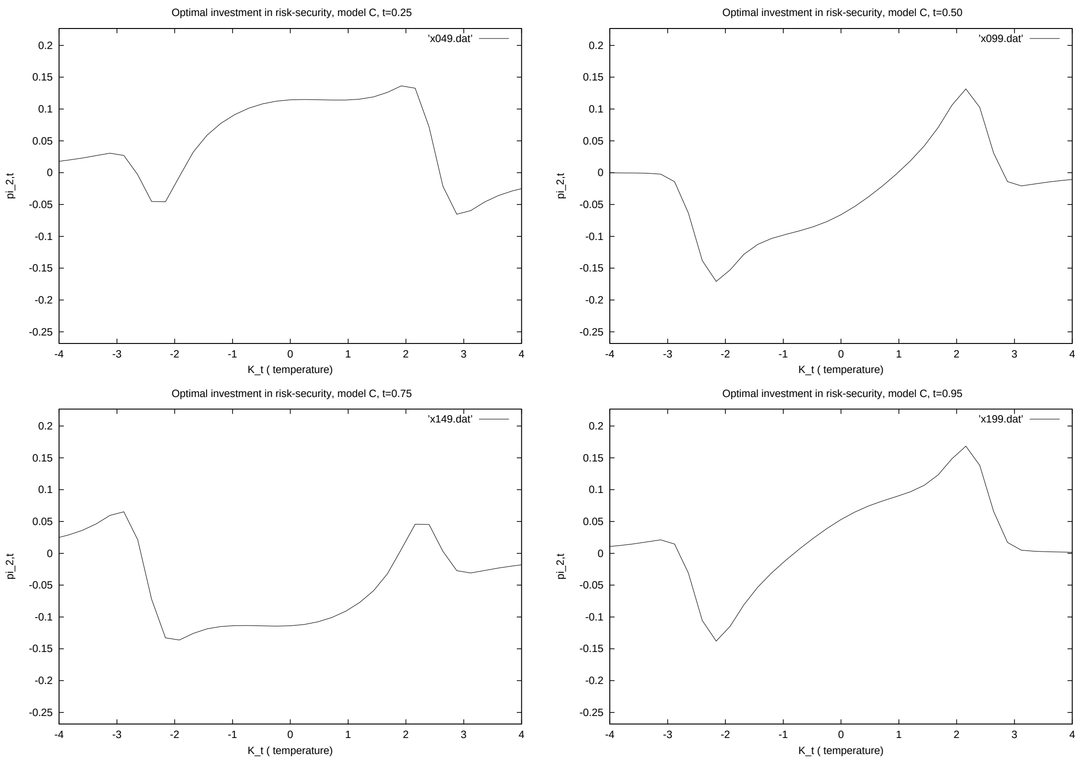

Figure 11: The optimal strategy for the farmer (model C).

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