Backward stochastic differential equations with time delayed generators - results and counterexamples

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Abstract

We deal with backward stochastic differential equations with time delayed generators. In this new type of equations, a generator at time t can depend on the values of a solution in the past, weighted with a time delay function for instance of the moving average type. We prove existence and uniqueness of a solution for a sufficiently small time horizon or for a sufficiently small Lipschitz constant of a generator. We give examples of BSDE with time delayed generators that have multiple solutions or that have no solutions. We show for some special class of generators that existence and uniqueness may still hold for an arbitrary time horizon and for arbitrary Lipschitz constant. This class includes linear time delayed generators, which we study in more detail. We are concerned with different properties of a solution of a BSDE with time delayed generator, including the inheritance of boundedness from the terminal condition, the comparison principle, the existence of a measure solution and the BMO martingale property. We give examples in which they may fail.
Keywords: backward stochastic differential equation, time delayed generator, contraction inequality, comparison principle, measure solution, BMO martingale.

1 Introduction

Backward stochastic differential equations have been introduced in [13]. Since then, they have been thoroughly studied in the literature, see [6] or [8] and references therein. The classical theory of BSDE driven by Brownian motions and with Lipschitz continuous generators has been extended in different directions. For instance, [10] discusses the existence of a solution in case the generator is of quadratic growth in the control variable; the existence of a solution for BSDE driven on a more general stochastic basis, created by Lévy processes resp. continuous martingales, is considered respectively in [2] and in [12]; a theory of BSDE with random time horizon is investigated in [3].

In this paper we study a new class of backward stochastic differential equation, the dynamics of which is given by

$$Y(t) = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z(s)dW(s), \quad t \in [0, T].$$

Here, a generator f at time s depends arbitrary on the past values of a solution $(Y_s, Z_s) = (Y(s+u), Z(s+u))_{-T \le u \le 0}$. They can be called backward stochastic differential equations with time delayed generators. This type of equations has been investigated for the first time very recently in [4], where only a special form of time delay in Z is considered, namely $f(s, y_s, z_s) = \int_{-T}^0 g(s+u, z(s+u))\alpha(du)$ with a measure α. The authors prove that in this case, there exists a unique solution on $[0, T]$ for $T = 1$.

We aim at providing some contributions to a general theory of BSDE in which the time delayed generators satisfy Lipschitz conditions. We are interested in existence and uniqueness results and in properties of solutions. We prove that a unique solution exists provided that the generator's Lipschitz constant is sufficiently small or the evolution is constrained to a sufficiently small time horizon. For cases of more general Lipschitz constants or time horizons we give examples of BSDE that have multiple solutions or no solutions at all. Following [4], we also study BSDE with time delayed generators independent of y, and fulfilling Lipschitz conditions. We show that a unique solution exists if the delay measure α is supported on $[-\gamma, 0]$ with a sufficiently small time delay γ. Moreover, in the case of a linear time delayed generator, which fits into the framework of [4], we derive an explicit solution to our BSDE.

We further consider properties of solutions of time delayed BSDE, such as the inheritance of boundedness from the terminal condition, the comparison principle, measure solutions and the BMO martingale property. All these concepts have turned out to be very useful in the theory of BSDE without delay, see [8] and [6]. We find that without requiring additional assumptions these well-known properties, which hold in the classical setting, may fail for a solution of a time delayed BSDE. We are only able to prove that the *BMO* martingale property holds in the case of linear time delayed generators independent of Y.

We would like to point out that except [4] the only paper we are aware of that deals with BSDE with time delayed generators is [7]. In [7] a forward-backward system of stochastic differential equations is considered in which the time delay appears in the forward component, and not in the backward one. This setting is completely different from the one considered in [4] and here. We would like to recall that forward stochastic differential equations with time delays, called functional stochastic differential equations, have been studied extensively in the literature. See for example [11], [15] and references therein.

Finally, we would like to refer the reader to the accompanying paper [5], where existence and uniqueness of a solution of a BSDE driven by a Brownian motion and a Poisson random measure and with time delayed generator is discussed, together with its Malliavin's differentiability, both with respect to the continuous as well as the jump component.

This paper is structured as follows. In Section 2, we deal with uniqueness and existence of a solution of a backward stochastic differential equation with time delayed generator. Counterexamples showing that we cannot obtain unique solutions in a more general setting are given in Section 3. Linear time delayed generators, depending only on the control variable z, are studied in Section 4, together with the inheritance of boundedness from the terminal condition and the *BMO* property. Section 5 investigates the concepts of measure solution and the comparison principle.

2 Existence and uniqueness of a solution

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ and a finite time horizon $T < \infty$. We assume that the filtration $\mathbb{F}$ is the natural filtration generated by a Brownian motion $W := (W(t), 0 \le t \le T)$, augmented by all sets of $\mathbb{P}$-measure zero.

We shall work with the following topological vector spaces.

Definition 2.1. *1. Let $L^2_{-T}(\mathbb{R})$ denote the space of measurable functions $z : [-T, 0] \to \mathbb{R}$ satisfying*

$$\int_{-T}^{0} |z(t)|^2 dt < \infty.$$

2. Let $L^\infty_{-T}(\mathbb{R})$ denote the space of bounded, measurable functions $y : [-T, 0] \to \mathbb{R}$ satisfying

$$\sup_{t \in [-T,0]} \left|y(t)\right|^2 < \infty.$$

3. *For $p \geq 2$, let $\mathbb{L}^p(\mathbb{R})$ denote the space of $\mathcal{F}_T$-measurable random variables $\xi : \Omega \to \mathbb{R}$ satisfying*
$$\mathbb{E}\big[|\xi|^p\big] < \infty.$$

4. *Let $\mathbb{H}^2_T(\mathbb{R})$ denote the space of $\mathbb{F}$-predictable processes $Z : \Omega \times [0,T] \to \mathbb{R}$ satisfying*
$$\mathbb{E}\big[\int_0^T |Z(t)|^2 dt\big] < \infty.$$

5. *Finally, let $\mathbb{S}^2_T(\mathbb{R})$ denote the space of $\mathbb{F}$-adapted, product measurable processes $Y : \Omega \times [0,T] \to \mathbb{R}$ satisfying*
$$\mathbb{E}\big[\sup_{t\in[0,T]} |Y(t)|^2\big] < \infty.$$

The spaces $\mathbb{H}^2_T(\mathbb{R})$ and $\mathbb{S}^2_T(\mathbb{R})$ are endowed with the norms
$$\|Z\|^2_{\mathbb{H}^2_T} = \mathbb{E}\big[\int_0^T e^{\beta t}|Z(t)|^2 dt\big],$$
$$\|Y\|^2_{\mathbb{S}^2_T} = \mathbb{E}\big[\sup_{t\in[0,T]} e^{\beta t}|Y(t)|^2\big],$$
with some $\beta > 0$.

As usual, by λ we denote Lebesgue measure on $([-T,0] \times \mathcal{B}([-T,0]))$, where $\mathcal{B}([-T,0])$ stands for the Borel sets of $[-T,0]$. In the sequel let us simply write $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ for $\mathbb{S}^2_T(\mathbb{R}) \times \mathbb{H}^2_T(\mathbb{R})$.

We shall deal with the existence and uniqueness of a solution $(Y,Z) := (Y(t), Z(t))_{0\leq t\leq T}$ of a backward stochastic differential equation with time delayed generator, the dynamics of which is given by
$$Y(t) = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T, \tag{2.1}$$
where the generator f at time $s \in [0,T]$ depends on the past values of the solution denoted by $Y_s := (Y(s+u))_{-T\leq u\leq 0}$ and $Z_s := (Z(s+u))_{-T\leq u\leq 0}$. We always set $Z(t) = 0$ and $Y(t) = Y(0)$ for $t < 0$.

We investigate (2.1) under the following assumptions:

(A1) $\xi \in \mathbb{L}^2(\mathbb{R})$ for the terminal variable ξ,

(A2) the generator $f : \Omega \times [0,T] \times L^\infty_{-T}(\mathbb{R}) \times L^2_{-T}(\mathbb{R}) \to \mathbb{R}$ is product measurable, $\mathbb{F}$-adapted and Lipschitz continuous in the sense that for some probability measure α on $([-T,0]\times\mathcal{B}([-T,0]))$

$$\begin{aligned}
&|f(t,y_t,z_t)-f(t,\tilde{y}_t,\tilde{z}_t)|^2\\
&\leq K\Big(\int_{-T}^{0}|y(t+u)-\tilde{y}(t+u)|^2\alpha(du)\\
&\quad+\int_{-T}^{0}|z(t+u)-\tilde{z}(t+u)|^2\alpha(du)\Big),
\end{aligned}$$

holds for $\mathbb{P}\times\lambda$-a.e. $(\omega,t)\in\Omega\times[0,T]$ and for any $(y_t,z_t),(\tilde{y}_t,\tilde{z}_t)\in L^\infty_{-T}(\mathbb{R})\times L^2_{-T}(\mathbb{R})$.

(A3) $\mathbb{E}\big[\int_0^T|f(t,0,0)|^2dt\big]<\infty$,

(A4) $f(t,.,.)=0$ for $t<0$.

We remark that $f(t,0,0)$ in **(A3)** should be understood as a value of the generator $f(t,y_t,z_t)$ at $y(t+u)=z(t+u)=0,-T\leq u\leq 0$. We would like to point out that the assumption **(A4)** in fact allows us to take $Y(t)=Y(0)$ and $Z(t)=0$ for $t<0$, as a solution of (2.1). Examples of generators could be linear functions of the form $f(t,y_t,z_t)=K\int_0^t z(s)ds$ or $f(t,y_t,z_t)=Kz(t-r), 0\leq t\leq T$ with a fixed time delay r, as studied in more detail in Section 4.

Note that for $(Y,Z)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})$ the generator is well-defined and $\mathbb{P}$-a.s integrable as

$$\begin{aligned}
\int_0^T&|f(t,Y_t,Z_t)|^2dt\leq 2\int_0^T|f(t,0,0)|^2dt\\
&+2K\Big(\int_0^T\int_{-T}^0|Y(t+u)|^2\alpha(du)dt+\int_0^T\int_{-T}^0|Z(t+u)|^2\alpha(du)dt\Big)\\
=\;&2\int_0^T|f(t,0,0)|^2dt\\
&+2K\int_{-T}^0\int_u^{T+u}|Y(v)|^2dv\alpha(du)+2K\int_{-T}^0\int_u^{T+u}|Z(v)|^2dv\alpha(du)\\
\leq\;&2\int_0^T|f(t,0,0)|^2dt+2K\big(T\sup_{v\in[0,T]}|Y(v)|^2+\int_0^T|Z(v)|^2dv\big)<\infty. \qquad (2.2)
\end{aligned}$$

To justify this, we apply Fubini's theorem, change the variables, use the assumption that $Z(t)=0$ and $Y(t)=Y(0)$ for $t<0$ and the fact that the probability measure α integrates to 1.

We first state some a priori estimates.

Lemma 2.1. *Let* $(Y,Z)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})$ *and* $(\tilde{Y},\tilde{Z})\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})$ *denote solutions of* (2.1) *with corresponding parameters* (ξ,f) *and* $(\tilde{\xi},\tilde{f})$ *which satisfy the* *assumptions **(A1)-(A4)**. Then the following inequalities hold*

$$\begin{aligned}
&\|Z-\tilde Z\|^2_{\mathbb{H}^2}\\
&\leq \; e^{\beta T}\mathbb{E}\big[|\xi-\tilde\xi|^2\big]+\frac{1}{\beta}\mathbb{E}\big[\int_0^T e^{\beta t}|f(t,Y_t,Z_t)-\tilde f(t,\tilde Y_t,\tilde Z_t)|^2dt\big],
\end{aligned}\tag{2.3}$$

$$\begin{aligned}
&\|Y-\tilde Y\|^2_{\mathbb{S}^2}\\
&\leq \; 8e^{\beta T}\mathbb{E}\big[|\xi-\tilde\xi|^2\big]+8T\mathbb{E}\big[\int_0^T e^{\beta t}|f(t,Y_t,Z_t)-\tilde f(t,\tilde Y_t,\tilde Z_t)|^2dt\big].
\end{aligned}\tag{2.4}$$

Proof:
The a priori estimates are classical in the theory of BSDEs without time delay and can be extended to our setting. The inequality (2.3) follows from Lemma 3.2.1 in [8]. In order to prove the second inequality, first notice that for all $t\in[0,T]$

$$\begin{aligned}
&e^{\frac{\beta}{2}t}|Y(t)-\tilde Y(t)|\\
&\leq \; e^{\frac{\beta}{2}T}\mathbb{E}\big[|\xi-\tilde\xi|^2|\mathcal{F}_t\big]+\mathbb{E}\big[\int_0^T e^{\frac{\beta}{2}t}|f(s,Y_s,Z_s)-\tilde f(s,\tilde Y_s,\tilde Z_s)|ds|\mathcal{F}_t\big].
\end{aligned}$$

Applying Doob's martingale inequality and Cauchy-Schwarz' inequality provides the second estimate. □

We state the main theorem of this section.

Theorem 2.1. *Assume that **(A1)-(A4)** hold. For a sufficiently small time horizon T or for a sufficiently small Lipschitz constant K, the backward stochastic differential equation* (2.1) *has a unique solution $(Y,Z)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})$.*

Proof:
To prove existence and uniqueness of a solution, we follow the classical idea by constructing a Picard scheme and show its convergence. See Theorem 2.1 in [6] or Theorem 3.2.1 in [8].

Let $Y^0(t)=Z^0(t)=0$ and define recursively for $n\in\mathbb{N}$

$$\begin{aligned}
&Y^{n+1}(t)=\xi\\
&\quad+\int_t^T f(s,Y^n_s,Z^n_s)ds-\int_t^T Z^{n+1}(s)dW(s)\quad 0\leq t\leq T.
\end{aligned}\tag{2.5}$$

Step 1) Given $(Y^n,Z^n)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})$, the equation (2.5) has a unique solution $(Y^{n+1},Z^{n+1})\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})$.
Based on the inequality (2.2), we can conclude that

$$\begin{aligned}
&\mathbb{E}\big[\int_0^T |f(t,Y^n_t,Z^n_t)|^2dt\big]\\
&\leq \; \mathbb{E}\big[\int_0^T |f(t,0,0)|^2dt\big]+2K\big(T\|Y^n\|_{\mathbb{S}^2}+\|Z^n\|_{\mathbb{H}^2}\big)<\infty.
\end{aligned}$$

As in the case of a BSDE without a time delay, the martingale representation provides a unique process $Z^{n+1} \in \mathbb{H}^2(\mathbb{R})$ such that

$$\xi + \int_0^T f(t, Y_t^n, Z_t^n)dt = \mathbb{E}\big[\xi + \int_0^T f(t, Y_t^n, Z_t^n)dt\big] + \int_0^T Z^{n+1}(t)dW(t),$$

and we take Y^{n+1} as a continuous version of

$$Y^{n+1}(t) = \mathbb{E}\big[\xi|\mathcal{F}_t\big] + \mathbb{E}\big[\int_t^T f(s, Y_s^n, Z_s^n)ds|\mathcal{F}_t\big], \quad 0 \leq t \leq T.$$

Similarly, as in Lemma 2.1, Doob's inequality, Cauchy-Schwarz' inequality and the estimates (2.2) yield that $Y^{n+1} \in \mathbb{S}^2(\mathbb{R})$.
In Step 2) we show the convergence of the sequence (Y^n, Z^n) in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$. The estimates (2.3) and (2.4) give the inequality

$$\begin{aligned} &\big\|Y^{n+1} - Y^n\big\|_{\mathbb{S}^2}^2 + \big\|Z^{n+1} - Z^n\big\|_{\mathbb{H}^2}^2 \\ &\quad \leq (8T + \frac{1}{\beta})\mathbb{E}\big[\int_0^T e^{\beta t}|f(t, Y_t^n, Z_t^n) - f(t, Y_t^{n-1}, Z_t^{n-1})|^2 dt\big]. \end{aligned} \tag{2.6}$$

By applying the Lipschitz condition **(A2)**, Fubini's theorem, changing the variables and using the assumption that $Y^n(s) = Y^n(0)$ and $Z^n(s) = 0$ for $s < 0$ and all $n \geq 0$, we can derive

$$\begin{aligned} &\mathbb{E}\big[\int_0^T e^{\beta t}|f(t, Y_t^n, Z_t^n) - f(t, Y_t^{n-1}, Z_t^{n-1})|^2 dt\big] \\ &\leq \; K\mathbb{E}\big[\int_0^T e^{\beta t}\int_{-T}^0 |Y^n(t+u) - Y^{n-1}(t+u)|^2\alpha(du)dt \\ &\quad + \int_0^T e^{\beta t}\int_{-T}^0 |Z^n(t+u) - Z^{n-1}(t+u)|^2\alpha(du)dt\big] \\ &= \; K\mathbb{E}\big[\int_{-T}^0 e^{-\beta u}\int_0^T e^{\beta(t+u)}|Y^n(t+u) - Y^{n-1}(t+u)|^2 dt\alpha(du) \\ &\quad + \int_{-T}^0 e^{-\beta u}\int_0^T e^{\beta(t+u)}|Z^n(t+u) - Z^{n-1}(t+u)|^2 dt\alpha(du)\big] \\ &= \; K\mathbb{E}\big[\int_{-T}^0 e^{-\beta u}\int_u^{T+u} e^{\beta v}|Y^n(v) - Y^{n-1}(v)|^2 dv\alpha(du) \\ &\quad + \int_{-T}^0 e^{-\beta u}\int_u^{T+u} e^{\beta v}|Z^n(v) - Z^{n-1}(v)|^2 dv\alpha(du)\big] \\ &\leq \; K\int_{-T}^0 e^{-\beta u}\alpha(du)\big(T\big\|Y^n - Y^{n-1}\big\|_{\mathbb{S}^2}^2 + \big\|Z^n - Z^{n-1}\big\|_{\mathbb{H}^2}^2\big). \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$\begin{aligned} &\big\|Y^{n+1} - Y^n\big\|_{\mathbb{S}^2}^2 + \big\|Z^{n+1} - Z^n\big\|_{\mathbb{H}^2}^2 \\ &\quad \leq \delta(T, K, \beta, \alpha)\big(\big\|Y^n - Y^{n-1}\big\|_{\mathbb{S}^2}^2 + \big\|Z^n - Z^{n-1}\big\|_{\mathbb{H}^2}^2\big), \end{aligned} \tag{2.8}$$

with

$$\delta(T,K,\beta,\alpha) = (8T + \frac{1}{\beta})K \int_{-T}^{0} e^{-\beta u}\alpha(du)\max\{1,T\}.$$

For $\beta = \frac{1}{T}$ we have

$$\delta(T,K,\beta,\alpha) \le 9TKe\max\{1,T\}.$$

For a sufficiently small T or for a sufficiently small K, the inequality (2.8) is a contraction, and there exists a unique limit $(Y,Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ of a converging sequence $(Y^n, Z^n)_{n\in\mathbb{N}}$, which satisfies the fixed point equation

$$Y(t) = \mathbb{E}\big[\xi|\mathcal{F}_t\big] + \mathbb{E}\Big[\int_t^T f(s, Y_s, Z_s)ds|\mathcal{F}_t\Big], \quad 0 \le t \le T.$$

Step 4) Define a solution $\bar{Y}$ of (2.1) as a continuous version of

$$\bar{Y}(t) = \mathbb{E}\big[\xi|\mathcal{F}_t\big] + \mathbb{E}\Big[\int_t^T f(s, Y_s, Z_s)ds|\mathcal{F}_t\Big], \quad 0 \le t \le T,$$

where (Y,Z) is the limit constructed in Step 3. □

Theorem 2.1 triggers the immediate question: is it possible to obtain existence and uniqueness for a bigger time horizon T and/or an arbitrary Lipschitz constant K? In the following section we show that such an extension is not possible. However, for a special class of generators Theorem 2.1 may be generalized, as we now show.

Theorem 2.2. *Assume that* ***(A1)-(A4)*** *hold and that the generator is independent of* y_t*, i. e. for* $t \in [0,T]$ *we have* $f(t, y_t, z_t) = f(t, z_t)$*. Let the measure* α *be supported on the interval* $[-\gamma, 0]$*. For a sufficiently small time delay* γ *the backward stochastic differential equation* (2.1) *has a unique solution* $(Y,Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$*.*

Proof:
The proof is very similar to the previous one. Note that in this case, based on (2.6) and (2.7), we have

$$\big\|Z^{n+1} - Z^n\big\|^2_{\mathbb{H}^2} \le \delta(T,K,\beta,\alpha)\big\|Z^n - Z^{n-1}\big\|^2_{\mathbb{H}^2},$$

with

$$\delta(T,K,\beta,\alpha) = \frac{1}{\beta}K\int_{-\gamma}^{0} e^{-\beta u}\alpha(du) \le \frac{Ke^{\beta\gamma}}{\beta}\int_{-\gamma}^{0}\alpha(du) = \frac{Ke^{\beta\gamma}}{\beta},$$

which is smaller than 1 for sufficiently big β and small γ. This proves the convergence of $(Z^n)_{n\in\mathbb{N}}$. To get the convergence of $(Y^n)_{n\in\mathbb{N}}$, notice again that, by (2.6) and (2.7)

$$\big\|Y^{n+1} - Y^n\big\|^2_{\mathbb{S}^2} \le 8T\delta(T,K,\beta,\alpha)\big\|Z^n - Z^{n-1}\big\|^2_{\mathbb{H}^2}.$$

□

Finally, to complete our presentation of the current state of knowledge on BSDE with time delayed generators, we shall recall a theorem proved in [4] recently.

Theorem 2.3. *Assume that $\xi \in \mathbb{L}^{2+\epsilon}(\mathbb{R})$ for some $\epsilon > 0$ and that **(A2)**-**(A4)** hold with respect to a generator of the form*

$$f(t, z_t) = \int_{-T}^{0} g(t+u, z(t+u))\alpha(du), \quad z \in L^2_{-T}(\mathbb{R}),$$

where α is a finite measure. The BSDE (2.1) *has a unique solution $(Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$.*

We remark that Theorem 2.3 is a slight extension of the theorem stated in [4] from $T = 1$ to an arbitrary T. Moreover, one can prove the result of 2.3 under weaker integrability assumptions concerning ξ, by replacing in the proof the Cauchy-Schwarz inequality with the Hölder inequality. This allows for $\xi \in \mathbb{L}^{2+\epsilon}(\mathbb{R})$.

3 Non-uniqueness and multiple solutions

In this section we discuss examples of BSDE with time delayed generators that fail to have solutions or have more than one. This confirms that there is a natural boundary for extensions of the "local" existence and uniqueness result from Theorem 2.1, and that one cannot expect to have existence and uniqueness for an arbitrary time horizon T and an arbitrary Lipschitz constant K without additional requirements.

Example 1

Let us first investigate the backward stochastic differential equation with the following generator of Lipschitz constant $K > 0$ and of fixed time delay

$$Y(t) = \xi + \int_t^T KY(s-T)ds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T. \tag{3.1}$$

Using the notions of the previous section, (3.1) can be rewritten as

$$\begin{aligned} Y(t) &= \xi \\ &+ \int_t^T \int_{-T}^0 KY(s+u)\mathbf{1}_{[0,\infty)}(s)\,\alpha(du)ds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T, \end{aligned}$$

with Dirac measure α concentrated at the point T. The equation (3.1) is clearly equivalent to

$$Y(t) = \xi + K(T-t)Y(0) - \int_t^T Z(s)dW(s), \quad 0 \le t \le T. \tag{3.2}$$

For $t = 0$ we arrive at

$$(1 - TK)Y(0) = \xi - \int_0^T Z(s)dW(s),$$

and integrating on both sides produces the condition

$$\mathbb{E}[\xi] = (1 - TK)Y(0).$$

We consider three cases.
Case 1) $TK < 1$.
Define Z as the unique square integrable process from the martingale representation of $\xi \in \mathbb{L}^2(\mathbb{R})$ given by

$$\xi = \mathbb{E}[\xi] + \int_0^T Z(s)dW(s),$$

and the process Y, according to (3.2), by

$$\begin{aligned} Y(t) &= \mathbb{E}[\xi] + K(T-t)Y(0) + \int_0^t Z(s)dW(s) \\ &= Y(0)(1 - tK) + \int_0^t Z(s)dW(s) \\ &= \frac{1 - tK}{1 - TK}\mathbb{E}[\xi] + \int_0^t Z(s)dW(s), \quad 0 \le t \le T. \end{aligned} \tag{3.3}$$

The pair $(Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ is the unique solution of (3.1) on $[0, T]$. In fact, suppose there were another solution $(\tilde{Y}, \tilde{Z})$ of (3.1) on $[0, T]$. Writing the difference of (3.2) for the two solutions, and using $(1 - TK)Y(0) = \mathbb{E}[\xi] = (1 - TK)\tilde{Y}(0)$ for getting $Y(0) = \tilde{Y}(0)$, we obtain

$$\int_0^T (Z(s) - \tilde{Z}(s))dW(s) = 0, \quad \mathbb{P} - a.s.,$$

hence $Z = \tilde{Z}$, whence finally $Y = \tilde{Y}$.
Case 2) $TK = 1$ and $\mathbb{E}[\xi] \neq 0$.
The condition $\mathbb{E}[\xi] = (1 - TK)Y(0)$ is not satisfied and therefore equation (3.1) does not have any solution.
Case 3) $TK = 1$ and $\mathbb{E}[\xi] = 0$.
As in case 1), define Z as the unique square integrable process appearing in the martingale representation of $\xi \in \mathbb{L}^2(\mathbb{R})$, and the process Y as

$$Y(t) = Y(0)(1 - tK) + \int_0^t Z(s)dW(s), \quad 0 \le t \le T, \tag{3.4}$$

with an arbitrary $Y(0) \in \mathbb{L}^2(\mathbb{R})$ which is $\mathcal{F}_0$-measurable. Any pair $(Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ satisfying (3.4) is a solution of (3.1) on $[0, T]$.

Example 2

Next, again let $K \in \mathbb{R}$, so that $|K|$ stands for the Lipschitz constant of the time delayed generator, we study the backward stochastic differential equation

$$Y(t) = \xi + \int_t^T \int_0^s KY(u)duds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T. \tag{3.5}$$

With the notation of the previous section the equation is of the form

$$\begin{aligned} Y(t) = \xi & \\ & + \int_t^T \int_{-T}^0 KTY(s+u)\mathbf{1}\{s+u \ge 0\}\alpha(du)ds \\ & - \int_t^T Z(s)dW(s), \quad 0 \le t \le T, \end{aligned}$$

with a uniform measure α on $[-T, 0]$. Changing the order of integration in the generator term and calculating the difference $Y(t) - Y(0)$ yields

$$\begin{aligned} Y(t) &= Y(0) \\ & -K \int_0^t (t-s)Y(s)ds + \int_0^t Z(s)dW(s), \quad 0 \le t \le T. \end{aligned} \tag{3.6}$$

In the sequel we construct a solution of (3.5). We comment on the main steps and leave details of the tedious but simple calculations to the reader.

Consider for a moment the deterministic integral equation corresponding to (3.6)

$$y(t) = y(0) - K \int_0^t (t-s)y(s)ds + h(t), \tag{3.7}$$

with a twice continuously differentiable function $h \in \mathcal{C}^2(\mathbb{R})$ such that $h(0) = 0$ and with a given initial condition $y(0)$. By differentiating, we obtain the nonhomogeneous linear second order differential equation

$$y^{''}(t) + Ky(t) = h^{''}(t). \tag{3.8}$$

The fundamental solution of the homogeneous part of (3.8) is well known and its general form is

$$y(t) = Ae^{\sqrt{-K}t} + Be^{-\sqrt{-K}t},$$

where A, B are constants, and $\sqrt{-K}$ for $K > 0$ is understood as a complex number. It is easy to check that the following formula gives a general solution of the inhomogeneous equation (3.8):

$$\begin{aligned} y(t) &= Ae^{\sqrt{-K}t} + Be^{-\sqrt{-K}t} \\ & + \int_0^t \frac{h^{''}(s)e^{-\sqrt{-K}s}}{2\sqrt{-K}} dse^{\sqrt{-K}t} - \int_0^t \frac{h^{''}(s)e^{\sqrt{-K}s}}{2\sqrt{-K}} dse^{-\sqrt{-K}t}. \end{aligned}$$

Integrating by parts twice gives

$$
\begin{aligned}
y(t) &= Ae^{\sqrt{-K}t} + Be^{-\sqrt{-K}t} + h(t) \\
&\quad + \int_0^t h(s)e^{-\sqrt{-K}s}ds\frac{\sqrt{-K}}{2}e^{\sqrt{-K}t} - \int_0^t h(s)e^{\sqrt{-K}s}ds\frac{\sqrt{-K}}{2}e^{-\sqrt{-K}t}. \qquad (3.9)
\end{aligned}
$$

One can further check that the part of the solution (3.9) containing h satisfies the integral equation (3.7) even without any differentiability assumptions concerning h. One can finally derive the following conditions, under which (3.9) solves the integral equation (3.7):

$$
\begin{cases} A = B, \\ A + B = y(0). \end{cases}
$$

Returning to our backward stochastic differential equation, it is straightforward to replace h with $\int_0^{\cdot} Z(s)dW(s)$, and to conclude that a solution of (3.6) must be of the form

$$
\begin{aligned}
Y(t) &= \frac{Y(0)}{2}\big(e^{\sqrt{-K}t} + e^{-\sqrt{-K}t}\big) + \int_0^t Z(s)dW(s) \\
&\quad + \int_0^t\int_0^s Z(u)dW(u)e^{-\sqrt{-K}s}ds\frac{\sqrt{-K}}{2}e^{\sqrt{-K}t} \\
&\quad - \int_0^t\int_0^s Z(u)dW(u)e^{\sqrt{-K}s}ds\frac{\sqrt{-K}}{2}e^{-\sqrt{-K}t}, \quad 0 \le t \le T.
\end{aligned}
$$

By applying Fubini's theorem for stochastic integrals, see Theorem 4.65 in [14], we finally derive

$$
Y(t) = \frac{Y(0)}{2}\big(e^{\sqrt{-K}t} + e^{-\sqrt{-K}t}\big) + \frac{1}{2}\int_0^t Z(s)\big(e^{\sqrt{-K}(t-s)} + e^{-\sqrt{-K}(t-s)}\big)dW(s),
$$

for $t \in [0,T]$, and $Y(0)$ is determined by

$$
\mathbb{E}[\xi] = \frac{Y(0)}{2}\big(e^{\sqrt{-K}T} + e^{-\sqrt{-K}T}\big).
$$

Case 1) Let us assume that $K < 0$. In this case the unique solution $(Y,Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ of (3.5) is given by

$$
\begin{aligned}
Y(t) &= \mathbb{E}[\xi]\frac{e^{\sqrt{-K}t} + e^{-\sqrt{-K}t}}{e^{\sqrt{-K}T} + e^{-\sqrt{-K}T}} \\
&\quad + \frac{1}{2}\int_0^t Z(s)\big(e^{\sqrt{-K}(t-s)} + e^{-\sqrt{-K}(t-s)}\big)dW(s), \\
Z(t) &= \frac{2M(t)}{e^{\sqrt{-K}(T-t)} + e^{-\sqrt{-K}(T-t)}},
\end{aligned}
$$

for $0 \le t \le T$, where $M := (M(t))_{0\le t\le T}$ is the unique square integrable process appearing in the martingale representation of $\xi \in \mathbb{L}^2(\mathbb{R})$, namely

$$\xi = \mathbb{E}[\xi] + \int_0^T M(t)dW(t).$$

Case 2) Let us assume now that $K > 0$. This case is more interesting, as it allows for uniqueness, nonexistence and multiplicity of solutions. By Euler's formula

$$Y(t) = Y(0)\cos(t\sqrt{K}) + \int_0^t \cos((t-s)\sqrt{K})Z(s)dW(s), \quad 0 \le t \le T,$$

with

$$\mathbb{E}[\xi] = Y(0)\cos(T\sqrt{K}).$$

Case 2.1). $T\sqrt{K} < \frac{\pi}{2}$.
The unique solution $(Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ of (3.5) is given by

$$Y(t) = \mathbb{E}[\xi]\frac{\cos(t\sqrt{K})}{\cos(T\sqrt{K})} + \int_0^t \frac{\cos((t-s)\sqrt{K})}{\cos((T-s)\sqrt{K})}M(s)dW(s), \quad 0 \le t \le T,$$

where M is the unique process arising from the martingale representation of $\xi \in \mathbb{L}^2(\mathbb{R})$.
Case 2.2). $T\sqrt{K} = \frac{\pi}{2}$ and $\mathbb{E}[\xi] \neq 0$.
Equation (3.5) does not have any solution, since condition $\mathbb{E}[\xi] = Y(0)\cos(T\sqrt{K})$ is not satisfied.
Case 2.3). $T\sqrt{K} = \frac{\pi}{2}$ and $\mathbb{E}[\xi] = 0$.
Equation (3.5) may not have any solution, or may have multiple solutions. Consider again the representation

$$\xi = \int_0^T M(s)dW(s),$$

and put

$$Z(t) = \frac{M(t)}{\cos((T-t)\sqrt{K})}\mathbf{1}\{t > 0\}, \quad 0 \le t \le T.$$

Case 2.3.a) If $\mathbb{E}[\int_0^T |Z(s)|^2 ds] = +\infty$, then equation (3.5) does not have any solution.
Case 3b) If $\mathbb{E}[\int_0^T |Z(s)|^2 ds] < \infty$, then equation (3.5) has multiple solutions $(Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ given by

$$Y(t) = Y(0)\cos(t\sqrt{K}) + \int_0^t \cos((t-s)\sqrt{K})Z(s)dW(s), \quad 0 \le t \le T,$$

with an arbitrary $Y(0) \in \mathbb{L}^2(\mathbb{R})$ which is $\mathcal{F}_0$-measurable.
To make the example complete, take $K = 1$ and notice that for $\xi = \int_0^{\frac{\pi}{2}} \cos(\frac{\pi}{2} - s)dW(s)$ we have multiple solutions, whereas for $\xi = W(T)$ we don't have any solution, since $\mathbb{E}[\int_0^{\frac{\pi}{2}} |\frac{1}{\cos(\frac{\pi}{2}-t)}|^2 ds] = +\infty$.

4 BSDEs with linear time delayed generators

In this section we investigate in more details the following backward stochastic differential equation with a linear time delayed generator

$$\begin{aligned} Y(t) &= \xi \\ &+ \int_t^T \int_{-T}^0 g(s+u)Z(s+u)\alpha(du)ds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T, \end{aligned} \tag{4.1}$$

with

(A5) a measurable, uniformly bounded function $g : [0,T] \to \mathbb{R}$ and the assumption $g(t) = 0$ for $t < 0$.

As for the measure α, we are particularly interested in the two extreme cases in which α is uniform or a Dirac measure.

For the linear equations (4.1) it is possible to describe solutions explicitly.

Theorem 4.1. *Assume that $\xi \in \mathbb{L}^{2+\epsilon}(\mathbb{R})$ for some $\epsilon > 0$, and* ***(A5)*** *holds. The backward stochastic differential equation* (4.1) *has a unique solution* $(Y,Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$, *where Z is the process appearing in the martingale representation*

$$\xi = \mathbb{E}^{\mathbb{Q}}[\xi] + \int_0^T Z(s)dW^{\mathbb{Q}}(s), \tag{4.2}$$

under the equivalent probability measure $\mathbb{Q}$ given by the density

$$\begin{aligned} &\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} \\ &= \exp\Big(\int_0^T \alpha((s-T,0])g(s)dW(s) - \frac{1}{2}\int_0^T \alpha^2((s-T,0])g^2(s)ds\Big), \end{aligned} \tag{4.3}$$

and the process Y defined by

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\big[\xi|\mathcal{F}_t\big] + \int_0^t \alpha((s-T,s-t])Z(s)ds, \quad 0 \le t \le T. \tag{4.4}$$

Proof:
First we notice that the generator of the equation (4.1) is Lipschitz continuous in the sense of **(A2)** from Section 2, since for λ-a.e. $t \in [0,T]$

$$\begin{aligned} &\Big|\int_{-T}^0 g(t+u)z(t+u)\alpha(du) - \int_{-T}^0 g(t+u)\tilde{z}(t+u)\alpha(du)\Big|^2 \\ &\le \int_{-T}^0 g^2(t+u)\alpha(du)\int_{-T}^0 \big|z(t+u)-\tilde{z}(t+u)\big|^2\alpha(du) \\ &\le G\int_{-T}^0 \big|z(t+u)-\tilde{z}(t+u)\big|^2\alpha(du), \end{aligned}$$

where we apply Cauchy-Schwarz' inequality, and G denotes the uniform bound on g. By recalling Theorem 2.3 we can conclude that there exists a unique solution $(Y, Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$ of (4.1).

Notice that by applying Fubini's theorem, changing the variables and using the assumption that g vanishes for $t < 0$, we can derive an alternative form of the integral of the generator

$$\begin{aligned}\int_t^T \int_{-T}^0 g(s+u)Z(s+u)\alpha(du)ds &= \int_{-T}^0 \int_t^T g(s+u)Z(s+u)ds\alpha(du)\\ &= \int_{-T}^0 \int_{(t-u)\vee 0}^{T-u} g(v)Z(v)dv\alpha(du)\\ &= \int_0^T \int_{v-T}^{(v-t)\wedge 0} g(v)Z(v)\alpha(du)dv,\end{aligned}$$

for $0 \le t \le T$. This allows us to rewrite the BSDE (4.1) as

$$\begin{aligned}Y(t) &= \xi + \int_0^T \alpha((s-T,(s-t)\wedge 0])g(s)Z(s)ds - \int_t^T Z(s)dW(s)\\ &= \xi + \int_0^t \alpha((s-T,s-t])g(s)Z(s)ds\\ &\quad - \int_t^T Z(s)\big(dW(s) - \alpha((s-T,0])g(s)ds\big), \quad 0 \le t \le T.\end{aligned}$$

The measure defined in (4.3) is an equivalent probability measure because $\int_0^T \alpha^2((s-T,0\rangle)g^2(s)ds$ is finite, and hence Novikov's condition is satisfied. We can therefore deal with the following equation under the measure $\mathbb{Q}$

$$\begin{aligned}Y(t) &= \xi\\ &\quad + \int_0^t \alpha((s-T,s-t\rangle)g(s)Z(s)ds - \int_t^T Z(s)dW^{\mathbb{Q}}(s)\big), \quad 0 \le t \le T, \end{aligned} \tag{4.5}$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion on $[0,T]$. The assumption that $\xi \in \mathbb{L}^{2+\epsilon}(\mathbb{R})$, for some $\epsilon > 0$, under the measure $\mathbb{P}$, yields that $\xi \in \mathbb{L}^{2+\frac{\epsilon}{2}}(\mathbb{R})$ under the measure $\mathbb{Q}$. This is justified by the inequality

$$\mathbb{E}^{\mathbb{Q}}\big[|\xi|^{2+\frac{\epsilon}{2}}\big] \le \big(\mathbb{E}^{\mathbb{P}}\big[\big(\frac{d\mathbb{Q}}{d\mathbb{P}}\big)^{\frac{4+2\epsilon}{\epsilon}}\big]\big)^{\frac{\epsilon}{4+2\epsilon}}\big(\mathbb{E}^{\mathbb{P}}\big[|\xi|^{2+\epsilon}\big]\big)^{\frac{2+\frac{\epsilon}{2}}{2+\epsilon}} < \infty,$$

which uses that the density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ possesses moments of all orders. Define (Y,Z) according to (4.2) and (4.4). Clearly, (Y,Z) is a solution of (4.5). The martingale representation theorem in $\mathbb{L}^{2+\frac{\epsilon}{2}}(\mathbb{R})$ under $\mathbb{Q}$ provides Z such that $\mathbb{E}^{\mathbb{Q}}\big[\big(\int_0^T |Z(s)|^2 ds\big)^{1+\frac{\epsilon}{4}}\big] < \infty$, see Theorem 5.1 in [6]. The process Z is also square integrable under $\mathbb{P}$, as is seen by

$$\mathbb{E}^{\mathbb{P}}\big[\int_0^T |Z(s)|^2 ds\big] \le \big(\mathbb{E}^{\mathbb{Q}}\big[\big(\frac{d\mathbb{P}}{d\mathbb{Q}}\big)^{\frac{4+\epsilon}{\epsilon}}\big]\big)^{\frac{\epsilon}{4+\epsilon}}\big(\mathbb{E}^{\mathbb{Q}}\big[\big(\int_0^T |Z(s)|^2 ds\big)^{1+\frac{\epsilon}{4}}\big]\big)^{\frac{4}{4+\epsilon}} < \infty.$$

It is standard to prove that $Y \in \mathbb{S}^2(\mathbb{R})$. We can conclude now that (Y,Z) defined by (4.2) and (4.4) is the unique solution of (4.1). $\square$

We can state two corollaries.

Corollary 4.1. *Under the assumptions of Theorem 4.1, the equation with a delay distributed according to an uniform measure*

$$Y(t) = \xi + \int_t^T \int_0^s KZ(u)duds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T,$$

has a unique solution $(Y,Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$, *with* Z *given by the martingale representation*

$$\xi = \mathbb{E}^{\mathbb{Q}}[\xi] + \int_0^T Z(s)dW^{\mathbb{Q}}(s),$$

under the equivalent probability measure

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \exp\big(K\int_0^T (T-s)dW(s) - K^2\frac{1}{2}\int_0^T (T-s)^2 ds\big),$$

and the process Y *defined by*

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\big[\xi|\mathcal{F}_t\big] + (T-t)K\int_0^t Z(s)ds, \quad 0 \le t \le T.$$

The proof is a straightforward application of Theorem 4.1.

Corollary 4.2. *Under the assumptions of Theorem 4.1, the equation with a delay distributed according to Dirac measure at the point* $r \in [0,T]$

$$Y(t) = \xi + \int_t^T KZ(s-r)ds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T,$$

has a unique solution $(Y,Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$, *given by the following statements.*

1. *On the interval* $[0, T-r)$, *define* Z *as the process arising in the martingale representation*

$$\mathbb{E}[\xi|\mathcal{F}_{T-r}] = \mathbb{E}^{\mathbb{Q}}\big[\mathbb{E}[\xi|\mathcal{F}_r]\big] + \int_0^{T-r} Z(s)dW^{\mathbb{Q}}(s),$$

under the equivalent probability measure

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_{T-r}} = \exp\big(KW(T-r) - \frac{1}{2}K^2(T-r)\big),$$

and

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\big[\mathbb{E}[\xi|\mathcal{F}_{T-r}]|\mathcal{F}_t\big] + K\int_{(t-r)\wedge 0}^t Z(s)ds, \quad 0 \le t < T-r.$$

2. *On the interval* $[T-r,T]$*, define* Z *as the process arising in the martingale representation*

$$\xi = \mathbb{E}[\xi|\mathcal{F}_{T-r}] + \int_{T-r}^{T} Z(s)dW(s),$$

and

$$Y(t) = \mathbb{E}[\xi|\mathcal{F}_t] + K\int_{(t-r)\wedge 0}^{T-r} Z(s)ds, \quad T-r \leq t \leq T.$$

Proof:

First we notice that in the case of a Dirac measure concentrated at r we have $\alpha((s-T,0]) = \mathbf{1}_{[0,T-r)}(s)$. We conclude that $\alpha((s-T,0]) = 0$ for $s \in [T-r,T]$ and the $\mathbb{Q}$-Brownian motion is the $\mathbb{P}$-Brownian motion on the interval $[T-r,T]$ (given $\mathcal{F}_{T-r}$). As $\alpha((s-T,0]) = 1$ for $s \in [0,T-r)$ we can define the corresponding density process $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t}$. Moreover, notice that $\alpha((s-T,s-t]) = \mathbf{1}_{[t-r,T-r)}(s)$.

Consider $t \in [0,T-r)$. By taking the conditional expectation under $\mathbb{P}$, we derive

$$Y(t) = \mathbb{E}\big[\xi\big|\mathcal{F}_{T-r}\big] + K\int_0^t \mathbf{1}_{[t-r,T-r)}(s)\, Z(s)ds - \int_t^{T-r} Z(s)dW^{\mathbb{Q}}(s).$$

Defining (Y,Z) as in the the first part of the statement, we get a solution on $[0,T-r)$.

Consider $t \in [T-r,T]$. We now have to deal with the equation

$$Y(t) = \xi + K\int_0^t \mathbf{1}_{[t-r,T-r)}(s)\, Z(s)ds - \int_t^T Z(s)dW(s).$$

Defining (Y,Z) as in the second part of the statement, we obtain a solution on $[T-r,T]$. One can verify, similarly as in the proof of Theorem 4.1, that the solution constructed in this way belongs to the right space: $(Y,Z) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$. This completes the proof. □

We next consider the *BMO* (bounded mean oscillation) property of the stochastic integral process $\int ZdW$ of the control component Z of the solution, which is important for solutions of BSDE without time delay in the generator. We discuss the question whether this property continues to hold if the generator possesses some linear time delay impact, such as in (4.1).

It is well-known, see Lemma 3.1.2 in [8], that for a BSDE with generator not subject to a time delay, a terminal condition ξ that is $\mathbb{P}$-a.s. bounded, and satisfying appropriate assumptions, the integral process $\int ZdW$ is a *BMO* martingale. It is a rather easy exercise to prove this result for a linear Lipschitz generator without a time delay. We show that for linear BSDEs with time delayed generators, this property still holds.

Recall that a stochastic integral process $(\int_0^t Z(s)dW(s))_{0\leq t\leq T}$ is a *BMO* martingale iff

$$\sup_\tau \mathbb{E}\big[\big|\int_\tau^T Z(s)dW(s)\big|^2\big|\mathcal{F}_\tau\big] = \sup_\tau \mathbb{E}\big[\int_\tau^T Z^2(s)ds\big|\mathcal{F}_\tau\big] < \infty, \quad \mathbb{P}-a.s.,$$

where the supremum is taken over all stopping times τ with respect to $\mathbb{F}$ and bounded by T, and $\mathcal{F}_\tau$ denotes the σ-algebra of the τ-past.

Lemma 4.1. *Assume that ξ is $\mathbb{P}$-a.s. bounded and **(A5)** holds. The backward stochastic differential equation* (4.1) *has a unique solution $(Y,Z)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})$, and the integral process $(\int_0^t Z(s)dW(s))_{0\leq t\leq T}$ is a BMO martingale.*

Proof:
Take two stopping times $\tau_1,\tau_2\leq T$ such that $0\leq\tau_2-\tau_1\leq\delta$ holds $\mathbb{P}$-a.s., with a sufficiently small constant δ to be specified later in the proof. The martingale representation (4.2) gives

$$\mathbb{E}^{\mathbb{Q}}\big[\xi\big|\mathcal{F}_{\tau_2}\big] = \mathbb{E}^{\mathbb{Q}}\big[\xi\big|\mathcal{F}_{\tau_1}\big] + \int_{\tau_2}^{\tau_1} Z(s)dW(s) - \int_{\tau_2}^{\tau_1} Z(s)\alpha((s-T,0])g(s)ds.$$

Now take a stopping time θ such that $\theta\leq\tau_1$ holds $\mathbb{P}$-a.s. As $Z\in\mathbb{H}^2(\mathbb{R})$, we have

$$\begin{aligned}
\mathbb{E}\big[\int_{\tau_1}^{\tau_2}|Z(s)|^2ds\big|\mathcal{F}_\theta\big] &= \mathbb{E}\big[\big|\int_{\tau_1}^{\tau_2} Z(s)dW(s)\big|^2\big|\mathcal{F}_\theta\big]\\
&= \mathbb{E}\big[\big|\mathbb{E}^{\mathbb{Q}}\big[\xi\big|\mathcal{F}_{\tau_2}\big] - \mathbb{E}^{\mathbb{Q}}\big[\xi\big|\mathcal{F}_{\tau_1}\big] - \int_{\tau_2}^{\tau_1} Z(s)\alpha((s-T,0])g(s)ds\big|^2\big]\\
&\leq 4\big(C+\delta G\mathbb{E}\big[\int_{\tau_1}^{\tau_2}|Z(s)|^2ds\big|\mathcal{F}_\theta\big]\big), \quad \mathbb{P}-a.s.,
\end{aligned}$$

where C denotes the uniform bound of ξ and G the one of g. For a sufficiently small $\delta<\frac{1}{4G}$ we have

$$\mathbb{E}\big[\int_{\tau_1}^{\tau_2}|Z(s)|^2ds\big|\mathcal{F}_\theta\big]\leq M, \quad \mathbb{P}-a.s.$$

holds with a finite constant M (independent of the stopping times).
Now let $\delta<\frac{1}{4G}$, and take an arbitrary stopping time τ. Define $\tau_k=(\tau+k\delta)\wedge T$, $k\geq 0$. Then $(\tau_k)_{k\geq 0}$ is a sequence of stopping times with respect to $\mathbb{F}$ such that $\tau_k-\tau_{k-1}\leq\delta$, and $\tau_k-\tau_{k-1}=0$ if $k\geq N=[\frac{T}{\delta}]+1$. We can deduce from the inequality proved before

$$\mathbb{E}\big[\int_\tau^T|Z(s)|^2ds\big|\mathcal{F}_\tau\big] = \sum_{i=1}^N\mathbb{E}\big[\int_{\tau_{i-1}}^{\tau_i}|Z(s)|^2ds\big|\mathcal{F}_\tau\big]\leq N\,M, \quad \mathbb{P}-a.s.$$

This proves the BMO property. □

For generators without time delay, it is well known that the solution component Y inherits uniform boundedness from the terminal condition ξ, see Proposition 2.1 in [10]. We shall now exhibit an example showing that this does not pertain if the generator has a linear delay dependence, as in Corollary 4.1.

Example 4

Consider first the local martingale $M(t) = \int_0^t \frac{2}{(1-s)^3} dW^{\mathbb{Q}}(s)$, $t \in [0,1)$, under the measure $\mathbb{Q}$ defined in Corollary 4.1. Let

$$\tau = \inf\{t \geq 0 : |M(t)| \geq 1\} \wedge 1.$$

We start by showing that τ can take values arbitrarily close to 1 with positive probability for $\mathbb{Q}$, hence also for the equivalent $\mathbb{P}$. This claim follows from classical results. In fact, by time change with the quadratic variation

$$\langle M \rangle_t = \int_0^t \frac{4}{(1-s)^6} ds = \frac{4q(t)}{5(1-t)^5}, \quad t \in [0,1),$$

with $q(t) = 1-(1-t)^5$, $t \in [0,1)$, $(M(t))_{0 \leq t < 1}$ has the same law as $(B(\frac{4q(t)}{5(1-t)^5}))_{0 \leq t < 1}$ with a $\mathbb{Q}$-Brownian motion B. Defining

$$\sigma = \inf\{t \geq 0 : |B(t)| \geq 1\},$$

we obtain that under $\mathbb{Q}$, σ has the same law as $\frac{4q(\tau)}{5(1-\tau)^5}$. Since σ can take values arbitrarily close to ∞ with positive probability, τ can take values arbitrarily close to 1 with positive probability.

Consider the linear BSDE

$$Y(t) = \xi + \int_t^1 \int_0^s Z(u) du ds - \int_t^1 Z(s) dW(s), \quad 0 \leq t \leq 1,$$

and define $\xi = M_\tau$. Then ξ is bounded, and we have the stochastic integral representation

$$\xi = \int_0^1 \frac{2}{(1-s)^3} 1_{[0,\tau)}(s) dW^{\mathbb{Q}}(s).$$

Therefore, Corollary 4.1 yields the solution

$$\begin{aligned} Z(t) &= \frac{2}{(1-t)^3} 1_{[0,\tau)}(t), \quad 0 \leq t \leq 1, \\ Y(t) &= \mathbb{E}^{\mathbb{Q}}[M_\tau | \mathcal{F}_t] + (1-t) \int_0^t Z(s) ds \\ &= \mathbb{E}^{\mathbb{Q}}[M_\tau | \mathcal{F}_t] + (1-t) \frac{(\tau \wedge t)(2 - \tau \wedge t)}{1 - (\tau \wedge t)^2}, \quad 0 \leq t \leq 1. \end{aligned}$$

Take an arbitrary constant $C > 0$. We can find $u \in [0, 1)$ such that $\frac{u(2-u)}{1-u} > C - 1$. As τ takes values arbitrarily close to 1 with positive probability, $\mathbb{P}(\tau \geq u) > 0$ and with positive probability

$$(1-u)\frac{(\tau \wedge u)(2 - \tau \wedge u)}{1 - (\tau \wedge u)^2} = \frac{u(2-u)}{1-u} > C - 1,$$

$$Y(u) > C.$$

We can conclude that for an arbitrarily large C there exists $u \in [0, 1)$ such that the process Y at time u crosses C with positive probability, $\mathbb{P}(Y(u) > C) > 0$. This establishes the lack of uniform boundedness for Y.

5 The comparison principle and measure solutions

The concepts of comparison principle and measure solutions play an important role in the theory of BSDE without time delays. In this section we first show by an example that they cannot be extended to time delayed BSDE. We shall see that the failure of the properties can be traced back to a sign change of the control process Z, and consequently show that they continue to hold on stochastic intervals on which Z stays away from 0. For a statement of the comparison principle, we refer the reader to Theorem 2.2 in [6], and for the concept of a measure solution to the paper [1].

Example 5

We first give an example exhibiting a failure of the comparison principle. Consider the linear backward stochastic differential equation with time delayed generator

$$Y(t) = \xi + \int_t^T \int_0^s Z(u)duds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T. \tag{5.1}$$

Take $\tilde{\xi} = 0$. The corresponding solution of (5.1) is $\tilde{Y}(t) = \tilde{Z}(t) = 0, 0 \leq t \leq T$. To compare, take $\xi = \big(W(T) - \frac{T^2}{2}\big)^2$. By applying Corollary 4.1 we can construct the corresponding solution (Y, Z) of (5.1). The martingale representation of ξ under the measure $\mathbb{Q}$ with the $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}(t) = W(t) + \frac{(T-t)^2}{2} - \frac{T^2}{2}, 0 \leq t \leq T$, yields

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\big(W(T) - \frac{T^2}{2}\big)^2|\mathcal{F}_t] &= \mathbb{E}^{\mathbb{Q}}[W^{\mathbb{Q}}(T)^2|\mathcal{F}_t] \\
&= W^{\mathbb{Q}}(t)^2 + (T-t) \\
&= T + \int_0^t 2W^{\mathbb{Q}}(s)\, dW^{\mathbb{Q}}(s),
\end{aligned}$$

and we can derive the solution

$$Y(t) = W^{\mathbb{Q}}(t)^2 + (T-t) + (T-t)\int_0^t 2W^{\mathbb{Q}}(t)ds.$$

Clearly, $\tilde{\xi} \leq \xi$ holds $\mathbb{P}$-a.s. It is straightforward to note that $\tilde{Y}(0) \leq Y(0)$ holds $\mathbb{P}$-a.s. However, we claim that for any $t \in (0,T)$ we have $\mathbb{Q}\big(\tilde{Y}(t) > Y(t)\big) > 0$, which, by equivalence of the measures, contradicts the comparison principle under $\mathbb{P}$.

To prove that $\mathbb{Q}\big(\tilde{Y}(t) > Y(t)\big) > 0$, it is sufficient to show that the conditional law of $\int_0^t W^{\mathbb{Q}}(s)ds$ given $W^{\mathbb{Q}}(t) = x$ is unbounded for any $t \in (0,T)$ and any $x \in \mathbb{R}$. This can be verified under $\mathbb{P}$ for W instead of $W^{\mathbb{Q}}$ as well. First recall, see Chapter 5.6.B in [9], that the conditional law of $W(s)$ given $W(t) = x$ is nondegenerate Gaussian, for $0 < s < t$, for any $(t,x) \in (0,T) \times \mathbb{R}$, and that the process $[0,t] \ni s \mapsto W(s)$ is a Brownian bridge from 0 to x conditional on $\{W_t = x\}$. We have the convergence

$$\int_0^t W(s)ds = \lim_{N\to\infty} \frac{\sum_{i=1}^N W(t_i^N)}{N}, \quad \mathbb{P}-a.s. \quad \text{and in} \quad L^2,$$

for a sequence of equidistant partitions $0 = t_0 < t_1^N < ... < t_N^N = t$ of $[0,t]$. As the L^2-limit of a Gaussian sequence $\int_0^t W(s)ds$ is Gaussian. It is straightforward to show that the variance of $\int_0^t W(s)ds$ conditional on $\{W_t = x\}$ is strictly positive. We conclude that the conditional law of $\int_0^t W(s)ds$ given $\{W(t) = x\}$ is unbounded for any $t \in (0,T)$ and any $x \in \mathbb{R}$.

A failure of the comparison principle indicates that it may also not always be possible to represent a solution of a BSDE with a time delayed generator as a measure solution. Recall, that for a BSDE without time delay and with a Lipschitz continuous generator independent of Y, a unique square integrable solution can always be represented as a conditional expectation of the terminal value under an appropriate probability measure (a *measure solution*), see Theorem 1.1 in [1]. The following example shows that this property may fail for the solution of a time delayed BSDE.

Example 6

Consider again the linear backward stochastic differential equation

$$Y(t) = \xi + \int_t^T \int_0^s Z(u)duds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T, \tag{5.2}$$

with $\xi = \int_0^T \cos t dW(t)$. An easy calculation shows that there exists a unique square integrable solution given by

$$\begin{aligned} Z(t) &= \cos t, \quad 0 \leq t \leq T, \\ Y(t) &= \int_0^t \cos s dW(s) + \cos t - \cos T, \quad 0 \leq t \leq T. \end{aligned}$$

To describe a possible measure solution, for $T < \frac{\pi}{2}$ rewrite the equation (5.2) as

$$Y(t) = \xi + \int_t^T \cos s\big(dW(s) - \tan s ds\big), \quad 0 \leq t \leq T.$$

One can define the equivalent probability measure

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \exp\big(\int_0^T \tan s dW(s) - \frac{1}{2}\int_0^T \tan^2 s ds\big),$$

and the unique measure solution under this measure

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\big[\xi|\mathcal{F}_t\big], \quad 0 \le t \le T.$$

Consider now the case $T = \frac{\pi}{2}$. If there were a measure solution on $[0, \frac{\pi}{2}]$ under some probability measure $\mathbb{Q}$ then

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\big(\int_0^t \tan s dW(s) - \frac{1}{2}\int_0^t \tan^2 s ds\big),$$

for any $0 \le t < \frac{\pi}{2}$, and the following limiting relation would hold

$$\lim_{t\to\frac{\pi}{2}} \exp\big(\int_0^t \tan s dW(s) - \frac{1}{2}\int_0^t \tan^2 s ds\big) = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_{\frac{\pi}{2}}}.$$

We show that this limit is not a probability density.
Define a sequence of points $0 = t_0 < t_1 < ... < t_n < \cdots < \frac{\pi}{2}$, for $n \in \mathbb{N}$, such that

$$\int_{t_{i-1}}^{t_i} \tan^2 s ds = 1, \quad \forall i \in \mathbb{N},$$

and $\lim_{n\to\infty} t_n = \frac{\pi}{2}$. The sequence of random variables $(X_i)_{i\in\mathbb{N}}$ defined by $X_i = \int_{t_{i-1}}^{t_i} \tan s dW(s)$ is i.i.d. with standard Gaussian laws. The strong law of large numbers implies

$$\begin{aligned}
&\lim_{t\to\frac{\pi}{2}} \exp\big(\int_0^t \tan s dW(s) - \frac{1}{2}\int_0^t \tan^2 s ds\big) \\
&= \lim_{n\to\infty} \exp\big(\int_0^{t_n} \tan s dW(s) - \frac{1}{2}\int_0^{t_n} \tan^2 s ds\big) \\
&= \lim_{n\to\infty} \exp\big(X_1 + \cdots + X_n - \frac{1}{2}n\big) \\
&= \lim_{n\to\infty} \exp\big(n(\frac{X_1 + \cdots + X_n}{n} - \frac{1}{2})\big) \\
&= 0, \quad \mathbb{P} - a.s.
\end{aligned}$$

This shows that an equivalent probability measure cannot be defined on $[0, \frac{\pi}{2}]$. In summary we have established a BSDE with a time delayed generator, for which there is a unique square integrable solution, whereas a measure solution fails to exist.

One observation we can draw from the preceding two examples is that the comparison principle may not hold and measure solutions may fail to exist, while the (continuous) control process Z can cross 0. In the following two Theorems we shall exclude the approach of the difference of two control processes resp. one control process to 0 or ∞ by stopping them before passages of small or large thresholds happen. We shall prove that on the corresponding stochastic intervals, the comparison principle holds, and a measure solution exist.

Theorem 5.1. *Consider the backward stochastic differential equations* (2.1) *with generators $f, \tilde{f}$ and corresponding terminal values $\xi, \tilde{\xi}$ satisfying the assumptions **(A1)-(A4)**. Let (Y, Z) and $(\tilde{Y}, \tilde{Z})$ denote the associated unique solutions in $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$. For $n \in \mathbb{N}$ define the stopping time $\tau_n = \inf\{t \geq 0 : |Y(t) - \tilde{Y}(t)| \vee |Z(t) - \tilde{Z}(t)| \leq \frac{1}{n}$ or $|Y(t) - \tilde{Y}(t)| \vee |Z(t) - \tilde{Z}(t)| \geq n\} \wedge T$. Suppose that*

- $Y(\tau_n) \geq \tilde{Y}(\tau_n), \mathbb{P}$*-a.s.,*
- $\delta f(t, y_t, z_t) := f(t, y_t, z_t) - \tilde{f}(t, y_t, z_t) \geq 0, \quad t \in [0, T], (y_t, z_t) \in L^{\infty}_{-T}(\mathbb{R}) \times L^2_{-T}(\mathbb{R})$.

Then, $Y(t) \geq \tilde{Y}(t)$ holds $\mathbb{P}$-a.s on $[0, \tau_n]$.

Proof:
We follow the idea from the proof of Theorem 2.2 in [6]. For $t \in [0, T]$ let

$$\delta Y(t) = Y(t) - \tilde{Y}(t), \quad \delta Z(t) = Z(t) - \tilde{Z}(t),$$

$$\delta f(s, \tilde{Y}_s, \tilde{Z}_s) = f(s, \tilde{Y}_s, \tilde{Z}_s) - \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s),$$

and

$$\triangle_y f(t) = \frac{f(t, Y_t, Z_t) - f(t, \tilde{Y}_t, Z_t)}{Y(t) - \tilde{Y}(t)}, \quad \triangle_z f(t) = \frac{f(t, \tilde{Y}_t, Z_t) - f(t, \tilde{Y}_t, \tilde{Z}_t)}{Z(t) - \tilde{Z}(t)}.$$

We can derive

$$\begin{aligned} \delta Y(t) &= \delta Y(\tau_n) + \int_t^{\tau_n} \big(\triangle_y f(s) \delta Y(s) + \triangle_z f(s) \delta Z(s) + \delta f(s, \tilde{Y}_s, \tilde{Z}_s)\big) ds \\ &\quad - \int_t^T \delta Z(s) dW(s), \quad 0 \leq t \leq \tau_n. \end{aligned}$$

By rewriting these expressions and changing the measure we obtain

$$\begin{aligned} \delta Y(t) &= \delta Y(\tau_n) e^{\int_t^{\tau_n} \triangle_y f(s) ds} + \int_t^{\tau_n} \delta f(s, \tilde{Y}_s, \tilde{Z}_s) e^{\int_t^s \triangle_y f(u) du} ds \\ &\quad - \int_t^{\tau_n} \delta Z(s) e^{\int_t^s \triangle_y f(u) du} dW^{\mathbb{Q}}(s), \quad 0 \leq t \leq \tau_n, \end{aligned} \tag{5.3}$$

with the equivalent probability measure $\mathbb{Q}$ defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_{\tau_n}} = \exp\Big(\int_0^{\tau_n} \triangle_z f(s) dW(s) - \frac{1}{2} \int_0^{\tau_n} \big(\triangle_z f(s)\big)^2 ds\Big).$$

Since $\delta Z \in \mathbb{H}^2(\mathbb{R})$ under the measure $\mathbb{P}$, the density $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_{\tau_n}}$ is square integrable under the measure $\mathbb{P}$, and $t \mapsto \triangle_y f(t)$ is a.s. uniformly bounded up to time τ_n, we can use Cauchy-Schwarz' inequality to obtain

$$\mathbb{E}^{\mathbb{Q}}\Big[\sqrt{\int_0^{\tau_n} \big|\delta Z(s)e^{\int_0^s \triangle_y f^1(u)du}\big|^2 ds}\Big] < \infty,$$

and $\big(\int_0^t \delta Z(s)e^{\int_0^s \triangle_y f^1(u)du} dW^{\mathbb{Q}}(s)\big)_{0\leq t\leq \tau_n}$ is a $\mathbb{Q}$-martingale with vanishing expectation, see Theorem 3.28 in [9]. Taking the conditional expectation with respect to $\mathcal{F}_t$ on both sides of equation (5.3) under the measure $\mathbb{Q}$, we get the desired result. $\square$

Theorem 5.2. *Consider the backward stochastic differential equation* (2.1) *with the generator* $f(t, y_t, z_t) = f(t, z_t), t \geq 0, (y_t, z_t) \in L^{\infty}_{-T}(\mathbb{R}) \times L^2_{-T}(\mathbb{R})$, *and the corresponding terminal value* ξ *satisfying the assumptions* ***(A1)-(A4)***. *Let* (Y, Z) *denote the associated unique solution in* $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R})$. *For* $n \in \mathbb{N}$ *define the stopping time* $\tau_n = \inf\{t \geq 0; |Z(t)| \leq \frac{1}{n} \text{ or } |Z(t)| \geq n\} \wedge T$. *Then, there exists a unique equivalent probability measure* $\mathbb{Q}$ *restricted to* $[0, \tau_n]$ *such that*

$$Y(t) = \mathbb{E}^{\mathbb{Q}}\big[Y(\tau_n)|\mathcal{F}_t\big],$$

holds for all $t \in [0, \tau_n]$, $\mathbb{P}$-*a.s.*

Proof:
The proof requires a change of measure argument, just as in the preceding proof. Details are omitted. $\square$

We remark that if Y is $\mathbb{P}$-a.s. bounded, one can define $\tau_n^m = \inf\{t \geq 0 : |Z(t)| \leq \frac{1}{n} \text{ or } |Z(t)| \geq m\} \wedge T$ for $n, m \in \mathbb{N}$ and show that the corresponding family of measures $\big(\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_{\tau_n^m}}\big)_{m\in\mathbb{N}}$ is uniformly integrable. Compare the proof of Theorem 1.1 [1]. In this case a unique measure solution can be defined on $[0, \tau_n^{\infty}]$ with $\tau_n^{\infty} = \inf\{t \geq 0; |Z(t)| \leq \frac{1}{n}\} \wedge T$.

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References

[1] Ankirchner, S., Imkeller, P., Popier, A. (2008) On measure solutions of backward stochastic differential equations. Preprint.

[2] Becherer, D. (2006) Bounded solutions to backward SDE's with jumps for utility optimization and indifference pricing. Ann. Appl. Probab. **16**, 2027-2054.

[3] Blanchet-Scalliet, C., Eyraud-Loisel, A., Royer-Carenzi, M. (2008). Hedging of defaultable claims using BSDE with uncertian time horizon. Preprint.

[4] Buckdahn, R., Imkeller, P. (2009) Backward stochastic differential equations with time delayed generator. Preprint.

[5] Delong, Ł., Imkeller, P. (2009) On Malliavin's differentiability of time delayed BSDEs driven by Brownian motions and Poisson random measures. Preprint.

[6] El Karoui, N., Peng, S., Quenez, M.C., (1997) Backward stochastic differential equations in finance. Math. Finance **7**, 1-71.

[7] Fuhrman, M., Masiero, F., Tessitore, G. (2008) Stochastic equations with delay: optimal control via BSDEs and regular solutions of Hamilton-Jacobi-Bellman equations. Preprint.

[8] Imkeller, P. (2008) *Malliavin's Calculus and Applications in Stochastic Control and Finance.* IM PAN Lectures Notes, Warsaw.

[9] Karatzas, I., Shreve, S. (1988) *Brownian Motion and Stochastic Calculus.* Springer-Verlag.

[10] Kobylanski, M. (2000) Backward stochastic differential equations and partial differential equations with quadratic growth. Ann. Probab. **28**, 558-602.

[11] Mohammed, S.E.A. (1984) *Stochastic functional Differential Equations.* Pitman.

[12] Morlais, M. A. (2009) Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem. Finance Stoch. **13**, 121-150.

[13] Pardoux, E. Peng, S. (1990) Adapted solution of a backward stochastic differential equation. Systems Control Lett. **14**, 55-61.

[14] Protter, P. (1992) *Stochastic Integration and Differential Equations.* Springer-Verlag.

[15] Xu, D., Yang, Z., Huang,Y. (2008) Existence-uniqueness and continuation theorems for stochastic functional differential equations. J. Differential Equations **254**, 1681-1703.