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The hierarchy of exit times of Lévy-driven Langevin equations

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Abstract. In this paper we consider the first exit problem of an overdamped Lévy driven particle in a confining potential. We survey results obtained in recent years from our work on the Kramers' times for dynamical systems of this type with Lévy perturbations containing heavy, and exponentially light jumps, and compare them to the well known case of dynamical systems with Gaussian perturbations. It turns out that exits induced by Lévy processes with jumps are always essentially faster than Gaussian exits.

1 Introduction

Dynamical systems subject to small random perturbations play an important role both in the physics and mathematics literature. Many interesting questions relate to the problem of the first exit from and the corresponding problem of transition between their domains of attraction of stable equilibria. Random noise makes stable equilibria become meta-stable and largely determines their asymptotic dynamic properties. The study of perturbations by white Gaussian noise has the longest history (see e.g. [1, 2]), and richest bibliography. The standard mathematical reference on this subject is [3].

Recently non-Gaussian, in particular jump Lévy noise has been receiving increasing attention in the study of many systems of sciences and economics. Lévy noise with heavy tails (Lévy flights) is observed for instance in Greenland ice core measurements (see [4]), and therefore used to model important qualitative features of paleo-climatic processes by low-dimensional dynamical systems. In biology Lévy flights are observed for example in the behavioral pattern of certain species such as albatrosses [5] or anchovies [6]. They are used to account for the uncertainties in price fluctuations in dynamical models of financial markets [7]. Lévy flights also naturally appear in particle evolutions along polymer chains [8, 9].

In this paper we give a purely probabilistic description of a noisy Lévy particle in an external potential. More precisely we investigate equations of motion of overdamped particles perturbed by small discontinuous noise processes. In the limit of small noise intensity we describe the exit law from a potential well, the analogue of Kramers' law for Gaussian diffusions. The main results presented in this paper account for a complete description of exit time rates for equations driven by Lévy processes with heavy-tailed (polynomial) and light-tailed (sub- or super-exponential) jump measures.

We observe different patterns of exit. For algebraic or sub-exponentially light jumps, the exit coincides with one big enough jump by which the particle penetrates through the barrier. For super-exponentially light jumps, the particle makes finitely many smaller jumps that add up

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to trigger the exit. Opposite to the Gaussian case, the logarithmic rates of the exit times turn out not to depend on the height of the potential barrier, but only on the distance between the location of the stable minimum and the boundary of its domain of attraction. Our results specify the dependence of the mean first exit time both on the small parameter and the geometry of the domain.

2 Object of study

We consider the following one-dimensional Langevin equation with additive Lévy noise:

$$dX_t^\varepsilon = -U'(X_t^\varepsilon)\,dt + \varepsilon dL_t, \quad X_0^\varepsilon = x, \quad t \geq 0. \tag{2.1}$$

We make the following assumptions on the components of this equation.

- $x \in [-a, b]$, $0 < a, b < \infty$;
- U is smooth enough and has 'parabolic' form, that is $xU'(x) \geq 0$, $U'(x) = 0$ iff $x = 0$, $U''(0) > 0$, $U'(-a), U'(b) \neq 0$;
- $\varepsilon > 0$ is the noise intensity;
- L is a symmetric Lévy process with characteristic function

$$\mathbf{E}e^{i\lambda L_t} = \exp\Big[-t\Big(\sigma^2\frac{\lambda^2}{2} + \int\Big(e^{i\lambda y} - 1 - \frac{y}{1+y^2}\Big)\nu(dy)\Big)\Big], \quad t \geq 0. \tag{2.2}$$

It is a sum of a Brownian motion with the variance σ^2 and an independent pure jump process governed by a jump measure ν satisfying the usual conditions $\int_{\mathbb{R}} \frac{y^2}{1+y^2}\nu(dy) < \infty$ and $\nu(\{0\}) = 0$.

Examples of Lévy processes:

1. If $\nu \equiv 0$ and $\sigma = 1$, then L is the standard Brownian motion.
2. If $\sigma = 0, \alpha \in (0, 2)$, and

$$\nu(dy) = \frac{dy}{|y|^{1+\alpha}}, \quad y \neq 0, \tag{2.3}$$

then L is a Lévy flights process (symmetric α-stable Lévy process). In this case, the characteristic function has the form

$$\mathbf{E}e^{i\lambda L_t} = e^{-tc(\alpha)|\lambda|^\alpha}, \quad \lambda \in \mathbb{R}, \tag{2.4}$$

with the normalizing constant $c(\alpha) = 2\int_0^\infty \frac{1-\cos y}{y^{1+\alpha}}\,dy$.
3. For $\nu(dy) = \frac{1}{2}(\delta_{-1}(dy) + \delta_1(dy))$, $\sigma = 0$, the process L is a Poisson process with symmetric jumps of size ± 1.
4. Let W be a symmetric random variable with a distribution function $F(x) = \mathbf{P}(W \leq x)$, $P(W = 0) = 0$. Let $c > 0$ and

$$\nu(dy) = cdF(y). \tag{2.5}$$

Then with $\sigma = 0$, L is a compound Poisson process with jumps distributed according to F and the exponentially distributed inter-jump times with intensity c:

$$\mathbf{E}e^{i\lambda L_t} = \exp\Big(c\int(e^{i\lambda y} - 1)\,dF(y)\Big). \tag{2.6}$$

If $\sigma \neq 0$ then L is a so-called jump diffusion, that is a sum of a compound Poisson process and an independent Brownian motion.

The solution X^ε of the SDE driven by εL is also a jump process with the same jumps as εL and a deterministic drift force towards the stable point 0 given by the gradient $-U'$. We consider the problem of the first exit of X^ε from the interval $[-a,b]$ in the limit of small noise $\varepsilon\to 0$. More precisely, we define the first exit time by

$$\sigma_x(\varepsilon)=\inf\{t\geq 0:\ X_t^\varepsilon(x)\notin[-a,b]\}. \tag{2.7}$$

For Gaussian forcing, the paths of X^ε are continuous. In this case we can speak about the first hitting time of the level $-a$ or b, whence $X^\varepsilon_{\sigma(\varepsilon)}=-a$ or $X^\varepsilon_{\sigma(\varepsilon)}=b$. If L is a pure jump process, X^ε hits the boundary $-a$ and b with probability zero, and either $X^\varepsilon_{\sigma(\varepsilon)}<-a$ or $X^\varepsilon_{\sigma(\varepsilon)}>b$. In our study of the first exit problem, we focus on the dependence of exit times on the *heavyness* of the tails of the jump measure ν. Our goal consists in description of the characteristic time scales (Kramers' times) which correspond to the following types of tails of the jump measure ν.

1. $\nu=0$: here we recover the classical Kramers' exit time theory for diffusions;
2. $\nu[u,+\infty)\approx\frac{1}{u^\alpha}$, $\alpha\in(0,2)$: this is the case of the first exit of jump diffusions driven by Lévy flights;
3. $\nu[u,+\infty)\approx\frac{1}{u^r}$, $r>0$: this is the first exit of jump diffusions driven by Lévy processes with power heavy jumps, in particular Lévy flights and the so-called weakly tempered Lévy flights (see [10,11]) with jump measure

$$\nu(dy)=\frac{dy}{|y|^{1+\alpha}(1+y^2)^{\beta/2}},\quad \alpha\in(0,2),\quad \beta\geq 0. \tag{2.8}$$

   These processes have jumps as small as a Lévy flight of index α; their big jumps have index $r=\alpha+\beta>0$. In particular, such processes have all moments up to order r: $\mathbf{E}|L_t|^\gamma<\infty$ for all $0<\gamma<r, t\geq 0$.
4. $\nu[u,+\infty)\approx e^{-u^\alpha}$, $\alpha\in(0,1)$: this describes the case of sub-exponentially light jumps;
5. $\nu[u,+\infty)\approx e^{-u^\alpha}$, $\alpha>1$: this is the case of super-exponentially light jumps.
6. The case of a Lévy process with bounded jumps can be considered as the limiting case $\alpha=+\infty$.

Note that although the jump measure ν determines the law of L_t for all $t\geq 0$, the tails of the probability density function of L_t do not always coincide with the tails of ν. Indeed, if L is a Brownian motion, then $\nu=0$ but $\mathbf{P}(L_t>x)\sim\frac{1}{x}e^{-x^2/(2t)}$. However, for Lévy processes with regularly varying tails of the jump measure one can prove the relation $\mathbf{P}(L_t>x)/\nu([x,\infty))\to 1$ as $x\to\infty$ (see [12]).

3 Hierarchy of exit times

3.1 Kramers' times

If L is a standard Brownian motion, then Eq. (2.1) is a classical Langevin equation driven by Gaussian white noise. The small noise limit of this equation has been studied in numerous mathematical [13,14,3,15–17] and physical [18,2] papers.

The results on the first exit time from a potential well of Gaussian diffusions in the small noise limit can be summarized as follows. Assume for brevity, that the potential barrier at the right end b of the interval is lower than at the left end, i.e. $U(b)<U(-a)$. Then the mean exit time from the interval in the small noise limit, known as Kramers' time, is determined by the height of the potential barrier the Gaussian particle has to overcome, and is exponentially large in the small intensity parameter ε. Formally, we have

$$\mathbf{E}\sigma_x(\varepsilon)\approx\frac{\varepsilon\sqrt{\pi}}{U'(b)\sqrt{U''(0)}}\exp\Big(\frac{2U(b)}{\varepsilon^2}\Big),\quad \varepsilon\to 0. \tag{3.1}$$

Furthermore, the normalized first exit time is exponentially distributed in the limit of small noise:

$$\mathbf{P}\Big(\frac{\sigma_x(\varepsilon)}{\mathbf{E}\sigma_x(\varepsilon)} > t\Big) \to e^{-t}, \quad t \geq 0, \quad \varepsilon \to 0. \tag{3.2}$$

To exit, the diffusion overcomes the lowest potential barrier. Unpredictable exponentially long exit times depend on the potential's energy landscape, on the curvature of the potential at the stable equilibrium of the dynamical system, and on its slope at the point of the most probable exit.

3.2 Lévy flights and general heavy tails

The exit pattern becomes quite different if we consider Langevin equations driven by white Lévy noise with heavy tails. Assume that the Lévy measure has fat tails, i.e. with some $r > 0$ we have $\nu([u, +\infty)) \approx u^{-r}$, $u \to +\infty$. The case $r \in (0, 2)$ corresponds to the well known case of L évy flights.

The asymptotics of the first exit time was studied mathematically about thirty years ago in [19], then in [20,4] in the context of paleo-climatic modelling as well as in [21,22,**?**]. Recent mathematical results can be found in [23–25]. In the case of power heavy tails the mean exit time is governed by the index r and the distance between the stable equilibrium of the deterministic system and the boundaries of the interval. Formally, we have

$$\mathbf{E}\sigma_x(\varepsilon) \approx \frac{1}{\varepsilon^r}\Big[\frac{1}{a^r} + \frac{1}{b^r}\Big]^{-1}. \tag{3.3}$$

The probability law of the normalized first exit time is again exponential in the limit of small noise:

$$\mathbf{P}\Big(\frac{\sigma_x(\varepsilon)}{\mathbf{E}\sigma_x(\varepsilon)} > t\Big) \to e^{-t}, \quad \varepsilon \to 0. \tag{3.4}$$

The jump diffusion exits with a single big jump. Unpredictable polynomially long exit times depend only on the horizontal diameter of the considered domain.

3.3 Sub-exponential tails

Consider now Lévy forcing with essentially lighter sub-exponential big jumps governed by the Lévy measure $\nu([u, \infty)) \approx \exp(-u^\alpha)$ with $0 < \alpha < 1$. This and the subsequent super-exponential case have been treated in [26]. Assume that $b < a$.

In this setting, the exit pattern is similar to the one for jump measures with polynomial tails. We have

$$\mathbf{E}\sigma_x(\varepsilon) \propto \exp\Big(\frac{b^\alpha}{\varepsilon^\alpha}\Big), \tag{3.5}$$

where '$\propto$' stands for the logarithmic equivalence, i.e. we have $\varepsilon^\alpha \ln \mathbf{E}\sigma_x(\varepsilon) \to b^\alpha$. Unfortunately we were not able to determine the pre-factor in this asymptotic relationship. Furthermore, the law of the first exit time is also exponential in the following sense: for any $\delta > 0$ and ε small enough the estimates

$$\exp(-C_\varepsilon^{1-\delta} t) \leq \mathbf{P}(\sigma_x(\varepsilon) > t) \leq \exp(-C_\varepsilon^{1+\delta} t), \tag{3.6}$$

hold for all $t \geq 0$ with $C_\varepsilon = e^{-b^\alpha/\varepsilon^\alpha}$.

Similarly to the heavy–tail case, exits occur via one big jump.

3.4 Super-exponential tails

Finally, we consider the Lévy forcing with super-exponentially light tails for the jump measure, where we have $\nu(u,\infty)\approx\exp(-u^\alpha)$, $\alpha>1$. As before, assume that $b<a$. The mean exit time now has a quite different asymptotic behavior. It turns out that

$$\mathbf{E}\sigma_x(\varepsilon)\propto\exp\Big(b\cdot\alpha(\alpha-1)^{\frac{1}{\alpha}-1}\frac{|\ln\varepsilon|^{1-\frac{1}{\alpha}}}{\varepsilon}\Big). \tag{3.7}$$

It is interesting to notice that in the super-exponential case the dependence of the exponent of the mean exit time on the shortest distance b between the exit point and the stable equilibrium of the dynamical system becomes linear. Although the pre-factor in the formula for the first exit time is not known, the following estimates hold for its distribution:

$$\begin{aligned}&\exp(-D_\varepsilon^{1-\delta}t)\le\mathbf{P}(\sigma_x(\varepsilon)>t)\le\exp(-D_\varepsilon^{1+\delta}t),\quad t\ge0,\quad\delta>0,\\&D_\varepsilon=\exp\Big(-b\cdot\alpha(\alpha-1)^{\frac{1}{\alpha}-1}\frac{|\ln\varepsilon|^{1-\frac{1}{\alpha}}}{\varepsilon}\Big).\end{aligned} \tag{3.8}$$

The exit now is due to a big but finite number of relatively small jumps ‘climbing’ in one direction towards the point of exit b.

4 Heuristic derivation of the mean exit times for sub- and super-exponential jump tails

A detailed and rigorous proof of the derivation of the asymptotic behavior of the first exit time can be found in [26].

The backbone of the proof consists in the decomposition of the driving process L into two Lévy processes. Indeed, for $\varepsilon>0$ let $g=g_\varepsilon>0$ be a threshold. We consider the big jump process η^ε consisting of jumps of L, the absolute values of which are bigger than g_ε:

$$\eta^\varepsilon_t:=\sum_{s\le t}\Delta L_s\cdot\mathbb{I}(|\Delta L_s|\ge g_\varepsilon),\ t\ge0. \tag{4.1}$$

We denote the small jump process by

$$\xi^\varepsilon_t:=L_t-\eta^\varepsilon_t,\ t\ge0. \tag{4.2}$$

The processes η^ε and ξ^ε are independent Lévy processes with characteristic functions

$$\begin{aligned}\mathbf{E}e^{i\lambda\xi^\varepsilon_t}&=\exp\Big[\int_{|y|<g_\varepsilon}\Big(e^{i\lambda y}-1-\frac{y}{1+y^2}\Big)\nu(dy)\Big],\\\mathbf{E}e^{i\lambda\eta^\varepsilon_t}&=\exp\Big[\int_{|y|\ge g_\varepsilon}(e^{i\lambda y}-1)\nu(dy)\Big],\ t\ge0,\ \lambda\in\mathbb{R}.\end{aligned} \tag{4.3}$$

The process η^ε is a compound Poisson process with the jump intensity $\beta_\varepsilon:=2\nu([g_\varepsilon,\infty))=2e^{-g_\varepsilon^\alpha}$ and jump sizes W_i, $i\ge1$, with probability distribution function

$$\mathbf{P}(W_i\le w)=\frac{1}{\beta_\varepsilon}\int_{-\infty}^{w}\mathbb{I}(|y|\ge g_\varepsilon)\nu(dy),\ w\in\mathbb{R}. \tag{4.4}$$

The threshold g_ε will be chosen later; at the moment we impose the condition that $g_\varepsilon\to\infty$ and $\varepsilon g_\varepsilon\to0$ as $\varepsilon\to0$. Under this condition it can be shown that the process $\varepsilon\xi^\varepsilon$ can be seen as a small perturbation of the deterministic system $\dot{x}=-U'(x)$, and its impact on the exit behavior of the system can be neglected. We will mainly deal with the compound Poisson process $\varepsilon\eta^\varepsilon$.

Assume that we know the asymptotic behavior of the probability $p_\varepsilon \approx \mathbf{P}(\sigma_x(\varepsilon) \le T)$ for some T which does not depend on ε. Although this probability depends on the initial point x, it is clear that for ε small and T large enough, the process X^ε spends most of the time in a small neighborhood of the stable equilibrium of the dynamical system. So we can assume $p_\varepsilon \approx \mathbf{P}(\sigma_0(\varepsilon) \le T)$. Then the mean exit time $\mathbf{E}\sigma_x(\varepsilon)$ is obtained as follows:

$$\mathbf{E}\sigma_x(\varepsilon) \le \sum_{k=1}^{\infty} kT\mathbf{P}((k-1)T < \sigma_x(\varepsilon) \le kT) \approx \sum_{k=1}^{\infty} kT(1-p_\varepsilon)^{k-1}p_\varepsilon = \frac{T}{p_\varepsilon}, \tag{4.5}$$

$$\mathbf{E}\sigma_x(\varepsilon) \ge \sum_{k=1}^{\infty}(k-1)T\mathbf{P}((k-1)T < \sigma_x(\varepsilon) \le kT) \approx \sum_{k=1}^{\infty}(k-1)T(1-p_\varepsilon)^{k-1}p_\varepsilon = \frac{T(1-p_\varepsilon)}{p_\varepsilon} \approx \frac{T}{p_\varepsilon}, \tag{4.6}$$

and therefore all we have to do is determine the logarithmic rate of the probability p_ε.

4.0.1 Estimate from below

At this point, we argue that the exit occurs due to big jumps of the compound Poisson process $\varepsilon\eta^\varepsilon$. The number N_T of such jumps on the interval $[0, T]$ is Poisson distributed with parameter β_ε. Assume that $b < a$. We can show that the main contribution to the exit probability comes from the event on which the sum of big jumps in the direction of b arising in the interval $[0, T]$ exceeds b. Formally, for $k \ge 1$ and arbitrary $x_1^k, \dots, x_k^k \ge 0$ such that $x_1^k + \dots + x_k^k = b$ we have

$$\begin{aligned} p_\varepsilon &\approx \mathbf{P}\Big(\sum_{i=1}^{N_T} \varepsilon W_i \ge b\Big) = \sum_{k=1}^{\infty} \mathbf{P}(N_T = k)\mathbf{P}\Big(\sum_{i=1}^{k} \varepsilon W_i \ge b \Big| N_T = k\Big) \\ &\ge \sum_{k=1}^{\infty} \mathbf{P}(N_T = k)\mathbf{P}(\varepsilon W_1 \ge x_1^k, \dots, \varepsilon W_k \ge x_k^k) \\ &= \sum_{k=1}^{\infty} \mathbf{P}(N_T = k)\prod_{i=1}^{k} \mathbf{P}(\varepsilon W_i \ge x_i^k) \\ &= \sum_{k=1}^{\infty} \frac{(\beta_\varepsilon T)^k}{k!}\prod_{i=1}^{k} \frac{\nu([\max\{g_\varepsilon, \frac{x_i^k}{\varepsilon}\}, +\infty))}{\beta_\varepsilon^k} \\ &= \sum_{k=1}^{\infty} \frac{T^k}{k!}\exp\Big(-\sum_{i=1}^{k}\Big(\max\{g_\varepsilon, \frac{x_i^k}{\varepsilon}\}\Big)^\alpha\Big). \end{aligned} \tag{4.7}$$

Since this inequality holds for $k \ge 1$ and arbitrary $x_1^k, \dots, x_k^k \ge 0$ such that $x_1^k + \dots + x_k^k = b$, we conclude that

$$\begin{aligned} p_\varepsilon &\ge \sum_{k=1}^{\infty} \frac{T^k}{k!}\exp\Big(-\min_{x_1^k+\dots+x_k^k=b}\sum_{i=1}^{k}\Big(\max\{g_\varepsilon, \frac{x_i^k}{\varepsilon}\}\Big)^\alpha\Big) \\ &\approx \sum_{k=1}^{\infty} \frac{T^k}{k!}\exp\Big(-\min_{x_1^k+\dots+x_k^k=b}\sum_{i=1}^{k}\Big(\max\{0, \frac{x_i^k}{\varepsilon}\}\Big)^\alpha\Big). \end{aligned} \tag{4.8}$$

4.0.2 Estimate from above

The estimate from above is technically more complicated. However it can be shown that

$$\begin{aligned} p_\varepsilon &\le \sum_{k=1}^{\infty} \mathbf{P}\Big(\sum_{i=1}^{k} \varepsilon W_i \ge b \Big| N_T = k\Big)\mathbf{P}(N_T = k) \\ &\le k_\varepsilon \max_{1\le k\le k_\varepsilon} \mathbf{P}\Big(\sum_{i=1}^{k} \varepsilon W_i \ge b \Big| N_T = k\Big)\mathbf{P}(N_T = k) + \mathbf{P}(N_T \ge k_\varepsilon + 1) \qquad (4.9) \\ &\le k_\varepsilon \max_{1\le k\le k_\varepsilon} C(\varepsilon, k)\frac{T^k}{k!} \exp\Big(-\min_{x_1^k+\cdots+x_k^k=b} \sum_{i=1}^{k} \Big(\frac{x_i^k}{\varepsilon}\Big)^\alpha\Big) + \mathbf{P}(N_T \ge k_\varepsilon + 1) \end{aligned}$$

with some factors $C(\varepsilon, k)$ that increase only sub-exponentially fast and do not matter on the logarithmic scale on which our estimate is obtained. The threshold k_ε is also chosen to be an algebraic function of $1/\varepsilon$, so that $\mathbf{P}(N_T \ge k_\varepsilon + 1) \to 0$ faster than the first term in the estimate, whence the term will be neglected.

4.1 Optimization problem

To estimate p_ε we therefore have to solve a maximization problem. Omitting the sub-exponential pre-factors T^k, $C(\varepsilon, k)$ and k_ε, let us consider the main component of the estimates, namely

$$\max_{k\ge 1}\{\frac{1}{k!} \exp\Big(-\min_{x_1^k+\cdots+x_k^k=b} \sum_{i=1}^{k} \Big(\frac{x_i^k}{\varepsilon}\Big)^\alpha\Big)\} \quad \text{as} \quad \varepsilon \to 0. \qquad (4.10)$$

Let us first calculate the minimum in the exponent:

$$\min\Big\{\sum_{i=1}^{k} x_i^\alpha : \ x_i \ge 0, \ \sum_{i=1}^{k} x_i = \frac{b}{\varepsilon}\Big\}. \qquad (4.11)$$

Due to the *concavity* of the function $x \mapsto x^\alpha$ for $\alpha \in (0, 1)$, and its *convexity* for $\alpha > 1$ we obtain

$$\min\Big\{\sum_{i=1}^{k} x_i^\alpha : \ x_i \ge 0, \ \sum_{i=1}^{k} x_i = \frac{b}{\varepsilon}\Big\} = \begin{cases} (\frac{b}{\epsilon})^\alpha + 0\cdot(k-1) = (\frac{b}{\epsilon})^\alpha, \ \text{for } \alpha \in (0,1) \\ k(\frac{b}{k\epsilon})^\alpha = k^{1-\alpha}(\frac{b}{\epsilon})^\alpha \ \text{for } \alpha > 0. \end{cases} \qquad (4.12)$$

This solution can be obtained by the method of Lagrangian multipliers which reduces the minimization problem with constraint to a minimization in $x_1, \cdots, x_k$ of the function

$$f(x_1, \ldots, x_k, \lambda) = \sum_{i=1}^{k} x_i^\alpha - \lambda\Big(\sum_{i=1}^{k} x_i - \frac{b}{\varepsilon}\Big), \qquad (4.13)$$

where λ is the Lagrange multiplier. The global minimum of a continuous function is attained either in an inner point of its domain of definition, or on its boundary. The coordinate of the extremum in the interior of the domain is a solution of the system of equations

$$\begin{cases} \frac{\partial}{\partial x_i} f(x_1, \ldots, x_k, \lambda) = \alpha x_i^{\alpha-1} - \lambda = 0, \\ \frac{\partial}{\partial \lambda} f(x_1, \ldots, x_k, \lambda) = \sum_{i=1}^{k} x_i - \frac{b}{\varepsilon} = 0, \quad 1 \le i \le k. \end{cases} \qquad (4.14)$$

Solving this system gives the coordinate of the extremum $x_i^* = \frac{b}{k\varepsilon}, 1 \le i \le k$.

In case $\alpha \geq 1$, the function $x \mapsto x^\alpha$ is convex, and the interior point given by $x_1^* = \cdots = x_k^* = \frac{b}{k\varepsilon}$ is a global minimum of the optimization problem with the minimal value $k(\frac{b}{k\varepsilon})^\alpha$. If $\alpha \in (0,1)$, the function $x \mapsto x^\alpha$ is concave, and its minimum attained on the boundary of the domain, namely at one of the points given by $x_i = \frac{b}{\varepsilon}$, $x_j = 0$, $j \neq i$. The minimal value in this case is given by $\frac{b^\alpha}{\varepsilon^\alpha}$.

To determine the maximum (4.10) we will have to consider the cases $\alpha \in (0,1)$ und $\alpha > 1$ separately. Different estimates for this maximum will lead to quite different asymptotics of mean exit times for sub- and super-exponential tails.

4.2 Mean exit time, $0 < \alpha < 1$

For any $\delta > 0$ and ε small we obtain the following estimates. According to Eq. (4.8) we have

$$p_\varepsilon \geq \sum_{k=1}^\infty \frac{T^k}{k!} \exp\Big(- \min_{x_1^k+\cdots+x_k^k=b} \sum_{i=1}^k \Big(\frac{x_i^k}{\varepsilon}\Big)^\alpha \Big) \geq \exp\Big(-\frac{b^\alpha}{\varepsilon^\alpha}\Big) \sum_{k=1}^\infty \frac{T^k}{k!} \geq \exp\Big(-(1+\delta)\frac{b^\alpha}{\varepsilon^\alpha}\Big), \tag{4.15}$$

which together with Eq. (4.6) gives the lower bound for the mean exit time.

On the other hand, according to Eq. (4.9), and recalling the negligibility of the terms related to $C(\varepsilon,k)$ and the second sum, we get

$$p_\varepsilon \leq k_\varepsilon \max_{1\leq k\leq k_\varepsilon} C(\varepsilon,k)\frac{T^k}{k!} \exp\Big(- \min_{x_1^k+\cdots+x_k^k=b} \sum_{i=1}^k \Big(\frac{x_i^k}{\varepsilon}\Big)^\alpha \Big) + \mathbf{P}(N_T \geq k_\varepsilon + 1) \leq \exp\Big(-(1-\delta)\frac{b^\alpha}{\varepsilon^\alpha}\Big). \tag{4.16}$$

With the help of Eq. (4.5) this entails the upper estimate for the mean exit time.

The solution of the optimization problem has a clear interpretation. The exit from the interval occurs in one jump of the process εL of size bigger than b. All other exit patterns with higher numbers of smaller jumps are essentially less probable in the small noise limit.

4.3 Mean exit time, $\alpha > 1$

According to Eq. (4.8) we have

$$p_\varepsilon \geq \sum_{k=1}^\infty \frac{T^k}{k!} \exp\Big(-k\Big(\frac{b}{k\varepsilon}\Big)^\alpha\Big) \geq \max_k \frac{T^k}{k!} \exp\Big(-k\Big(\frac{b}{k\varepsilon}\Big)^\alpha\Big). \tag{4.17}$$

On the other hand,

$$p_\varepsilon \leq k_\varepsilon \max_{1\leq k\leq k_\varepsilon} C(\varepsilon,k)\frac{T^k}{k!} \exp\Big(-k\Big(\frac{b}{k\varepsilon}\Big)^\alpha\Big) + \mathbf{P}(N_T \geq k_\varepsilon + 1). \tag{4.18}$$

Again we do not discuss the contributions of the negligible components T^k, $C(k,\varepsilon)$ and k_ε, and maximize the function

$$\frac{1}{k!} \exp\Big(-k\Big(\frac{b}{k\varepsilon}\Big)^\alpha\Big) \tag{4.19}$$

over positive integer k. Recalling that $\frac{1}{\sqrt{2\pi}} e^{k(\ln k -1) - \frac12 k} = \frac{1}{\sqrt{2\pi k}} (\frac{k}{e})^k \leq k! \leq k^k = e^{k\ln k}$, and again only retaining the leading term on the logarithmic scale, it is enough to minimize the exponent

$$k\ln k + k\Big(\frac{b}{k\varepsilon}\Big)^\alpha \tag{4.20}$$

in k in the small noise limit $\varepsilon \to 0$. Let

$$f(y) = y\ln y + \varepsilon^{-\alpha} y^{1-\alpha} b^\alpha, \quad y > 0. \tag{4.21}$$

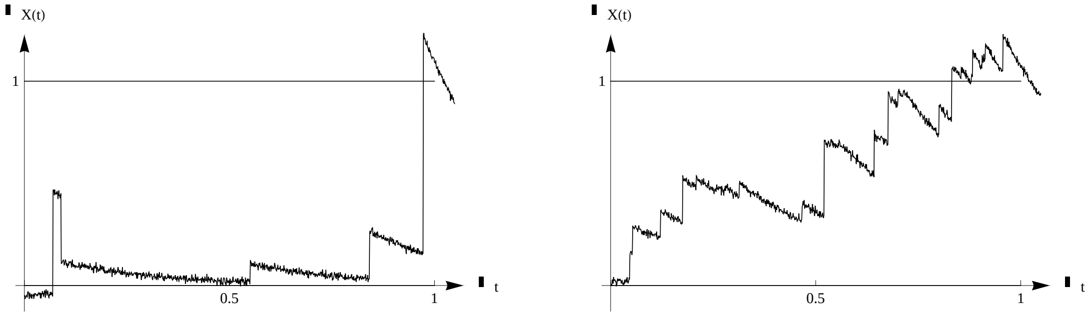

Fig. 1. Typical exit paths of the solutions of the Eq. 2.1 in case of sub-exponential tails of the jump measure, $\alpha \in (0,1)$, (l.) and super-exponential tails, $\alpha \in (1,\infty)$ (r.).

To estimate the infimum of this function carefully, we introduce the new variable $y = \frac{z}{\varepsilon}$. In these terms

$$f(y) = \frac{z}{\varepsilon}\ln\frac{z}{\varepsilon} + \varepsilon^{-\alpha}\Big(\frac{z}{\varepsilon}\Big)^{1-\alpha} b^\alpha = \frac{1}{\varepsilon}\Big(z(\ln z + |\ln\varepsilon|) + \frac{b^\alpha}{z^{\alpha-1}}\Big) = \frac{|\ln\varepsilon|}{\varepsilon}\Big(z\frac{\ln z + |\ln\varepsilon|}{|\ln\varepsilon|} + \frac{b^\alpha}{z^{\alpha-1}|\ln\varepsilon|}\Big). \tag{4.22}$$

The factor $\frac{\ln z + |\ln\varepsilon|}{|\ln\varepsilon|}$ converges to 1 uniformly in $z \leq o(|\ln\varepsilon|)$. Thus in the limit $\varepsilon \to 0$ we obtain

$$f(y) \approx \frac{|\ln\varepsilon|}{\varepsilon}\Big(z + \frac{b^\alpha}{z^{\alpha-1}|\ln\varepsilon|}\Big). \tag{4.23}$$

Minimizing the right-hand side yields the optimal exponent in the mean exit time:

$$f(y) \geq b\alpha(\alpha-1)^{\frac{1}{\alpha}-1}\frac{|\ln\varepsilon|^{1-\frac{1}{\alpha}}}{\varepsilon}. \tag{4.24}$$

We conclude that for any $\delta > 0$ and ε small we obtain the estimates

$$\exp\Big(-(1+\delta)b\alpha(\alpha-1)^{\frac{1}{\alpha}-1}\frac{|\ln\varepsilon|^{1-\frac{1}{\alpha}}}{\varepsilon}\Big) \leq p_\varepsilon \leq \exp\Big(-(1-\delta)b\alpha(\alpha-1)^{\frac{1}{\alpha}-1}\frac{|\ln\varepsilon|^{1-\frac{1}{\alpha}}}{\varepsilon}\Big), \tag{4.25}$$

which together with Eqs. (4.6) and (4.5) give the asymptotic bounds for the mean exit time. It is instructive to notice that the minimum of the function f above is obtained at $y \approx \frac{1}{\varepsilon}\Big(\frac{b(\alpha-1)}{|\ln\varepsilon|}\Big)^{1/\alpha}$. This means that the most probable exit path of the process X^ε consists of $k = \mathcal{O}(\varepsilon^{-1}|\ln\varepsilon|^{-1/\alpha})$ jumps of the size $\varepsilon W_i = \mathcal{O}(\varepsilon|\ln\varepsilon|^{1/\alpha})$ in the direction of b.

5 Conclusion

For dynamical systems perturbed by small Lévy noise, exits from domains of attraction induced by jumps are always faster than those induced by diffusive behavior. More formally, any non-Gaussian Lévy forcing of the form $\varepsilon L(t)$ induces exit times of the asymptotic size of

$$\mathbf{E}\sigma_x(\varepsilon) \lesssim \exp\Big(\frac{c|\ln\varepsilon|}{\varepsilon}\Big) \ll \exp\Big(\frac{c}{\varepsilon^2}\Big). \tag{5.1}$$

No forcing of the type εL can fill the essential gap between non-Gaussian and Gaussian exit time scales. Note that the jump *intensity* is not included in our scaling. For example, for the Poisson process $\varepsilon L^{(\varepsilon)}$ with the jump measure

$$\nu^\varepsilon(dy) = \frac{1}{2\varepsilon}(\delta_{-\varepsilon}(dy) + \delta_\varepsilon(dy)) \tag{5.2}$$

$\nu([u,\infty)) =$	$\frac{1}{u^r}$, $r > 0$	$\exp(-u^\alpha)$			Gaussian
		$\alpha \in (0,1)$	$\alpha \in (1,\infty)$	$\alpha = \infty$	
$\mathbf{E}\sigma_x(\varepsilon) \propto$	$\frac{1}{\varepsilon^r}\left[\frac{1}{a^r}+\frac{1}{b^r}\right]^{-1}$	$\exp\left(\frac{b^\alpha}{\varepsilon^\alpha}\right)$	$\exp\left(\frac{b\alpha(\alpha-1)^{\frac{1}{\alpha}-1}\lvert\ln\varepsilon\rvert^{1-\frac{1}{\alpha}}}{\varepsilon}\right)$	$\exp\left(\frac{b\lvert\ln\varepsilon\rvert}{\varepsilon}\right)$	$\exp\left(\frac{2U(b)}{\varepsilon^2}\right)$

Table 1. The hierarchy of mean exit times for a process X^ε defined by Eq. (2.1) from the domain $[-a, b]$, $b < a$, in dependence on the tails of the jump measure ν. For the Gaussian forcing we assume that $U(b) < U(a)$ and $U(0) = 0$.

the central limit theorem implies the convergence of $\varepsilon L^{(\varepsilon)}$ to a Brownian motion. For a stochastic perturbation of this form exit times would become similar to Kramers' times ([3]).

In the following diagram we summarize the different scenarios of the asymptotic behavior of exit times in the small noise limit $\varepsilon \to 0$ corresponding to jump measures ν with different tails $\nu([u,\infty))$, $u \to \infty$. The limiting case $\alpha = \infty$ of the Lévy noise with exponentially light jumps corresponds to the case of bounded jumps.

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References

1. L. S. Pontryagin, A. A. Andronov, and A. A. Vitt. O statističeskom rassmotrenii dinamičeskih sistem. *Zh. Eksper. Teor. Fiz.*, 3(3):165–180, 1933.
2. H. A. Kramers. Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica*, 7:284–304, 1940.
3. M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften*. Springer, second edition, 1998.
4. P. D. Ditlevsen. Observation of α-stable noise induced millenial climate changes from an ice record. *Geophysical Research Letters*, 26(10):1441–1444, May 1999.
5. G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphey, P. A. Prince, and H. E. Stanley. Lévy flight search patterns of wandering albatrosses. *Nature*, 381:413–415, 1996.
6. S. Bertrand, J. M. Burgos, F Gerlotto, and J. Atiquipa. Lévy trajectories of peruvian purse-seiners as an indicator of the spatial distribution of anchovy (*Engraulis ringens*). *ICES Journal of Marine Science*, 62(3):477–482, 2005.
7. R. N. Mantegna and H. E. Stanley. Scaling behaviour in the dynamics of an economic index. *Nature*, 376:46–49, 1995.
8. I. M. Sokolov, J. Mai, and A. Blumen. Paradoxical diffusion in chemical space for nearest–neighbor walks over polymer chains. *Physical Review Letters*, 79(5):857–860, 1997.
9. M. A. Lomholt, T. Ambjörnsson, and R. Metzler. Optimal target search on a fast folding polymer chain with volume exchange. *Physical Review Letters*, 95(26):260603, 2005.
10. R. N. Mantegna and H. E. Stanley. Stochastic process with ultraslow convergence to the Gaussian: the truncated Lévy flight. *Physical Review Letters*, 73(22):2946–2949, 1994.
11. I. M. Sokolov, A. V. Chechkin, and J. Klafter. Fractional diffusion equation for a power-law-truncated lévy process. *Physica A*, 336(3–4):245–251, 2004.
12. H. Hult and F. Lindskog. On regular variation for infinitely divisible random vectors and additive processes. *Advances in Applied Probability*, 38(1):134–148, 2006.
13. A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes I: Sharp asymptotics for capacities and exit times. *Journal of the European Mathematical Society*, 6(4):399–424, 2004.
14. M. V. Day. Mathematical approaches to the problem of noise–induced exit. In McEneaney, W. M. et al., editors, *Stochastic analysis, control, optimization and applications. A volume in honor of W. H. Fleming, on the occasion of his 70th birthday*, pages 269–287. Birkhäuser, Boston, 1999.

15. A. Galves, E. Olivieri, and M. E. Vares. Metastability for a class of dynamical systems subject to small random perturbations. *The Annals of Probability*, 15(4):1288–1305, 1987.
16. Z. Schuss. *Theory and applications of stochastic differential equations*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, 1980.
17. M. Williams. Asymptotic exit time distributions. *SIAM Journal on Applied Mathematics*, 42:149–154, 1982.
18. H. Eyring. The activated complex in chemical reactions. *The Journal of Chemical Physics*, 3:107–115, 1935.
19. V. V. Godovanchuk. Asymptotic probabilities of large deviations due to large jumps of a Markov process. *Theory of Probability and its Applications*, 26:314–327, 1982.
20. P. D. Ditlevsen. Anomalous jumping in a double-well potential. *Physical Review E*, 60(1):172–179, 1999.
21. A. Chechkin, O. Sliusarenko, R. Metzler, and J. Klafter. Barrier crossing driven by Lévy noise: Universality and the role of noise intensity. *Physical Review E*, 75:041101, 2007.
22. A. V. Chechkin, V. Yu. Gonchar, J. Klafter, and R. Metzler. Barrier crossings of a Lévy flight. *Europhysics Letters*, 72(3):348–354, 2005.
23. P. Imkeller and I. Pavlyukevich. First exit times of SDEs driven by stable Lévy processes. *Stochastic Processes and their Applications*, 116(4):611–642, 2006.
24. P. Imkeller and I. Pavlyukevich. Lévy flights: transitions and meta–stability. *Journal of Physics A: Mathematical and General*, 39:L237–L246, 2006.
25. P. Imkeller and I. Pavlyukevich. Metastable behaviour of small noise Lévy–driven diffusions. *ESAIM: Probaility and Statistics*, 12:412–437, 2008.
26. P. Imkeller, I. Pavlyukevich, and T. Wetzel. First exit times for Lévy-driven diffusions with exponentially light jumps. *The Annals of Probability*, 37(2):530–564, 2009.