First exit times for Lévy-driven diffusions with exponentially light jumps

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Abstract

We consider a dynamical system described by the differential equation $\dot{Y}_t = -U'(Y_t)$ with a unique stable point at the origin. We perturb the system by Lévy noise of intensity ε, to obtain the stochastic differential equation $dX_t^\varepsilon = -U'(X_{t-}^\varepsilon)dt + \varepsilon dL_t$. The process L is a symmetric Lévy process whose jump measure ν has exponentially light tails, $\nu([u,\infty)) \sim \exp(-u^\alpha)$, $\alpha > 0$, $u \to \infty$. We study the first exit problem for the trajectories of the solutions of the stochastic differential equation from the interval $[-1,1]$. In the small noise limit $\varepsilon \to 0$ we determine the law and the mean value of the first exit time, to discover an intriguing phase transition at the critical index $\alpha = 1$.

1 Introduction

In this paper a mathematically rigorous study of the first exit problem for jump-diffusions driven by small scale Lévy processes with rare big jumps is given. The problem under consideration can be outlined as follows. Consider a deterministic dynamical systems given by a differential equation $\dot{Y}_t = -U'(Y_t)$ which has a unique asymptotically stable state at the origin 0. We assume that the interval $[-1,1]$ belongs to the domain of attraction of 0, so that the solution trajectories of the deterministic part cannot leave this interval.

The situation becomes different if the dynamical system is perturbed by some noise of small intensity. The stable state becomes meta-stable, and exits from the interval become possible. The probabilistic characteristics of the first exit time such as its law or the mean value are determined by the nature of the noise and geometry of the potential U.

Unquestionably, perturbations by Brownian motion are by far best understood. The first exit problem for small Brownian perturbations was treated in a pioneering work by

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Kramers [18]. The main mathematical reference on this subject is the book [12] by Freidlin and Wentzell, in which the theory of large deviations for dynamical systems with small Brownian perturbations was developed as the main tool for exit problems. Among many other papers on this subject we mention a paper by Williams [26], a book [24] by Schuss, and a series of papers by Day [6, 7, 8] and Bovier et al. [2, 3].

In our particular case, the results obtained in very general geometric settings in the references cited above offer a simple explanation. It turns out that the length of the mean exit time is asymptotically given by e^{ζ/ε^2}, and its logarithmic rate ζ is determined by the lowest potential barrier a Gaussian particle has to overcome in order to exit. For instance if $U(-1) < U(1)$, the exit occurs with an overwhelming probability at -1, and $\zeta = 2(U(-1) - U(0))$. Moreover, the normalized exit time has a standard exponential law in the limit of small noise $\varepsilon \to 0$.

It is interesting to note that asymptotics of the Gaussian type are also obtained in a situation in which a random Markov perturbation possesses locally infinitely divisible laws with exponential moments of any order, while jump intensity increases and jump size decreases simultaneously with an appropriate rate along with the noise parameter ε tending to 0. A typical example of such a perturbation is given by a compensated Poisson process with jump size ε, and jump intensity $1/\varepsilon$, see [12, Chapter 5].

The situation becomes quite different for dynamical system perturbed by heavy tailed Lévy noise. There the big jumps begin to play the major role in the exit time dynamics. If the jump measure of the driving Lévy process has power tails, the mean exit time turns out to behave like a power function of the small noise amplitude. Moreover, the leading term of the average first exit time does not depend on the vertical parameters of the potential's geometry, the heights of potential barriers, but rather on horizontal ones such as the distances between the stable point and the domain boundary. Due to the presence of big jumps, the trajectories of the perturbed dynamical system leave the interval in one big jump, and do not climb up the potential barrier as in the Gaussian case.

Rough large deviation estimates and the asymptotics of the mean first exit time from a domain for a more general class of Markov processes with heavy tailed jumps were first obtained by Godovanchuk [13], whereas a general large deviations theory for Markov processes can be found in the book [25] by Wentzell. With different methods, Imkeller and Pavlyukevich [15, 14] recently described the fine asymptotics for the law and moments of exit times for jump diffusions driven by α-stable Lévy processes, and more generally by Lévy processes the jump measure of which has regularly varying tails. These asymptotic properties were used to show metastability properties for Lévy-driven jump diffusions in multi-well potentials (see [16, 14]). The techniques were further generalized to study simulated annealing with time nonhomogeneous jump processes, [21, 22].

Our recent interest in small noise jump diffusions arose from the acquaintance with the paper [10] by Ditlevsen. In an attempt to model paleoclimatic time series from the Greenland ice core by dynamical systems perturbed by noise, the author discovers an α-stable noise signal with $\alpha \approx 1.75$. In his setting, big jumps of the α-stable Lévy process are responsible for rapid catastrophic climate changes (the so-called Dansgaard-Oeschger events) observed in the Earth's northern hemisphere during the last glaciation. The appearance of a stable Lévy noise signal can be explained if the observed time series is interpreted as a mesoscopic limit of some more complicated climate dynamics.

Lévy driven stochastic dynamics has become a popular research field in physics, where stable non-Gaussian Lévy processes are often named *Lévy flights*. We draw the reader's attention to the topical review by Metzler and Klafter [20], where Lévy flights are discussed in detail from a physicists point of view.

The first exit problem (also called Kramers' or barrier crossing problem) is of central importance in the physical sciences. Small noise barrier crossing problems were studied on a physical level of rigor by Ditlevsen [9], Chechkin at al. [5, 4] and Dybiec et al. [11]. In particular, Chechkin at al. [4] present numerical experiments that strongly support the theoretical findings of [16, 14].

The relationship between Lévy and Gaussian exit time dynamics circumscribes another problem that has received a considerable deal of attendance in physics applications. The problem was first considered by Mantegna and Stanley in [19] and Koponen [17]. In order to see Gaussian type asymptotic behavior emerge in dynamical systems perturbed by Lévy processes, the authors suggest to either eliminate big jumps, or to make their appearance rare by modifying the jump measure to have exponentially light tails.

In this paper, we will study exit times of solutions of the stochastic differential equation

$$dX_t^\varepsilon = -U'(X_{t-}^\varepsilon)dt + \varepsilon dL_t \tag{1}$$

driven by a Lévy process L of (small) intensity ε whose jump measure ν is symmetric, and which contains a non-trivial Gaussian component. The argument and results obtained in [15, 14] for the heavy-tail jump measures when $\nu([u,\infty)) \sim u^{-r}$ with some $r > 0$ show that the exit occurs due to a single big jump of the order $1/\varepsilon$, and thus a mean exit time is of the order $\nu(\{|y| \geq 1/\varepsilon\})^{-1} \sim \varepsilon^{-r}$, $\varepsilon \to 0$. Moreover, one can see that the argument of [14] would hold also for Lévy measures with sub-exponential tails $\nu([u,+\infty)) \sim e^{-u^\alpha}$ for very small values $\alpha \ll 1$ leading to mean exit times of the order e^{1/ε^α}. Recalling that Gaussian exits occur on time scales of the order e^{1/ε^2} one may ask if a further reduction of the tail weight can lead to Gaussian dynamics as $\alpha \uparrow 2$, and whether Gaussian exit dynamics dominate after crossing the critical index, i.e. for $\alpha > 2$? This is the motivating question of this paper.

The result is very surprising. We show that big jumps *always* dominate, independently of how light they are. Looking at the results in more detail, the behavior of exit times encounters a phase transition at the critical value $\alpha = 1$ which marks the transition from *sub-exponential* to *super-exponential* dynamics.

Our arguments leading to the discovery of this transition can be outlined as follows. As in the case for power type tails in [15], for $\varepsilon > 0$ the process L is decomposed into a compound Poisson pure jump part η^ε with jumps of height larger than some critical level g_ε, and a remainder ξ^ε with jumps not exceeding this bound. The critical threshold g_ε has to be chosen individually according to whether the jump law possesses sub- or super-exponential tails. In the crucial estimate one has to show that to exit from the interval $[-1,1]$ around the stable fixed point 0 before some finite time T while never returning to a small interval $[-\delta,\delta]$ requires that either the increments of the small jump component ξ^ε exceed certain bounds for which probability can be shown to be small enough by suitable choice of g_ε, or that the sum of large jumps occurring before time T exceeds the bound $1-\delta$. Generally, the analysis reveals that large jumps are responsible for exits irrespective of whether we are in the sub- or the super-exponential regime.

To see the phase transition at $\alpha = 1$, let N_T be the random number of large jumps before time T, W_i the random variable describing large jump $i \in \mathbf{N}$. Then N_T has a Poisson law with expectation $\beta_\varepsilon T$, where $\beta_\varepsilon = \nu([-g_\varepsilon, g_\varepsilon]^c) = 2\exp(-x^\alpha)$. Hence, for n fixed, the probability that the sum of large jumps exceeds the bound $1-\delta$ can be estimated by

$$P(N_t > n) + \sum_{k=1}^{n} P(N_T = k) P(\sum_{i=1}^{k} |\varepsilon W_i| > 1 - \delta).$$

The further estimation of the first term in this expression makes use of Stirling's formula, and ultimately stipulates the choice of a large enough ε-dependent $n = n_\varepsilon$. The essential term to estimate for $1 \leq k \leq n_\varepsilon$ is

$$P(\sum_{i=1}^{k} |\varepsilon W_i| > 1 - \delta).$$

The law of the i.i.d. random variables $(|W_i|)_{i\in\mathbf{N}}$ being given by $\beta_\varepsilon^{-1} 2\nu|_{[g_\varepsilon,\infty[}$, one gets

$$\begin{aligned} P(\sum_{i=1}^{k} |\varepsilon W_i| > 1 - \delta) &\leq \beta_\varepsilon^{-k} \exp(-\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i \geq \frac{1-\delta}{\varepsilon}, x_i \in [g_\varepsilon, \infty[\}) \\ &= \beta_\varepsilon^{-k} \exp(-\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i = \frac{1-\delta}{\varepsilon}, x_i \in [g_\varepsilon, \infty[\}). \end{aligned}$$

The phase transition emerges when solving the minimization problem in the exponent of this estimate. From the choice of g_ε it is seen that the lower boundary for $x_i, 1 \leq i \leq k$, in the exponential rate can be taken 0. Now in the case of *sub-exponential tails*,

$$\inf\{\sum_{i=1}^{k} x_i^\alpha : \sum_{i=1}^{k} x_i = 1, x_i \geq 0\} = 1$$

is taken on the boundary of the simplex $\{(x_1, \cdots, x_n) : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$, and $x_i = \frac{1}{n}, 1 \leq i \leq n$, corresponds to the maximum of the function $(x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} x_i^\alpha$. For *super-exponential tails*, the inf is taken for $x_i = \frac{1}{n}, 1 \leq i \leq n$, and gives the the unique local minimum of the function $(x_1, \cdots, x_n) \mapsto \sum_{i=1}^{n} x_i^\alpha$ on the simplex. So the phase transition is due to the switching from *concavity* to *convexity* of the function

$$x \mapsto x^\alpha, \quad x \geq 0,$$

as α increases through 1. So, the surprising behavior of our jump diffusion with exponentially light jumps of degree α can roughly be summarized by the statement: big jumps of the Lévy process govern the asymptotic behavior in the sub- ($\alpha < 1$) as well as in the super- ($\alpha > 1$) exponential regimes, but in the latter one the cumulative action of several large jumps reminding the climbing of the potential well in the Gaussian regime, becomes important, while for $\alpha < 1$ the biggest jump alone governs the asymptotic behavior.

The material is organized as follows. In Section 2 we explain the setup of the problem and state our main results about the asymptotics of exit times. Section 3 contains our general

strategy of estimating the tails of the law of exit times from the knowledge of the tails of the jump measure. The concept of ε-dependent separation of small and large jump parts which already proved to be successful in [15] takes a central role, and is basic for the proof that, apart from a Gaussian part, small jumps do not alter the behavior of solution curves of the unperturbed dynamical system by much. This leaves the role of triggering exits to the large jump part the contribution of which receives a careful estimation. In the technical Sections 4 and 5, upper and lower bounds for the tails of the exit time laws are derived.

2 Object of study and main result

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space. We assume that the filtration satisfies the usual hypotheses in the sense of [23], i.e. it is right continuous, and $\mathcal{F}_0$ contains all the $\mathbf{P}$-null sets of $\mathcal{F}$.

For $\varepsilon > 0$ we consider solutions $X^\varepsilon = (X^\varepsilon_t)_{t\geq 0}$ of the one-dimensional stochastic differential equation

$$X^\varepsilon_t(x) = x - \int_0^t U'(X^\varepsilon_{s-}(x))\, ds + \varepsilon L_t, \quad x \in \mathbb{R}, \tag{2}$$

where L is a Lévy process and U is a potential function satisfying assumptions specified in the following. The principal goal of our investigation is the small noise behavior of X^ε, i.e. the behavior of the process as $\varepsilon \to 0$. More specifically, our interest is focused on the exit of X^ε from a neighborhood of the stable attracting point 0 of the real valued potential function U defined on $\mathbb{R}$. For this reason, besides assuming that U be continuously differentiable with derivative U', we only have to fix some minimal conditions on U concerning its properties in a neighborhood of 0. We shall work under the following assumption.

(U) $U(x) = U'(x) = 0$ if only if $x = 0$, U' is Lipschitz continuous, and $U'(x)x \geq 0$, $x \in \mathbb{R}$.

In order to state the conditions our Lévy process L is supposed to satisfy, let us recall that a positive Lebesgue measurable function l is slowly varying at infinity if $\lim_{u\to+\infty} l(\lambda u)/l(u) = 1$ for any $\lambda > 0$. For example, positive constants, logarithms and iterated logarithms are slowly varying functions.

(L1) L has a generating triplet (d, ν, μ) with a Gaussian variance $d \geq 0$, an arbitrary drift $\mu \in \mathbb{R}$ and a *symmetric* Lévy measure ν satisfying the usual condition $\int_{\mathbb{R}\setminus\{0\}}(y^2 \wedge 1)\, \nu(dy) < \infty$.

(L2) Let $f(u) := -\ln \nu([u, +\infty))$, $u > 0$. Then there is $\alpha > 0$ such that

$$f(u) = u^{-\alpha} l(u), \quad u \geq 1, \tag{3}$$

for some function l slowly varying at $+\infty$.

We say that the Lévy measure ν has *sub-exponential* or *super-exponential tails* with index α if $0 < \alpha < 1$ or $\alpha > 1$ in (L2), respectively. Typical examples of Lévy processes under consideration are given by a symmetric Lévy measures ν with tails

$$\nu([u, \infty)) = \exp(-u^\alpha), \quad \alpha > 0, \quad u \geq 1. \tag{4}$$

The family of strongly tempered stable processes with the jump measures

$$\nu(dy) = e^{-\lambda|y|^\alpha}|y|^{-1-\beta}\mathbb{I}\{y \neq 0\}\,dy, \quad \lambda > 0, \beta \in (0,2), \alpha > 0, \alpha \neq 1, \tag{5}$$

provides another example.

Since Lévy processes are semimartingales, and due to (U), the stochastic differential equation (2) possesses a strong solution defined for all $t \geq 0$. See [23] for the general theory. Moreover, the underlying deterministic equation ($\varepsilon = 0$) given by

$$Y_t(x) = x - \int_0^t U'(Y_s(x))\,ds \tag{6}$$

has a unique solution for any initial value $x \in \mathbb{R}$ and all $t \geq 0$. The position $x = 0$ of the minimum of U is a stable attractor for the dynamical system Y, i.e. for any $x \in \mathbb{R}$ we have $Y_t(x) \to 0$ as $t \to \infty$. It is clear that the deterministic solution $Y(x)$ does not leave the interval $[-1,1]$ for initial values $x \in (-1,1)$. The main object of study of this paper is the asymptotic law and the mean value of the first exit time of the jump-diffusion X^ε

$$\sigma_x(\varepsilon) = \inf\{t \geq 0 : |X_t^\varepsilon(x)| \geq 1\}, \quad x \in \mathbb{R}. \tag{7}$$

Our main findings are stated in the following theorems.

Theorem 2.1 (sub-exponential tails) *Let the jump measure ν of L be sub-exponential with index $0 < \alpha < 1$. Then for any $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ the following inequalities hold uniformly for $t \geq 0$:*

$$(1-\delta)\exp(-C_\varepsilon^{1-\delta}t) \leq \inf_{|x|\leq 1-\delta} \mathbf{P}(\sigma_x(\varepsilon) > t) \leq \sup_{|x|\leq 1} \mathbf{P}(\sigma_x(\varepsilon) > t) \leq \exp(-\tfrac{1}{2}C_\varepsilon t), \tag{8}$$

with $C_\varepsilon := 2\nu([\frac{1}{\varepsilon}, \infty))$. Consequently, for any $|x| < 1$ we have

$$\lim_{\varepsilon \to 0} f\left(\tfrac{1}{\varepsilon}\right) \ln \mathbf{E}\sigma_x(\varepsilon) = 1. \tag{9}$$

Theorem 2.2 (super-exponential tails) *Let the jump measure ν of L be super-exponential with index $\alpha > 1$. Let q_ε denote its ε-quantile, $q_\varepsilon := \sup\{u > 0 : \nu([u,\infty)) \geq \varepsilon\}$. Then for any $\delta > 0$ there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ the following inequalities hold uniformly for $t \geq 0$:*

$$(1-\delta)\exp(-D_\varepsilon^{1-\delta}t) \leq \inf_{|x|\leq 1-\delta} \mathbf{P}(\sigma_x(\varepsilon) > t) \leq \sup_{|x|\leq 1} \mathbf{P}(\sigma_x(\varepsilon) > t) \leq (1+\delta)\exp(-D_\varepsilon^{1+\delta}t), \tag{10}$$

where $D_\varepsilon = \exp(-d_\alpha \frac{|\ln \varepsilon|}{\varepsilon q_\varepsilon})$ and $d_\alpha = \alpha(\alpha-1)^{\frac{1}{\alpha}-1}$. Consequently, for any $|x| < 1$ we have

$$d_\alpha^{-1} \lim_{\varepsilon \to 0} \frac{\varepsilon q_\varepsilon}{|\ln \varepsilon|} \ln \mathbf{E}\sigma_x(\varepsilon) = 1. \tag{11}$$

It is instructive compare qualitatively the results of Theorems 2.1 and 2.2 with known results for exit times in the case in which L is a pure Brownian motion, or contains a symmetric jump component with regularly varying tails. We therefore briefly consider the mean exit times of Lévy-driven diffusions of four types. Then the following limiting relations hold and are uniform over all initial points x belonging to a compact subset $K \subset (-1, 1)$.

1. Power tails. Let L be such that $\nu([u, \infty)) = u^{-r}$, $u \geq 1$ for some $r > 0$. Then as was shown in [15] the mean exit time satisfies

$$2 \lim_{\varepsilon \to 0} \varepsilon^r \mathbf{E}\sigma_x(\varepsilon) = 1. \tag{12}$$

2. Sub-exponential tails. Assume that L is such that for some $\alpha \in (0, 1)$ we have $\nu([u, \infty)) = \exp(-u^\alpha)$, $u \geq 1$. Then Theorem 2.1 easily implies that

$$\lim_{\varepsilon \to 0} \varepsilon^\alpha \ln \mathbf{E}\sigma_x(\varepsilon) = 1. \tag{13}$$

3. Super-exponential tails. Assume that L is such that $\nu([u, \infty)) = \exp(-u^\alpha)$, $u \geq 1$, for some $\alpha \in (1, \infty)$. Then the ε-quantile $q_\varepsilon = |\ln \varepsilon|^{1/\alpha}$ and Theorem 2.2 entails that

$$d_\alpha^{-1} \lim_{\varepsilon \to 0} \varepsilon |\ln \varepsilon|^{\frac{1}{\alpha} - 1} \ln \mathbf{E}\sigma_x(\varepsilon) = 1. \tag{14}$$

4. Gaussian diffusion. Assume that L possesses the characteristic triplet $(1, 0, 0)$, i.e. L is a standard one-dimensional Brownian motion. Then the mean exit time depends on the height of the potential barrier at the interval ends, and

$$\tfrac{1}{2}(U(-1) \wedge U(1))^{-1} \lim_{\varepsilon \to 0} \varepsilon^2 \ln \mathbf{E}\sigma_x(\varepsilon) = 1. \tag{15}$$

First we note that cases 1 and 2 mathematically do not differ by much, since the mean exit time can be expressed by the same formula $\mathbf{E}\sigma_x(\varepsilon) \sim (2\nu([\frac{1}{\varepsilon}, \infty))^{-1}$. The gaps between 2. and 3., and 3. and 4. are much more surprising. The logarithmic rate of the expected first exit time drastically changes its behavior in the super-exponential case: jump lightness influences the mean exit time in a rather insignificant way. Even more surprising is the fact that we do not obtain Gaussian asymptotics even for light tails with $\alpha \geq 2$. This is underlined in an intriguing way through the form of the pre-factors: in the cases of perturbations with jumps they only depend on the distance between the stable equilibrium 0 and the interval boundary, whereas in the Gaussian case the heights of the potential barriers come into play. The phase transition between 3. and 4. raises another question, that remains unanswered: which types of noise in (2) can fill the gap between cases 3. and 4., in other words lead to mean exit times of the order $\exp(\varepsilon^{-\alpha})$ for $\alpha \in (1, 2)$?

3 The general strategy of estimation

In this section we shall prepare the proof of our main results by setting up a global strategy. It will mainly be based upon an appropriate *ε-dependent* separation of small and large jumps. Given this separation, we will decompose our jump diffusion into a component depending on the small jumps, which is described by trajectories that deviate by only little from the

trajectories of the deterministic equation converging to the stable equilibrium 0. The small size of this deviation will be sufficient to argue that exits from the interval considered can asymptotically only be due to large jumps. These ideas compel us to proceed along the following lines. In a first subsection, we shall give elementary but efficient estimates for tails of the exit laws on short time intervals. The second subsection is crucial: for a given noise intensity ε in (2) we separate jumps at a critical height g_ε which, as a function of ε, has to increase slowly enough for the system perturbed by the small jump part to create small deviations from deterministic behavior. A sensitive choice of g_ε is given later. In the third subsection, uniform estimates for the small jump part are provided, while in the last one the crucial tail estimates for the sums of finitely many big jumps creating exits by adding up are given.

3.1 Estimates on short time intervals

Let $X = (X_t(x))_{t\geq 0}$ be a time homogeneous Markov process starting in $x \in \mathbb{R}$ whose sample paths are right-continuous and have left limits, $I = [-1, 1]$. Consider an interval $J \subset I$ and consider the (Markovian) *first exit* time $\sigma_x := \inf\{t \geq 0 : X_t(x) \notin I\}$, and the (non-Markovian) time $\sigma_x^* := \sup\{t < \sigma : X_t(x) \in J\}$ which marks the *start of the exit from* J.

The following Lemmas allow to estimate the law of σ_x from the dynamics of the process X on relatively short time intervals. For the sake of simplicity of notation, we sometimes omit the subscript x in expressions containing the times σ_x and σ_x^*.

Lemma 3.1 *Let C and T be positive real numbers such that $CT < 1$.*
1. If $\inf_{x\in I} \mathbf{P}(\sigma_x \leq T) \geq CT$, the following estimate from above for $t \geq 0$ holds

$$\sup_{x\in I} \mathbf{P}(\sigma_x > t) \leq (1 - CT)^{-1} \exp(-Ct). \tag{16}$$

2. If $\sup_{x\in J} \mathbf{P}(\sigma_x^* \leq T) \leq CT$, the following estimate from below for $t \geq 0$ holds

$$\inf_{x\in J} \mathbf{P}(\sigma_x > t) \geq (1 - CT) \exp\left(\frac{\ln(1 - CT)}{T} t\right). \tag{17}$$

Proof: The proof of part 1 is a straightforward application of the strong Markov property and time homogeneity of X. In fact, choose an arbitrary $t > 0$ and let $k := [\frac{t}{T}]$. Then for any $x \in I$ we obtain the following chain of inequalities

$$\mathbf{P}(\sigma_x > t) \leq \mathbf{P}(\sigma_x > kT) \leq \left(\sup_{x\in I} \mathbf{P}(\sigma_x > T)\right)^k \leq (1 - CT)^{\frac{t}{T}-1} \leq (1 - CT)^{-1} \exp(-Ct). \tag{18}$$

In order to use similar arguments to prove part 2, we need to define a sequence of stopping times. Let $T_J^0 := 0$ and for any $n \geq 1$ let $T_J^n := \inf\{t \mid t \geq T_J^{n-1} + T, X_t \in J\}$. Obviously $\{\sigma \leq T_J^1\} = \{\sigma^* \leq T\}$ holds for any $x \in J$, and moreover $T_J^n \geq nT$ for any $n \geq 1$. Again choose an arbitrary $t > 0$ and let $k := [\frac{t}{T}]$. Then for any $x \in J$ we have

$$\begin{aligned} \mathbf{P}(\sigma_x > t) \geq \mathbf{P}(\sigma_x > T_J^{k+1}) &\geq \left(\inf_{x\in J} \mathbf{P}(\sigma_x > T_J^1)\right)^{k+1} \\ &\geq (1 - CT)^{\frac{t}{T}+1} \geq (1 - CT) \exp\left(\frac{\ln(1 - CT)}{T} t\right). \end{aligned} \tag{19}$$

■

3.2 Decomposition into small and large jump parts

In our separation of the jump part of the Lévy process L into a component for small and one for large jumps the latter will turn out to be a compound Poisson process. This makes large jumps relatively easily amenable to an individual investigation. Suppose that $g > 0$ is a *cutoff* height. We shall leave a particular choice of g to later parts of this paper, and for the moment use the cutoff height to define the g-dependent decomposition

$$L = \xi + \eta, \tag{20}$$

with jump measures $\nu_\xi = \nu|_{[-g,g]}$ and $\nu_\eta = \nu|_{[-g,g]^c}$. The resulting independent Lévy processes η and ξ possess generating triplets $(0, \nu_\eta, 0)$ and (d, ν_ξ, μ). For the compound Poisson part η and $k \geq 1$ we denote by S_k the arrival times of jumps ($S_0 = 0$), by $\tau_k = S_k - S_{k-1}$ its interjump periods, and by W_k the respective jump sizes, and note that $\beta = \nu_\eta(\mathbb{R})$ expresses its jump frequency, i.e. the inverse expected interjump time. Finally, we denote by $N_t = \sup\{k \geq 0 : S_k \leq t\}$ the number of jumps until time $t, t \geq 0$.

Lemma 3.1 reduces the main task of the proof of Theorem 2.1 and 2.2 to finding an appropriate $T > 0$ and estimating the probabilities to exit before T and to start an exit from a subinterval before T. For technical reasons, we have to reduce the interval $I = [-1, 1]$ a bit, and study exits from this subinterval. So for some $0 < \delta < \frac{1}{2}$ let $I_\delta^- := [-1 + \delta, 1 - \delta]$, and $\sigma^- := \inf\{t : X_t^\varepsilon \notin I_\delta^-\}$. Now take $J_\delta = [-\delta, \delta]$ as subinterval of I_δ^- and, according to Lemma 3.1 consider $\sigma^* := \sup\{t \leq \sigma^- : X_t^\varepsilon \in J_\delta\}$. The following auxiliary estimates intend to control the probability of $\{\sigma^* < T\}$ through finding a covering by sets of sufficiently small probability.

Lemma 3.2 *(i) Let $0 < \delta < \frac{1}{2}$ and $\varepsilon > 0$ be such that $\varepsilon g < \delta$, and $x \in I_\delta^-$. Let $m := \inf_{y \in I_\delta^- \setminus J_\delta} |U'(y)|$ and $\hat{T} > \frac{1}{m}$. Then for any $T > 0$ and $t > T + \hat{T}$ the following estimate holds*

$$\{\sigma^* < T, \sigma^- \geq t\} \subseteq \{t - S_{N_t} < \hat{T}\} \cup \Big\{ \sup_{r \leq \hat{T}} \varepsilon |\xi_{t-\hat{T}+r} - \xi_{t-\hat{T}}| \geq m\hat{T} - 1 \Big\} \tag{21}$$

(ii) For any $T \in [0, \sigma], x \in I$ we have the following estimate

$$|X_T| \leq |x| + \big(\sup - \inf\big)_{t \leq T} \varepsilon L_t. \tag{22}$$

Proof: *(i)* Choose $T > 0$, $\hat{T} > \frac{1}{m}$ and $t \geq T + \hat{T}$ arbitrarily. Consider the event $A := \{\sigma^* < T, \sigma^- \geq t, t - S_{N_t} \geq \hat{T}\}$. It is sufficient to show, that on A we have

$$\sup_{r \leq \hat{T}} \varepsilon |\xi_{t-\hat{T}+r} - \xi_{t-\hat{T}}| \geq m\hat{T} - 1. \tag{23}$$

First of all, by definition, on A the process εL cannot make jumps larger than εg during the time period $[t - \hat{T}, t]$ and does not leave $I_\delta^- \setminus J_\delta$ during the time period $[T, t]$. Further, by choice of $T, \hat{T}$, we have $[t - \hat{T}, \hat{T}] \subseteq [T, t]$ and $\varepsilon g \leq \delta$. Thus X^ε does not change its sign

during the time period $[t-\hat{T},\hat{T}]$. By definition we have $A \subseteq \{X_t^\varepsilon \neq 0\}$. Hence it is sufficient to consider the cases $A \cap \{X_t^\varepsilon > 0\}$ and $A \cap \{X_t^\varepsilon > 0\}$ separately. On the former we have

$$0 < X_t^\varepsilon = X_{t-\hat{T}}^\varepsilon - \int_{t-\hat{T}}^t U'(X_{s-}^\varepsilon)ds + \varepsilon(L_t - L_{t-\hat{T}}) \leq 1 - m\hat{T} + \sup_{s\leq\hat{T}} \varepsilon|\xi_{t-\hat{T}+s} - \xi_{t-\hat{T}}|. \quad (24)$$

Analogously, on $A \cap \{X_t^\varepsilon < 0\}$ we may estimate

$$0 > -1 + m\hat{T} - \sup_{s\leq\hat{T}} \varepsilon|\xi_{t-\hat{T}+s} - \xi_{t-\hat{T}}|. \quad (25)$$

This completes the proof of part *(i)*.

(ii) Let $\tau := \inf\{t \in [0,T] : X_s^\varepsilon > 0 \text{ for all } s \in [t,T)\}$. By construction $X_t^\varepsilon \geq 0$ and $U'(X_t^\varepsilon) \geq 0$ for any $t \in (\tau, T)$. Thus

$$\begin{aligned} X_T^\varepsilon = X_\tau^\varepsilon - \int_\tau^T U'(X_{t-}^\varepsilon)\,dt + \varepsilon(L_T - L_\tau) \leq X_\tau^\varepsilon + \varepsilon(L_T - L_\tau) \leq \begin{cases} x + \sup_{t\leq T} \varepsilon L_t, & \text{if } \tau = 0, \\ \varepsilon(L_T - L_{\tau-}), & \text{if } \tau > 0, \end{cases} \\ \leq |x| + \big(\sup - \inf\big)\varepsilon L_t. \end{aligned} \quad (26)$$

Analogously we have $X_T^\varepsilon \geq -|x| - \big(\sup - \inf\big)\varepsilon L_t$. ■

Corollary 3.1 *(i) Let $0 < \delta < \frac{1}{2}$ and $\varepsilon > 0$ be such that $\varepsilon g < \delta$, and $x \in I_\delta^-$. Let $m := \inf_{y\in I_\delta^- \setminus J_\delta} |U'(y)|$, and $T = \frac{2}{m}$. Then for any $k \geq 1$ the following inclusions hold:*

$$\{\sigma^* < T, \sigma^- \geq S_k\} \subseteq \{\sigma^* < T, \sigma^- \geq 2Tk \wedge S_k\} \subseteq \chi_k \cup \bigcap_{i=1}^k \{\tau_i \leq 2T\} \quad (27)$$

with

$$\chi_k = \bigcup_{i=0}^{k-1} \{\sup_{t\leq T} \varepsilon|\xi_{S_i+T+t} - \xi_{S_i+T}| \geq 1\}. \quad (28)$$

(ii) For any $x \in I$, $k \geq 1$ and $T > 0$ the following estimate holds:

$$\sup_{t<S_k\wedge T} |X_t| \leq |x| + \sum_{i=1}^{k-1} |\varepsilon W_i| + 2\sup_{t\leq T} |\varepsilon\xi_t|. \quad (29)$$

Proof: *(i)* Obviously, it suffices to prove the second inclusion in (27). We set $\hat{T} = T = \frac{2}{m}$, and let $\hat{\chi}_i^c =: \{\sup_{t\leq T} \varepsilon|\xi_{S_{i-1}+T+t} - \xi_{S_{i-1}+T}| < 1\}$, $i \geq 0$. Then we notice that

$$\begin{aligned} \{\sigma^* < T, \sigma^- \geq 2Tk \wedge S_k\} &\subseteq \{\sigma^* < T, \sigma^- \geq S_k\} \cup \{\sigma^* < T, \sigma^- \geq 2Tk, S_k > 2Tk\} \\ &\subseteq \{\sigma^* < T, \sigma^- \geq S_k\} \cup \bigcup_{i=0}^{k-1} \{\sigma^* < T, \sigma^- \geq 2Tk, N_{2Tk} = i\}. \end{aligned} \quad (30)$$

For $0 \le i \le k-1$ we have

$$\begin{aligned}\{\sigma^* < T, \sigma^- \ge 2Tk, N_{2Tk} = i\} \subseteq & \{\sigma^* < T, \sigma^- \ge 2Tk\} \cap \cup_{1\le j\le i}\{\tau_j > 2T, S_j \le 2Tk\} \\ & \subseteq \{\sigma^* < T, \sigma^- \ge S_{j-1} + 2T\} \cap \cup_{1\le j\le i}\{\tau_j > 2T\} \\ & \subseteq (\hat{\chi}_j \cap \{\tau_j \le 2T\}) \cap \cup\{\tau_j > 2T\} \\ & \subseteq \chi_k,\end{aligned} \tag{31}$$

where we applied Lemma 3.2 with $t = S_{j-1} + 2T$ to see that

$$\begin{aligned}\{\sigma^* < T, \sigma^- \ge S_{j-1} + 2T\} &\subseteq \{S_{j-1} + T - S_{N_{S_{j-1}+2T}} < 0\} \cup \hat{\chi}_j \\ &\subseteq \{N_{S_{j-1}+2T} > j-1\} \cup \hat{\chi}_j \subseteq \{\tau_j \le 2T\} \cup \hat{\chi}_j.\end{aligned} \tag{32}$$

Next we prove the inclusions

$$\{\sigma^* < T, \sigma^- \ge S_i\} \cap \hat{\chi}_i^c \subseteq \{\tau_i \le 2T\}, \quad 1 \le i \le k, \tag{33}$$

and taking intersections of their right- and left-hand-sides over i we obtain the proper covering for the set $\{\sigma^* < T, \sigma^- \ge S_k\}$. Consider the decomposition

$$\begin{aligned}\{\sigma^* < T, \sigma^- \ge S_i\} \cap \hat{\chi}_i^c \subseteq & \{\sigma^* < T, \sigma^- \ge S_{i-1} + 2T\} \cap \hat{\chi}_i^c \\ & \cup \{S_i \le \sigma^- < S_{i-1} + 2T\}.\end{aligned} \tag{34}$$

The second set in the previous formula is obviously contained in $\{\tau_i \le 2T\}$ while to study the first one we apply again Lemma 3.2*(i)* with $t = S_{i-1} + 2T$ to obtain

$$\Big(\{S_{i-1} + T - S_{N_{S_{i-1}+2T}} < 0\} \cup \hat{\chi}_i\Big) \cap \hat{\chi}_i^c \subseteq \{N_{S_{i-1}+2T} > i-1\} = \{\tau_i \le 2T\}. \tag{35}$$

(ii) The second part follows from the estimate

$$\big(\sup - \inf\big)\varepsilon L_t \le \big(\sup - \inf\big)\varepsilon\eta_t + \big(\sup - \inf\big)\varepsilon\xi_t \le \sum_{i=1}^{k-1} |\varepsilon W_i| + 2\sup_{t\le T} |\varepsilon\xi_t|. \tag{36}$$

■

3.3 Estimates of the small jump process ξ

In this subsection we shall give some estimates for the tails of the maximal fluctuation of the small jump component in the decomposition of noise derived in the previous subsection. In the statement we intend to keep the dependence on the parameters as general as possible, and as explicit as necessary later.

Lemma 3.3 *Let $\nu \ne 0$ be a symmetric Lévy measure, $b \ge 0$ and $\rho \in \mathbb{R}$. For $g \ge 1$ let $\xi = (\xi_t)_{t\ge 0}$ denote the Lévy process defined by the characteristic triple $(b, \nu|_{[-g,g]}, \rho)$. Then for any $\delta > 0$ there exist $u_0 > 0$ such that for $T > 0, g \ge 1, f > 0$ satisfying $\frac{f}{Tg} \ge u_0$ the following estimate holds:*

$$\mathbf{P}\Big(\sup_{t\le T} |\xi_t| > f\Big) \le \exp\Big(-(1-\delta)\frac{f}{g}\ln\frac{f}{gT}\Big). \tag{37}$$

Remark 3.1 *In particular, if we parameterise $g = g_\varepsilon$, $f = f_\varepsilon$ and $T = T_\varepsilon$ and assume that $\frac{f_\varepsilon}{g_\varepsilon T_\varepsilon} \to \infty$ as $\varepsilon \to 0$, then for every $\delta > 0$ there exists $\varepsilon_0 > 0$, such that (37) holds for any $0 < \varepsilon \le \varepsilon_0$.*

Proof: First let us consider the case $\rho = 0$. In this case for any $g > 0$ the process ξ is a martingale. For any $h > 0$, the reflection principle for symmetric Lévy processes and the Chebyshev inequality applied to the exponent of ξ_T yield

$$\mathbf{P}\left(\sup_{t\le T}|\xi_t| > f\right) \le 4\mathbf{P}(\xi_T > f) \le 4e^{-hf}\mathbf{E}e^{h\xi_T}. \tag{38}$$

The Lévy measure of ξ has bounded support. Thus the analytic extension of the characteristic function of ξ can be used to estimate $\mathbf{E}e^{h\xi_t}$. Denote $m := \int_{\mathbb{R}}(1\wedge y^2)\nu(dy) < \infty$, and let $h := \frac{1}{g}\ln\frac{f}{gT}$. Now recall that $g \ge 1$, to be able to choose u_0 large enough, such that $\frac{b}{2}h^2 \le \frac{b}{2}(\ln\frac{f}{gT})^2 \le m\frac{f}{gT}$ for any $\frac{f}{gT} \ge u_0$. The extension of the characteristic function of ξ yields the chain of inequalities

$$\begin{aligned}
\mathbf{E}e^{h\xi_T} &= \exp\left[\frac{b}{2}h^2T + T\int_{|y|\le g}\left(e^{hy} - 1 - hy\mathbb{I}\{|y|<1\}\right)\nu(dy)\right] \\
&= \exp\left[\frac{b}{2}h^2T + T\int_{|y|\le g}\left(hy\mathbb{I}\{|y|\ge 1\} + \sum_{k=2}^{\infty}\frac{(hy)^k}{k!}\right)\nu(dy)\right] \\
&\le \exp\left[\frac{b}{2}h^2T + T\int_{|y|\le g}\left(hg(1\wedge y^2) + \sum_{k=2}^{\infty}\frac{(hg)^k}{k!}(1\wedge y^2)\right)\nu(dy)\right] \\
&\le \exp\left[\frac{mf}{g} + Tm\exp(hg)\right] = \exp\left[\frac{2mf}{g}\right].
\end{aligned} \tag{39}$$

The statement of the Lemma for $\rho = 0$ follows immediately from (38) for sufficiently large u_0, such that $(2m + \frac{g}{f}\ln 4)/\ln(\frac{f}{gT}) < \delta$ for $\frac{f}{gT} \ge u_0$.

If $\rho \ne 0$, we apply the previous argument to the symmetric martingale $(\xi_t - \rho t)_{t\ge 0}$ and use the estimate

$$\mathbf{P}\left(\sup_{t\le T}|\xi_t| > f\right) \le \mathbf{P}\left(\sup_{t\le T}|\xi_t - \rho t| > f - |\rho|T\right) \le \mathbf{P}\left(\sup_{t\le T}|\xi_t - \rho t| > (1-\delta')f\right), \tag{40}$$

which holds for any $\delta' > 0$ and sufficiently large $\frac{f}{T} \ge \frac{f}{gT}$. ■

3.4 Tail estimates for the sum of big jumps of η

In this crucial subsection we shall give tail estimates for finite sums of jump heights by exponential rates depending on sums of logarithmic tails of the jump laws. The asymptotics of the exit times considered will later be seen to depend on the convexity properties of this sum of logarithmic tails.

Let again $\nu \ne 0$ be a symmetric Lévy measure. For $g \ge 1$ let $\eta = (\eta_t)_{t\ge 0}$ be the Lévy process defined by the characteristic triple $(0, \nu|_{[-g,g]^c}, 0)$. It is clear that η is a compound

Poisson process. Let W_k, $k \geq 0$, denote its jump sizes, with the convention $W_0 = 0$. The random variables W_k, $k \geq 1$, are i.i.d. and satisfy $|W_k| \geq g$.

For $u > 0$ denote $f(u) = -\ln \nu([g \vee u, \infty))$, with the convention $\ln 0 = -\infty$. Let $\beta := \nu([-g, g]^c) = 2\exp(f(g))$.

Lemma 3.4 *For $k \geq 1$ let r and g be such that $r > kg$. Then for any $\delta \in (0,1)$ such that $(1-\delta)r > kg$ the following estimate holds:*

$$\begin{aligned} \mathbf{P}\Big(\sum_{i=1}^{k} |W_i| > r\Big) \leq \frac{2^k}{\beta^k}\Big(2 + \frac{\ln r - \ln g}{\ln(1+\delta)}\Big)^k \\ \times \exp\Big(-\inf\Big\{\sum_{i=1}^{k} f(x_i) : \sum_{i=1}^{k} x_i = (1-\delta)r, x_i \in [g,r]\Big\}\Big). \end{aligned} \tag{41}$$

Proof: Denote $A := \{(x_1, \ldots, x_k) \in [g,r]^k : \sum_{i=1}^k x_i \geq r\}$. We have

$$\mathbf{P}\Big(\sum_{i=1}^{k} |W_i| > r\Big) \leq k\mathbf{P}(|W_1| \geq r) + \mathbf{P}((|W_1|, \ldots, |W_k|) \in A). \tag{42}$$

The first summand can be estimated as

$$\begin{aligned} \mathbf{P}(|W_1| \geq r) &\leq 2\beta^{-1}\exp(-f(r)) = 2^k\beta^{-k}\exp(-(f(r) + (k-1)f(g))) \\ &\leq 2^k\beta^{-k}\exp\Big(-\inf\Big\{\sum_{i=1}^{k} f(x_i) : \sum_{i=1}^{k} x_i \geq (1-\delta)r, x_i \in [g,r]\Big\}\Big). \end{aligned} \tag{43}$$

To estimate the second summand, we cover the set A by a union of parallelepipeds of a special form. Let $M := [\frac{\ln r - \ln g}{\ln(1+\delta)}]$ and consider a set of points

$$S := \{s_i, 0 \leq i \leq M\}, \quad s_i = (1+\delta)^i g, \tag{44}$$

such that $s_M \leq r$ and $(1+\delta)s_M > r$. Consider $(M+1)^k$ parallelepipeds of the type

$$P_{s_1,\ldots,s_k} := [s_1, (1+\delta)s_1] \times \cdots \times [s_k, (1+\delta)s_k], \quad s_1, \ldots, s_k \in S. \tag{45}$$

Obviously, the union of these $(M+1)^k$ parallelepipeds covers the cube $[g,r]^k$, and thus the set A. Let N denote the smallest covering of A by these parallelepipeds. If some $P_{s_1,\ldots,s_k} \in N$, i.e. $P_{s_1,\ldots,s_k} \cap A \neq \emptyset$, then

$$\max_{(x_1,\ldots,x_k) \in P_{s_1,\ldots,s_k}} \sum_{i=1}^{k} x_i = (1+\delta)\sum_{i=1}^{k} s_i \geq r \tag{46}$$

and thus $\sum_{i=1}^k s_i \geq (1+\delta)^{-1} r \geq (1-\delta) r$. Then

$$\begin{aligned}
\mathbf{P}\left((|W_1|,\dots,|W_k|) \in A\right) &\leq \sum_{P_{s_1,\dots,s_k} \in N} \prod_{j=1}^k \mathbf{P}\left(|W_j| \in [s_j, (1+\delta)s_j]\right) \\
&\leq (M+1)^k \max_{P_{s_1,\dots,s_k} \in N} \prod_{j=1}^k \mathbf{P}\left(|W_j| \in [s_j, (1+\delta)s_j]\right) \\
&\leq (M+1)^k \max_{P_{s_1,\dots,s_k} \in N} \prod_{j=1}^k \mathbf{P}\left(|W_1| \geq s_j\right) \\
&\leq 2^k (M+1)^k \beta^{-k} \max_{P_{s_1,\dots,s_k} \in N} \exp\Big(- \sum_{j=1}^k f(s_j) \Big) \\
&\leq 2^k (1+M)^k \beta^{-k} \exp\Big(- \inf\Big\{ \sum_{j=1}^k f(x_j) : \sum_{j=1}^k x_i \geq (1-\delta) r, x_j \in [g,r] \Big\} \Big).
\end{aligned} \tag{47}$$

The monotonicity of f and (43) lead to

$$\begin{aligned}
\mathbf{P}\Big(\sum_{i=1}^k |W_i| > r \Big) &\leq 2^k (k + (1+M)^k) \beta^{-k} \exp\Big(- \inf\Big\{ \sum_{i=1}^k f(x_i) : \sum_{i=1}^k x_i \geq (1-\delta) r, x_i \in [g,r] \Big\} \Big) \\
&= 2^k (k + (1+M)^k) \beta^{-k} \exp\Big(- \inf\Big\{ \sum_{i=1}^k f(x_i) : \sum_{i=1}^k x_i = (1-\delta) r, x_i \in [g,r] \Big\} \Big).
\end{aligned} \tag{48}$$

Finally, the elementary inequality $k + (1+M)^k \leq (2+M)^k$, $k \geq 1$, $M \geq 0$, applied to the prefactor completes the estimation. ■

3.5 A simple minimisation problem

Later in Sections 5.3.2 and 5.4.2 we will apply Lemma 3.4 to estimate the tails of sums of big jumps of the process L. In particular, we shall use the following result. Let $k \geq 1$ and $a > 0$. Then

$$\inf\Big\{ \sum_{i=1}^k x_i^\alpha \ : \ \sum_{i=1}^k x_i = a, 0 \leq x_i \leq a \Big\} \geq a^\alpha, \quad \text{if } 0 < \alpha \leq 1, \tag{49}$$

$$\inf\Big\{ \sum_{i=1}^k x_i^\alpha \ : \ \sum_{i=1}^k x_i = a, x_i \geq 0 \Big\} \geq k \Big(\frac{a}{k}\Big)^\alpha, \quad \text{if } \alpha \geq 1. \tag{50}$$

The inequality (49) is a direct consequence of the inequality $x_1^\alpha + \dots + x_k^\alpha \geq (x_1 + \dots + x_k)^\alpha$, $x_i \geq 0$, which in turn follows from the elementary inequality $1 + x^\alpha \geq (1+x)^\alpha$, $x \geq 0$, $\alpha \in (0,1]$.

The inequality (50) can be obtained by a straightforward application of the method of Lagrangian multipliers which implies that the local (and global) minimum is attained for $x_1 = \dots = x_k = a/k$.

The essentially different solutions to the minimization problems above are due to *convex* resp. *concave* behavior of the mapping $x \mapsto x^\alpha$ for $\alpha \in (0,1]$ resp. $\alpha \geq 1$.

4 The upper bounds

In this section we employ the first part of Lemma 3.1 in order to deduce upper bounds for the tails of the law of exit times presented in Theorems 2.1 and 2.2. This is done in separate arguments in the sub- and super-exponential case.

4.1 Sub-exponential tails. Proof of Theorem 2.1, upper bound

For the choice $g_\varepsilon := \frac{1}{2}\varepsilon^{-1}$, $\varepsilon \in (0,1]$, let us consider a decomposition $L = \xi^\varepsilon + \eta^\varepsilon$ as in section 3.2. Let z^ε be a solution of the corresponding SDE driven by small jumps, namely

$$z_t^\varepsilon = x - \int_0^t U'(z_{s-}^\varepsilon)ds + \varepsilon \xi_t^\varepsilon. \tag{51}$$

By construction we have $X_{\tau_1}^\varepsilon = z_{\tau_1}^\varepsilon + \varepsilon W_1$. Thus for any $\varepsilon \in (0,1]$, $T \in (0,1]$, and $x \in I$ the following estimate holds

$$\{\sigma < T\} \supseteq \{\tau_1 < T, X_{\tau_1}^\varepsilon \notin I\} = \{\tau_1 < T, W_1 \notin \varepsilon^{-1}(-1 - z_{\tau_1}^\varepsilon, 1 - z_{\tau_1}^\varepsilon)\}. \tag{52}$$

Since τ_1, W_1 and z^ε are independent, we get

$$\begin{aligned} \mathbf{P}_x(\sigma < T) &\geq \mathbf{P}(\tau_1 < T) \inf_{y \in \mathbb{R}} \mathbf{P}(W_1 \notin \varepsilon^{-1}(-1-y, 1-y)) \\ &= (1 - e^{-\beta_\varepsilon T})\beta_\varepsilon^{-1} \inf_{y \in \mathbb{R}} \nu_\eta^\varepsilon(\mathbb{R} \setminus \varepsilon^{-1}(-1-y, 1-y)). \end{aligned} \tag{53}$$

Symmetry of the jump measure implies $\inf_{y\in\mathbb{R}} \nu_\eta^\varepsilon(\mathbb{R}\setminus\varepsilon^{-1}(-1-y,1-y)) \geq \nu_\eta^\varepsilon([\varepsilon^{-1},\infty)) = \frac{1}{2}C_\varepsilon$. Using the elementary inequality $1 - e^{-x} \geq x - x^2/2$, $x \geq 0$, yields

$$\mathbf{P}_x(\sigma < T) \geq T\Big(1 - \frac{\beta_\varepsilon T}{2}\Big)\frac{C_\varepsilon}{2}. \tag{54}$$

We apply Lemma 3.1 (1) to obtain for any $t \geq 0$, $T \in (0,1]$ and ε small enough that

$$\sup_{x \in I} \mathbf{P}_x(\sigma > t) \leq \Big(1 - T\Big(1 - \frac{\beta_\varepsilon T}{2}\Big)\frac{C_\varepsilon}{2}\Big)^{-1} \exp\Big[-t\Big(1 - \frac{\beta_\varepsilon T}{2}\Big)\frac{C_\varepsilon}{2}\Big]. \tag{55}$$

The upper bound in Theorem 2.1 follows immediately by taking the infimum of the right-hand side of the latter estimate over $T \in (0,1]$.

4.2 Super-exponential tails. Proof of Theorem 2.2, upper bound

We need to show, that for any $\delta \in (0,1]$ there exists $T > 0$ and $\varepsilon_0 > 0$, such that for any $0 < \varepsilon < \varepsilon_0$ we have $TD_\varepsilon^{1+\delta} \leq \delta$ and the following estimate holds

$$\inf_{x \in I} \mathbf{P}_x(\sigma \leq T) \geq TD_\varepsilon^{1+\delta}, \tag{56}$$

with D_ε defined in Theorem 2.2. Then Lemma 3.1 yields the assertion.

Let $\delta' := \delta/7$, $M := \sup_{y\in I} |U'(y)|$, and let us choose $T \in (0, \frac{\delta'}{M} \wedge 1)$. Due to symmetry, it is sufficient to consider $x \in [0, 1)$. We start by remarking that $\sigma_x > T$ implies the inequality

$$1 > X_T = x - \int_0^T U'(X_{s-})ds + \varepsilon L_t \geq -MT + \varepsilon L_T. \tag{57}$$

Since $MT < \delta'$ we conclude $\{\sigma > T\} \subseteq \{\varepsilon L_T < 1 + \delta'\}$ and thus

$$\{\sigma \leq T\} \supseteq \{\varepsilon L_T \geq 1 + \delta'\}. \tag{58}$$

Recall that q_ε denotes the ε-quantile of $\nu([\cdot, \infty))$, the positive tail of the Lévy measure, and set

$$g_\varepsilon := (\alpha - 1)^{-1/\alpha} q_\varepsilon. \tag{59}$$

Since the exponent $f(u) := -\ln \nu([u, \infty))$ is regularly varying at $+\infty$ with index $\alpha > 1$, we have $\varepsilon g_\varepsilon \to 0$ as $\varepsilon \to 0$.

Consider a decomposition $L = \xi + \varphi^\varepsilon + \eta^\varepsilon$, where ξ, φ^ε and η^ε are Lévy processes having generating triplets $(d, \nu|_{(-\infty,1]}, \mu)$, $(0, \nu|_{(1,g_\varepsilon)}, 0)$, and $(0, \nu|_{[g_\varepsilon,\infty)}, 0)$ accordingly. If $N^{(\eta)}$ denotes the counting process of η^ε, we have $\varphi^\varepsilon_T > 0$ and $\eta^\varepsilon_T \geq g_\varepsilon N^{(\eta)}_T$. Inequality (58) yields the inclusion

$$\{\sigma \leq T\} \supseteq \{\varepsilon\xi_T \geq -\delta'\} \cap \{\varepsilon\eta^\varepsilon_T \geq 1 + 2\delta'\} \supseteq \{\varepsilon\xi_T \geq -\delta'\} \cap \{N^{(\eta)}_T \geq \tfrac{1+2\delta'}{\varepsilon g_\varepsilon}\}. \tag{60}$$

The random variables ξ_T and $N^{(\eta)}_T$ are independent and $\mathbf{P}(\varepsilon\xi_T \geq -\delta') \geq 1 - \delta'$ for ε small enough. Let $k_\varepsilon := [\frac{1+2\delta'}{\varepsilon g_\varepsilon}] + 1$. Moreover, using the definition of regularly varying functions, and the fact that $q_\varepsilon \to \infty$ as $\varepsilon \to 0$, we estimate for small ε:

$$|\ln \beta^{(\eta)}_\varepsilon| = f(g_\varepsilon) \leq (1 + \delta') \left(\frac{g_\varepsilon}{q_\varepsilon}\right)^\alpha f(q_\varepsilon) \leq \frac{1 + \delta'}{\alpha - 1} |\ln \varepsilon|. \tag{61}$$

In particular, this means that $k_\varepsilon/\beta^{(\eta)}_\varepsilon \to \infty$ as $\varepsilon \to 0$. Thus, with the help of the inequality $k! \leq 3k^{k+1/2}$, $k \geq 1$, and the previous estimate, we get for ε sufficiently small that

$$\begin{aligned} \mathbf{P}(\sigma \leq T) &\geq (1 - \delta')\mathbf{P}(N^{(\eta)}_T = k_\varepsilon) = (1 - \delta')\frac{(\beta^{(\eta)}_\varepsilon T)^{k_\varepsilon}}{k_\varepsilon!} \exp(-\beta^{(\eta)}_\varepsilon T) \\ &\geq \exp\left(-(1 + \delta')k_\varepsilon(\ln k_\varepsilon + |\ln \beta^{(\eta)}_\varepsilon|)\right). \end{aligned} \tag{62}$$

By definition of k_ε we have $\ln k_\varepsilon \leq |\ln \varepsilon|$ and $k_\varepsilon \leq \frac{1+3\delta'}{\varepsilon g_\varepsilon}$ for ε small enough. This leads to the final estimate

$$\begin{aligned} \mathbf{P}(\sigma \leq T) &\geq \exp\left(-(1 + \delta')^2 \frac{\alpha}{\alpha - 1} k_\varepsilon |\ln \varepsilon|\right) \\ &\geq \exp\left(-(1 + \delta')^2(1 + 3\delta')d_\alpha \frac{|\ln \varepsilon|}{\varepsilon q_\varepsilon}\right) \geq T D_\varepsilon^{1+\delta}. \end{aligned} \tag{63}$$

5 The lower bounds

5.1 General remarks

We will use the second part of Lemma 3.1 to establish the lower bound estimates for Theorems 2.1 and 2.2. Consider σ^- and σ^* as in Lemma 3.2. In section 5.3 we will show, that for some appropriately chosen $\delta_0 > 0$, $r \in (0,1]$, $T > 0$ and any $\delta \in (0, \delta_0)$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the following estimate holds:

$$\sup_{|x| \le \delta} \mathbf{P}_x(\sigma^* < T) \le T C_\varepsilon^{1-r\delta}, \tag{64}$$

in the sub-exponential case, and an analogous inequality with D_ε replacing C_ε in section 5.4 for the super-exponential one. Lemma 3.1 yields

$$\inf_{|x| \le \delta} \mathbf{P}_x(\sigma > t) \ge \inf_{|x| \le \delta} \mathbf{P}_x(\sigma^- > t) \ge (1-\delta) \exp(-C_\varepsilon^{1-r\delta} t) \tag{65}$$

for any $t \ge 0$ and ε sufficiently small.

The structures of the proofs providing the lower bounds in Theorem 2.1 and Theorem 2.2 are similar. To prove that inequality (64) holds, we consider a decomposition $L = \xi^\varepsilon + \eta^\varepsilon$ as in Section 3.2 with some appropriately chosen g_ε, such that $g_\varepsilon \to \infty$ and $\varepsilon g_\varepsilon \to 0$ as $\varepsilon \to 0$. In the subsections 5.3.1 respectively 5.4.1 we will use Lemma 3.2 and Corollary 3.1 to obtain an embedding of $\{\sigma^* < T\}$ in terms of sets described by the large and small jump parts ξ^ε and η^ε. In subsections 5.3.2 respectively 5.4.2 we will use Lemmas 3.3 and 3.4 to estimate the probabilities of the covering sets.

5.2 Reduction of starting values

In this section we will show that for any $\delta \in (0, \frac{1}{2}]$ the following estimate holds for small ε:

$$1 - \delta \le \frac{\inf_{x \in I_\delta^-} \mathbf{P}(\sigma_x > t)}{\inf_{|x| \le \delta} \mathbf{P}(\sigma_x > t)} \le 1. \tag{66}$$

Thus equation (65) yields

$$\inf_{x \in I_\delta^-} \mathbf{P}(\sigma > t) \ge (1-\delta) \exp(-C_\varepsilon^{1-r\delta} t) \ge (1-\delta) \exp(-C_\varepsilon^{1-\delta} t) \tag{67}$$

for any $t \ge 0$ and ε sufficiently small. This entails the lower bounds of the estimate (8). The estimate leading to (10) is obtained analogously.

Proof of estimate (66)**.** Obviously we only need to prove the first estimate from below. Let $\sigma_1 := \inf\{t \ge 0 : |X_t^\varepsilon(x)| \le \delta\}$ and $\sigma_2 := \inf\{t \ge 0 : |X_t^\varepsilon(x)| \ge 1 - \frac{\delta}{2}\}$. The strong Markov property and time homogeneity of X^ε yield for any $x \in I_\delta^-$

$$\mathbf{P}(\sigma_x > t) \ge \mathbf{P}(\sigma_x > \sigma_1 + t) \ge \mathbf{P}(\sigma_x > \sigma_1) \inf_{|x| \le \delta} \mathbf{P}(\sigma_x > t) \ge \mathbf{P}(\sigma_2 > \sigma_1) \inf_{|x| \le \delta} \mathbf{P}(\sigma_x > t). \tag{68}$$

Let $\tilde{m} := \min_{y \in I_{\delta/2}^-} |U'(y)|$ and $T := 2/\tilde{m}$. We have

$$\{\sigma_2 \le \sigma_1\} \subseteq \{\sigma_1 \ge T, \sigma_2 \ge T, \tau_1 > T\} \cup \{\sigma_2 < T \wedge \tau_1\} \cup \{\tau_1 \le T\}. \tag{69}$$

We choose g_ε such, that $g_\varepsilon \to \infty$ and $\varepsilon g_\varepsilon \to 0$ as $\varepsilon \to 0$. In analogy with (24) we get $\{\sigma_1 \geq T, \sigma_2 \geq T, \tau_1 > T\} \subseteq \{\sup_{t\leq T} |\varepsilon \xi^\varepsilon_t| > \tilde{m}T - 1 = 1\}$, and for any $x \in I^-_\delta$ the second part of Lemma 3.2 yields $\{\sigma_2 < T \wedge \tau_1\} \subseteq \{\big(\sup_{t<T\wedge\tau_1} - \inf\big)\varepsilon L_t \geq \frac{\delta}{2}\} \subseteq \{\sup_{t\leq T} |\varepsilon \xi^\varepsilon_t| \geq \frac{\delta}{4}\}$. Thus we have

$$\mathbf{P}(\sigma_2 < \sigma_1) \leq \mathbf{P}\left(\sup_{t\leq T} |\varepsilon \xi^\varepsilon_t| \geq \tfrac{\delta}{4}\right) + \mathbf{P}(\tau_1 \leq T) \leq \delta, \tag{70}$$

for ε small enough and (68) yields the assertion.

5.3 Sub-exponential tails. Proof of Theorem 2.2, lower bound

5.3.1 Estimate of beginning of exit

Let $J_\delta = [-\delta, \delta]$, and $m = \inf\{|U'(y)| : y \in I^-_\delta \setminus J_\delta\}$ as in Lemma 3.2. The following estimates would hold analogously for any choice $T > 0$ and $\hat{T} > \frac{1}{m}$. For simplicity we choose $\hat{T} = \frac{2}{m}$ and $T = \hat{T}$, such that $m\hat{T} - 1 = 1$ and $T + \hat{T} = 2T$. Let $k \geq 1$ and abbreviate

$$\chi_k := \bigcup_{i=0}^{k-1} \left\{ \sup_{t\leq T} \varepsilon |\xi^\varepsilon_{S_i+T+t} - \xi^\varepsilon_{S_i+T}| \geq 1 \right\}. \tag{71}$$

We have

$$\{\sigma^* < T\} \subseteq \chi_k \cup (\{\sigma^* < T, \sigma^- \geq S_k\} \setminus \chi_k) \cup \bigcup_{j=0}^{k-1} (\{\sigma^* < T, \sigma^- \in [S_j, S_{j+1})\} \setminus \chi_k). \tag{72}$$

The first part of Corollary 3.1 yields that for all $1 \leq j \leq k$ the following inclusion is valid:

$$\{\sigma^* < T, \sigma^- \geq S_j\} \subseteq \chi_k \cup \bigcap_{i=1}^{j} \{\tau_i \leq 2T\}. \tag{73}$$

Let us next fix $1 \leq j \leq k-1$ to prove

$$\{\sigma^* < T, \sigma^- \in [S_j, S_{j+1})\} \subseteq \chi_k \cup \{\sigma^* < T, \sigma^- \in [S_j, S_{j+1} \wedge (S_j + 2T))\}. \tag{74}$$

To see this inclusion, note first that $\sigma^- \in [S_j, S_j + 2T)$ and $S_{j+1} > S_j + 2T$ clearly implies $\sigma^- \in [S_j, S_{j+1} \wedge (S_j + 2T))$. Also, if $\sigma^- \in [S_j, S_{j+1})$ and $S_{j+1} \leq S_j + 2T$, we conclude $\sigma^- \in [S_j, S_{j+1} \wedge (S_j + 2T))$. It remains to argue for the case $\sigma^- \in [S_j + 2T, S_{j+1})$ and $S_{j+1} > S_j + 2T$. In this case, we may apply Lemma 3.2*(i)* with $t = S_j + 2T$, and the fact that $N_{S_j+2T} = j$ on the set $\{S_{j+1} > S_j + 2T\}$. It implies

$$\begin{aligned}
&\{\sigma^* < T, \sigma^- \in [S_j + 2T, S_{j+1}), S_{j+1} > S_j + 2T\} \\
&\subseteq \{\sigma^* < T, \sigma^- \geq S_j + 2T\} \cap \{S_{j+1} > S_j + 2T\} \\
&\subseteq \Big(\{S_j + 2T - S_{N_{S_j+2T}} < T\} \cup \chi_k\Big) \cap \{S_{j+1} > S_j + 2T\} \\
&\subseteq \Big(\{S_j + T - S_{N_{S_j+2T}} < 0\} \cap \{S_{j+1} > S_j + 2T\}\Big) \cup \chi_k \\
&= \emptyset \cup \chi_k = \chi_k.
\end{aligned} \tag{75}$$

This proves (74). An application of Corollary 3.1*(ii)* further yields for any $x \in J_\delta$ and $1 \le j \le k-1$

$$\begin{aligned}\{\sigma^* < T, \sigma^- \in [S_j, S_{j+1})\} \setminus \chi_k &\subseteq \bigcap_{i=1}^{j}\{\tau_i \le 2T\} \cap \{\sigma^- < S_{j+1} \wedge (S_j + 2T)\} \\ &\subseteq \bigcap_{i=1}^{j}\{\tau_i \le 2T\} \cap \{\sigma^- < S_{j+1} \wedge 2kT\} \\ &\subseteq \bigcap_{i=1}^{j}\{\tau_i \le 2T\} \cap \Big(\Big\{\sum_{i=1}^{j}|\varepsilon W_i| \ge 1-3\delta\Big\} \cup \Big\{\sup_{t\le 2kT}|\varepsilon\xi_t^\varepsilon| \ge \tfrac{\delta}{2}\Big\}\Big).\end{aligned} \tag{76}$$

Putting all arguments in (72) together, we obtain

$$\begin{aligned}\{\sigma^* < T\} \subseteq \chi_k &\cup \bigcap_{i=1}^{k}\{\tau_i \le 2T\} \cup \Big\{\sup_{t\le 2kT}|\varepsilon\xi_t^\varepsilon| \ge \tfrac{\delta}{2}\Big\} \\ &\cup \bigcup_{j=1}^{k-1}\left(\bigcap_{i=1}^{j}\{\tau_i \le 2T\} \cap \Big\{\sum_{i=1}^{j}|\varepsilon W_i| \ge 1-3\delta\Big\}\right).\end{aligned} \tag{77}$$

5.3.2 The lower bound via beginning of exit

Let $\delta_0 := \frac{1}{3}\alpha(1-\alpha)$. Pick $\delta \in (0, \delta_0)$ and let

$$g_\varepsilon := \varepsilon^{-(1-\alpha-\delta)} \quad \text{and} \quad k = k_\varepsilon := [\varepsilon^{-(\alpha-\delta)}]. \tag{78}$$

By definition we have $\varepsilon k_\varepsilon g_\varepsilon \to 0$ as $\varepsilon \to 0$. Then (77) yields

$$\begin{aligned}\mathbf{P}_x(\sigma^* < T) \le \mathbf{P}(\chi_{k_\varepsilon}) &+ \mathbf{P}(\tau_1 \le 2T)^{k_\varepsilon} + \mathbf{P}\Big(\sup_{t\le 2k_\varepsilon T}|\varepsilon\xi_t^\varepsilon| \ge \tfrac{\delta}{2}\Big) \\ &+ \sum_{k=1}^{k_\varepsilon-1}\mathbf{P}(\tau_1 \le 2T)^k \cdot \mathbf{P}\Big(\sum_{i=1}^{k}|\varepsilon W_i| \ge 1-3\delta\Big).\end{aligned} \tag{79}$$

In the next steps we estimate the summands of the previous formula.

1. We first apply the strong Markov property of ξ^ε and Lemma 3.3 with $2k_\varepsilon T$ instead of T and $\frac{\delta}{2\varepsilon}$ instead of f to the first and third term on the right hand side of (79) to obtain for ε sufficiently small

$$\begin{aligned}\mathbf{P}(\chi_{k_\varepsilon}) + \mathbf{P}\Big(\sup_{t\le 2k_\varepsilon T}|\varepsilon\xi_t^\varepsilon| \ge \tfrac{\delta}{2}\Big) &\le k_\varepsilon \mathbf{P}\Big(\sup_{t\le T}|\varepsilon\xi_t^\varepsilon| \ge 1\Big) + \mathbf{P}\Big(\sup_{t\le 2k_\varepsilon T}|\varepsilon\xi_t^\varepsilon| \ge \tfrac{\delta}{2}\Big) \\ &\le (k_\varepsilon+1)\mathbf{P}\Big(\sup_{t\le 2k_\varepsilon T}|\varepsilon\xi_t^\varepsilon| \ge \tfrac{\delta}{2}\Big) \\ &\le (k_\varepsilon+1)\exp\Big(-(1-\delta)\frac{\delta}{2\varepsilon g_\varepsilon}\ln\frac{\delta}{4T\varepsilon k_\varepsilon g_\varepsilon}\Big) \\ &\le \exp\Big(-\frac{1}{\varepsilon g_\varepsilon}\Big) = \exp(-\varepsilon^{-(\alpha+\delta)}) < TC_\varepsilon.\end{aligned} \tag{80}$$

2. We next deal with the second term on the right hand side of (79). Recall that the logarithm of the Lévy measure's tail $f(u)$, $u > 0$, is a regularly varying function with index $\alpha \in (0,1)$. Since $g_\varepsilon \to \infty$ as $\varepsilon \to 0$, we can choose ε sufficiently small, such that $f(g_\varepsilon) \geq g_\varepsilon^{\alpha-\delta}$ and the following estimate holds:

$$\begin{aligned}\mathbf{P}(\tau_1 \leq 2T)^{k_\varepsilon} &\leq (2T\beta_\varepsilon)^{k_\varepsilon} = (4Te^{-f(g_\varepsilon)})^{k_\varepsilon} \leq (4Te^{-g_\varepsilon^{\alpha-\delta}})^{k_\varepsilon} \\ &\leq \exp\left(-(1-\delta)\varepsilon^{-(1-\alpha-\delta)(\alpha-\delta)-(\alpha-\delta)}\right) \leq \exp\left(-(1-\delta)\varepsilon^{-(\alpha+\alpha(1-\alpha)-3\delta)}\right).\end{aligned} \tag{81}$$

For the first inequality in the chain (81) we hereby use that the law of τ_1 is exponential with mean β_ε^{-1}, while $\alpha \in (0,1)$ is needed for the last. Thus the hypothesis $\delta < \frac{1}{3}\alpha(1-\alpha)$ yields $\mathbf{P}(\tau_1 \leq 2T)^{k_\varepsilon} < TC_\varepsilon$ for ε sufficiently small.

3. We finally fix $1 \leq k \leq k_\varepsilon - 1$ to treat a summand of the last term on the right hand side of (79). By definition we have $\varepsilon g_\varepsilon k_\varepsilon \to 0$ as $\varepsilon \to 0$. Thus ε can be chosen sufficiently small, such that by Lemma 3.4 applied with $r = \frac{1-3\delta}{\varepsilon}$, to estimate $\mathbf{P}\left(\sum_{i=1}^k |\varepsilon W_i| \geq 1-3\delta\right)$, we get the following inequalities

$$\begin{aligned}&\max_{1\leq k\leq k_\varepsilon-1} \mathbf{P}(\tau_1 \leq 2T)^k \cdot \mathbf{P}\Big(\sum_{i=1}^k |\varepsilon W_i| \geq 1-3\delta\Big) \leq \max_{1\leq k\leq k_\varepsilon-1} (2T\beta_\varepsilon)^k \cdot \mathbf{P}\Big(\sum_{i=1}^k |\varepsilon W_i| \geq 1-3\delta\Big) \\ &\leq \max_{1\leq k\leq k_\varepsilon-1} \Big(8T + 4T\frac{|\ln\varepsilon| - \ln g_\varepsilon}{\ln(1+\delta)}\Big)^k \\ &\qquad \times \exp\Big(-\inf\Big\{\sum_{i=1}^k f(x_i) \,:\, \sum_{i=1}^k x_i = \tfrac{(1-3\delta)(1-\delta)}{\varepsilon}, x_i \in [g_\varepsilon, \tfrac{1-3\delta}{\varepsilon}]\Big\}\Big) \\ &\leq |\ln\varepsilon|^{2k_\varepsilon} \max_{1\leq k\leq k_\varepsilon-1} \exp\Big(-\inf\Big\{\sum_{i=1}^k f(x_i) \,:\, \sum_{i=1}^k x_i = \tfrac{(1-3\delta)(1-\delta)}{\varepsilon}, x_i \in [g_\varepsilon, \tfrac{1}{\varepsilon}]\Big\}\Big).\end{aligned} \tag{82}$$

Note that for the second inequality, Lemma 3.4 tells us to minimize on jumps $x_i \in [g_\varepsilon, \frac{1-3\delta}{\varepsilon}]$. The minimizer on the intervals $[g_\varepsilon, \frac{1}{\varepsilon}]$ we take is smaller, so gives an upper estimate. For the crucial estimate of the exponential rate, we call upon Potters' bound for the regularly varying function f (see [1, Theorem 1.5.6]). Since $g_\varepsilon \to \infty$ as $\varepsilon \to 0$, it provides the following estimate for all $x \in [g_\varepsilon, \frac{1}{\varepsilon}]$ and ε sufficiently small:

$$f(x) \geq (1-\delta) f(\varepsilon^{-1})(\varepsilon x)^{\alpha+\delta}. \tag{83}$$

Therefore we get

$$\begin{aligned}&\inf\Big\{\sum_{i=1}^k f(x_i) \,:\, \sum_{i=1}^k x_i = (1-3\delta)(1-\delta)\varepsilon^{-1}, x_i \in [g_\varepsilon, \varepsilon^{-1}]\Big\} \\ &\quad \geq (1-\delta) f(\varepsilon^{-1})\varepsilon^{\alpha+\delta} \inf\Big\{\sum_{i=1}^k x_i^{\alpha+\delta} \,:\, \sum_{i=1}^k x_i = (1-3\delta)(1-\delta)\varepsilon^{-1}, x_i \in [g_\varepsilon, \varepsilon^{-1}]\Big\} \\ &\quad \geq (1-\delta) f(\varepsilon^{-1})\varepsilon^{\alpha+\delta}((1-3\delta)(1-\delta)\varepsilon^{-1})^{\alpha+\delta} \geq (1-5\delta) f(\varepsilon^{-1}).\end{aligned} \tag{84}$$

For obtaining the second inequality in the preceding chain, we have to recall that $\alpha+\delta<1$, and use the inequality (49). So the estimation in (82) may be completed by

$$\begin{aligned}\max_{1\le k\le k_\varepsilon-1}\mathbf{P}(\tau_1\le 2T)^k\cdot\mathbf{P}\Big(\sum_{i=1}^{k}|\varepsilon W_i|\ge 1-3\delta\Big)&\le\exp(-(1-5\delta)f(\varepsilon^{-1})+2k_\varepsilon\ln|\ln\varepsilon|)\\&\le\exp(-(1-6\delta)f(\varepsilon^{-1}))\le TC_\varepsilon^{1-7\delta},\end{aligned}\tag{85}$$

which again holds for ε small enough. Collecting the bounds we obtained for the terms in (79), we finally get for ε small enough

$$\mathbf{P}_x(\sigma^*<T)\le 2TC_\varepsilon+(k_\varepsilon-1)TC_\varepsilon^{1-7\delta}\le TC_\varepsilon^{1-8\delta}.\tag{86}$$

So we just have to rename the δ we started with by $\delta'=\delta/8$, to obtain the desired upper bound, and complete the proof of Theorem 2.1.

5.4 Super-exponential tails. Proof of Theorem 2.2, lower bound

5.4.1 Estimate of beginning of exit

Again, we start by covering the crucial set $\{\sigma^*<T\}$ by sets described in terms of the small and large jump parts. Let $k\ge 1$. Let m, $\hat{T}$, T and χ_k be defined as in Section 5.3.1. Then with help of Corollary 3.1*(i)* we obtain

$$\begin{aligned}\{\sigma^*<T\}&=\{\sigma^*<T,\sigma^-<2Tk\wedge S_k\}\cup\{\sigma^*<T,\sigma^-\ge 2Tk\wedge S_k\}\\&\subseteq\{\sigma^*<T,\sigma^-<2Tk\wedge S_k\}\cup\chi_k\cup\bigcap_{i=1}^{k}\{\tau_i\le 2T\}.\end{aligned}\tag{87}$$

Define the set

$$\bar{\chi}_k:=\bigcup_{i=1}^{k-1}\{\varepsilon W_i\ge\delta\}.\tag{88}$$

We notice that on the event $\{\sigma^*<T,\sigma^-<2Tk\wedge S_k\}\cap\bar{\chi}_k^c$ the estimate

$$\big(\sup-\inf\big)_{t\le\sigma^-}\varepsilon L_t\ge m(\sigma^--\sigma^*)\tag{89}$$

holds. To see this, we use similar arguments as in the proof of Lemma 3.2*(i)*. Indeed, on the event $\{\sigma^*<T,\sigma^-<2Tk\wedge S_k\}\cap\bar{\chi}_k^c$ the process X^ε does not change its sign during the time interval $[\sigma^*,\sigma^-]$, and for any $t\in(\sigma^*,\sigma^-)$ we have $|U'(X_{t-})|\ge m$. Further we have that if $0<X_{\sigma^*-}\le\delta$ then $X_{\sigma^-}\ge 1-\delta$, and $X_{\sigma^-}=X_{\sigma^*-}+\int_{\sigma^*}^{\sigma^-}U'(X_{t-})dt+\varepsilon(L_{\sigma^-}-L_{\sigma^*-})$, and thus

$$\big(\sup-\inf\big)_{t\le\sigma^-}\varepsilon L_t\ge(1-2\delta)+m(\sigma^--\sigma^*)\ge m(\sigma^--\sigma^*).\tag{90}$$

The case of negative values X_{σ^*-} is considered analogously.

This and Lemma 3.2*(ii)* lead to the following estimate for $k \geq 1$ (recall $mT = 2$):

$$\begin{aligned}
&\{\sigma^* < T, \sigma^- < 2Tk \wedge S_k\} \cap \bar{\chi}_k^c \\
&\subseteq \{\sigma^- < 4T\} \cup \bigcup_{i=2}^{k-1} \Big(\Big\{ \sigma^* < T, \sigma^- \in [2iT, 2(i+1)T], \sigma^- < S_k \Big\} \cap \bar{\chi}_k^c \Big) \\
&\subseteq \Big\{ \big(\sup - \inf\big)_{t<4T} \varepsilon L_t \geq 1 - 2\delta \Big\} \cup \bigcup_{i=2}^{k-1} \Big\{ \big(\sup - \inf\big)_{t<2(i+1)T} \varepsilon L_t \geq (2i-1)Tm \Big\} \\
&\subseteq \Big\{ \big(\sup - \inf\big)_{t<4T} \varepsilon L_t \geq 1 - 2\delta \Big\} \cup \bigcup_{i=2}^{k-1} \Big\{ \sum_{j=0}^{i} \big(\sup - \inf\big)_{t<2T} \varepsilon (L_{2jT+t} - L_{2jT}) \geq i+1 \Big\} \\
&\subseteq \Big\{ \big(\sup - \inf\big)_{t<4T} \varepsilon L_t \geq 1 - 2\delta \Big\} \cup \bigcup_{i=0}^{k-1} \Big\{ \big(\sup - \inf\big)_{t<2T} \varepsilon (L_{2iT+t} - L_{2iT}) \geq 1 \Big\}.
\end{aligned} \tag{91}$$

In particular, this entails

$$\{\sigma^* < T\} \subseteq \chi_k \cup \bar{\chi}_k \cup \bigcap_{i=1}^{k} \{\tau_i \leq 2T\} \cup \bigcup_{i=0}^{k-1} \Big\{ \big(\sup - \inf\big)_{t\leq 4T} \varepsilon (L_{2iT+t} - L_{2iT}) \geq 1 - 2\delta \Big\}. \tag{92}$$

5.4.2 The lower bound via beginning of exit

Let $\delta \in (0, \frac{1}{2})$ be fixed, let $q_\varepsilon := \sup\{u > 0 : \nu([u, \infty)) \geq \varepsilon\}$ denote the ε-quantile of the Lévy measure ν, and set

$$g_\varepsilon := \frac{q_\varepsilon}{3} \quad \text{and} \quad k_\varepsilon := \left\lceil \frac{|\ln \varepsilon|}{\varepsilon} \right\rceil. \tag{93}$$

In particular, since the tails of ν are super-exponential, $\varepsilon g_\varepsilon \to 0$ as $\varepsilon \to 0$.

We shall also use a simple estimate of $d_\alpha := \inf_{y>0}(y^{-\alpha+1} + y) = \alpha(\alpha-1)^{\frac{1}{\alpha}-1}$. It is easy to see that

$$1 = \inf_{y>0}(y^{-\alpha+1} \vee y) \leq d_\alpha \leq (y^{-\alpha+1} + y)|_{y=1} = 2. \tag{94}$$

Next, we estimate the probabilities of the events in (92).

1. We use the inequality $d_\alpha \leq 2$ and Lemma 3.3 with $f = \frac{1}{\varepsilon}$ and to obtain for ε small that

$$\begin{aligned}
\mathbf{P}(\xi_k) &\leq k_\varepsilon \mathbf{P}\Big(\sup_{t\leq 2T} |\varepsilon \xi_t^\varepsilon| \geq 1 \Big) \leq k_\varepsilon \exp\Big(- \frac{1-\delta}{\varepsilon g_\varepsilon} \ln \frac{1}{2T\varepsilon g_\varepsilon} \Big) \\
&\leq k_\varepsilon \exp\Big(- 3(1-2\delta)\varepsilon^{-1} q_\varepsilon^{-1} |\ln \varepsilon| \Big) < T D_\varepsilon.
\end{aligned} \tag{95}$$

2. To deal with the second term, recall that $f(u) = -\ln \nu([u, \infty))$ is regularly varying with $\alpha > 1$ at infinity. So we have

$$\mathbf{P}(\bar{\chi}_{k_\varepsilon}) \leq k_\varepsilon \mathbf{P}(|\varepsilon W_1| \geq \delta) \leq 2k_\varepsilon \beta_\varepsilon^{-1} e^{-f(\delta/\varepsilon)} \leq \exp(-(\delta/\varepsilon)^{\alpha-\delta} + 2|\ln \varepsilon| + \ln k_\varepsilon) \leq T D_\varepsilon. \tag{96}$$

3. Since $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$ we have for sufficiently small $\varepsilon > 0$

$$\mathbf{P}(\cap_{i=1}^{k_\varepsilon} \{\tau_i \leq 2T\}) \leq \mathbf{P}(\tau_1 \leq 2T)^{k_\varepsilon} \leq (2T\beta_\varepsilon)^{k_\varepsilon} \leq \exp(-k_\varepsilon) < T D_\varepsilon. \tag{97}$$

4. The estimate for the last union in (92) is the most important part of the proof. Since for any $0 \le i \le k-1$ the processes $(L_{2iT+t} - L_{2iT})_{t\ge 0}$ are independent and have the same law as $L = (L_t)_{t\ge 0}$, its enough to work with the original process L. We prove the following Lemma.

Lemma 5.1 *Let the jump measure ν of L be symmetric and super-exponential with index $\alpha > 1$. Then for any $T > 0$, $a > 0$, and $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ the following estimate holds:*

$$\mathbf{P}\left(\big(\sup_{t\le T} - \inf_{t\le T}\big)\varepsilon L_t > a\right) \le D_\varepsilon^{(1-\delta)a}. \tag{98}$$

Applying Lemma 5.1 with $a = 1 - 2\delta$ to the last event in (92) we get

$$k_\varepsilon \mathbf{P}\left(\big(\sup_{t\le 4T} - \inf_{t\le 4T}\big)\varepsilon L_t \ge 1 - 2\delta\right) \le k_\varepsilon D_\varepsilon^{1-3\delta} \le T D_\varepsilon^{1-4\delta} \tag{99}$$

for sufficiently small ε. This completes the proof of Theorem 2.2.

Proof of Lemma 5.1. Due to monotonicity, it is sufficient to consider $\delta \in (0, \frac{1\wedge a}{4})$. Let $r := 8\alpha$. We shall prove that for any such δ there exists $\varepsilon_0 > 0$, such that for every $0 < \varepsilon < \varepsilon_0$ the estimate $\mathbf{P}\Big(\big(\sup_{t\le T} - \inf_{t\le T}\big)\varepsilon L_t > a\Big) \le D_\varepsilon^{(1-r\delta)a}$ holds. This entails the asserted inequality.

Consider a decomposition $L = \xi^\varepsilon + \eta^\varepsilon$ as in Section 3.2 with the threshold

$$g_\varepsilon := \frac{\delta q_\varepsilon}{6}. \tag{100}$$

Note that this g_ε is different from its counterpart defined in (93) at the beginning of this subsection, and will be only used in the proof of the Lemma.

Since $\big(\sup_{t\le T} - \inf_{t\le T}\big)\varepsilon\xi_t^\varepsilon \le 2\sup_{t\le T}|\varepsilon\xi_t^\varepsilon|$, we have for any $n \ge 1$

$$\begin{aligned}
&\Big\{\big(\sup_{t\le T} - \inf_{t\le T}\big)\varepsilon L_t \ge a\Big\} \subseteq \Big\{\big(\sup_{t\le T} - \inf_{t\le T}\big)\varepsilon\xi_t^\varepsilon \ge \delta a\Big\} \cup \Big\{\big(\sup_{t\le T} - \inf_{t\le T}\big)\varepsilon\eta_t^\varepsilon \ge (1-\delta)a\Big\} \\
&\subseteq \Big\{\sup_{t\le T}|\varepsilon\xi_t^\varepsilon| \ge \tfrac{1}{2}\delta a\Big\} \cup \{N_T > n\} \cup \bigcup_{k=1}^{n}\Big\{N_T = k, \sum_{i=1}^{k}|\varepsilon W_i| > (1-\delta)a\Big\}.
\end{aligned} \tag{101}$$

The goal of the next steps consists in estimating the probabilities of the events figuring in the second line of (101) with an appropriately chosen n, namely with

$$n := n_\varepsilon = \left[\frac{3a}{q_\varepsilon\varepsilon}\right]. \tag{102}$$

1. We apply Lemma 3.3 with $f = \frac{\delta a}{2\varepsilon}$, and $2T$ instead of T to get

$$\begin{aligned}
\mathbf{P}\left(\sup_{t\le T}|\varepsilon\xi_t^\varepsilon| \ge \frac{\delta a}{2}\right) &\le \exp\left(-(1-\delta)\frac{3a}{\varepsilon q_\varepsilon}\ln\frac{3a}{T\varepsilon q_\varepsilon}\right) \\
&\le \exp\left(-(1-2\delta)\frac{3a}{\varepsilon q_\varepsilon}|\ln\varepsilon|\right) < D_\varepsilon^a
\end{aligned} \tag{103}$$

for ε small enough.
2. To estimate $\mathbf{P}(N_T > n_\varepsilon)$ we will use Stirling's formula. By choice of g_ε and n_ε we have $\varepsilon g_\varepsilon \to 0$, $\frac{\ln n_\varepsilon}{|\ln \varepsilon|} \to 1$, and $\beta_\varepsilon T \le 1$ as $\varepsilon \to 0$. Thus for ε sufficiently small the estimate $n_\varepsilon! \ge \exp(n_\varepsilon(\ln n_\varepsilon - 1)) \ge \exp((1-\delta)n_\varepsilon |\ln \varepsilon|)$ holds, and we get

$$\begin{aligned}\mathbf{P}(N_T > n_\varepsilon) &\le \sum_{k=n_\varepsilon}^{\infty} \frac{(\beta_\varepsilon T)^k}{k!} \le (1+\delta)\frac{(\beta_\varepsilon T)^{n_\varepsilon}}{n_\varepsilon!} \\ &\le \exp(-(1-\delta)n_\varepsilon|\ln\varepsilon|) \le \exp\left(-(1-2\delta)\frac{3a}{\varepsilon q_\varepsilon}|\ln\varepsilon|\right) < D_\varepsilon^a.\end{aligned} \tag{104}$$

3. As in Section 5.3, the crucial ingredient which produces the phase transition at $\alpha = 1$ comes from the following estimate for the exponential rate of sums of large jumps. Recall that $f(u) = -\ln \nu([u, +\infty))$, $u > 0$, is regularly varying with index $\alpha > 1$. By choice of parameters we have $\varepsilon g_\varepsilon n_\varepsilon \to \frac{\delta}{2} < (1-\delta)^2 a$ as $\varepsilon \to 0$. Thus Lemma 3.4 with $r = \frac{(1-\delta)a}{\varepsilon}$ can be used to estimate $\mathbf{P}\left(\sum_{i=1}^k |\varepsilon W_i| \ge (1-\delta)a\right)$ for $1 \le k \le n_\varepsilon - 1$. Hence for ε sufficiently small the following estimate holds uniformly for all $1 \le k < n_\varepsilon$:

$$\begin{aligned}&\mathbf{P}\Big(\sum_{i=1}^k |\varepsilon W_i| \ge (1-\delta)a\Big) \\ &\le \beta_\varepsilon^{-k}|\ln\varepsilon|^{2k} \exp\Big(-\inf\Big\{\sum_{i=1}^k f(x_i) \,:\, \sum_{i=1}^k x_i = \tfrac{(1-\delta)^2 a}{\varepsilon}, x_i \in [g_\varepsilon, \tfrac{(1-\delta)a}{\varepsilon}]\Big\}\Big).\end{aligned} \tag{105}$$

Again we invoke Potter's bound to estimate the negative of the exponential rate. Choose $\tilde\delta$ sufficiently small, such that $\alpha - \tilde\delta > 1$ and $(2\delta)^{\tilde\delta} \ge 1 - \tilde\delta|\ln(2\delta)| > 1 - \delta$. For sufficiently small ε the estimate $f(x) \ge (1-\delta) f(g_\varepsilon)(\frac{x}{g_\varepsilon})^{\alpha-\tilde\delta}$ then holds for any $x \ge g_\varepsilon$. Thus for any $1 \le k < n_\varepsilon$ we get

$$\begin{aligned}&\inf\Big\{\sum_{i=1}^k f(x_i) \,:\, \sum_{i=1}^k x_i = \tfrac{(1-\delta)^2 a}{\varepsilon}, x_i \in [g_\varepsilon, \tfrac{(1-\delta)a}{\varepsilon}]\Big\} \\ &\ge (1-\delta)\frac{f(g_\varepsilon)}{g_\varepsilon^{\alpha-\tilde\delta}} \inf\Big\{\sum_{i=1}^k x_i^{\alpha-\tilde\delta} \,:\, \sum_{i=1}^k x_i = \tfrac{(1-\delta)^2 a}{\varepsilon}, x_i > 0\Big\} \\ &= (1-\delta)\frac{f(g_\varepsilon)}{g_\varepsilon^{\alpha-\tilde\delta}} k \left(\frac{(1-\delta)^2 a}{\varepsilon k}\right)^{\alpha-\tilde\delta} \\ &\ge (1-3\alpha\delta) f(g_\varepsilon) k \left(\frac{a}{\varepsilon g_\varepsilon k}\right)^{\alpha-\tilde\delta} \\ &\ge (1-3\alpha\delta) f(g_\varepsilon) n_\varepsilon \left(\frac{a}{\varepsilon g_\varepsilon n_\varepsilon}\right)^{\alpha-\tilde\delta} \left(\frac{n_\varepsilon}{k}\right)^{\alpha-\tilde\delta-1} \\ &\ge (1-5\alpha\delta) 3^{-\alpha} |\ln\varepsilon| n_\varepsilon \left(\frac{n_\varepsilon}{k}\right)^{\alpha-\tilde\delta-1}.\end{aligned} \tag{106}$$

In the crucial step from the second to the third line of the inequality chain we use that the relation $\alpha - \tilde\delta > 1$ imposes that the function $x \mapsto x^{\alpha-\tilde\delta}$ is *convex*, and therefore the minimum

is taken for the choice $x_i = \frac{(1-\delta)^2 a}{k\varepsilon}$, $0 \le i \le k$, due to (50). The same conditions imply the following inequalities which are used in the last line of the chain. In fact, for sufficiently small ε we obtain

$$f(g_\varepsilon) \ge (1-\delta)\left(\frac{g_\varepsilon}{q_\varepsilon}\right)^\alpha f(q_\varepsilon) \ge (1-\delta)\left(\frac{\delta}{6}\right)^\alpha |\ln\varepsilon| \tag{107}$$

and

$$\left(\frac{a}{\varepsilon g_\varepsilon n_\varepsilon}\right)^{\alpha-\tilde\delta} \ge \left(\frac{2}{\delta}\right)^{-(\alpha-\tilde\delta)} \ge (1-\delta)\left(\frac{2}{\delta}\right)^\alpha. \tag{108}$$

Summarising our conclusions, we may continue the estimate in (105) by the inequality

$$\begin{aligned}\mathbf{P}\Big(\sum_{i=1}^k |\varepsilon W_i| \ge (1-\delta)a\Big) &\le \beta_\varepsilon^{-k}\exp\Big(-(1-5\alpha\delta)3^{-\alpha}|\ln\varepsilon| n_\varepsilon\Big(\frac{n_\varepsilon}{k}\Big)^{\alpha-\tilde\delta-1} + n_\varepsilon \ln|\ln\varepsilon|\Big)\\ &\le \beta_\varepsilon^{-k}\exp\Big(-(1-6\alpha\delta)3^{-\alpha}|\ln\varepsilon| n_\varepsilon\Big(\frac{n_\varepsilon}{k}\Big)^{\alpha-\tilde\delta-1}\Big),\end{aligned} \tag{109}$$

again valid for ε small enough.

It remains to include the probabilities $\mathbf{P}(N_T = k)$ for $1 \le k < n_\varepsilon$ into our estimates. For this purpose, we shall estimate $\max_{1\le k<n_\varepsilon} \mathbf{P}(N_T = k)\mathbf{P}\left(\sum_{i=1}^k |\varepsilon W_i| > (1-\delta)a\right)$. This will be done by looking separately at the cases $k \in A$ and $k \in B$, where

$$A := \{1 \le k < n_\varepsilon : (\tfrac{n_\varepsilon}{k})^{\alpha-\tilde\delta-1} > 3^\alpha\}, \quad B := \{1 \le k < n_\varepsilon : (\tfrac{n_\varepsilon}{k})^{\alpha-\tilde\delta-1} \le 3^\alpha\}. \tag{110}$$

For any $1 \le k < n_\varepsilon$ the estimate $\mathbf{P}(N_T = k) \le \mathbf{P}(\tau_1 \le T)^k \le (\beta_\varepsilon T)^k$ is valid. So we obtain for small enough ε

$$\begin{aligned}\max_{k\in A}\mathbf{P}(N_T = k)\mathbf{P}\Big(\sum_{i=1}^k |\varepsilon W_i| > (1-\delta)a\Big) &\le T^{n_\varepsilon}\exp(-(1-6\alpha\delta)|\ln\varepsilon| n_\varepsilon)\\ &\le \exp\Big(-(1-7\alpha\delta)3a\frac{|\ln\varepsilon|}{\varepsilon q_\varepsilon}\Big) < D_\varepsilon^a.\end{aligned} \tag{111}$$

Finally we consider $k \in B$. By choice of parameters and definition of B we have $\inf_{k\in B}\ln k \ge (1-\delta)|\ln\varepsilon|$ for sufficiently small ε. Hence for ε small, again by means of Stirling's formula

$$\begin{aligned}\mathbf{P}(N_T = k) \le \frac{(\beta_\varepsilon T)^k}{k!} &\le \beta_\varepsilon^k \exp(-k(\ln k - 1 - \ln T)) \le \beta_\varepsilon^k \exp(-(1-2\delta)k|\ln\varepsilon|)\\ &\le \beta_\varepsilon^k \exp\Big(-(1-2\delta)n_\varepsilon|\ln\varepsilon|\frac{k}{n_\varepsilon}\Big).\end{aligned} \tag{112}$$

This combines with our estimate for the rate of sums of big jumps to the inequality

$$\begin{aligned}&\max_{k\in B}\mathbf{P}(N_T = k)\mathbf{P}\Big(\sum_{i=1}^k |\varepsilon W_i| > (1-\delta)a\Big)\\ &\le \exp\Big(-(1-6\alpha\delta)|\ln\varepsilon| n_\varepsilon\Big[3^{-\alpha}\Big(\frac{n_\varepsilon}{k}\Big)^{\alpha-\tilde\delta-1} + \frac{k}{n_\varepsilon}\Big]\Big)\\ &\le \exp\Big(-(1-6\alpha\delta)|\ln\varepsilon| n_\varepsilon \inf_{y>0}[3^{-\alpha}y^{-(\alpha-\tilde\delta-1)} + y]\Big).\end{aligned} \tag{113}$$

It is easy to see that $\inf_{y>0}[3^{-\rho}y^{-(\rho-1)}+y] = \frac{1}{3}\rho(\rho-1)^{-(1-\frac{1}{\rho})}$ holds for any $\rho > 1$. The mapping $\rho \mapsto \frac{1}{3}\rho(\rho-1)^{-(1-\frac{1}{\rho})}$ is continuous on $(1,\infty)$. Thus $\tilde{\delta}$ can be chosen sufficiently small for the following estimate to hold

$$\begin{aligned}\inf_{y>0} 3^{-\alpha}y^{-(\alpha-\tilde{\delta}-1)}+y &\geq 3^{-\tilde{\delta}}\inf_{y>0} 3^{-(\alpha-\tilde{\delta})}y^{-(\alpha-\tilde{\delta}-1)}+y \\ &\geq (1-\delta)\inf_{y>0} 3^{-\alpha}y^{-(\alpha-1)}+y = \frac{1-\delta}{3}d_\alpha.\end{aligned} \tag{114}$$

So we finally get the inequality

$$\begin{aligned}\max_{k\in B}\mathbf{P}(N_T=k)\mathbf{P}\Big(\sum_{i=1}^{k}|\varepsilon W_i| > (1-\delta)a\Big) &\leq \exp\Big(-(1-6\alpha\delta)\frac{d_\alpha}{3}|\ln\varepsilon|n_\varepsilon\Big) \\ &\leq \exp\Big(-(1-7\alpha\delta)ad_\alpha\frac{|\ln\varepsilon|}{\varepsilon q_\varepsilon}\Big) \\ &\leq D_\varepsilon^{(1-7\alpha\delta)a}.\end{aligned} \tag{115}$$

Now combine this with our estimate on A, and take into account that we have n_ε summands of the two types. But by its choice, the factor n_ε being a power function of ε does not change the exponential asymptotics in the limit $\varepsilon \to 0$. This completes the proof of both Lemma 5.1 and Theorem 2.2. ■

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