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THE ASYMPTOTIC STABILITY OF WEAKLY PERTURBED TWO DIMENSIONAL HAMILTONIAN SYSTEMS

Ludwig Arnold
Institut für Dynamische Systeme
Universität Bremen
28334 Bremen
GERMANY
arnold@math.uni-bremen.de

Peter Imkeller
Institut für Mathematik
Humboldt-Universität zu Berlin
10099 Berlin
GERMANY
imkeller@mathematik.hu-berlin.de

N. Sri Namachchivaya
Nonlinear Systems Group
University of Illinois at Urbana-Champaign
Urbana, IL 61801
USA
sri@nsgsun.aae.uiuc.edu

ABSTRACT

The purpose of this work is to obtain an approximation for the top Lyapunov exponent, the exponential growth rate, of the response of a single-well Kramers Oscillator driven by either a multiplicative or an additive white noise process. To this end, we consider the equations of motion as dissipative and noisy perturbations of a two-dimensional Hamiltonian system. A perturbation approach is used to obtain explicit expressions for the exponent in the presence of small intensity noise and small dissipation. We show analytically that the top Lyapunov exponent is positive, and for small values of noise intensity $\sqrt{\varepsilon}$ and dissipation ε the exponent grows proportional to $\varepsilon^{\frac{1}{3}}$.

1 Introduction

No theorem has had so direct and powerful an influence upon the study of stochastic stability of noisy dynamical systems as the multiplicative ergodic theorem (MET) of Oseledec [13], which established the existence of (typically) finitely many deterministic exponential growth rates called *Lyapunov exponents*. The stability of linear stochastic systems based on MET has been well established [3,1] and the top Lyapunov exponent can be evaluated explicitly with relative ease when the noisy perturbations and dissipation are weak [2,15].

The primary concern in the analysis of nonlinear dynamical systems is the determination and prediction of steady-state or stationary motions and their corresponding stability. The challenge has been to extend the existing techniques in order to explicitly evaluate the top Lyapunov exponent of nonlinear systems with noise, and in particular additive white noise.

For example, many engineering systems under additive white noise excitations can be expressed as

$$\ddot{x}^i_t + \beta_i \dot{x}^i_t + \frac{\partial U}{\partial x^i}(x_t) = \xi_i(t), \quad i = 1, 2, ..., n, \tag{1}$$

where $\xi_i(t)$'s are stationary stochastic processes, β_i's represent the damping in each mode, and U is the potential. Under the assumptions that $\xi_i(t)$'s are uncorrelated Gaussian processes, and the ratio of the spectral density of each excitation, $\xi_i(t)$, to the corresponding damping, β_i, is the same, i.e.,

$$\gamma = \frac{\beta_i}{\kappa_{ii}} \quad \text{for all } i,$$

where $\mathbf{E}[\xi_i(t+\tau)\xi_i(t)] = 2\kappa_{ii}\delta(\tau)$, the stationary probability density of (1) can be easily written as

$$p(x, \dot{x}) = C \exp\left\{-\gamma\left[\frac{1}{2}\sum_{i=1}^{n}(\dot{x}^i)^2 + U(x)\right]\right\}$$

Such stationary probability densities exist for an even larger class of multi-dimensional nonlinear systems and there is a vast engineering literature that deals with the determination of such stationary measures (see for example, Lin and Cai [14]). However, there are no concrete results on the sign of the top Lyapunov exponents corresponding to these stationary measures. Hence, their stability is not known. The challenge therefore, is to explicitly evaluate the top Lyapunov exponents of these stationary measures. It is this need and challenge that we shall address in this paper.

Schimansky-Geier and Herzel [17] were the first to consider numerically the Lyapunov exponents of a two dimensional nonlinear system under additive noise. Their work was devoted to the effect of noise on the Kramers Oscillator

$$\ddot{x}_t + \varepsilon \dot{x}_t + U'(x_t)) = \sqrt{2\varepsilon}\,\xi(t), \tag{2}$$

where $U(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4, \quad a, b > 0$, with double-well potential. It was shown [17] that the top Lyapunov exponent is positive, i.e.,

$$\lambda(\varepsilon) > 0 \quad \text{for } \varepsilon \text{ not too large}$$

Hence, an additive noise induces an unstable stationary measure. Our task in this paper is to show analytically this remarkable observation for (2) as well for similar systems with multiplicative noise.

Kramers Oscillator with double-well potential, considered by Schimansky-Geier and Herzel [17], has multiple fixed points, one of which is connected to itself by a homoclinic orbit. The procedure presented here relies upon an implicit assumption that the instantaneous frequency of the unperturbed motion ($\varepsilon = 0$) must be non-zero or the periods of oscillations or rotations are finite. Hence, a subtle treatment is necessary in a neighborhood of the homoclinic orbit where the unperturbed orbits have arbitrarily long periods. In order to remedy this problem, two different models, one which is valid away from the homoclinic orbit, the other valid in a boundary layer about the homoclinic orbit should be introduced and it is beyond the scope of this paper. Thus, we do not consider it fruitful to attempt to make a general theory for all types of two dimensional nonlinear Hamiltonians. Rather, we restrict our development to the case for *Hamiltonians with isolated single elliptic fixed point*, i.e., a weakly perturbed oscillator with a single-well potential. However, the analysis developed in this paper could be extended with some effort to provide analogous theorems pertaining to Hamiltonians with multiple fixed points. The versatility of the method presented here, will make this method to be adopted to such situations.

Before we proceed further, we should mention in this context some well-known results pertaining to one-dimensional nonlinear stochastic systems. It has been shown that the two point motion of a one dimensional nonlinear stochastic system is unique. More precisely if a noisy one dimensional equation,

$$\dot{x}_t = f(x_t) + g(x_t)\xi(t) \tag{3}$$

has a stationary invariant measure

$$p(x) = \frac{N}{g(x)} \exp\left\{ \int^x \frac{2f(\eta)}{g^2(\eta)} d\eta \right\}, \tag{4}$$

then as in Arnold [1], the Lyapunov exponent is

$$\lambda = -2 \int_0^\infty \left[\frac{f(x)}{g(x)} \right]^2 p(x)\, dx. \tag{5}$$

The Lyapunov exponent is always negative provided $f(x) \neq 0$. Similar results are also presented by Leng et al. [11].

In the absence of dissipation and random perturbations ($\varepsilon = 0$), system (2) is integrable (Hamiltonian). Unperturbed Hamiltonian dynamics provides amazingly successful descriptions of the nonlinear dynamics and its mathematical theory [6] has evolved alongside the physical understanding, to a point of high sophistication. The underpinning of the method presented here is a separation of scales. The slowly varying coordinate is the value of the Hamiltonian and the quickly varying coordinate is the position (or angle) in the appropriate level set of the Hamiltonian. Here we present a general, effective, systematic approach to determine the asymptotic sample stability of weakly perturbed (dissipatively and stochastically) two dimensional nonlinear Hamiltonian systems. Random perturbations of Hamiltonian systems are of great interest, particularly, in the study of noisy nonlinear mechanical systems. Randomly-perturbed Hamiltonian system on $\mathbb{R}^2$ with multiple fixed points are considered by Freidlin and Wentzell [10] in the contex of stochastic averaging and by Freidlin and Wentzell [9] in the contex of large deviations techniques.

In Section 2, we state the mathematical structure of the problem. In section 3, we introduce the concept of action-angle variables [6], apply the classical results of symplectic transformation and derive the evolution of the action-angle variables. In section 4, due to the nilpotent structure of the linear variational equations, Pinsky and Wihstutz [16] re-scaling is used in the linear variational equations to derive the Furstenberg-Khasminskii formula. In sections 5 and 6 we appeal to the results of Sri Namachchivaya and Van Rossel [15] and Imkeller and Lederer [12] to evaluate the first term in the asymptotic expansion of the top Lyapunov exponent.

2 Statement of the Problem

We consider an idealized particle moving in a symmetric single well potential described by a function U defined on $\mathbb{R}$. The Hamiltonian of the system will be given by

$$H(x,y) = U(x) + \frac{y^2}{2}, \quad x,y \in \mathbb{R}.$$

and it is assumed that the Hamiltonian has *an isolated elliptic fixed point*. The purpose of this paper is to examine the asymptotic sample stability of this nonlinear system under random and dissipative perturbations. We restrict to this class of potentials from the beginning to make the calculations of the top Lyapunov exponent become less cumbersome. The particular set of global variables discussed in the subsequent sections of this paper will shed light on this restriction. Formally, we assume

$$U \geq 0, \quad U(0) = 0, \quad U(x) = U(-x), x \in \mathbb{R},$$

and $x \mapsto U(x)$ strictly increasing on $\mathbb{R}_+$. The motion of the corresponding Hamiltonian system is periodic returning to the same point $x,y \in \mathbb{R}$ in the phase space after a period $T(x,y)$. For each $x,y \in \mathbb{R}$, define the return time

$$T(x,y) =: \inf\{t > 0 : \xi_t(z) = z\} \tag{6}$$

where $\xi_t(z)$ is the Hamiltonian flow for all $(x,y) = z \in \mathbb{R}^2$. It is clear that T depends solely on $H(x,y)$ and that it is nonnegative on $\mathbb{R}^2 \setminus \{0\}$. Thus we start out with a Hamiltonian energy function with a very simple structure.

Assumption 2.1. (Hamiltonian)*: We assume that $H : \mathbb{R}^2 \to \mathbb{R}$ is $\mathbf{C}^\infty$ and nonnegative. We assume also that $H(x,y) = 0$ if and only if $x = 0, y = 0$. Secondly, $H(x,y) = H(-x,-y)$ for all $x,y \in \mathbb{R}$. Thirdly, we assume that*

$$A =: D^2 H(0)$$

is positive-definite. Finally, we assume that for each $h > 0$, the set $H^{-1}(h)$ is connected and of finite 1-dimensional Hausdorff measure.

We assume that the particle is weakly damped and weakly perturbed by a white noise process. The primary concern is the determination of the stability of the stationary invariant measures, which are the stochastic analogue of steady state solutions in nonlinear determinsic systems. The perturbations are scaled by appropriate powers of ε, ($\varepsilon << 1$), in order to obtain the effect of the damping and the noise at the same order. To this end, random perturbation of a two-dimensional Hamiltonian system, with an isolated elliptic fixed point, is precisely given by

$$\begin{aligned} dx_t &= y_t\, dt, \\ dy_t &= (-\varepsilon y_t - U'(x_t))\, dt + \sqrt{2\varepsilon}\,\sigma(x_t, y_t) \circ dW_t. \end{aligned} \tag{7}$$

Here $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is supposed to be a smooth function of sublinear growth. Equation (7) represents the random vibration of single degree of freedom mechanical systems under either parametric or additive white noise excitations. Hence, the typical examples that we consider are given by the *additive noise* case, i.e., $\sigma(x,y) = \sigma = const$, which has been studied extensively in the literature (see for example, Bolotin [8]), or by the *multiplicative noise* coupled to the displacement, i.e. $\sigma(x,y) = x$, or the velocity, i.e. $\sigma(x,y) = y$. Our aim is to obtain an asymptotic expansion of the top Lyapunov exponent of the random dynamical system described in (7) by making use of the prescribed scaling.

3 Problem Formulation in Action-Angle Coordinates

The random motions consist of fast rotations along the unperturbed trajectories of the deterministic system and slow motion across these trajectories. The nature of our system thus suggests a set of coordinates which splits the two components of motion: *action-angle coordinates*. They are commonly used in the classical perturbation theory of mechanical systems (see Arnold [6]). The *action* part is defined by the area enclosed by the level curves of H. Hence, it captures the slow component of the motion. Whereas the *angle* part describes uniform motion along the level curves, and is therefore related with the fast component.

To this end, we need to transform $H(x,y)$ by means of a canonical transformation into new variables (I, ϕ action-angle) such that the new Hamiltonian is a constant, $h(I)$ and the angle coordinate ϕ increases by 2π after each complete period $T(x,y) = T(I)$ of the motion. To introduce these variables, following Arnold [6], we work with the generating function $S(I,x)$, determined by the requirements

$$y = \frac{\partial S}{\partial x}(I,x), \quad \phi = \frac{\partial S}{\partial I}(I,x), \quad H(x, \frac{\partial S}{\partial x}(I,x)) = h(I), \tag{8}$$

$I = I(h)$ is a function of the possible values h of H. The Hamilton-Jacobi equation in (8) is solved for the generating function $S(I,x)$ by letting

$$S(I,x) = \int_{-x_0(I)}^{x} y(I,\xi)\, d\xi, \quad -x_0(I) \leq x \leq x_0(I),$$

where

$$y(I,x) = \sqrt{2(h(I) - U(x))}.$$

It is immediately obvious that $S(I,x_0(I)) = \pi I$. Hence, following Arnold [6], we introduce the transformation

$$\begin{aligned}\phi &= \frac{\partial}{\partial I}\int_{-x_0(I)}^{x}\sqrt{2(h(I)-U(\xi))}\,d\xi \\ &= h'(I)\frac{\partial}{\partial h}\int_{-x_0(I)}^{x}\sqrt{2(h-U(\xi))}\,d\xi \\ &= \omega(I)\int_{-x_0(I)}^{x}\frac{1}{\sqrt{2(h(I)-U(\xi))}}\,d\xi,\end{aligned} \tag{9}$$

and

$$y(\phi,I) = \pm\sqrt{2(h(I)-U(x(\phi,I)))}. \tag{10}$$

The main point behind the method that is developed here is to use the geometric structure of the unperturbed integrable Hamiltonian problem in order to develop an appropriate set of "coordinates" for studying the perturbed problem. Now that we have developed such symplectic coordinates, let us use (9) and (10) to give some information on the Jacobian of the transformation $(x,y)\mapsto(\phi,I)$ which is essential in deriving the perturbed equations in the new variables (ϕ,I).

Lemma 3.1. *For $I>0$ we have*

$$\frac{\partial x}{\partial\phi} = \frac{y}{\omega}, \quad \frac{\partial y}{\partial\phi} = -\frac{U'(x)}{\omega}. \tag{11}$$

Moreover,

$$\omega = U'(x)\frac{\partial x}{\partial I} + y\frac{\partial y}{\partial I}. \tag{12}$$

In particular,

$$\frac{\partial x}{\partial\phi}\frac{\partial y}{\partial I} - \frac{\partial x}{\partial I}\frac{\partial y}{\partial\phi} = 1,$$

i.e. the transformation belongs to a symplectic form.

Proof:
Straight forward. □

Lemma 3.2. *For $I>0, \phi\in]0,\pi[$ define*

$$\beta(\phi,I) = \int_{\frac{\pi}{2}}^{\phi}\Big[\frac{1}{y^2(\xi,I)} - \frac{\omega'(I)}{\omega^2(I)}\Big]\,d\xi,$$

for $\phi\in[0,\frac{\pi}{2}[$

$$\alpha_0(\phi,I) = \int_0^{\phi}\Big[\frac{U''(x(\xi,I))}{U'(x(\xi,I))^2} - \frac{\omega'(I)}{\omega^2(I)}\Big]\,d\xi,$$

and for $\phi\in]\frac{\pi}{2},\pi]$

$$\alpha_\pi(\phi,I) = \int_\pi^{\phi}\Big[\frac{U''(x(\xi,I))}{U'(x(\xi,I))^2} - \frac{\omega'(I)}{\omega^2(I)}\Big]\,d\xi.$$

Then we may write for $I>0$ and $\phi\in]-\pi,\pi[$

$$\frac{\partial(x,y)}{\partial(\phi,I)} = \begin{bmatrix}\frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial I}\\ \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial I}\end{bmatrix} = \begin{bmatrix}\frac{y}{\omega} & y\beta\\ -\frac{U'(x)}{\omega} & \frac{\omega}{y} - U'(x)\,\beta\end{bmatrix}, \tag{13}$$

for $\phi\in[0,\frac{\pi}{2}[$

$$\frac{\partial(x,y)}{\partial(\phi,I)} = \begin{bmatrix}\frac{y}{\omega} & \frac{\omega}{U'(x)} + y\alpha_0\\ -\frac{U'(x)}{\omega} & -U'(x)\,\alpha_0\end{bmatrix}, \tag{14}$$

and for $\phi\in]\frac{\pi}{2},\pi]$

$$\frac{\partial(x,y)}{\partial(\phi,I)} = \begin{bmatrix}\frac{y}{\omega} & \frac{\omega}{U'(x)} + y\alpha_\pi\\ -\frac{U'(x)}{\omega} & -U'(x)\,\alpha_\pi\end{bmatrix}. \tag{15}$$

Moreover, for $I>0, \phi\in[-\pi,\pi]$ we have

$$\begin{aligned}\frac{\partial x}{\partial\phi}(-\phi,I) &= -\frac{\partial x}{\partial\phi}(\phi,I), \frac{\partial y}{\partial\phi}(-\phi,I) = \frac{\partial y}{\partial\phi}(\phi,I),\\ \frac{\partial x}{\partial I}(-\phi,I) &= \frac{\partial x}{\partial I}(\phi,I), \frac{\partial y}{\partial I}(-\phi,I) = -\frac{\partial y}{\partial I}(\phi,I).\end{aligned} \tag{16}$$

Proof:
Let us first treat the case $-x_0(I)\le x<0, y\ge 0$ which corresponds to $I>0, \phi\in[0,\frac{\pi}{2}[$. Integrating (9) by parts and then differentiating with respect to I, we obtain

$$-\phi\frac{\omega'}{\omega^2} = \frac{1}{y}\Big[-\frac{\omega}{U'(x)} + \frac{\partial x}{\partial I}\Big] - \omega\int_{-x_0}^{x}\frac{U''(\xi)}{U'(\xi)^2}\frac{1}{\sqrt{2(h(I)-U(\xi))}}\,d\xi.$$

Solving this equation for $\frac{\partial x}{\partial I}$ and noting that by Lemma 3.1 we have $d\xi = \frac{\partial\xi}{\partial\phi}\,d\phi = \frac{y}{\omega}\,d\phi$ yields the requested formula for $\frac{\partial x}{\partial I}$.

In case $-x_0(I) < x < x_0(I), y \geq 0$ corresponding to $I > 0, \phi \in]0,\pi[$ symmetry allows us to write the alternative of (9)

$$\phi = \frac{\pi}{2} + \omega \int_0^{x(\phi,I)} \frac{1}{\sqrt{2(h(I)-U(\xi))}}\, d\xi.$$

Now differentiate with respect to I to get

$$\left(\frac{\pi}{2} - \phi\right)\frac{\omega'}{\omega^2} = \frac{1}{y}\frac{\partial x}{\partial I} - \omega \int_0^{x} \frac{1}{\sqrt{2(h(I)-U(\xi))}^3}\, d\xi.$$

This equation is again solved for $\frac{\partial x}{\partial I}$, and the integration in x is replaced by an integration in ϕ. This gives (13).

The case $0 < x < x_0(I), y \geq 0$ is treated as the first case. Finally, (16) is obvious from the definitions. □

The symplectic property of our coordinate change immediately allows to give formulae for the inverse of the Jacobian. This is an additional advantage of using canonical transformation.

Lemma 3.3. *We have for $I > 0$ and $\phi \in]-\pi,\pi[$*

$$\frac{\partial(\phi,I)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \\ \frac{\partial I}{\partial x} & \frac{\partial I}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\omega}{y} - U'(x)\,\beta & -y\beta \\ \frac{U'(x)}{\omega} & \frac{y}{\omega} \end{bmatrix}, \tag{17}$$

for $\phi \in [0,\frac{\pi}{2}[$

$$\frac{\partial(\phi,I)}{\partial(x,y)} = \begin{bmatrix} -U'(x)\,\alpha_0 - \frac{\omega}{U'(x)} & -y\alpha_0 \\ \frac{U'(x)}{\omega} & \frac{y}{\omega} \end{bmatrix}, \tag{18}$$

and for $\phi \in]\frac{\pi}{2},\pi]$

$$\frac{\partial(\phi,I)}{\partial(x,y)} = \begin{bmatrix} -U'(x)\,\alpha_\pi - \frac{\omega}{U'(x)} & -y\alpha_\pi \\ \frac{U'(x)}{\omega} & \frac{y}{\omega} \end{bmatrix}, \tag{19}$$

Moreover, for $(x,y) \neq (0,0)$ we have

$$\begin{aligned} \frac{\partial\phi}{\partial x}(x,-y) &= -\frac{\partial\phi}{\partial x}(x,y), \;\; \frac{\partial I}{\partial y}(x,-y) = \frac{\partial I}{\partial x}(x,y), \\ \frac{\partial\phi}{\partial y}(x,-y) &= \frac{\partial\phi}{\partial y}(x,y), \;\; \frac{\partial I}{\partial y}(x,-y) = -\frac{\partial I}{\partial y}(x,y). \end{aligned} \tag{20}$$

Proof:
This follows directly from Lemma 3.2 and the fact that the Jacobian has determinant 1 due to the symplectic character of the transformation. □

We are now in a position to describe our basic equations (7) in action-angle variables. Differentiating the action-angle variables and making use of Lemma 3.2 and Lemma 3.3 yields,

$$\begin{aligned} dI_t =& \frac{U'(x_t)}{\omega(I_t)}\, y_t\, dt + \frac{y_t}{\omega(I_t)}[-\varepsilon y_t - U'(x_t)]\, dt \\ &+ \sqrt{2\varepsilon}\, \frac{y_t}{\omega(I_t)}\, \sigma(x_t,y_t) \circ dW_t \\ \stackrel{\text{def}}{=}& \varepsilon\, f_I(\phi_t,I_t)\, dt + \sqrt{2\varepsilon} g_I(\phi_t,I_t) \circ dW_t, \end{aligned} \tag{21}$$

$$\begin{aligned} d\phi_t =& \frac{\partial y_t}{\partial I}\, y_t\, dt - \frac{\partial x_t}{\partial I}[-\varepsilon y_t - U'(x_t)]\, dt \\ &- \sqrt{2\varepsilon}\, \frac{\partial x_t}{\partial I}\, \sigma(x_t,y_t) \circ dW_t \\ \stackrel{\text{def}}{=}& \omega(I_t)\, dt + \varepsilon\, f_\phi(\phi_t,I_t)\, dt + \sqrt{2\varepsilon} g_\phi(\phi_t,I_t) \circ dW_t, \end{aligned} \tag{22}$$

where the vector fields appearing in (21) and (22) are renamed as

$$\begin{aligned} f_I(\phi,I) &= -\frac{y^2(\phi,I)}{\omega(I)}\;,\; f_\phi(\phi,I) = y\,\frac{\partial x}{\partial I}(\phi,I), \\ g_I(\phi,I) &= \frac{(y\,\sigma(x,y))\,(\phi,I)}{\omega(I)}\;,\; g_\phi(\phi,I) = -\left(\frac{\partial x}{\partial I}\,\sigma(x,y)\right)(\phi,I), \end{aligned}$$

$I \geq 0, \phi \in [-\pi,\pi]$, to simplify notation in the decomposition of the infinitesimal generator in the following section. For the linearization of our system we need the Jacobian of the vector fields. For convenience we change the order of ϕ and I and the Jacobian is given by

$$A^f = \begin{bmatrix} A^f_{11} & A^f_{12} \\ A^f_{21} & A^f_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_I}{\partial I} & \frac{\partial f_I}{\partial \phi} \\ \frac{\partial f_\phi}{\partial I} & \frac{\partial f_\phi}{\partial \phi} \end{bmatrix}, \tag{23}$$

$$A^g = \begin{bmatrix} A^g_{11} & A^g_{12} \\ A^g_{21} & A^g_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial g_I}{\partial I} & \frac{\partial g_I}{\partial \phi} \\ \frac{\partial g_\phi}{\partial I} & \frac{\partial g_\phi}{\partial \phi} \end{bmatrix}. \tag{24}$$

Calculations using the preceding Lemmas yield the formulae for each element of the above matrices A^f and A^g. Furstenberg-Khasminskii formula for the top Lyapunov exponent is derived in the next section.

4 Furstenberg-Khasminskii Formula

Following the notation of the preceding section we shall now consider the stochastic system in action-angle variables given by in (21) and (22). Our aim is to obtain an asymptotic expansion of the top Lyapunov exponent of the random dynamical system described by (21) and (22). For this purpose we have to study its linearization. Let us denote the linearized variables by (X,Y) and keeping track of the notation introduced in the preceding section, we have

$$\begin{bmatrix} dX_t \\ dY_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \omega'(I_t) & 0 \end{bmatrix} \begin{bmatrix} X_t \\ Y_t \end{bmatrix} dt + \varepsilon A^f(\phi_t, I_t) \begin{bmatrix} X_t \\ Y_t \end{bmatrix} dt + \sqrt{2\varepsilon} A^g(\phi_t, I_t) \begin{bmatrix} X_t \\ Y_t \end{bmatrix} \circ dW_t. \tag{25}$$

Because of the special structure of the zeroth order terms in equations (21) and (22), the linear variational equations (25) naturally exhibit a nilpotent structure. This nilpotent form is responsible for the main results on the asymptotic expansion of its top Lyapunov exponent to be developed in this and the following sections. Since the invariant measure of the angular part of the linearization (nilpotent) trivializes in one direction, we appeal to the results of Pinsky and Wihstutz [16]. Accordingly, the variables (X,Y) are rescaled with a certain fractional power of ε, i.e.,

$$X = \varepsilon^{\frac{1}{3}} u, \quad Y = v,$$

in order to see the correct asymptotics. In the rescaled variables we obtain the equation

$$\begin{bmatrix} du_t \\ dv_t \end{bmatrix} = \begin{bmatrix} \varepsilon A^f_{11} & \varepsilon^{\frac{2}{3}} A^f_{12} \\ \varepsilon^{\frac{1}{3}} \omega' + \varepsilon^{\frac{4}{3}} A^f_{21} & \varepsilon A^f_{22} \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix} dt + \sqrt{2} \begin{bmatrix} \varepsilon^{\frac{1}{2}} A^g_{11} & \varepsilon^{\frac{1}{6}} A^g_{12} \\ \varepsilon^{\frac{5}{6}} \omega' + \varepsilon^{\frac{1}{2}} A^g_{21} & \varepsilon A^g_{22} \end{bmatrix} \begin{bmatrix} u_t \\ v_t \end{bmatrix} \circ dW_t.$$

We next apply the Khasminskii transformation so that the above linear equation is decomposed into radial and angular part. This provides the most convenient setting for the description of the top Lyapunov exponent by means of the so-called Furstenberg-Khasminskii formula. Write

$$u = r\cos\theta, \quad v = r\sin\theta.$$

Then the angular component described by the process $\theta_t, t \in \mathbb{R}$, satisfies the stochastic differential equation

$$d\theta_t = h^\theta_0(\phi_t, I_t, \theta_t)\, dt + h^\theta_1(\phi_t, I_t, \theta_t) \circ dW_t, \tag{26}$$

where for $I \geq 0, \phi \in [-\pi, \pi], \theta \in [0, \pi]$ we have

$$\begin{aligned} h^\theta_0(\phi, I, \theta) =& \varepsilon^{\frac{1}{3}} \omega'(I) \cos^2\theta - \varepsilon^{\frac{2}{3}} A^f_{12}(\phi, I) \sin^2\theta \\ &+ \varepsilon (A^f_{22} - A^f_{11})(\phi, I) \sin\theta\cos\theta + \varepsilon^{\frac{4}{3}} A^f_{21}(\phi, I) \cos^2\theta, \\ h^\theta_1(\phi, I, \theta) =& \sqrt{2}\, [-\varepsilon^{\frac{1}{6}} A^g_{12}(\phi, I) \sin^2\theta \\ &+ \varepsilon^{\frac{1}{2}} (A^g_{22} - A^g_{11})(\phi, I) \sin\theta\cos\theta + \varepsilon^{\frac{5}{6}} A^g_{21}(\phi, I) \cos^2\theta]. \end{aligned}$$

For the rest of this section we shall be concerned with a calculation of the scaled decomposition of the infinitesimal generator of our 3-dimensional system given by (21) and (22) and (26) as well as the functional of the radial part appearing in the representation of Lyapunov exponents in formulas of the Furstenberg-Khasminskii type. .

Appropriately adding the drift and the diffusion parts, finally yields the infinitesimal generator L^ε of our system (21) and (22) and (26) as

Theorem 4.1. *Define*

$$L_0 = \omega \frac{\partial}{\partial\phi}, \tag{27}$$

$$L_1 = [\omega' \cos^2\theta + 2(A^g_{12})^2 \sin^3\theta\cos\theta] \frac{\partial}{\partial\theta} + [(A^g_{12})^2 \sin^4\theta] \frac{\partial^2}{\partial\theta^2}, \tag{28}$$

$$\begin{aligned} L_2 =& -[(A^f_{12} + g_I A^g_{121} + g_\phi A^g_{122}) \sin^2\theta \\ &+ A^g_{12}(A^g_{22} - A^g_{11})(3\sin^2\theta\cos^2\theta - \sin^4\theta)] \frac{\partial}{\partial\theta} \\ &- [2A^g_{12}(A^g_{22} - A^g_{11}) \sin^3\theta\cos\theta] \frac{\partial^2}{\partial\theta^2} \\ &- [2g_I A^g_{12} \sin^2\theta] \frac{\partial^2}{\partial I \partial\theta} - [2g_\phi A^g_{12} \sin^2\theta] \frac{\partial^2}{\partial\phi\partial\theta}, \end{aligned} \tag{29}$$

$$\begin{aligned} L_3 =& [f_I + g_I A^g_{11} + g_\phi A^g_{12}] \frac{\partial}{\partial I} + [f_\phi + g_I A^g_{21} + g_\phi A^g_{22}] \frac{\partial}{\partial\phi} \\ &+ [(A^f_{22} - A^f_{11} + g_I(A^g_{221} - A^g_{111}) \\ &+ g_\phi(A^g_{222} - A^g_{112})) \sin\theta\cos\theta + ((A^g_{22} - A^g_{11})^2 \\ &- 2A^g_{12}A^g_{21})(\sin\theta\cos^3\theta - \sin^3\theta\cos\theta)] \frac{\partial}{\partial\theta} \\ &+ g_I^2 \frac{\partial^2}{\partial I^2} + g_\phi^2 \frac{\partial^2}{\partial\phi^2} + 2g_I g_\phi \frac{\partial^2}{\partial I \partial\phi} \\ &+ [((A^g_{22} - A^g_{11})^2 - 2A^g_{12}A^g_{21}) \sin^2\theta\cos^2\theta] \frac{\partial^2}{\partial\theta^2} \end{aligned} \tag{30}$$

$$+[2g_I(A^g_{22}-A^g_{11})\sin\theta\cos\theta]\frac{\partial^2}{\partial I\partial\theta}$$
$$+[2g_\phi(A^g_{22}-A^g_{11})\sin\theta\cos\theta]\frac{\partial^2}{\partial\phi\partial\theta},$$
$$L_4=[A^f_{21}+g_IA^g_{211}+g_\phi A^g_{212})\cos^2\theta \tag{31}$$
$$+A^g_{21}(A^g_{22}-A^g_{11})(\cos^4\theta-3\sin^2\theta\cos^2\theta]\frac{\partial}{\partial\theta}$$
$$+\left[2A^g_{21}(A^g_{22}-A^g_{11})\sin\theta\cos^3\theta\right]\frac{\partial^2}{\partial\theta^2}$$
$$+[2g_IA^g_{21}\cos^2\theta]\frac{\partial^2}{\partial I\partial\theta}+[2g_\phi A^g_{21}\cos^2\theta]\frac{\partial^2}{\partial\phi\partial\theta},$$
$$L_5=[2\,(A^g_{21})^2\,\sin\theta\cos^3\theta]\frac{\partial}{\partial\theta}+[(A^g_{21})^2\cos^4\theta]\frac{\partial^2}{\partial\theta^2}. \tag{32}$$

Then we have

$$L^\varepsilon=L_0+\varepsilon^{\frac{1}{3}}L_1+\varepsilon^{\frac{2}{3}}L_2+\varepsilon L_3+\varepsilon^{\frac{4}{3}}L_4+\varepsilon^{\frac{5}{3}}L_5.$$

To represent Lyapunov exponents, we shall make use of a formula of Furstenberg-Khasminskii. In this formula, the following functional of the radial part of the linearization has to be integrated with the invariant measure of our system. Due to the regularity properties of our vector fields, we know that there exists an invariant density p_ε. In this case, the formula of Furstenberg-Khasminskii (see Arnold [1]) states that the leading Lyapunov exponent λ_ε of our system satisfies

$$\lambda_\varepsilon=\int_{[-\pi,\pi]\times\mathbb{R}_+\times[0,\pi]}Q^\varepsilon(\phi,I,\theta)\;p_\varepsilon(\phi,I,\theta)\;d\phi\,dI\,d\theta.$$

As for the infinitesimal generator, our asymptotic analysis requires that we decompose Q^ε into fractional powers of $\varepsilon^{\frac{1}{3}}$. Similar calculations as for the generator yield

Theorem 4.2. *Define*

$$Q_1(.,\theta)=\omega'\sin\theta\cos\theta-(A^g_{12})^2\,\sin^2\theta(\cos^2\theta-\sin^2\theta), \tag{33}$$
$$Q_2(.,\theta)=A^f_{12}\sin\theta\cos\theta+(g_IA^g_{121}+g_\phi A^g_{122})\sin\theta\cos\theta \tag{34}$$
$$+A^g_{12}(A^g_{22}-A^g_{11})\,\sin\theta\cos\theta(\cos^2\theta-3\sin^2\theta),$$
$$Q_3(.,\theta)=A^f_{11}\cos^2\theta+A^f_{22}\sin^2\theta \tag{35}$$
$$+(A^g_{22}-A^g_{11})^2\,2\sin^2\theta\cos^2\theta+A^g_{12}A^g_{21}(\cos^2\theta-\sin^2\theta)^2$$
$$+(g_IA^g_{111}+g_\phi A^g_{112})\cos^2\theta+(g_IA^g_{221}+g_\phi A^g_{222})\sin^2\theta,$$
$$Q_4(.,\theta)=A^f_{21}\sin\theta\cos\theta \tag{36}$$
$$+A^g_{21}(A^g_{22}-A^g_{11})\,\sin\theta\cos\theta(3\cos^2\theta-\sin^2\theta)$$
$$+(g_IA^g_{211}+g_\phi A^g_{212})\sin\theta\cos\theta,$$
$$Q_5(.,\theta)=(A^g_{21})^2\,\cos^2\theta(\cos^2\theta-\sin^2\theta). \tag{37}$$

Then we have

$$Q^\varepsilon=\varepsilon^{\frac{1}{3}}Q_1+\varepsilon^{\frac{2}{3}}Q_2+\varepsilon Q_3+\varepsilon^{\frac{4}{3}}Q_4+\varepsilon^{\frac{5}{3}}Q_5.$$

5 Asymptotic Expansion

We construct a formal expansion of the invariant measure, i.e.,

$$p_\varepsilon=p_0+\varepsilon^{\frac{1}{3}}p_1+\varepsilon^{\frac{2}{3}}p_2+\cdots+\varepsilon^{\frac{N}{3}}p_N+\cdots$$

Substituting this expansion and the expansion for L^ε into the Fokker-Planck equation yields the following sequence of Poisson equations to be solved for p_0, p_1, p_2, ... :

$$\begin{aligned}L_0^*p_0&=0\\L_0^*p_1&=-L_1^*p_0\\L_0^*p_2&=-L_1^*p_1-L_2^*p_0\\L_0^*p_3&=-L_1^*p_2-L_2^*p_1-L_3^*p_0\\&\vdots\end{aligned}$$

This yields the following expression for the maximal Lyapunov exponent:

$$\lambda^\varepsilon=\varepsilon^{\frac{1}{3}}\langle Q_1,p_0\rangle+\varepsilon^{\frac{2}{3}}\left[\langle Q_2,p_0\rangle+\langle Q_1,p_1\rangle\right]+\cdots$$

As in [15], a proof that this expansion is, in fact, asymptotic begins with the construction of the *adjoint problem*

$$L^\varepsilon f_\varepsilon=Q_\varepsilon-\Lambda_\varepsilon \tag{38}$$

with $Q^\varepsilon, L^\varepsilon$ as defined above and

$$\begin{aligned}f^\varepsilon&=f_0+\varepsilon^{\frac{1}{3}}f_1+\varepsilon^{\frac{2}{3}}f_2+\varepsilon f_3+\cdots+\varepsilon^{\frac{N}{3}}f_N,\\\Lambda^\varepsilon&=\Lambda_0+\varepsilon^{\frac{1}{3}}\Lambda_1+\varepsilon^{\frac{2}{3}}\Lambda_2+\varepsilon\Lambda_3+\cdots+\varepsilon^{\frac{N}{3}}\Lambda_N.\end{aligned}$$

Contrary to the usual form, we allow $\Lambda^\varepsilon, \Lambda_i, i\geq 0$ to be functions of I alone. By using Theorems 4.1 and 4.2, and identifying terms in the corresponding expansion following from (38) then produces a set of Poisson-Type equations. Hence, Λ_i's are chosen so that the sequence of equations

$$\begin{aligned} L_0 f_0 &= -\Lambda_0 \qquad (39)\\ L_0 f_1 &= Q_1 - \Lambda_1 - L_1 f_0 \\ L_0 f_2 &= Q_2 - \Lambda_2 - L_1 f_1 - L_2 f_0 \\ &\vdots \\ L_0 f_N &= -\Lambda_N - \sum_{i=1}^{i=5} L_i f_{N-i} \end{aligned}$$

are solvable. Next we define the truncated density $\tilde{p}^\varepsilon = p_0 + \varepsilon^{\frac{1}{3}} p_1 + \varepsilon^{\frac{2}{3}} p_2 + \cdots + \varepsilon^{\frac{N}{3}} p_N$ and assume $\nu(I)$ as $I-$marginal of both p_ε and $\tilde{p}^\varepsilon$. Then, the error $\langle Q_\varepsilon, p_\varepsilon\rangle - \langle Q_\varepsilon, \tilde{p}^\varepsilon\rangle$ introduced by truncating λ^ε at an arbitrary order $N \geq 0$ can be evaluated as in [15]. Suppose that the functions $p_0, p_1, \cdots, p_N$ and $f_0, f_1, \cdots, f_N$ are constructed such that all inner products in the expressions for the error are well defined and bounded, then it it can be shown as in [15] that the expansion for a fixed $N \geq 0$ is a valid asymptotic expansion. In the subsequent section, we compute the leading term

$$\lambda_1 = \langle Q_1, p_0\rangle \qquad (40)$$

along with the estimate of the remainder term in the asymptotic expansion of the top Lyapunov exponent.

6 Calculation of the First Term λ_1

In this section we shall compute the leading terms in the asymptotic expansion of the top Lyapunov exponent of our system, based on its representation in the Furstenberg-Khasminskii formula. Due to nondegeneracy, the invariant measure of the three dimensional system has a density p^ε which is the unique lift of the density $\nu(I)$ of the $I-$motion. The latter does not depend on ε as follows immediately from Theorem 4.1. The density $\nu(I)$ is given as the solution of the adjoint equation

$$-\frac{d}{dI}[(\overline{f_I} + \overline{g_I A_{11}^g} + \overline{g_\phi A_{12}^g})\nu] + \frac{d^2}{dI^2}[\overline{g_I^2}\nu] = 0, \qquad (41)$$

where for convenience the average of functions k over $\phi \in [-\pi,\pi]$ is denoted by $\overline{k}$. We can easily calculate $\nu(I)$ for the three cases we are mostly interested in, i.e., $\sigma = const$ or $\sigma(x,y) = x$ or y, for $x,y \in \mathbb{R}$.

Lemma 6.1. *Let $c \geq 0$ be given such that*

$$c_1 = \int_0^\infty \exp\left(-\int_c^I \omega(J)\left[\frac{\overline{y^2}(J)}{\overline{y^2\sigma^2}(J)} + \frac{\overline{y\sigma\sigma_y}(J)}{\overline{y^2\sigma^2}(J)}\right] dJ\right) dI < \infty.$$

Then

$$\nu(I) = c_1 \exp\left(-\int_c^I \omega(J)\left[\frac{\overline{y^2}(J)}{\overline{y^2\sigma^2}(J)} + \frac{\overline{y\sigma\sigma_y}(J)}{\overline{y^2\sigma^2}(J)}\right] dJ\right), \qquad (42)$$

gives the marginal density in I of p^ε. In particular, if $\sigma = constant$ (i.e., additive noise), we have

$$\nu(I) = c_1 \exp\left\{-\frac{h(I)}{\sigma^2}\right\}, \quad I \geq 0, \qquad (43)$$

Proof:
Solution of (41). □

Since for the convergence of our algorithm the following condition,

$$(\mathbf{F}) \quad \omega'(I) > 0 \quad \text{for a.e. } I \geq 0,$$

is important, we shall make this general assumption throught out this paper.

For reasons which will become clear, in the computation of the leading term in the asymptotic expansion of the top Lyapunov exponent, we shall solve for a density $p_0(I,\theta)$ which satisfies both $L_0^* p_0 = 0$, the zeroth order term in the expansion of the Fokker-Planck equation, and $\overline{L_1}^* p_0 = 0$, the solvability of the first order term in the expansion, i.e.,

$$L_0^* p_1 = -L_1^* p_0.$$

Proposition 6.1. *Denote $a(I) = \overline{(A_{12}^g)^2}(I)$, $\alpha(I) = \frac{\omega'(I)}{3a(I)}$, and*

$$c(I) = \left(\frac{\omega'(I)}{a(I)}\right)^{\frac{2}{3}}\left(\frac{3}{2}\right)^{\frac{5}{6}} \frac{1}{\Gamma(\frac{1}{6})\Gamma(\frac{1}{2})}.$$

Let

$$p_0(I,\theta) = c(I)\frac{1}{\sin^2\theta} \times \int_\theta^\pi \exp(-\alpha(I)[\cot^3\theta - \cot^3 y])\frac{1}{\sin^2 y}dy \cdot \nu(I). \qquad (44)$$

Then we have

$$L_0^* p_0 = 0, \quad \overline{L_1}^* p_0 = 0,$$

and p_0 possesses ν as $I-$marginal.

Proof:
Proof is given in Arnold *et al.* [5]. □

We now start our asymptotic analysis with Ansatz of *adjoint expansion* (38). In order to obtain the first term in the asymptotic expansion of the top Lyapunov exponent, the first three of (39) have to be analyzed carefully in the sequel. They are given by

$$L_0 f_0 = -\Lambda_0, \tag{45}$$
$$L_0 f_1 + L_1 f_0 = Q_1 - \Lambda_1, \tag{46}$$
$$L_0 f_2 + L_1 f_1 + L_2 f_0 = Q_2 - \Lambda_2. \tag{47}$$

We first obtain

$$\begin{aligned} \int \Lambda_0 p_0 d(\phi, I, \theta) &= -\int L_0 f_0 \, p_0 d(\phi, I, \theta) \\ &= \int f_0 L_0^* p_0 \, d(\phi, I, \theta) = 0, \end{aligned} \tag{48}$$

since $L_0^* p_0 = 0$. This expresses the fact that the zeroth term λ_0 in the development of λ^ε vanishes. Moreover, we have

$$L_0 f_0 = \omega \frac{\partial}{\partial \phi} f_0 = -\Lambda_0,$$

hence for $I \geq 0, \phi \in [-\pi, \pi], \theta \in [0, \pi]$

$$f_0(\phi, I, \theta) = -\Lambda_0(I)\, \phi + g(I, \theta).$$

But by periodicity in ϕ, this in turn implies that

$$\Lambda_0 = 0. \tag{49}$$

Hence f_0 is just a function of I and θ. Let us next use this knowledge to analyze (46). Since $L_0^* p_0 = 0 = \overline{L_1}^* p_0$ we get

$$\begin{aligned} 0 &= -\int f_1 \, L_0^* p_0 \, d(\phi, I, \theta) \\ &= \int L_0 f_1 \, p_0 \, d(\phi, I, \theta) \\ &= \int [Q_1 - \Lambda_1 - L_1 f_0] \, p_0 \, d(\phi, I, \theta) \\ &= \int [Q_1 - \Lambda_1] \, p_0 \, d(\phi, I, \theta) - \int f_0 \overline{L_1}^* p_0 \, d(I, \theta) \\ &= \int [Q_1 - \Lambda_1] \, p_0 \, d(\phi, I, \theta). \end{aligned} \tag{50}$$

Equation (50) gives us the leading term in the development of the top Lyapunov exponent of our system. It can also be interpreted as the solvability condition for (46).

Theorem 6.1. *We have*

$$\lambda_1 = (\frac{3}{2})^{\frac{1}{3}} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})} \int_0^\infty \omega'(I)^{\frac{2}{3}} \, a(I)^{\frac{1}{3}} \, \nu(I) \, dI. \tag{51}$$

In particular, $\lambda_1 > 0$.

Proof:
Making use of theorem 4.2, and the idea (given in Imkeller and Lederer [12]) of splitting off $\overline{Q_1}$ into a part which lies in the range of $\overline{L_1}$, we prove the theorem. The complete proof is given in Arnold *et al.* [5]. □

The difficult part of these calculations is to show that the expansion is, in fact, asymptotic. So that the computational algorithim that is developed here is indeed convergent. For this, we need the estimation of the remainder terms in our asymptotic expansion, i.e., we need some more information on f's. The proof that such an *algorithm of computation is convergent* are presented in Arnold *et al.* [5].

7 Conclusions

In this paper we extend the work by Arnold and Imkeller [4] on the Kramers oscillator. To this end, we made use of the classical results on action-angle variables [6], and more recent results on Lyapunov exponents by Arnold, Papanicolaou and Wihstutz [2], Pinsky and Wihstutz [16], Sri Namachchivaya and Van Rossel [15] and Imkeller and Lederer [12]. An asymptotic expansion for the maximal Lyapunov exponent, the exponential growth rate of the response of single-well Kramers Oscillator driven by either an additive or multiplicative white noise process was constructed. However, only the first term of the asymptotic expansion was analytically evaluated. Based on this, it was shown that the top Lyapunov exponent is positive, and for small values of noise intensity $\sqrt{\varepsilon}$ and dissipation ε the exponent grows proportional to $\varepsilon^{\frac{1}{3}}$. Similar results are also reported by Baxendale and Goukasian [7] for the multiplicative case, where calculations were done using the coordinates suggested by Sowers [18]. *Due to the page limitation of these proceedings*, we only presented the main results, and the proofs of the main theorem and the fact that such an *algorithm of computation is convergent* are presented in Arnold *et al.* [5].

In closing, it seems appropriate to make the following remarks regarding the implications of the positive top Lyapunov exponent of the stationary measure for the Kramers Oscillator. Since the corresponding Markov process $(x_t, \dot{x}_t)$ generated by (2) (so-called one-point motion of the Kramers Oscillator) is positive recurrent, the stationary measure can be viewed as the occupation measure, i.e., the proportion of time spent by a typical solution of (2) in the volume element $dx\,dy$. The top Lyapunov exponents which deal with stability on the other hand, is determined by the behavior of two neighboring orbits or the two point motion of the Kramers Oscillator. In this context, the positivity of the top Lyapunov exponent is remarkable, because it implies while for each initial condition the solution trajectory asymptotically approach the volume element in the state space giving raise to a nontrivial stationary measure, the distance between any two initial conditions will grow at an exponential fast rate. Furthermore, as the growth of the square volume under the solution flow is determined by the sum of the two Lyapunov exponents which is $-\varepsilon$ and thus negative. Hence, as the t goes to ∞ the original square volume under the solution flow will shrink, but will be continuously stretched in one direction (and folded in a complicated manner).

In addition, a positive Lapunov exponent is also an indication of the fact that via Pesin's entropy formula, the system under the stationary measure has positive entropy.

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