Additional utility of insiders with imperfect dynamical information

José M. Corcuera
Universitat de Barcelona, Facultat de Matemàtiques, Gran Via de les Corts
Catalanes 585, 08007 Barcelona, Spain

Peter Imkeller
Humboldt-Universität zu Berlin, Institut für Mathematik,
Unter den Linden 6, 10999 Berlin, Germany

Arturo Kohatsu-Higa
Universitat Pompeu Fabra, Department of Economics
and Business, Ramón Trias Fargas 25-27, 08005 Barcelona, Spain

David Nualart
Universitat de Barcelona, Facultat de Matemàtiques, Gran Via de les Corts
Catalanes 585, 08007 Barcelona, Spain

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Abstract

In this paper we consider a market driven by a Wiener process where there is an insider and a regular trader. The insider has privileged information which has been deformed by an independent noise vanishing as the revelation time approaches. At this time, the information of every trader is the same.

We obtain the semimartingale decomposition of the original Wiener process under dynamical enlargement of the filtration, and we prove that if the rate at which the additional noise in the insider's information vanishes is slow enough then there is no arbitrage and the additional utility of the insider is finite.

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1 Introduction

Financial markets inherently have asymmetry of information. That is, there are different types of traders whose behavior is induced by different types of information they possess (or not). In the classical setting for financial markets one assumes that all traders share the same information which allows the study of non-arbitrage and equilibrium conditions. In this article we are interested in finding settings where a continuous time financial market can accept continuous differences in information between traders. These differences can disappear at certain revelation times. Our aim in this paper is to show that one can construct such markets and still achieve non-arbitrage under some restrictions on the trading strategies of the informed agents.

One of the aspects of this complex problem is to study the effects of changes in information of different agents. One can find literature in mathematical economics as well as in stochastic process theory dealing with problems of this nature. In the latter area the most frequently used techniques are based on the enlargement of filtrations. A basic reference on this topic is the series of papers in the Séminaire de Calcul Stochastique (1982/83) of the University Paris VI published in 1985.

In the past few years we have seen expanding interest in this area. Articles where the enlargement of filtrations technique is applied to portfolio optimization of an insider are Karatzas and Pikovsky (1996), Imkeller (1996, 1997), Amendinger et al. (1998), Grorud and Pontier (1998) and Imkeller et al. (2001). The setup in most of these works is to consider two small agents who optimize their logarithmic utility. One considers the difference of utility between these two agents supposing that one of them is better informed than the other. One of the important conclusions of this body of work is that if the information is generated by the initial knowledge of the value of a random variable then the additional utility is the relative entropy of this random variable with respect to the original probability measure, see for instance remark 2.5 in Karatzas and Pikovsky (1996).

In most of these results the extra information of the insider can be classified into two types. In the first, the insider has direct access to the price of the underlying at some time in the future T. In this case the utility difference is infinite and there is arbitrage which is realized at this future time T. In the second, the insider knows the price of the underlying up to a perturbation by a remaining independent noise which is constant through the time interval $[0,T]$ (see Karatzas and Pikovsky (1996) and Amendinger et al. (1998)). In this case the additional utility is finite. Nevertheless we encounter the somewhat odd situation that the level of information is the same even at the revelation time T.

In this article we propose to study situations in which the insider knows a functional of the underlying deformed by an independent noise process which tends to zero as T approches. This has obvious relations with the actual evolution of information in markets. Mathematically, this shows the need to develop a *dynamical enlargement of filtrations*. In our case this will follow from a projection of the decomposition obtained through Jacod's theorem.

Other approaches to insider's effects in financial markets are made by Kyle (1985), and Back (1992) in the context of an equilibrium theory, with different kind of traders acting in the market, and by Baudoin (2001), where the true model of stock prices is partially observed and where the insider's extra information consists of the knowledge about the law of some functional of the future prices of stocks. Even though, in this latter case, enlargement of filtrations techniques are not relevant due to the type of additional information being considered, the author establishes the relationship, via Girsanov's theorem, with the case where the extra information is the value of a functional of the future prices of stocks.

We treat different examples. The results in all our examples state that if the rate at which the blurring noise disappears is suficiently slow then there will be a finite additional logarithmic utility and no arbitrage. Therefore this allows the construction of a stable market where insiders and regular traders coexist.

This situation corresponds to a more natural situation where the information retained by the insider is improving as times evolves. The minimum rate at which the noise has to go to zero in order to achieve absence of arbitrage could also be interpreted as the necessary noise that has to be generated by noise traders, see Back (1992), in order not to reveal to "market makers" the information the "true" insider possesses.

For simplicity, we restrict our analysis to the case of a one-dimensional model, although our approach can be easily extended to a multidimensional framework.

The paper is organized in the following way. In Section 2 we analyze the effects of the privileged information on the optimal portfolios and on the viability of the market. Section 3 is devoted to finding a formula for the compensator of the Wiener process under the new filtration. In Section 4 we apply the previous results to several examples.

2 Gain and arbitrage possibilities under additional information

Consider, for the sake of simplicity, a Black-Scholes model with one risky stock $S = \{S_t,\ 0 \leq t \leq T\}$. Namely, S satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad 0 \leq t \leq T, \tag{1}$$

with some parameters $\sigma > 0, \mu \in \mathbb{R}$ and initial condition S_0, and where $W = \{W_t, 0 \leq t \leq T\}$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{A}, P)$.

We write $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ to denote the P-completed natural filtration generated by the Wiener process. We also consider a riskless stock $B_t = \exp\{rt\}$, where r is the instantaneous interest rate.

Assume that the additional information until time t is given by a family of random variables $\{L_s\ , s \leq t\}$. Suppose that these random variables have the following structure:

$$L_t = G(X, Y_t),$$

where X is an $\mathcal{F}_T$-measurable random variable, the process $Y = \{Y_t, 0 \leq t \leq T\}$ is independent of the σ-algebra $\mathcal{F}_T$, and $G : \mathbb{R}^2 \to \mathbb{R}$ is a given measurable function. Write $\mathcal{H} = (\mathcal{H}_t)_{t\in[0,T]}$ to denote the usual augmentation of the filtration $(\mathcal{F}_t \vee \sigma(L_s, s \leq t))_{t\in[0,T]}$ (P-completed and right-continuous).

The filtration $\mathcal{F}$ gives the regular trader's evolution of knowledge, whereas the enlarged filtration $\mathcal{H}$ describes the insider's filtration. The random variable X contains the additional information available to the privileged trader, and the random variables Y_t represent a different (independent) noise that perturbs this additional information. Therefore one expects in general that $Y_T = 0$ and that the variance of the noise should decrease to zero as the revelation time T approaches.

If under the new filtration $\mathcal{H}$, $W_t = W_t^* + \int_0^t \beta_s ds,\ t \in [0, T]$, where

$W = (W_t^*)_{t\in[0,T]}$ is an $\mathcal{H}$-Brownian motion and $\beta = (\beta_t)_{t\in[0,T]}$ is an $\mathcal{H}$-progressively measurable process, we can write

$$dS_t = (\mu + \sigma\beta_t)S_t dt + \sigma S_t dW_t^*, \quad 0 \le t \le T.$$

Then the market for the insider is simply a market with a different drift. Therefore utility optimization or arbitrage possibilities for the insider can be studied as utility optimization or arbitrage possibilities in a market with different dynamics.

For instance, if we take the logarithmic utility function and we try to maximize the expected utility of the terminal wealth of traders, for fixed initial wealth, the difference between regular traders and insiders

$$\max_{\pi\in \mathrm{I}} E(\ln(\mathcal{W}_T^\pi)) - \max_{\pi\in \mathrm{R}} E(\ln(\mathcal{W}_T^\pi)),$$

can be obtained by solving the optimization problems with the two dynamics. Here $\mathcal{W}_T^\pi$ is the value of the portfolio π at T, and I and R are, respectively, the sets of admissible portfolios for insiders and regular traders. The solution can be found in Amendinger et al. (1998), and this difference is given by

$$\frac{1}{2}E(\int_0^T \beta_t^2 dt), \tag{2}$$

where the optimal portfolio for insiders is such that the amount of money invested in the risky asset is given by

$$\left(\frac{\mu - r}{\sigma^2} + \frac{\beta_t}{\sigma}\right)\mathcal{W}_t^\pi. \tag{3}$$

We now consider arbitrage opportunities for insiders. As usual, we shall say that a portfolio π is an *arbitrage opportunity* if $\mathcal{W}_0^\pi = 0$ and $P\{\mathcal{W}_T^\pi \ge 0\} = 1$ with $P\{\mathcal{W}_T^\pi > 0\} > 0$. To avoid "doubling strategies" we shall impose the condition $\mathcal{W}_t^\pi \ge C$ a.s. for some constant $C \in \mathbb{R}$ and for any $t \in [0,T]$, that is we only admit the so called *tame portfolios.*

Suppose that $\int_0^T \beta_t^2 dt < \infty$ a.s and that there exists a probability measure Q^* equivalent to P such that $W_t = W_t^* + \int_0^t \beta_s ds$, $t \in [0,T]$, is an $\mathcal{H}$-Brownian motion. Then,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad 0 \le t \le T,$$

where W is an $\mathcal{H}$-Brownian motion with respect to Q^*. Now, we know that there exists a probability measure Q equivalent to Q^* such that

$$dS_t = rS_t dt + \sigma S_t d\hat{W}_t, \quad 0 \le t \le T,$$

where $\hat{W}$ is an $\mathcal{H}$-Brownian motion with respect to Q. By composing the two steps we have a probability measure Q equivalent to P, such that

$$\hat{W}_t = W_t^* + \int_0^t (\frac{\mu - r}{\sigma} + \beta_s) ds, \quad 0 \le t \le T,$$

is an $\mathcal{H}$-Brownian motion with respect to Q, in other words, there is a risk neutral measure for insiders. Then, according to Corollary 2 in Levental and Skorohod (1995) insiders will not have arbitrage opportunities in $\mathcal{H}$ with tame portfolios. So, we simply have to know whether or not there is a probability measure Q^* equivalent to P such that $W_t = W_t^* + \int_0^t \beta_s ds$, $t \in [0,T]$, is an $\mathcal{H}$-Brownian motion with respect to Q^*.

3 Formulae for the compensator

We have seen in the previous section that the characteristics of the insider's view of market are determined by the decomposition $W_t = W_t^* + \int_0^t \beta_s ds$, where W^* is an $\mathcal{H}$-Brownian motion and the drift β is an $\mathcal{H}$-progressively mesurable process. Then the problem is to find β. A useful fact is that if we know the drift for the case $L_t = X$ we can obtain the drift for the general case. In fact, we have the following proposition.

Proposition 1 *Let X be an $\mathcal{F}_T$-measurable random variable and assume that there exists an integrable, $\mathcal{F} \vee \sigma(X)$-progressively measurable process $\alpha = \{\alpha_t, t \in [0,T)\}$, such that $W - \int_0^\cdot \alpha_s ds$ is an $\mathcal{F} \vee \sigma(X)$-Brownian motion. Then $W - \int_0^\cdot E(\alpha_s|\mathcal{H}_s) ds$ is an $\mathcal{H}$-Brownian for an appropriate version of $E(\alpha_s|\mathcal{H}_s)$.*

Proof. Since $Y = \{Y_t, 0 \le t \le T\}$ is independent of $\mathcal{F}_T$, then $\bar{W}_t = W_t - \int_0^t \alpha_s ds$ is a $\mathcal{J}$-Brownian motion, with $\mathcal{J} = (\mathcal{F}_t \vee \sigma(X) \vee \sigma(Y_s, s \le t))_{t \in [0,T)}$. We have that $E(\bar{W}_t|\mathcal{H}_t) = W_t - \int_0^t E(\alpha_s|\mathcal{H}_s) ds$, where we can consider an $\mathcal{H}$-progressively measurable version of $E(\alpha_s|\mathcal{H}_s)$, $s \in [0,T)$ (see Dellacherie and

Meyer (1980), page 113), also since $L_t = G(X, Y_t)$, $\mathcal{H}_t \subset \mathcal{J}_t$, and $E(\bar{W}_t|\mathcal{H}_t)$, $t \in [0, T)$, will be an $\mathcal{H}$-martingale. In fact, for $0 \leq s < t < T$

$$E(E(\bar{W}_t|\mathcal{H}_t)|\mathcal{H}_s) = E(\bar{W}_t|\mathcal{H}_s) = E(E(\bar{W}_t|\mathcal{J}_s)|\mathcal{H}_s) = E(\bar{W}_s|\mathcal{H}_s).$$

Finally Lévy's characterization theorem implies the result. ■

According to the preceding proposition, given an $\mathcal{F}_T$-mesurable X, it is enough to compute the compensator α with respect to $\mathcal{F} \vee \sigma(X)$.The following proposition shows how to calculate α if we have a certain integral representation of any functional of X, and the Clark-Ocone formula will allow us, in many cases, to compute explicitly this representation.

Proposition 2 *Suppose that X is an $\mathcal{F}_T$-measurable random variable. Assume that there exists a $\mathcal{B}_{[0,T]} \otimes \mathcal{F}_T$-measurable process $\xi = \{\xi_t, t \in [0, T]\}$ such that $\int_0^T E(|\xi_t|)dt < \infty$, and that for any measurable and bounded function f we have*

$$f(X) = E(f(X)) + \int_0^T \Phi_t^f dW_t$$

$$\Phi_t^f = E\left(f(X)\xi_t|\mathcal{F}_t\right), \tag{4}$$

for almost all (t, ω). Then, $W_t - \int_0^t \alpha_s ds$ is an $\mathcal{F} \vee \sigma(X)$-Brownian motion, $t \in [0, T]$ where

$$\alpha_t = E(\xi_t | \mathcal{F}_t \vee \sigma(X))$$

for an appropriate version of the conditional expectation.

Proof. Let f be a bounded measurable function on $\mathbb{R}$, let $s \leq t$ and $A \in \mathcal{F}_s$. Set $F = f(X)$. Then we have

$$\begin{aligned}
&E\left((W_t - W_s)\mathbf{1}_A F\right) = E\left((W_t - W_s)\mathbf{1}_A f(X)\right) \\
&= E\left(\mathbf{1}_A \int_s^t E\left(f(X)\xi_u|\mathcal{F}_u\right) du\right) = E\left(\mathbf{1}_A \int_s^t f(X)\xi_u du\right) \\
&= E\left(\mathbf{1}_A \int_s^t f(X)\, E(\xi_u | \mathcal{F}_u \vee \sigma(X))du\right) = E\left(\mathbf{1}_A F \int_s^t \alpha_u du\right).
\end{aligned}$$

Here, as in the proof of the previous proposition, we are considering the $\mathcal{F} \vee \sigma(X)$-progressively measurable version of $E(\xi_t | \mathcal{F}_t \vee \sigma(X))$, $0 \leq t \leq T$. Finally, Lévy's characterization theorem and the condition $\int_0^T |\xi_t|dt < \infty$ a.s. imply the result. ■

Remark 3 *Note that if in Proposition 1* $\alpha_t = E(\xi_t|\mathcal{F}_t \vee \sigma(X))$, $\beta_t = E(\alpha_t|\mathcal{H}_t) = E(\xi_t|\mathcal{H}_t)$ *for adequate versions.*

In fact, denote $Y = (Y_{s_1}, ..., Y_{s_n})$ and $E_Y(\cdot)$ the conditional expectation fixing $Y = (y_{s_1}, ..., y_{s_n})$. Fix $t \in [0,T]$ such that $E(|\xi_t|) < \infty$ and $B \in \mathcal{F}_t$, h a bounded measurable function on $\mathbb{R}^n$, $s_1 \leq \cdots \leq s_n \leq t$, and set $H = h(L_{s_1}, \ldots, L_{s_n})$. Then, we have

$$\begin{aligned}
&E_Y\left(\xi_t \mathbf{1}_B H\right) \\
&= E\left(\xi_t \mathbf{1}_B h(G(X, y_{s_1}), \ldots, G(X, y_{s_n})\right) \\
&= E(E(\xi_t|\mathcal{F}_t \vee \sigma(X))\mathbf{1}_B h(G(X, y_{s_1}), \ldots, G(X, y_{s_n})) \\
&= E\left(\alpha_t \mathbf{1}_B(X) h(G(X, y_{s_1}), \ldots, G(X, y_{s_n})\right) \\
&= E_Y\left(\alpha_t(X)\mathbf{1}_B H\right).
\end{aligned}$$

In general (4) is obtained through an integration by parts and therefore the process ξ need not be $\mathcal{F}$-adapted. The next proposition gives a general formula for β in the case of additive noise. In the sequel we consider the particular case where $G(x,y) = x + y$ and $Y_t = Z_{T-t}$, Z being a continuous process with independent increments whose marginal Z_t has density q_t.

Proposition 4 *Suppose that the assumptions of Proposition 1 are fulfilled. Let for* $t \in [0,T]$ *the random variables* L_t *be given by* $L_t = X + Y_t$. *Then we have for* $t \in [0,T]$

$$\beta_t = \frac{\int_{\mathbb{R}} \alpha_t(x)\, q_{T-t}(L_t - x)\, P_t(dx)}{\int_{\mathbb{R}} q_{T-t}(L_t - x)\, P_t(dx)},$$

where we denote by $P_t(dx)$ *a regular version of the conditional law of the random variable* X *given the* σ*-field* $\mathcal{F}_t$.

Proof. For $t \in [0,T]$ we may write, using the independence of $\mathcal{F}_T$ and Y

$$\begin{aligned}
\beta_t &= E(\alpha_t(X)|\mathcal{F}_t \vee \sigma(L_s : s \leq t)) \\
&= E(\alpha_t(X)|\mathcal{F}_t \vee \sigma(L_t) \vee \sigma(Y_t - Y_s : s \leq t)) \\
&= E(\alpha_t(X)|\mathcal{F}_t \vee \sigma(L_t)).
\end{aligned}$$

Let Q_t be a regular version of the conditional distribution of $(X, X + Y_t)$ given $\mathcal{F}_t$. Then for $C \in \mathcal{B}(\mathbb{R}^2)$

$$Q_t(C) = \int_{\mathbb{R}^2} \mathbf{1}_C(x, x+y)\, q_{T-t}(y)\, P_t(dx)\, dy = \int_{\mathbb{R}^2} \mathbf{1}_C(x, l)\, q_{T-t}(l-x)\, P_t(dx)\, dl.$$

Hence for $A \in \mathcal{B}(\mathbb{R})$

$$P(X \in A|\mathcal{F}_t \vee \sigma(L_t)) = \frac{\int_A q_{T-t}(L_t - x)\, P_t(dx)}{\int_{\mathbb{R}} q_{T-t}(L_t - x)\, P_t(dx)}, \tag{5}$$

and we obtain

$$E(\alpha_t(X)|\mathcal{F}_t \vee \sigma(L_t)) = \frac{\int_{\mathbb{R}} \alpha_t(x)\, q_{T-t}(L_t - x)\, P_t(dx)}{\int_{\mathbb{R}} q_{T-t}(L_t - x)\, P_t(dx)}. \tag{6}$$

■

4 Examples

Example 1 Let $L_t = X + \tilde{W}_{g(T-t)}$, where $X = F(W_T)$, $F : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with $E(F'(W_T)^2) < \infty$ and $g : [0,T] \to [0,+\infty)$ is a strictly increasing bounded function with $g(0) = 0$. It is well known that if $X = W_T$, then $\alpha_t(X) = \frac{W_T - W_t}{T-t}$, see for instance Jeulin (1980), p. 49. Hence, by Proposition 1

$$\beta_t = E(\frac{W_T - W_t}{T-t}|\mathcal{H}_t).$$

Now, for $t \in [0,T]$ we may write, using the conditional independence of $\{W_r, r < s\}$ and $\sigma(L_r, r \leq s)$ given W_s and the independence of W and $\widetilde{W}$

$$\begin{aligned} E(\frac{W_T - W_t}{T-t}|\mathcal{H}_t) &= E(\frac{W_T - W_t}{T-t}|W_t, \sigma(L_s : s \leq t)) \\ &= E(\frac{W_T - W_t}{T-t}|W_t, F(W_T) + Y_t), \end{aligned}$$

where $Y_t = \tilde{W}_{g(T-t)}$. Let Q_t be the regular conditional distribution of $(W_T - W_t, F(W_T) + Y_t)$ given $W_t = x$. Then for $C \in \mathcal{B}(\mathbb{R}^2)$

$$\begin{aligned} Q_t(C) &= \int_{\mathbb{R}^2} \mathbf{1}_C(y, F(x+y) + z)\, \phi_{g(T-t)}(z)\, \phi_{T-t}(y)\, dzdy \\ &= \int_{\mathbb{R}^2} \mathbf{1}_C(y, w)\, \phi_{g(T-t)}(w - F(x+y))\, \phi_{T-t}(y)\, dwdy. \end{aligned}$$

Hence for $A \in \mathcal{B}(\mathbb{R})$, $t < T$

$$P(W_T - W_t \in A | W_t, L_t) = \frac{\int_A \phi_{g(T-t)}(L_t - F(W_t + y))\, \phi_{T-t}(y)\, dy}{\int_{\mathbb{R}} \phi_{g(T-t)}(L_t - F(W_t + y))\, \phi_{T-t}(y)\, dy}$$

and

$$\beta_t = \frac{\int_{\mathbb{R}} y\phi_{g(T-t)}(L_t - F(W_t + y))\, \phi_{T-t}(y)\, dy}{(T-t)\int_{\mathbb{R}} \phi_{g(T-t)}(L_t - F(W_t + y))\, \phi_{T-t}(y)\, dy}.$$

Notice that $\frac{y}{T-t}\phi_{T-t}(y) = -\phi'_{T-t}(y)$. Hence, integrating by parts yields

$$\begin{aligned}\beta_t &= \frac{\int_{\mathbb{R}}(L_t - F(W_t + y))\, F'(W_t + y)\phi_{g(T-t)}(L_t - F(W_t + y))\, \phi_{T-t}(y)\, dy}{g(T-t)\int_{\mathbb{R}} \phi_{g(T-t)}(L_t - F(W_t + y))\, \phi_{T-t}(y)\, dy} \\ &= \frac{1}{g(T-t)} E(Y_t F'(W_T) | W_t, F(W_T) + Y_t).\end{aligned}$$

Hence, applying Cauchy-Schwarz's inequality we obtain

$$E(\beta_t^2) \leq \frac{1}{g(T-t)^2} E(Y_t^2 F'(W_T)^2) = \frac{1}{g(T-t)} E(F'(W_T)^2).$$

Therefore we conclude,

$$E(\int_0^T \beta_t^2 dt) \leq E(F'(W_T)^2) \int_0^T \frac{dt}{g(t)} \ ,$$

and $E(\int_0^T \beta_t^2 dt) < \infty$ if $\int_0^T \frac{dt}{g(t)} < \infty$. This condition is satisfied, for instance, in the case $g(s) = Ks^p$ with $0 < p < 1, K > 0$.

Let us take

$$X = \log(S_T) = \log S_0 + \tilde{\mu}T + \sigma W_T,$$

with $\tilde{\mu} = \mu - \sigma^2/2$. Then F is a linear function and as a consequence,

$$\beta_t = \frac{\sigma^2(W_T - W_t) + \sigma \tilde{W}_{g(T-t)}}{\sigma^2(T-t) + g(T-t)} = \frac{\sigma(L_t - \log(S_t) - \tilde{\mu}(T-t))}{\sigma^2(T-t) + g(T-t)},$$

and

$$E\int_0^T \beta_t^2 dt = \int_0^T \frac{\sigma^2}{\sigma^2(T-t) + g(T-t)} dt.$$

Since β is a Gaussian process $E(\int_0^T \beta_t^2 dt) < \infty$ is a sufficient condition (see Liptser and Shiryaev (1997)) to guarantee the existence of the equivalent martingale mesure Q^* (see section 2), then if we take $g(s) = Ks^p$ with $0 < p < 1$, there are no arbitrage opportunities. But if $g(s) = Ks^p$ with $p \geq 1$ we have, by the law of the iterated logarithm and because W and $\tilde{W}$ are independent, that

$$\left(\frac{\sigma^2(W_T - W_t) + \sigma\tilde{W}_{g(T-t)}}{\sigma^2(T-t) + g(T-t)}\right)^2 = O((T-t)^{-1}\log\log(1/(T-t))) \quad \text{a.s., when } t \uparrow T,$$

and since

$$\int_0^T t^{-1}\log\log(1/t)dt = \infty$$

$$\int_0^T \beta_t^2 dt = \infty \quad \text{a.s.}$$

Levental and Skorohod (1995) have constructed an arbitrage opportunity for this situation as exhibited in the proof of Theorem 1 of their paper.

Moreover, note that if $0 < p < 1$,

$$\frac{\sigma^2(W_T - W_t) + \sigma\tilde{W}_{g(T-t)}}{\sigma^2(T-t) + g(T-t)} = O((T-t)^{-p/2}\sqrt{(}\log\log(1/(T-t)^p))) \text{ a.s., when } t \uparrow T.$$

Then $\lim_{t\to T}|\beta_t| = \infty$ a.s and by (3), the insider becomes a large trader when the revelation time T approaches.

Example 2. Let $X = M = \max_{0\leq t\leq T} W_t$, $F(x,y) = x + y$, $Y_t = \tilde{W}_{g(T-t)}$ and $L_t = M + \tilde{W}_{g(T-t)}, t \in [0,T]$. Set $M_t = \max_{0\leq s\leq t} W_s$, $t \in [0,T]$, and

$$\beta_{t,T} = \max_{t\leq s\leq T}(W_s - W_t).$$

Then

$$M = M_t \vee (\beta_{t,T} + W_t).$$

For any bounded and measurable function f on $\mathbb{R}$ we can write

$$\begin{aligned} f(M) &= f(M)\left(\mathbf{1}_{\{M=M_t\}} + \mathbf{1}_{\{M>M_t\}}\right) \\ &= f(M_t)\mathbf{1}_{\{M=M_t\}} + f(\beta_{t,T} + W_t)\,\mathbf{1}_{\left\{\beta_{t,T}+W_t>M_t\right\}}. \end{aligned}$$

¿From this decomposition we can find a regular version of the conditional law of M given $\mathcal{F}_t, t \in [0,T]$. Indeed,

$$\begin{aligned} E(f(M)|\mathcal{F}_t) &= E\left(f(M_t)\mathbf{1}_{\{M=M_t\}} + f(\beta_{t,T} + W_t)\,\mathbf{1}_{\left\{\beta_{t,T}+W_t>M_t\right\}}|\mathcal{F}_t\right) \\ &= f(M_t)R_{T-t}(M_t - W_t) \;+ \int_{M_t-W_t}^{\infty} f(x+W_t)r_{T-t}(x)dx, \end{aligned}$$

where r_t and R_t denote the density and the distribution function, respectively, of the maximum of the Wiener process in the interval $[0, t]$, which are given by

$$r_t(x) = \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{x^2}{2t}\right), \quad x > 0$$

and

$$R_t(y) = \int_0^y r_t(x)dx, \quad y \geq 0.$$

Hence,

$$P_t(dx) = \delta_{M_t}(dx) R_{T-t}(M_t - W_t) \;+\; r_{T-t}(x - W_t)\mathbf{1}_{(M_t,\infty)}(x)dx. \qquad (7)$$

Then, it is known, see Jeulin (1980), p. 49, that

$$\alpha_t(x) = -\frac{r_{T-t}(M_t - W_t)}{R_{T-t}(M_t - W_t)} 1(x = M_t) + \frac{x - W_t}{T - t}\mathbf{1}_{(M_t,\infty)}(x).$$

Then, Proposition 4 allows to deduce the following representation of β. We obtain for $t \in [0, T]$

$$\beta_t = \frac{-r_{T-t}(M_t - W_t)\, q_{T-t}(L_t - M_t) + \int_{M_t}^{\infty} r_{T-t}(x - W_t)\, \frac{x-W_t}{T-t}\, q_{T-t}(L_t - x)dx}{R_{T-t}(M_t - W_t)\, q_{T-t}(L_t - M_t) + \int_{M_t}^{\infty} r_{T-t}(x - W_t)\, q_{T-t}(L_t - x)dx}. \qquad (8)$$

Now similar techniques as in the previous example will be used in order to assess the integrability properties of β. First of all, we note that

$$r_{T-t}(x - W_t)\, \frac{x - W_t}{T - t} = -\frac{\partial}{\partial x} r_{T-t}(x - W_t), \quad x > W_t.$$

We use this formula to integrate by parts the second expression in the numerator of the representation of β_t. The result obviously is

$$\begin{aligned}
&\int_{M_t}^{\infty} r_{T-t}(x - W_t)\, \frac{x - W_t}{T - t}\, q_{T-t}(L_t - x)dx \\
&= r_{T-t}(M_t - W_t)\, q_{T-t}(L_t - M_t) \\
&+ \frac{1}{g(T-t)} \int_{M_t}^{\infty} r_{T-t}(x - W_t)\, (L_t - x)\, q_{T-t}(L_t - x)\, dx.
\end{aligned}$$

Substituting this in (8) gives the alternative representation

$$\beta_t = \frac{1}{g(T-t)} \frac{\int_{M_t}^{\infty} r_{T-t}(x-W_t)\,(L_t-x)\,q_{T-t}(L_t-x)dx}{R_{T-t}(M_t-W_t)\,q_{T-t}(L_t-M_t)+\int_{M_t}^{\infty} r_{T-t}(x-W_t)\,q_{T-t}(L_t-x)dx}.$$

Then from (5) and (7) we have that

$$\beta_t = \frac{1}{g(T-t)}E(Y_t\mathbf{1}_{\{M>M_t\}}|\mathcal{H}_t) = \frac{1}{g(T-t)}E(Y_t\mathbf{1}_{\{M>M_t\}}|\mathcal{F}_t\vee\sigma(L_t)).$$

Applying Cauchy-Schwarz's inequality yields

$$E(\beta_t^2) \leq \frac{1}{g(T-t)^2}E(Y_t^2\mathbf{1}_{\{M>M_t\}}) \leq \frac{1}{g(T-t)^2}E(Y_t^2) = \frac{1}{g(T-t)}.$$

Again $E(\int_0^T \beta_t^2 dt) < \infty$ if $\int_0^T \frac{dt}{g(t)} < \infty$. As before, this condition is satisfied in the case $g(s) = Ks^p$ with $0 < p < 1, K > 0$.

A sufficient condition to guarantee the existence of Q^* is the Novikov condition:

$$E(\exp\{\frac{1}{2}\int_0^T \beta_t^2 dt\}) < \infty.$$

In our example we have

$$\begin{aligned}
\beta_t^2 &\leq \frac{1}{g(T-t)^2}E(Y_t^2|\mathcal{F}_t\vee\sigma(L_t)) \\
&\leq \frac{1}{g(T-t)^2}E(\sup_{0\leq t\leq T} Y_t^2|\mathcal{F}_t\vee\sigma(L_t)).
\end{aligned}$$

Then writing $U_t = E(\sup_{0\leq t\leq T} Y_t^2|\mathcal{F}_t\vee\sigma(L_t))$, and if $\int_0^T \frac{1}{g(T-t)^2}dt < \infty$ we have

$$\begin{aligned}
&E(\exp\{\frac{1}{2}\int_0^T \beta_t^2 dt\}) \\
&\leq E(\exp\{\frac{1}{2}\int_0^T \frac{U_t}{g(T-t)^2}dt\}) \\
&\leq E(\exp\{\frac{\sup_{0\leq t\leq T} U_t}{2}\int_0^T \frac{1}{g(T-t)^2}dt\}) \\
&\leq E(\sup_{0\leq t\leq T}(\exp\{\frac{U_t}{4}\int_0^T \frac{1}{g(T-t)^2}dt\})^2) \\
&\leq 4E(\exp\{\frac{U_T}{2}\int_0^T \frac{1}{g(T-t)^2}dt\}) < \infty,
\end{aligned}$$

since

$$U_T = E(\sup_{0\le t\le T}(Y_t^2)|\mathcal{F}_T \vee \sigma(L_T)) = E(\sup_{0\le t\le T}(Y_t^2)|\mathcal{F}_T) = E(\sup_{0\le t\le T}(Y_t^2)) < \infty.$$

Note that $\int_0^T \frac{1}{g(T-t)^2} dt < \infty$ is satisfied, for instance, in case $g(s) = Ks^p$ with $0 < p < 1/2$ and $K > 0$. In these cases there are no arbitrage opportunities.

Example 3. Let $(X_t)_{t\in[0,T]}$ be a one dimensional time homogeneous Markov process with transition density $p_t(x,y), x, y \in \mathbb{R}, t \in [0,T]$, determined by a stochastic differential equation

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t. \tag{9}$$

This example can be seen as a generalization of example 1, disregarding the case where the function F is not one-to-one, and it is an example where Proposition 2 applies in a natural way. The stochastic equation (9) can be consider as a generalization of the Black-Scholes model, where we assume implicitly the existence of a riskless asset B that evolves as

$$dB_t = B_t r_t dt, \quad B_0 = 1,$$

where r is an $\mathcal{F}$-adapted process that represents the instantaneous interest rate. Suppose that

$$\int_0^T \left(\frac{b(X_t) - r_t}{\sigma(X_t)}\right)^2 dt < \infty \quad \text{a.s.}$$

Then we could argue as before to study the arbitrage opportunities in the insider filtration, but we shall only consider the utility gain of the insider.

Assume that the density function is continuously differentiable in x and y and that there is a function $\gamma_t(y,x), x, y \in \mathbb{R}$, which is also continuously differentiable in x and y such that we have

$$\frac{\partial}{\partial y} p_t(y,x) = \gamma_t(y,x)\,\frac{\partial}{\partial x} p_t(y,x), \quad x, y \in \mathbb{R}, \quad t \in [0,T].$$

Let then $X = X_T$, the final value of the Markov process. Let $L_t = X + \tilde{W}_{g(T-t)}, t \in [0,T]$, and use the notations of Example 1. This time the Markov property yields for $A \in \mathcal{B}(\mathbb{R}), t \in [0,T]$ the equation

$$P(X \in A|\mathcal{F}_t) = \int_A p_{T-t}(X_t, x)\,dx,$$

whence the conditional density of X given $\mathcal{F}_t$ is given by

$$P_t(dx) = p_{T-t}(X_t, x)dx, \quad x \in \mathbb{R}.$$

Now we compute $\alpha_t(x)$ by Proposition 2. In fact, by the Clark-Ocone formula, we have

$$\begin{aligned}
\Phi_t^f &= E\left(D_t f(X_T)\right|\mathcal{F}_t) = D_t E\left(f(X_T)\right|\mathcal{F}_t) \\
&= D_t \int_{\mathbb{R}} f(x) p_{T-t}(X_t, x)\, dx \\
&= \int_{\mathbb{R}} f(x) D_t p_{T-t}(X_t, x)\, dx \\
&= \int_{\mathbb{R}} f(x)\sigma(X_t)\, \gamma_t(X_t, x)\, \frac{\partial}{\partial x} p_{T-t}(X_t, x) dx. \\
&= E(f(X_T)\sigma(X_t)\, \gamma_t(X_t, X_T)\, \frac{\partial}{\partial x} p_{T-t}(X_t, X_T)|\mathcal{F}_t)
\end{aligned}$$

Therefore we may take

$$\xi_t = \sigma(X_t)\, \gamma_t(X_t, X_T)\, \frac{\partial}{\partial x} p_{T-t}(X_t, X_T)$$

and

$$\alpha_t(x) = \sigma(X_t)\, \gamma_t(X_t, x)\, \frac{\partial}{\partial y} \log p_{T-t}(X_t, x).$$

One can use Proposition 4 to obtain the compensator β. First we use integration by parts to compute the numerator

$$\begin{aligned}
&\int_{\mathbb{R}} \alpha_t(x) q_{T-t}(L_t - x)\, P_t(dx) \\
&= \sigma(X_t) \int_{\mathbb{R}} q_{T-t}(L_t - x)\, \gamma_t(X_t, x)\, \frac{\partial}{\partial x} p_{T-t}(X_t, x)\, dx \\
&= \sigma(X_t) \int_{\mathbb{R}} p_{T-t}(X_t, x)\, [\frac{\partial}{\partial x}\gamma_t(X_t, x) + \gamma_t(X_t, x)\, \frac{L_t - x}{g(T-t)}]\, q_{T-t}(L_t - x)\, dx \\
&= \sigma(X_t) \int_{\mathbb{R}} [\frac{\partial}{\partial x}\gamma_t(X_t, x) + \gamma_t(X_t, x)\, \frac{L_t - x}{g(T-t)}]\, q_{T-t}(L_t - x)\, P_t(dx). \qquad (10)
\end{aligned}$$

Hence,

$$\beta_t = \frac{\sigma(X_t) \int_{\mathbb{R}} [\frac{\partial}{\partial x}\gamma_t(X_t, x) + \gamma_t(X_t, x)\, \frac{L_t - x}{g(T-t)}]\, q_{T-t}(L_t - x)\, P_t(dx)}{\int_{\mathbb{R}}\, q_{T-t}(L_t - x)\, P_t(dx)}.$$

Let us next apply Cauchy-Schwarz's inequality to the result of the integration by parts appearing in (10) and integrate with respect to the conditional law of L_t given $\mathcal{F}_t$. We obtain

$$
\begin{aligned}
&E(\beta_t^2) \\
&= \int_{\mathbf{R}} E\left(\frac{(\int_{\mathbf{R}} \alpha_t(x) q_{T-t}(y-x) P_t(dx))^2}{\int_{\mathbf{R}} q_{T-t}(y-x)\, P_t(dx)}\right)\, dy \\
&\leq E\left(\int_{\mathbf{R}}\int_{\mathbf{R}} \sigma(X_t)^2 [\frac{\partial}{\partial x}\gamma_t(X_t,x) + \gamma_t(X_t,x)\,\frac{y-x}{g(T-t)}]^2\, q_{T-t}(y-x)\, P_t(dx)\, dy\right) \\
&= E\left(\sigma(X_t)^2\,[\frac{\partial}{\partial x}\gamma_t(X_t,X_T) + \gamma_t(X_t,X_T)\,\frac{Y_t}{g(T-t)}]^2\right).
\end{aligned}
$$

Therefore we conclude

$$
E(\int_0^T \beta_t^2 dt) \leq \int_0^T E\left(\sigma(X_t)^2\,[\frac{\partial}{\partial x}\gamma_t(X_t,X_T) + \gamma_t(X_t,X_T)\,\frac{Y_t}{g(T-t)}]^2\right)\, dt,
$$

and

$$
E(\int_0^T \beta_t^2 dt) < \infty
$$

if

$$
g(s) = K s^p \text{ with } 0 < p < 1, K > 0,
$$

and

$$
\sup_{0\leq t\leq T} E(\sigma(X_t)^2\,\gamma_t(X_t,X_T)^2) < \infty, \quad \sup_{0\leq t\leq T} E(\sigma(X_t)^2\,\frac{\partial}{\partial x}\gamma_t(X_t,X_T)^2) < \infty.
$$

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