Additional Logarithmic Utility of an Insider

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January 29, 1998

Abstract

In this paper, we consider a security market in which two investors on different information levels maximize their expected logarithmic utility from terminal wealth. While the ordinary investor's portfolio decisions are based on a public information flow, the insider possesses from the beginning extra information about the outcome of some random variable G, e.g., the future price of a stock. We solve the two optimization problems explicitly and rewrite the insider's additional expected logarithmic utility in terms of a relative entropy. This allows us to provide simple conditions on G for the finiteness of this additional utility and to show that it is basically given by the entropy of G.

Key Words: utility maximization, insider trading, initial enlargement of filtrations, relative entropy, entropy

AMS 1991 Subject Classification: Primary 90A09, 60H30. Secondary 60G44, 90A35, 94A17.

1 Introduction

In the past decades, an extensive mathematical theory using martingale techniques has been developed for the problems of derivative pricing, utility maximization of investors and equilibrium theory in security market models. One of the salient features of this theory is its assumption of one common information flow on which the portfolio decisions of all economic agents are based. In this paper, we attempt to widen the scope of the martingale approach by studying a utility maximization problem in a security market with two types of investors on different information levels.

Despite its practical importance, this question has only quite recently been addressed in the literature. The first thorough mathematical study is a paper by Pikovsky and

Karatzas (1996) whose methods and results strongly inspired much of the developments presented here. In particular, we follow their lead in the modelling of additional information and by considering two investors with logarithmic utility functions. While the *ordinary economic agent* makes his portfolio decisions according to the 'public' information flow $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, the *insider* possesses from the beginning additional information about the outcome of some random variable G and therefore has the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ with $\mathcal{G}_t = \bigcap\limits_{\varepsilon>0} \big(\mathcal{F}_{t+\varepsilon} \vee \sigma(G)\big)$ at his disposal. For instance, the insider may know the price of a stock at time T, or the price range of a stock at time T, or the price of a stock at time T distorted by some noise, etc. In this framework, the following questions arise: How should the insider trade on the security market to optimally exploit his extra information? What is the insider's additional utility arising from his extra knowledge?

In this paper, we solve the optimization problems for the two investors by adapting ideas of Karatzas, Lehoczky, Shreve and Xu (1991) to our framework and so obtain a first expression for the insider's additional expected logarithmic utility. This extends results of Pikovsky and Karatzas (1996) and Elliott, Geman and Korkie (1997) from the case of a complete model with a Brownian filtration $\mathbb{F}$ to an incomplete market. Building on results about initial enlargements of filtrations by Jacod (1985) and Föllmer and Imkeller (1993), we then rewrite the additional expected logarithmic utility in terms of the relative entropy of the objective probability measure P with respect to a new probability measure $\widetilde{P}_t$ that we call $[0,t]$-*insider martingale measure* or $[0,t]$-*martingale preserving measure under initial enlargement*. In the case of a complete market studied by Pikovsky and Karatzas (1996), this allows us to systematically analyze the additional expected utility. We provide simple conditions on G for the finiteness of the additional utility, show that it is basically given by the entropy of G and thereby solve a number of previously open problems raised by Pikovsky and Karatzas (1996).

The paper is organized as follows. Section 2 is exclusively concerned with the mathematical theory of initial enlargement of filtrations. We first recall some results of Jacod (1985) which show that a continuous local $\mathbb{F}$-martingale K remains a semimartingale for the filtration $\mathbb{G}^\circ = (\mathcal{G}_t)_{t\in[0,T)}$ if the regular conditional distributions of G given $\mathcal{F}_t$ are absolutely continuous with respect to the law of G for all $t \in [0,T)$. Moreover, Jacod (1985) presents the canonical decomposition of K in $\mathbb{G}^\circ$ which involves the conditional density processes p^ℓ, $\ell \in \text{range}(G)$. By adapting arguments of Föllmer and Imkeller (1993), we prove that $1/p^G$ is a $\mathbb{G}^\circ$-martingale and thus defines a family of probability measures $\widetilde{P}_t$ on $(\Omega, \mathcal{G}_t)$ for $t < T$, provided that the regular conditional distributions of G given $\mathcal{F}_t$ are *equivalent* to the law of G. Furthermore, we show that any (local) $(\mathbb{F}, P)$-martingale is a (local) $(\mathbb{G}, \widetilde{P}_t)$-martingale on $[0,t]$ for $t < T$; this justifies calling $\widetilde{P}_t$ the *martingale preserving probability measure under initial enlargement*. We give examples for the calculation of p^ℓ and the absolute continuity and equivalence conditions, respectively, and conclude section 2 by showing that the $\mathbb{F}^\circ$-martingale p^ℓ and the $\mathbb{G}^\circ$-martingale $1/p^G$ can be written as stochastic exponentials of a particular form. This provides the key tool for the subsequent

sections.
Section 3 introduces a general incomplete security market model with continuous prices. We therein consider an ordinary investor who has the filtration $\mathbb{F}$ as his information flow, and an insider whose portfolio decisions are based on the larger filtration $\mathbb{G}$. The investors' goal is to maximize the expected logarithmic utility of terminal wealth by trading in the security market. After solving these optimization problems, we compare the maximal expected logarithmic utilities of the two investors. By using the theoretical results from section 2, we obtain a new alternative expression for the insider's additional expected utility involving the relative entropy of the probability measure P with respect to the $[0, t]$-insider martingale measure $\widetilde{P}_t$.
In section 4, we consider a *complete* security market and calculate the terminal additional expected logarithmic utility of an insider for a wide class of random variables G, thereby generalizing some of the results of Pikovsky and Karatzas (1996). If G is $\mathcal{F}_T$-measurable, the insider's additional expected logarithmic utility turns out to be an expression one could call the *entropy of the initial enlargement*; see Yor (1985). If G is even of finite entropy, the additional utility simply consists of the entropy of G, while it becomes infinite if G is of infinite entropy.

Convention: Section Assumptions are imposed throughout the respective sections.

2 Some Results on Initial Enlargements of Filtrations

This section collects some known and some new results about initial enlargements of filtrations. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions of right-continuity and completeness. $T \in (0, \infty]$ is a fixed time horizon, and we assume that $\mathcal{F}_0$ is trivial. For some $\mathcal{F}$-measurable random variable G with values in a Polish space $(U, \mathcal{U})$, we define the enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ by

$$\mathcal{G}_t := \bigcap_{\epsilon>0} \Big(\mathcal{F}_{t+\epsilon} \vee \sigma(G)\Big) \qquad , \qquad t \in [0, T].$$

Furthermore, we introduce the notations $\mathbb{F}^\circ := (\mathcal{F}_t)_{t\in[0,T)}$ and $\mathbb{G}^\circ := (\mathcal{G}_t)_{t\in[0,T)}$; note the distinction between $[0, T]$ and $[0, T)$. Throughout this section, let $K = (K_t)_{t\in[0,T]} = \Big(K_t^1, \ldots, K_t^d\Big)^*_{t\in[0,T]}$ be a d-dimensional continuous local $\mathbb{F}$-martingale with quadratic variation $\langle K\rangle = (\langle K^i, K^j\rangle)_{i,j=1,\ldots,d}$.

2.1 A Summary of Fundamentals

Most of the general theory presented in this subsection goes back to Jacod (1985), who formulated his results under the following crucial assumption:

Section Assumption 2.1 *There exists a σ-finite measure η on $(U,\mathcal{U})$ such that for all $t \in [0,T)$, the regular conditional distribution of G given $\mathcal{F}_t$ is absolutely continuous with respect to η for P-almost all $\omega \in \Omega$, i.e.,*

$$P[G \in \cdot|\mathcal{F}_t](\omega) \ll \eta(\cdot) \qquad \text{for } P\text{-a.a. } \omega \in \Omega. \tag{1}$$

Before recalling those results of Jacod (1985) used in the sequel, we need some more notation. Let $\widehat{\Omega} := \Omega \times U$, $\widehat{\mathcal{F}}_t := \bigcap_{\epsilon>0}(\mathcal{F}_{t+\epsilon} \otimes \mathcal{U})$ and $\widehat{\mathbb{F}}^\circ := (\widehat{\mathcal{F}}_t)_{t\in[0,T)}$, and denote by $\mathcal{O}(\widehat{\mathbb{F}}^\circ)$ and $\mathcal{P}(\widehat{\mathbb{F}}^\circ)$ the optional and predictable σ-fields on $\widehat{\Omega} \times [0,T)$, respectively. Note that $\mathcal{P}(\widehat{\mathbb{F}}^\circ) = \mathcal{P}(\mathbb{F}^\circ) \otimes \mathcal{U}$; see (1.7) of Jacod (1985). The following lemma provides a 'nice' version of the conditional density process q^ℓ resulting from (1).

Lemma 2.1 *(Lemme 1.8 and Corollaire 1.11 of Jacod (1985))*

1. *There exists a nonnegative $\mathcal{O}(\widehat{\mathbb{F}}^\circ)$-measurable function $(\omega,\ell,t) \mapsto q_t^\ell(\omega)$ which is right-continuous with left limits in t and such that*

   (a) *for all $\ell \in U$, q^ℓ is an $\mathbb{F}^\circ$-martingale, the processes q^ℓ, q_-^ℓ are strictly positive on $[\![0, T^\ell [\![$, and $q^\ell = 0$ on $[\![T^\ell, T [\![$, where*

   $$T^\ell := \inf\left\{t \geq 0 \middle| q_{t-}^\ell = 0\right\} \wedge T; \tag{2}$$

   (b) *for all $t \in [0,T)$, the measure $q_t^\ell(\cdot)\,\eta(d\ell)$ on $(U,\mathcal{U})$ is a version of the conditional distribution $P[G \in d\ell|\mathcal{F}_t]$.*

2. *$T^G = T$ P-a.s.*

The conditional density process q^ℓ is the key to the study of continuous local $\mathbb{F}$-martingales in the enlarged filtration $\mathbb{G}^\circ$. The following theorem shows that under Section Assumption 2.1, every continuous local $\mathbb{F}$-martingale is a $\mathbb{G}^\circ$-semimartingale, and explicitly gives its canonical decomposition.

Theorem 2.2 *(Théorème 2.1 of Jacod (1985))*
For $i = 1,\ldots,d$, there exists a $\mathcal{P}(\widehat{\mathbb{F}}^\circ)$-measurable function $(\omega,\ell,t) \mapsto \left(k_t^\ell(\omega)\right)^i$ such that

$$\langle q^\ell, K^i\rangle = \int (k^\ell)^i q_-^\ell \, d\langle K^i\rangle. \tag{3}$$

For every such function k^i, we have:

1. $\displaystyle\int_0^t \left|(k_s^G)^i\right| \, d\langle K^i\rangle_s < \infty$ *P-a.s. for all $t \in [0,T)$, and*

2. *K^i is a $\mathbb{G}^\circ$-semimartingale, and the continuous local $\mathbb{G}^\circ$-martingale in its canonical decomposition is*

$$\widetilde{K}_t^i := K_t^i - \int_0^t (k_s^G)^i \, d\langle K^i\rangle_s \qquad , \qquad t \in [0,T). \tag{4}$$

Remark: If the absolute continuity condition (1) holds for all $t \in [0,T]$, then $\widetilde{K}$ is even a local $\mathbb{G}$-martingale; this will be used later.

Before we make extensive use of the preceding result, we normalize the conditional density process q^ℓ. Since $\mathcal{F}_0$ is trivial, we have

$$\int_B P[G \in d\ell] = P[G \in B] = P[G \in B|\mathcal{F}_0] = \int_B q_0^\ell \, \eta(d\ell)$$

for all $B \in \mathcal{U}$. By choosing U smaller if necessary, we can therefore assume that $q_0^\ell > 0$ for all $\ell \in U$, and so we obtain for P-a.a. ω and all $t \in [0,T)$

$$P[G \in B|\mathcal{F}_t](\omega) = \int_B q_t^\ell(\omega) \, \eta(d\ell) = \int_B p_t^\ell(\omega) \, P[G \in d\ell],$$

where

$$p_t^\ell(\omega) := \frac{q_t^\ell(\omega)}{q_0^\ell}. \tag{5}$$

Clearly, we can take p as the process q appearing in Lemma 2.1 and Theorem 2.2; this corresponds to choosing for η the law of G.

By Lemma 2.1, the first time p^G hits 0 is P-a.s. equal to T so that we can consider the process $1/p^G$ on $[0,T)$. This process will play a pivotal role in the sequel. If the regular conditional distributions of G given $\mathcal{F}_t$ are *equivalent* to the law of G, then $1/p^G$ turns out to be a positive $\mathbb{G}^\circ$-martingale starting from 1 and thus defines a probability measure $\widetilde{P}_t$ on $(\Omega, \mathcal{G}_t)$ for all $t \in [0,T)$. $\widetilde{P}_t$ coincides with P on $\mathcal{F}_t$, and the σ-algebras $\mathcal{F}_t$ and $\sigma(G)$ become independent under $\widetilde{P}_t$. We show these properties in the next proposition which is a variant of results on p.578 of Föllmer and Imkeller (1993). Basically, we just have to transfer their arguments from their Wiener space framework to our present situation.

Proposition 2.3 *Suppose that the regular conditional distributions of G given $\mathcal{F}_t$ are equivalent to the law of G for all $t \in [0,T)$, i.e., for all $\ell \in U$, the process $(p_t^\ell)_{t\in[0,T)}$ is strictly positive P-a.s. Then:*

1. *$\dfrac{1}{p^G}$ is a $\mathbb{G}^\circ$-martingale.*
2. *For $t \in [0,T)$, the σ-algebras $\mathcal{F}_t$ and $\sigma(G)$ are independent under the probability measure*

$$\widetilde{P}_t(A) := \int_A \frac{1}{p_t^G} \, dP \qquad \text{for } A \in \mathcal{G}_t, \tag{6}$$

i.e., for $A_t \in \mathcal{F}_t$ and $B \in \mathcal{U}$,

$$\widetilde{P}_t\left[A_t \cap \{G \in B\}\right] = P[A_t]P[G \in B] = \widetilde{P}_t[A_t]\widetilde{P}_t[G \in B]. \tag{7}$$

Proof: To prove (7), fix $A_t \in \mathcal{F}_t$ and $B \in \mathcal{U}$. By conditioning on $\mathcal{F}_t$, we obtain

$$E\left[\mathbf{I}_{A_t \cap \{G\in B\}} \frac{1}{p_t^G}\right] = E\left[\mathbf{I}_{A_t} E\left[\mathbf{I}_{\{G\in B\}} \frac{1}{p_t^G}\middle|\mathcal{F}_t\right]\right] = \int_{A_t} E\left[\mathbf{I}_{\{G\in B\}} \frac{1}{p_t^G}\middle|\mathcal{F}_t\right](\omega) \, P(d\omega).$$

The definition of $p_t^\ell(\omega)$ yields

$$E\left[\mathbf{I}_{\{G\in B\}}\frac{1}{p_t^G}\bigg|\mathcal{F}_t\right](\omega) = \int\limits_B \frac{1}{p_t^\ell(\omega)}\, p_t^\ell(\omega)\, P[G\in d\ell] = P[G\in B],$$

and so we get the first equality in (7). The second follows by choosing $A_t = \Omega$ or $B = U$. Now fix $0 \le s \le t < T$ and choose $A \in \mathcal{G}_s$ of the form $A = A_s \cap \{G \in B\}$ with $A_s \in \mathcal{F}_s$ and $B \in \mathcal{U}$. Then we obtain by (7) and by reversing the above argument that

$$\begin{aligned} E\left[\mathbf{I}_A \frac{1}{p_t^G}\right] &= P[A_s]P[G\in B] \\ &= E\Big[\mathbf{I}_{A_s} P[G\in B]\Big] \\ &= \int\limits_{A_s}\int\limits_B \frac{1}{p_s^\ell(\omega)} p_s^\ell(\omega)\, P[G\in d\ell]\, P(d\omega) \\ &= E\left[\mathbf{I}_{A_s} E\left[\mathbf{I}_{\{G\in B\}}\frac{1}{p_s^G}\bigg|\mathcal{F}_s\right]\right] \\ &= E\left[\mathbf{I}_A \frac{1}{p_s^G}\right]. \end{aligned}$$

By a monotone class and right-continuity argument, this extends to arbitrary sets $A \in \mathcal{G}_s$. Hence $1/p^G$ is a $\mathbb{G}^\circ$-martingale and (6) defines indeed a probability measure on $(\Omega, \mathcal{G}_t)$.

q.e.d.

Definition 2.4 *Let $t \in [0,T)$. The probability measure $\widetilde{P}_t$ on $(\Omega,\mathcal{G}_t)$ defined by (6) is called the $[0,t]$-martingale preserving measure under initial enlargement of filtration, or in the context of financial mathematics, the $[0,t]$-insider martingale measure.*

The above terminology is justified by the next result.

Theorem 2.5 *Suppose that the regular conditional distributions of G given $\mathcal{F}_t$ are equivalent to the law of G for all $t \in [0,T)$. For fixed $t \in [0,T)$, any (local) $(P,\mathbb{F})$-martingale L on $[0,t]$ is then a (local) $(\widetilde{P}_t,\mathbb{G})$-martingale on $[0,t]$, hence also a (local) $(\widetilde{P}_t,\mathbb{F})$-martingale on $[0,t]$.*

Proof: Because (7) implies that G is independent of $\mathcal{F}_t$ under $\widetilde{P}_t$ and $\widetilde{P}_t = P$ on $\mathcal{F}_t$, it follows easily that a $(P,\mathbb{F})$-martingale on $[0,t]$ is also a $(\widetilde{P}_t,\mathbb{G})$-martingale on $[0,t]$ and therefore also a $(\widetilde{P}_t,\mathbb{F})$-martingale on $[0,t]$. Since $\mathbb{F}^\circ$-stopping times are also $\mathbb{G}^\circ$-stopping times, any localizing sequence (T_n) for some L with respect to $(P,\mathbb{F})$ on $[0,t]$ will then also localize L with respect to $(\widetilde{P}_t,\mathbb{G})$ and $(\widetilde{P}_t,\mathbb{F})$ on $[0,t]$. **q.e.d.**

2.2 Examples for the Calculation of p^G

This subsection illustrates the preceding results by several examples for G that will be used again later. These examples show that the absolute continuity assumption (1) is typically only satisfied for $t \in [0,T)$ so that p_t^ℓ is only defined on $[0,T)$.

Example 2.6 Let G be the endpoint W_T of a one-dimensional $I\!\!F$-Brownian motion W. Then we have for all $t < T$

$$\begin{aligned} P[W_T \in d\ell | \mathcal{F}_t] &= P[W_T - W_t + W_t \in d\ell | \mathcal{F}_t] \\ &= P[W_T - W_t \in d\ell - y]\Big|_{y=W_t} \\ &= \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(\ell - W_t)^2}{2(T-t)}\right) d\ell \\ &= p_t^\ell \, P[W_T \in d\ell], \end{aligned}$$

where $p_t^\ell = \sqrt{\frac{T}{T-t}} \exp\left(-\frac{(\ell - W_t)^2}{2(T-t)} + \frac{\ell^2}{2T}\right)$, $\ell \in I\!\!R$, is strictly positive for all $t < T$. Furthermore, applying Itô's formula to $(\ell - W_t)^2/(T-t)$ gives

$$p_t^\ell = \mathcal{E}\left(\int \frac{\ell - W_s}{T-s} \, dW_s\right)_t .$$

In this example, the conditional law of G given $\mathcal{F}_t$ is therefore not only absolutely continuous with respect to the law of G, but even equivalent to it for all $t \in [0,T)$. On the other hand, the conditional law of W_T given $\mathcal{F}_T$ is obviously the point mass in $W_T(\omega)$ and therefore not absolutely continuous with respect to the law of W_T.

Example 2.7 Let G be a random variable with values in a countable set U such that $P[G = \ell] > 0$ for all $\ell \in U$. Then every $A \in \sigma(G)$ is of the form $A = \bigcup_{\ell \in J} \{G = \ell\}$ for some $J \subseteq U$. Therefore we have

$$P[G \in A | \mathcal{F}_t] = \sum_{\ell \in J} P[G = \ell | \mathcal{F}_t] = \sum_{\ell \in J} p_t^\ell P[G = \ell] = \int_A p_t^\ell \, P[G \in d\ell]$$

for all $t \in [0,T]$, where $p_t^\ell = \frac{P[G = \ell | \mathcal{F}_t]}{P[G = \ell]}$, and so $P[G \in \cdot | \mathcal{F}_t]$ is absolutely continuous with respect to the law of G for all $t \in [0,T]$. Thus we obtain by Theorem 2.2 and the subsequent remark that every local $I\!\!F$-martingale is a $I\!\!G$-semimartingale and therefore Theorem 1 of Meyer (1978). However, the conditional laws of G given $\mathcal{F}_t$ are equivalent to the law of G on $\mathcal{F}_t$ for $t < T$ only if $P[G = \ell | \mathcal{F}_t] > 0$ P-a.s. for all $\ell \in U$. Moreover, there is certainly no equivalence on $\mathcal{F}_T$ if G is $\mathcal{F}_T$-measurable, because in this case $P[G = \ell | \mathcal{F}_T] = \mathbf{I}_{\{G = \ell\}}$ is zero with positive probability (unless G is a constant).
As a special case, consider the situation in which G describes whether the endpoint of a one-dimensional $I\!\!F$-Brownian motion lies in some given interval, i.e., $G := \mathbf{I}_{\{W_T \in [a,b]\}}$ for some $a < b$. Then we have $p_t^1 = \frac{P[G = 1 | \mathcal{F}_t]}{P[G = 1]}$ and $p_t^0 = \frac{1 - P[G = 1 | \mathcal{F}_t]}{1 - P[G = 1]}$, and a similar computation as in Example 2.6 yields

$$P[G = 1 | \mathcal{F}_t] = \frac{1}{\sqrt{2\pi(T-t)}} \int_a^b \exp\left(-\frac{(u - W_t)^2}{2(T-t)}\right) du \quad , \quad t \in [0,T),$$

and $P[G=1] = P[G=1|\mathcal{F}_0] = \Phi\big(b/\sqrt{T}\big) - \Phi\big(a/\sqrt{T}\big)$, where Φ is the standard normal distribution function. Hence, $P[G \in \cdot|\mathcal{F}_t]$ is absolutely continuous with respect to the law of G for $t \in [0,T]$ and equivalent to the law of G only for all $t \in [0,T)$.

2.3 Writing $1/p^G$ as a Stochastic Exponential

This subsection shows that under the assumptions of Proposition 2.3, the processes p^ℓ and $1/p^G$ can be written as stochastic exponentials of a particular form. More precisely, the $\mathbb{F}^\circ$-martingale p^ℓ is the stochastic exponential of the sum of a stochastic integral with respect to K with integrand κ^ℓ and an orthogonal local $\mathbb{F}^\circ$-martingale, whereas the $\mathbb{G}^\circ$-martingale $1/p^G$ can be written as a stochastic exponential of the sum of a stochastic integral of κ^G with respect to $\widetilde{K}$ and an orthogonal local $\mathbb{G}^\circ$-martingale. To do this, we first prove a structure condition on the finite variation term appearing in the canonical decomposition of K in $\mathbb{G}^\circ$.

Lemma 2.8 *Under Section Assumption 2.1, there exists an $\mathbb{R}^d$-valued, $\mathcal{P}(\mathbb{F}^\circ)\otimes\mathcal{U}$-measurable process $(\kappa_t^\ell)_{t\in[0,T)}$ such that for all $\ell \in U$,*

$$\int_0^t d\langle K\rangle_s\,\kappa_s^\ell = \begin{pmatrix} \int_0^t (k_s^\ell)^1\,d\langle K^1\rangle_s \\ \vdots \\ \int_0^t (k_s^\ell)^d\,d\langle K^d\rangle_s \end{pmatrix}, \qquad t\in[0,T). \tag{8}$$

Proof: Take an increasing $\mathbb{F}^\circ$-predictable process B such that $\langle K^i\rangle \ll B$ for $i=1,\dots,d$. Then we obtain $\langle K^i, K^j\rangle = \int b_s^{ij}\,dB_s$ for a matrix-valued $\mathbb{F}^\circ$-predictable process b, and so (8) amounts to finding a $\mathcal{P}(\mathbb{F}^\circ)\otimes\mathcal{U}$-measurable solution κ to the system of equations

$$\sum_{j=1}^d b_s^{ij}\,(\kappa_s^\ell)^j = (k_s^\ell)^i\,b_s^{ii} \qquad \text{for } i=1,\dots,d \text{ and all } s\in[0,T).$$

Since each $(k^\ell)^i$ is $\mathcal{P}(\widehat{\mathbb{F}}^\circ)$-measurable by Theorem 2.2 and $\mathcal{P}(\widehat{\mathbb{F}}^\circ) = \mathcal{P}(\mathbb{F}^\circ)\otimes\mathcal{U}$, this is clearly possible. **q.e.d.**

For the subsequent developments, we need a weak integrability condition on κ; see Delbaen and Schachermayer (1995) for its relation to absence of arbitrage.

Section Assumption 2.2 *The process κ from Lemma 2.8 satisfies*

$$\int_0^T (\kappa_s^\ell)^*\,d\langle K\rangle_s\,\kappa_s^\ell < \infty \qquad P\text{-a.s. for all } \ell\in U.$$

Remark: A standard argument shows that the process κ^G is $\mathcal{P}(\mathbb{G}^\circ)$-measurable, and so the stochastic integral $\int(\kappa^G)^*\,d\widetilde{K}$ is well-defined under Section Assumption 2.2. For each $\ell \in U$, the process κ^ℓ is unique up to nullsets with respect to $P \times \langle K \rangle$, and so the stochastic integrals $\int (\kappa^\ell)^* \, dK$ and $\int (\kappa^G)^* \, d\widetilde{K}$ do not depend on the choice of κ. Finally, we can now write $\widetilde{K} := \left(\widetilde{K}^1, \dots, \widetilde{K}^d\right)^*$ more compactly as

$$\widetilde{K} = K - \int d\langle K \rangle \, \kappa^G.$$

Proposition 2.9

1. *Suppose that the regular conditional distributions of G given $\mathcal{F}_t$ are equivalent to the law of G for all $t \in [0, T)$. Then there exists a local $\mathbb{G}^\circ$-martingale $\widetilde{L}$ null at 0 which is orthogonal to $\widetilde{K}$ from (4) (i.e., $\langle \widetilde{K}^i, \widetilde{L} \rangle = 0$ for $i = 1, \dots, d$) and such that*

$$\frac{1}{p_t^G} = \mathcal{E}\left(- \int \left(\kappa_s^G\right)^* \, d\widetilde{K}_s + \widetilde{L} \right)_t \quad , \quad t \in [0, T). \tag{9}$$

2. *Fix $\ell \in U$. If $p_{T^\ell -} > 0$ P-a.s., then there exists a local $\mathbb{F}^\circ$-martingale L^ℓ null at 0 which is orthogonal to K and such that*

$$p_t^\ell = \mathcal{E}\left(\int (\kappa_s^\ell)^* \, dK_s + L^\ell \right)_t \quad , \quad t \in [0, T). \tag{10}$$

Proof:

1. Since $1/p^G$ is a strictly positive $\mathbb{G}^\circ$-martingale, there exists a local $\mathbb{G}^\circ$-martingale $\widetilde{O}$ null at 0 such that $1/p^G = \mathcal{E}(\widetilde{O})$. Because of the continuity of $\widetilde{K}$, we can write $\widetilde{O}$ as $\widetilde{O} = \int \widetilde{h}^* \, d\widetilde{K} + \widetilde{L}$ with a $\mathbb{G}^\circ$-predictable process $\widetilde{h}$ satisfying $\int_0^t \widetilde{h}_s^* \, d\langle \widetilde{K} \rangle_s \, \widetilde{h}_s < \infty$ P-a.s. for all $t \in [0, T)$ and a local $\mathbb{G}^\circ$-martingale $\widetilde{L}$ null at 0 which is orthogonal to $\widetilde{K}$; see Ansel and Stricker (1993a). This yields

$$\langle p^G, K^i \rangle = \left\langle \frac{1}{\mathcal{E}(\widetilde{O})}, K^i \right\rangle = - \int p_-^G \left(d\langle K \rangle \, \widetilde{h} \right)^i$$

by using the continuity of $\widetilde{K}$, the orthogonality of $\widetilde{K}$ and $\widetilde{L}$ and Itô's formula. On the other hand, we can also compute $\langle p^G, K^i \rangle$ with the help of Theorem 2.2, and we want to use this to identify $\widetilde{h}$ as $-\kappa^G$. Leaving aside measurability questions for the moment, we simply replace ℓ by G in (3) and obtain

$$\langle p^G, K^i \rangle = \int p_-^G (k^G)^i \, d\langle K^i \rangle \tag{11}$$

$$= \int p_-^G \left(d\langle K \rangle \, \kappa^G \right)^i \tag{12}$$

from Lemma 2.8. Hence we conclude that $\int d\langle K \rangle \, \widetilde{h} = - \int d\langle K \rangle \, \kappa^G$ and therefore $\int \widetilde{h}^* \, d\widetilde{K} = - \int (\kappa^G)^* \, d\widetilde{K}$ by the preceding remark. Plugging this into $\widetilde{O}$ yields (9), and so it only remains to justify (11) and (12). Since $p : (\omega, \ell, t) \mapsto p_t^\ell(\omega)$ is a measurable function, Proposition 2 of Stricker and Yor (1978) implies the existence of a version of $\langle p^\ell, K^i \rangle$ which is measurable in ℓ, and we denote this again by $\langle p^\ell, K^i \rangle$.

Since $\int (k^\ell)^i p^\ell_- \, d\langle K^i \rangle$ is well-defined by (3), and since k^ℓ and p^ℓ are measurable in ℓ, Lemma 2 of Stricker and Yor (1978) now yields the existence of a version of $\int (k^\ell)^i p^\ell_- \, d\langle K^i \rangle$ which is measurable in ℓ. This justifies (11). Moreover, Lemma 2 of Stricker and Yor (1978) also implies the existence of versions of $\int (k^\ell)^i \, d\langle K^i \rangle$ and $\int \left(d\langle K \rangle \, \kappa^\ell \right)^i$ which are measurable in ℓ and thus the existence of a version of $\int p^\ell_- \left(d\langle K \rangle \, \kappa^\ell \right)^i$ which is measurable in ℓ. This justifies (12) and completes the proof of the first assertion.

2. The properties of p^ℓ in Lemma 2.1 and the condition $p_{T\ell-} > 0$ P-a.s. guarantee by Exercise 6.1 of Jacod (1979) the existence of a local $I\!\!F^\circ$-martingale O such that $p^\ell = \mathcal{E}(O)$. The rest of the proof then proceeds as in the first part; it actually becomes even simpler since there are no measurability problems for fixed ℓ.

q.e.d.

Remark: If the regular conditional distributions of G given $\mathcal{F}_t$ are equivalent to the law of G for all $t \in [0, T)$, then the condition in the second part of Proposition 2.9 is automatically satisfied for all $\ell \in U$.

The next result gives an explicit expression for $\widetilde{L}$ in (9) in terms of L^G if p^ℓ is continuous for all $\ell \in U$. As a consequence, we obtain then in particular that $1/p^G$ can be written as a stochastic exponential of a stochastic integral with respect to $\widetilde{K}$, if we have in addition a martingale representation theorem for the filtration $I\!\!F$. This happens for instance in a complete financial market, and we shall come back to this case in section 4.

Corollary 2.10

1. *If p^ℓ is continuous and strictly positive for all $\ell \in U$, then*

$$\frac{1}{p_t^G} = \mathcal{E}\left(-\int (\kappa_s^G)^* \, d\widetilde{K}_s - L^G + \langle L^G \rangle \right)_t \quad , \qquad t \in [0, T). \tag{13}$$

In particular, $\widetilde{L}$ from (9) is given by

$$\widetilde{L}_t = -L_t^G + \langle L^G \rangle_t \quad , \qquad t \in [0, T). \tag{14}$$

2. *In particular, if $p^\ell = \mathcal{E}\left(\int (\kappa^\ell)^* \, dK \right)$ for all $\ell \in U$, then*

$$\frac{1}{p_t^G} = \mathcal{E}\left(-\int (\kappa^G)^* \, d\widetilde{K} \right)_t \quad , \qquad t \in [0, T).$$

Proof: Since p^ℓ is strictly positive, (10) implies for all $\ell \in U$ that

$$\int \frac{1}{p^\ell} \, dp^\ell = \int (\kappa^\ell)^* \, dK + L^\ell = \int (\kappa^\ell)^* \, d\widetilde{K} + \int (\kappa^\ell)^* \, d\langle \widetilde{K} \rangle \, \kappa^G + L^\ell. \tag{15}$$

Thus, the continuity of p^ℓ implies for all $\ell \in U$ that L^ℓ is continuous and by (10) that

$$\begin{aligned} \frac{1}{p^\ell} &= \exp\Big(-\int (\kappa^\ell)^*\, dK + \frac{1}{2}\int (\kappa^\ell)^*\, d\langle K\rangle\, \kappa^\ell - L^\ell + \frac{1}{2}\langle L^\ell\rangle\Big) \\ &= \exp\Big(-\int (\kappa^\ell)^*\, d\widetilde{K} - \int (\kappa^\ell)^*\, d\langle K\rangle\, \kappa^G + \frac{1}{2}\int (\kappa^\ell)^*\, d\langle K\rangle\, \kappa^\ell - L^\ell + \frac{1}{2}\langle L^\ell\rangle\Big). \quad (16) \end{aligned}$$

We first show the measurability on $\Omega \times [0,T) \times U$ of all the terms appearing in (16). By Lemma 2.8, κ^ℓ is $\mathcal{P}(\mathbb{F}^\circ)\otimes\mathcal{U}$-measurable, thus $\mathcal{O}(\mathbb{G}^\circ)\otimes\mathcal{U}$-measurable. Since $\int (\kappa^\ell)^*\, d\langle K\rangle\, \kappa^\ell$ is locally integrable thanks to Section Assumption 2.2 and the continuity of K, Theorem 2 of Stricker and Yor (1978) (plus the note added in proof on p.133) implies the existence of an $\mathcal{O}(\mathbb{G}^\circ) \otimes \mathcal{U}$-measurable version of $\int (\kappa^\ell)^* d\widetilde{K}$. Hence Proposition 2 of Stricker and Yor (1978) yields that $\int (\kappa^\ell)^* d\widetilde{K}$ and $\int (\kappa^\ell)^* d\langle K\rangle\, \kappa^G = \left\langle \int (\kappa^\ell)^* d\widetilde{K}, \int (\kappa^G)^* d\widetilde{K}\right\rangle$ are measurable functions on $\Omega \times [0,T) \times U$. Since $1/p^\ell$ is $\mathcal{P}(\mathbb{F}^\circ) \otimes \mathcal{U}$-measurable by Lemma 2.1 and continuous, hence locally bounded, Theorem 1 of Stricker and Yor (1978) implies the existence of an $\mathcal{O}(\mathbb{F}^\circ) \otimes \mathcal{U}$-measurable version of $\int 1/p^\ell\, dp^\ell$. Hence we obtain that L^ℓ has a measurable version, since all other terms in (15) have one. Finally, $\langle L^\ell\rangle$ has a measurable version by Proposition 2 of Stricker and Yor (1978).
Thanks to these measurability properties, we can now substitute G for ℓ in (16) to obtain

$$\frac{1}{p^G} = \exp\Big(-\int (\kappa^G)^*\, d\widetilde{K} - \frac{1}{2}\int (\kappa^G)^*\, d\langle K\rangle\, \kappa^G - L^G + \frac{1}{2}\langle L^G\rangle\Big),$$

hence (13). Comparing this to (9) yields (14) by the uniqueness of the stochastic exponential. **q.e.d.**

3 Utility Maximization

In this section, we first explain the optimization problems faced by two investors with different information. We then solve these problems explicitly and use the results of section 2 to rewrite the utility gain of the better informed investor in a form that is more suitable for further analysis.

3.1 The Model

The uncertainty of the security market is described by our given probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. We fix a d-dimensional continuous local $\mathbb{F}$-martingale $M = (M^1, \ldots, M^d)^*$ and a d-dimensional predictable process $\alpha = (\alpha^1, \ldots, \alpha^d)^*$ with

$$E\left[\int_0^T \alpha_s^*\, d\langle M\rangle_s\, \alpha_s\right] < \infty. \quad (17)$$

The *discounted prices* $X = (X^1, \ldots, X^d)^*$ of d stocks are then assumed to evolve according to the stochastic differential equations

$$dX_t^i = X_t^i \left(dM_t^i + \sum_{j=1}^d \alpha_t^j \, d\langle M^i, M^j \rangle_t \right) \qquad , \qquad t \in [0,T], \, i = 1, \ldots, d,$$

with $X_0^i > 0$. In addition to the *ordinary economic agent* whose information flow is given by the filtration $I\!F$, we also want to consider an *insider* who is better informed. His extra information is the knowledge at time 0 of the outcome of some $\mathcal{F}$-measurable random variable G. For instance, G might be the price of a stock at time T, or the price of a stock at time T distorted by some noise, or the value of some external source of uncertainty, etc. Technically, G will have values in a Polish space $(U, \mathcal{U})$, and the information flow of the insider is described by the larger filtration $I\!\!G = (\mathcal{G}_t)_{t \in [0,T]}$ given as in section 2 by

$$\mathcal{G}_t = \bigcap_{\epsilon > 0} \Big(\mathcal{F}_{t+\epsilon} \vee \sigma(G) \Big) \qquad , \qquad t \in [0,T].$$

We shall also assume that M is a $I\!\!G^\circ$-semimartingale and that its canonical decomposition can be constructed as in Theorem 2.2. More precisely, we make the

Section Assumption 3.1 *M is a $I\!\!G^\circ$-semimartingale, and the local $I\!\!G^\circ$-martingale $\widetilde{M}$ in its canonical $I\!\!G^\circ$-decomposition has the form*

$$\widetilde{M}_t^i = M_t^i - \int_0^t (m_s^G)^i \, d\langle M^i \rangle_s \qquad , \qquad t \in [0,T), \, i = 1, \ldots, d, \tag{18}$$

where $m = (m_t^\ell)$ has the same measurability and integrability properties as k in Theorem 2.2.

As in Lemma 2.8, we obtain a $\mathcal{P}(I\!F^\circ) \otimes \mathcal{U}$-measurable process $(\mu_t^\ell)_{t \in [0,T)}$ such that

$$\int_0^t d\langle M \rangle_s \, \mu_s^\ell = \begin{pmatrix} \int_0^t (m_s^\ell)^1 \, d\langle M^1 \rangle_s \\ \vdots \\ \int_0^t (m_s^\ell)^d \, d\langle M^d \rangle_s \end{pmatrix} \qquad , \qquad t \in [0,T). \tag{19}$$

Thus we can write $\widetilde{M}$ more compactly as $\widetilde{M} = M - \int d\langle M \rangle \, \mu^G$, and so the discounted stock price evolution from the insider's point of view is

$$\begin{aligned} dX^i &= X^i \left(d\widetilde{M}^i + (m^G)^i \, d\langle M^i \rangle + \sum_{j=1}^d \alpha^j \, d\langle M^i, M^j \rangle \right) \\ &= X^i \left(d\widetilde{M}^i + \Big(d\langle M \rangle \, (\alpha + \mu^G) \Big)^i \right) \qquad , \qquad i = 1, \ldots, d. \end{aligned}$$

We now impose

Section Assumption 3.2 $E\left[\int_0^T (\mu_s^G)^* \, d\langle M\rangle_s \, \mu_s^G\right] < \infty.$

This allows us in particular to extend $\widetilde{M}$ to the closed interval $[0,T]$ by defining $\widetilde{M}_T := \lim_{t\to T} \widetilde{M}_t$. Note here that we do not assume that $\widetilde{M}$ is a local $\mathbb{G}$-martingale on $[0,T]$.

Remark: Our framework includes the classical incomplete market model studied by Karatzas, Lehoczky, Shreve and Xu (1991) where the filtration $\mathbb{F}$ is generated by an n-dimensional Brownian motion W. In our notation, they have $M = \int \sigma \, dW$ for a $d \times n$-matrix-valued $\mathbb{F}$-progressively measurable process $\sigma = (\sigma_t)_{t\in[0,T]}$ with full rank $d \le n$ for every $t \in [0,T]$, and $\alpha = (\sigma\sigma^*)^{-1}(b - r\mathbf{1_d})$ for $\mathbb{F}$-progressively measurable processes b ($\mathbb{R}^d$-valued) and r ($\mathbb{R}$-valued) such that $\int_0^T |b_t| \, dt < \infty$ and $\int_0^T |r_t| \, dt \le$ const. P-a.s. Furthermore, their standing assumption 5.1 imposes exactly our condition (17). As a special case of the model of Karatzas, Lehoczky, Shreve and Xu (1991), Pikovsky and Karatzas (1996) consider in their study of insider trading the complete market model with $d = n$; this is therefore included in our framework as well.

Definition 3.1 *Let $t \in [0,T]$, $x > 0$ and denote by $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$ a generic filtration.*

1. *An $\mathbb{H}$-portfolio process up to time t is an $\mathbb{R}^d$-valued $\mathbb{H}$-predictable process $\pi = (\pi_s)_{s\in[0,t]}$ such that $\int_0^t \pi_s^* \, d\langle M\rangle_s \, \pi_s < \infty$ P-a.s.*

2. *For an $\mathbb{H}$-portfolio process π, the discounted wealth process $V(x,\pi)$ is defined by $V_0(x,\pi) = x$ and*

$$dV_s(x,\pi) = \sum_{i=1}^d \pi_s^i V_s(x,\pi) \, \frac{dX_s^i}{X_s^i} \qquad \text{for } s \in [0,t]. \tag{20}$$

3. *The class of admissible $\mathbb{H}$-portfolio processes up to time t is defined by*

$$\mathcal{A}_{\mathbb{H}}(x,t) := \left\{\pi \,\middle|\, \pi \text{ is an } \mathbb{H}\text{-portfolio process and } E[\log^- V_t(x,\pi)] < \infty\right\}. \tag{21}$$

As usual, π_t^i describes the proportion of total wealth at time t invested in asset i, and (20) is the familiar self-financing condition; see for instance Pikovsky and Karatzas (1996). For a strategy $\pi \in \mathcal{A}_{\mathbb{H}}(x,t)$ with $x > 0$, the wealth process is strictly positive and explicitly given by

$$V_s(x,\pi) = x\mathcal{E}\left(\int \sum_{i=1}^d \pi_u^i \frac{dX_u^i}{X_u^i}\right)_s = x\mathcal{E}\left(\int \pi_u^* \, dM_u + \int \pi_u^* \, d\langle M\rangle_u \, \alpha_u\right)_s \tag{22}$$

for $s \in [0,t]$. From the point of view of the insider, this can also be written as

$$V_s(x,\pi) = x\mathcal{E}\left(\int \pi_u^* \, d\widetilde{M}_u + \int \pi_u^* \, d\langle M\rangle_u \, (\alpha_u + \mu_u^G)\right)_s \quad , \qquad s \in [0,t]. \tag{23}$$

Definition 3.2 (Optimization Problems)
Let the initial wealth $x > 0$ and the time horizon $t \in [0, T]$ be given.

1. *The ordinary economic agent's optimization problem is to solve:*

$$\max_{\pi \in \mathcal{A}_{\mathbb{F}}(x,t)} E[\log V_t(x, \pi)].$$

2. *The insider's optimization problem is to solve:*

$$\max_{\pi \in \mathcal{A}_{\mathbb{G}}(x,t)} E[\log V_t(x, \pi)].$$

While it is not the most general case, assuming a logarithmic utility function will enable us to exploit the exponential structure of the wealth process and to obtain fairly explicit results in the next section.

3.2 Solution of the Logarithmic Utility Maximization Problems

Let us first give an easy argument under more restrictive integrability assumptions to motivate the construction of the solutions. (23) gives for $\pi \in \mathcal{A}_{\mathbb{G}}(x, t)$

$$\begin{aligned} \log V_t(x, \pi) &= \log x + \int_0^t \pi_s^* \, d\widetilde{M}_s + \int_0^t \pi_s^* \, d\langle M \rangle_s \left(\alpha_s + \mu_s^G - \frac{1}{2} \pi_s \right) \\ &= \log x + \int_0^t \pi_s^* \, d\widetilde{M}_s + \frac{1}{2} \int_0^t (\alpha_s + \mu_s^G)^* \, d\langle M \rangle_s \, (\alpha_s + \mu_s^G) \\ &\quad - \frac{1}{2} \int_0^t (\alpha_s + \mu_s^G - \pi_s)^* \, d\langle M \rangle_s \, (\alpha_s + \mu_s^G - \pi_s) \end{aligned} \tag{24}$$

by adding and subtracting the term $\frac{1}{2} \int_0^t (\alpha_s + \mu_s^G)^* \, d\langle M \rangle_s \, (\alpha_s + \mu_s^G)$. If we now had $E\left[\int_0^t \pi_s^* \, d\langle M \rangle_s \, \pi_s\right] < \infty$, the local $\mathbb{G}^\circ$-martingale $\int \pi_s^* \, d\widetilde{M}_s$ would be a true martingale and hence would have expectation zero. Then $\pi_s = \alpha_s + \mu_s^G$, $s \le t$, would be an optimal strategy for the insider up to time t, yielding a maximal expected utility up to time t of $\log x + \frac{1}{2} E\left[\int_0^t (\alpha_s + \mu_s^G)^* \, d\langle M \rangle_s (\alpha_s + \mu_s^G)\right]$. Setting $\mu^G \equiv 0$, we could similarly get the optimal strategy and maximal expected utility of the ordinary agent.
We now show that even in our larger class of admissible strategies, the solution of the optimization problems is of the above form. Our argument exploits the close connection between logarithmic optimization problems and the minimal martingale density processes.

Definition 3.3 *The process $\widehat{Z}^{\mathbb{F}} = (\widehat{Z}_s^{\mathbb{F}})_{s \in [0,T]}$ defined by*

$$\widehat{Z}_s^{\mathbb{F}} := \mathcal{E}\left(- \int \alpha_u^* \, dM_u \right)_s \quad , \quad s \in [0, T] \tag{25}$$

is called the $\mathbb{F}$-minimal martingale density, and the process $\widehat{Z}^{\mathbb{G}} = (\widehat{Z}_s^{\mathbb{G}})_{s \in [0,T]}$ defined by

$$\widehat{Z}_s^{\mathbb{G}} := \mathcal{E}\left(- \int (\alpha_u + \mu_u^G)^* \, d\widetilde{M}_u \right)_s \quad , \quad s \in [0, T] \tag{26}$$

is called the $\mathbb{G}$-minimal martingale density.

Note that (17) and Section Assumption 3.2 imply that $\int(\alpha_s + \mu_s^G)^* \, d\widetilde{M}_s$ is well-defined and a local $\mathbb{G}$-martingale on $[0,T]$, and that both minimal martingale densities are strictly positive.

Proposition 3.4

1. *For $t \in [0,T]$, the processes $\widehat{Z}^{\mathbb{F}} X^i$, $i = 1, \ldots, d$, and $\widehat{Z}^{\mathbb{F}} V(x,\pi)$ with $\pi \in \mathcal{A}_{\mathbb{F}}(x,t)$ are local $\mathbb{F}$-martingales on $[0,t]$.*

2. *For $t \in [0,T)$, the processes $\widehat{Z}^{\mathbb{G}} X^i$, $i = 1, \ldots, d$, and $\widehat{Z}^{\mathbb{G}} V(x,\pi)$ with $\pi \in \mathcal{A}_{\mathbb{G}}(x,t)$ are local $\mathbb{G}$-martingales on $[0,t]$. If $\widetilde{M}$ is a local $\mathbb{G}$-martingale, this even holds for $t \in [0,T]$.*

Proof: The first part is well known and can be found in Ansel and Stricker (1992), (1993b) or Schweizer (1995). The second claim is similarly obtained by applying Itô's formula to get

$$
\begin{aligned}
d(\widehat{Z}^{\mathbb{G}} X^i) &= X^i \, d\widehat{Z}^{\mathbb{G}} + \widehat{Z}^{\mathbb{G}} \, dX^i + d\langle \widehat{Z}^{\mathbb{G}}, X^i \rangle \\
&= X^i \, d\widehat{Z}^{\mathbb{G}} + \widehat{Z}^{\mathbb{G}} X^i \, d\widetilde{M}^i \\
&\quad + \widehat{Z}^{\mathbb{G}} X^i \left((m^G)^i \, d\langle M^i \rangle + \sum_{j=1}^d \alpha^j \, d\langle M^i, M^j \rangle - d\left\langle \int (\alpha + \mu^G)^* \, d\widetilde{M}, M^i \right\rangle \right).
\end{aligned}
$$

Since

$$
d\left\langle \int (\mu^G)^* \, d\widetilde{M}, M^i \right\rangle = \sum_{j=1}^d (\mu^G)^j \, d\langle M^j, M^i \rangle = \left(d\langle M \rangle \, \mu^G \right)^i = (m^G)^i \, d\langle M^i \rangle
$$

by (19), we have

$$
d(\widehat{Z}^{\mathbb{G}} X^i) = X^i \, d\widehat{Z}^{\mathbb{G}} + \widehat{Z}^{\mathbb{G}} X^i \, d\widetilde{M}^i,
$$

and this shows that $\widehat{Z}^{\mathbb{G}} X^i$ is a local $\mathbb{G}^\circ$-martingale, and even a local $\mathbb{G}$-martingale if $\widetilde{M}^i$ is. The remaining assertions are proved in a similar way. **q.e.d.**

The next result gives explicit solutions for the two investors' optimization problems.

Theorem 3.5

1. *Fix a time horizon $t \in [0,T]$. An optimal strategy up to time t for the ordinary economic agent is then given by*

$$
\pi_s^{ord} := \alpha_s \quad , \quad 0 \le s \le t, \tag{27}
$$

and the corresponding maximal expected logarithmic utility up to time t is

$$
E\left[\log V_t(x, \pi^{ord})\right] = \log x + \frac{1}{2} E\left[\int_0^t \alpha_s^* \, d\langle M \rangle_s \, \alpha_s \right]. \tag{28}
$$

2. *Fix a time horizon $t \in [0,T)$. An optimal strategy up to time t for the insider is then given by*

$$\pi_s^{opt} := \alpha_s + \mu_s^G \quad , \qquad 0 \leq s \leq t, \tag{29}$$

and the corresponding maximal expected logarithmic utility up to time t is

$$E[\log V_t(x, \pi^{opt})] = \log x + \frac{1}{2} E\left[\int_0^t \alpha_s^* \, d\langle M \rangle_s \, \alpha_s\right] + \frac{1}{2} E\left[\int_0^t (\mu_s^G)^* \, d\langle M \rangle_s \, \mu_s^G\right]. \tag{30}$$

3. *The insider's maximal expected logarithmic utility up to the terminal time T is at least*

$$\log x + \frac{1}{2} E\left[\int_0^T \alpha_s^* \, d\langle M \rangle_s \, \alpha_s\right] + \frac{1}{2} E\left[\int_0^T \left(\mu_s^G\right)^* \, d\langle M \rangle_s \, \mu_s^G\right]. \tag{31}$$

4. *If $\widetilde{M}$ is a local $\mathbb{G}$-martingale, then an optimal strategy up to the terminal time T for the insider is given by*

$$\pi_s^{opt} := \alpha_s + \mu_s^G \quad , \qquad 0 \leq s \leq T, \tag{32}$$

and the corresponding maximal expected logarithmic utility up to time T is

$$\log x + \frac{1}{2} E\left[\int_0^T \alpha_s^* \, d\langle M \rangle_s \, \alpha_s\right] + \frac{1}{2} E\left[\int_0^T \left(\mu_s^G\right)^* \, d\langle M \rangle_s \, \mu_s^G\right]. \tag{33}$$

Proof: We omit the proof for the 'ordinary agent part' because it is a copy of the other parts with $\mu^G \equiv 0$ and $\widehat{Z}^{\mathbb{F}}$ instead of $\widehat{Z}^{\mathbb{G}}$.

1. For concave C^1-functions u such that u' has an inverse I, we recall the inequality

$$u(a) \leq u\Big(I(b)\Big) - b\Big(I(b) - a\Big) \qquad \text{for all } a, b.$$

If we fix $t \in [0,T)$ and $\pi \in \mathcal{A}_{\mathbb{G}}(x,t)$, we thus obtain with $u = \log$, $a = V_t(x,\pi)$ and $b = y\widehat{Z}_t^{\mathbb{G}}$ for some constant $y > 0$ that

$$\log V_t(x,\pi) \leq \log \frac{1}{y\widehat{Z}_t^{\mathbb{G}}} - y\widehat{Z}_t^{\mathbb{G}} \left(\frac{1}{y\widehat{Z}_t^{\mathbb{G}}} - V_t(x,\pi) \right) = -\log y - \log \widehat{Z}_t^{\mathbb{G}} - 1 + y\widehat{Z}_t^{\mathbb{G}} V_t(x,\pi).$$

Since $V(x,\pi)$ is nonnegative and $\widehat{Z}^{\mathbb{G}} V(x,\pi)$ is by Proposition 3.4 a local $\mathbb{G}^\circ$-martingale, hence a $\mathbb{G}^\circ$-supermartingale starting in x, we get

$$E[\log V_t(x,\pi)] \leq -1 - \log y - E[\log \widehat{Z}_t^{\mathbb{G}}] + yx \tag{34}$$

for all $\pi \in \mathcal{A}_{\mathbb{G}}(x,t)$ and $y > 0$. To find an optimal portfolio, it is therefore enough to find $\pi \in \mathcal{A}_{\mathbb{G}}(x,t)$ and $y > 0$ such that equality holds in (34). We claim that π^{opt} defined by (29) and $y = 1/x$ will do. Indeed, (24) yields

$$\begin{aligned} \log V_t(x,\pi^{opt}) &= \log x + \int_0^t (\alpha_s + \mu_s^G)^* \, d\widetilde{M}_s + \frac{1}{2} \int_0^t (\alpha_s + \mu_s^G)^* \, d\langle M \rangle_s \, (\alpha_s + \mu_s^G) \\ &= -\log y - \log \widehat{Z}_t^{\mathbb{G}} \end{aligned}$$

by (26), and so we get equality in (34). Moreover, π^{opt} is in $\mathcal{A}_{\mathbb{G}}(x,t)$ due to (17), Section Assumption 3.2, and the fact that $\widehat{Z}_t^{\mathbb{G}} > 0$ P-a.s. so that $\log^-(\widehat{Z}_t^{\mathbb{G}}) = 0$ P-a.s.

2. Thanks to (17), Doob's inequality implies that both $\sup_{0\le s\le t}|\int_0^s \alpha_u^* \, dM_u|$ and $\sup_{0\le s\le t}\left|\int_0^s \alpha_u^* \, d\widetilde{M}_u\right|$ are integrable so that $\int \alpha_u^* \, dM_u$ and $\int \alpha_u^* \, d\widetilde{M}_u$ are martingales (with respect to $\mathbb{F}$ and $\mathbb{G}$, respectively) on $[0,t]$. By the definitions of $\widetilde{M}$ and μ^G, we therefore obtain

$$0 = E\left[\int_0^t \alpha_u^* \, dM_u - \int_0^t \alpha_u^* \, d\widetilde{M}_u\right] = E\left[\int_0^t \alpha_u^* \, d\langle M\rangle_u \, \mu_u^G\right],$$

and (30) follows by squaring out $(\alpha+\mu^G)^* \, d\langle M\rangle \, (\alpha+\mu^G)$.

3. We now show (31). Up to time $t < T$, let the insider choose the portfolio $\pi^{opt} = \alpha + \mu^G$ as in (29). At time t, he then invests his wealth in the riskless asset and keeps it there up to time T so that his strategy is given by

$$\widehat{\pi}_s := \pi_s^{opt}\mathbf{I}_{\{s\le t\}} \qquad , \qquad s\in[0,T].$$

This implies that $\widehat{\pi} \in \mathcal{A}_{\mathbb{G}}(x,T)$ and $V_T(x,\widehat{\pi}) = V_t(x,\pi^{opt})$ for every $t < T$. Therefore the insider's maximal expected utility is at least

$$\sup_{t<T} E[\log V_t(x,\pi^{opt})] = \log x + \frac{1}{2}E\left[\int_0^T \alpha_s^* \, d\langle M\rangle_s \, \alpha_s\right] + \frac{1}{2}E\left[\int_0^T (\mu_s^G)^* \, d\langle M\rangle_s \, \mu_s^G\right]$$

by (30) and monotone convergence, and this proves (31).

4. If $\widetilde{M}$ is a local $\mathbb{G}$-martingale, then $\widehat{Z}^{\mathbb{G}}V(x,\pi)$ is by Proposition 3.4 a local $\mathbb{G}$-martingale, hence a $\mathbb{G}$-supermartingale. Thus we can repeat steps 1 and 2 with $t = T$ to complete the proof. **q.e.d.**

Definition 3.6

1. *The insider's additional expected logarithmic utility up to time $t\in[0,T]$ is defined by*

$$\sup_{\pi\in\mathcal{A}_{\mathbb{G}}(x,t)} E[\log V_t(x,\pi)] - \sup_{\pi\in\mathcal{A}_{\mathbb{F}}(x,t)} E[\log V_t(x,\pi)] \qquad , \qquad t\in[0,T].$$

2. *The insider's utility gain up to time $t\in[0,T]$ is defined by*

$$E[a_t] \qquad \textit{with} \quad a_t := \frac{1}{2}\int_0^t (\mu_s^G)^* \, d\langle M\rangle_s \, \mu_s^G. \tag{35}$$

Before we proceed to rewrite the utility gain, let us comment on the above terminology. Under Section Assumption 3.2, $E[a_t]$ coincides with the insider's additional expected logarithmic utility up to any time $t < T$. A look at the above proofs reveals that this remains true even if we only have $E[a_t] < \infty$ for all $t < T$ and $E[a_T] = \infty$. Examples in Pikovsky and Karatzas (1996) and section 4 show the latter situation to be typical. Even under this weakened assumption, the same argument as for (31) gives $E[a_T]$ as a lower bound for the insider's additional expected utility up to time T, and thus the two quantities also coincide for $E[a_T] = \infty$. If $E[a_T] < \infty$, they agree again by (33) if $\widetilde{M}$ is a local $\mathbb{G}$-martingale; we shall see examples of this in subsections 4.1 and 4.3. The only case where a discrepancy between the two quantities could arise is if $E[a_T] < \infty$ and $\widetilde{M}$ is a local $\mathbb{G}^\circ$-martingale, but not a local $\mathbb{G}$-martingale. It would be interesting to see an example of this type analyzed in more detail.

3.3 p^G-Representation of the Insider's Utility Gain

Although (35) provides an explicit expression for the insider's utility gain in terms of μ^G, it is not really useful in that form. Even in fairly simple examples, (35) is rather hard to evaluate; this is illustrated by the results of Pikovsky and Karatzas (1996) where closed-form solutions or upper bounds for (35) are obtained only in some special cases and after sometimes rather cumbersome calculations. In this section, we therefore derive an alternative expression for the utility gain which can be evaluated simply and explicitly for a large class of examples for G. By applying the results of section 2 to the continuous local $\mathbb{F}$-martingale M, we compute the utility gain $E[a_t]$ for all $t < T$ and then let t increase to T. This requires

Section Assumption 3.3 *The regular conditional distributions of G given $\mathcal{F}_t$ are P-a.s. equivalent to the law of G for all $t \in [0,T)$, and the process of Radon-Nikodym derivatives is P-a.s. continuous in t. More precisely, we assume that there exists a strictly positive, continuous $\mathcal{P}(\mathbb{F}^\circ) \otimes \mathcal{U}$-measurable process $(p_t^\ell)_{t\in[0,T)}, \ell \in U$, such that for all $B \in \mathcal{U}$, we have*

$$P[G \in B|\mathcal{F}_t](\omega) = \int_B p_t^\ell(\omega)\, P[G \in d\ell] \qquad \text{for } P\text{-a.a. } \omega \text{ and all } t \in [0,T).$$

Note that by Theorem 2.2, Section Assumption 3.3 implies Section Assumption 3.1.

Remark: Intuitively, the assumption that the conditional laws of G given $\mathcal{F}_t$ are equivalent to the law of G for $t < T$ means that at all times prior to T, the insider has an informational advantage over the ordinary agent since the latter sees all outcomes of G as possible before time T. More vaguely, the outcome of G is not revealed to the public before T.

Applying Proposition 2.9 to M gives the existence of a continuous local $\mathbb{G}^\circ$-martingale $\widetilde{N}$ null at 0 and orthogonal to M such that

$$\frac{1}{p_t^G} = \mathcal{E}\left(-\int (\mu_s^G)^*\, d\widetilde{M}_s + \widetilde{N}\right)_t \quad , \qquad t \in [0,T). \tag{36}$$

Continuity and orthogonality of $\widetilde{M}$ and $\widetilde{N}$ therefore yield for $t \in [0,T)$

$$-\log p_t^G = -\int_0^t (\mu_s^G)^*\, d\widetilde{M}_s + \widetilde{N}_t - \frac{1}{2}\int_0^t (\mu_s^G)^*\, d\langle M\rangle_s\, \mu_s^G - \frac{1}{2}\langle \widetilde{N}\rangle_t. \tag{37}$$

If the expectations of all terms were finite and if $\widetilde{N}$ was not only a local $\mathbb{G}^\circ$-martingale, but a true $\mathbb{G}^\circ$-martingale, the utility gain could obviously be written as

$$E[a_t] = \frac{1}{2}E\left[\int_0^t (\mu_s^G)^*\, d\langle M\rangle_s\, \mu_s^G\right] = E[\log p_t^G] - \frac{1}{2}E[\langle \widetilde{N}\rangle_t].$$

The next result will provide such a relation even without the assumption that $\int (\mu_s^G)^*\, d\widetilde{M}_s$ and $\widetilde{N}$ are $\mathbb{G}^\circ$-martingales. For its formulation, we recall that for two probabilities P and Q on $(\Omega, \mathcal{A})$, the *relative entropy* of P with respect to Q on $\mathcal{A}$ is defined as

$$H_{\mathcal{A}}(P|Q) := \begin{cases} E_P\left[\log \left.\frac{dP}{dQ}\right|_{\mathcal{A}}\right] & \text{, if } P \ll Q \text{ on } \mathcal{A} \\ +\infty & \text{, otherwise.} \end{cases}$$

It is well known that $H_{\mathcal{A}}(P|Q)$ is always nonnegative, equal to 0 if and only if $P = Q$ on $\mathcal{A}$, and increasing in $\mathcal{A}$.

Theorem 3.7

1. *The insider's utility gain up to time $t \in [0,T)$ satisfies the relation*

$$E[a_t] + \frac{1}{2}E[\langle \widetilde{N}\rangle_t] = E[\log p_t^G] = H_{\mathcal{G}_t}(P|\widetilde{P}_t) \quad , \qquad t \in [0,T), \tag{38}$$

where $\widetilde{P}_t$ is the $[0,t]$-insider martingale measure defined in Proposition 2.3.

2. *The insider's utility gain up to the terminal time T satisfies the relation*

$$E[a_T] + \frac{1}{2}E[\langle \widetilde{N}\rangle_T] = \lim_{t\to T} E[\log p_t^G] = \lim_{t\to T} H_{\mathcal{G}_t}(P|\widetilde{P}_t). \tag{39}$$

Proof: Let (T_n) be an increasing sequence of $\mathbb{G}^\circ$-stopping times such that all terms in (37) are bounded when stopped at T_n and such that $\left(\int (\mu_s^G)^*\, d\widetilde{M}_s\right)^{T_n}$ and $\widetilde{N}^{T_n}$ are $\mathbb{G}^\circ$-martingales. For each $t \in [0,T)$, we then have

$$E\left[\log p_{T_n\wedge t}^G\right] = \frac{1}{2}E\left[\int_0^{T_n\wedge t} (\mu_s^G)^*\, d\langle M\rangle_s\, \mu_s^G\right] + \frac{1}{2}E\left[\langle \widetilde{N}\rangle_{T_n\wedge t}\right],$$

and as $n \to \infty$, the right-hand side increases to $\frac{1}{2}E\left[\int_0^t (\mu_s^G)^* \, d\langle M\rangle_s \, \mu_s^G\right] + \frac{1}{2}E[\langle \widetilde{N}\rangle_t]$ by monotone convergence. Thus it only remains to show that

$$E\left[\log p^G_{T_n \wedge t}\right] \longrightarrow E[\log p^G_t] \qquad \text{as } n \to \infty.$$

Since $1/p^G$ is a $\mathbb{G}^\circ$-martingale by Proposition 2.3, the optional stopping theorem implies that $E\left[1/p^G_{T_n\wedge t}\middle|\mathcal{G}_{T_{n-1}\wedge t}\right] = 1/p^G_{T_{n-1}\wedge t}$, and so $(1/p^G_{T_n\wedge t})_{n\in\mathbb{N}}$ is a sequence of probability densities on $(\mathcal{G}_{T_n\wedge t})_{n\in\mathbb{N}}$ with limit $1/p^G_t$. Hence we can apply Lemma 2 of Barron (1985) to the sequence $p^G_{T_n\wedge t} = \frac{dP}{d\widetilde{P}_t}\Big|_{\mathcal{G}_{T_n\wedge t}}$, $n \in \mathbb{N}$, to conclude that

$$\lim_{n\to\infty} E\left[\log p^G_{T_n\wedge t}\right] = E[\log p^G_t] = H_{\mathcal{G}_t}(P|\widetilde{P}_t)$$

by (6). This proves the first assertion, and the second follows by monotone convergence.

q.e.d.

In the special case where $\widetilde{N}$ vanishes, the insider's utility gain is just the relative entropy of P with respect to the $[0,t]$-insider martingale measure. This happens for instance if we have a martingale representation theorem for the filtration $\mathbb{F}$.

Corollary 3.8 *If $p^\ell = \mathcal{E}\left(\int (\mu^\ell)^* \, dM\right)$ for each $\ell \in U$, the insider's utility gain*

1. *up to time $t \in [0,T)$ is given by*

$$E[a_t] = E[\log p^G_t] = H_{\mathcal{G}_t}(P|\widetilde{P}_t) \qquad , \qquad t \in [0,T). \tag{40}$$

2. *up to the terminal time T is given by*

$$E[a_T] = \lim_{t\to T} E[\log p^G_t] = \lim_{t\to T} H_{\mathcal{G}_t}(P|\widetilde{P}_t). \tag{41}$$

Proof: By Corollary 2.10, we have $1/p^G = \mathcal{E}\left(-\int (\mu^G)^* \, d\widetilde{M}\right)$. This means in particular that $\widetilde{N} \equiv 0$, and so the assertions follow from (38) and (39). **q.e.d.**

4 Explicit Calculations of the Insider's Additional Expected Logarithmic Utility

In this section, we systematically analyze the insider's additional expected logarithmic utility. For simplicity, we consider the case where the insider's utility gain up to time $t \in [0,T)$ is given by

$$E[a_t] = E\left[\int_0^t (\mu_s^G)^* \, d\langle M\rangle_s \, \mu_s^G\right] = E[\log p^G_t] = H_{\mathcal{G}_t}(P|\widetilde{P}_t) \tag{42}$$

and up to the terminal time T by

$$E[a_T] = E\left[\int_0^T (\mu_s^G)^* \, d\langle M\rangle_s \, \mu_s^G\right] = \lim_{t\to T} E[\log p_t^G] = \lim_{t\to T} H_{\mathcal{G}_t}(P|\widetilde{P}_t). \tag{43}$$

We can then obtain results which only depend on the structure of the additional information G and not on the decomposition of M in $\mathbb{G}^\circ$. This is the key point which allows us to simplify and generalize results obtained by Pikovsky and Karatzas (1996).
Let us now summarize the assumptions needed in chapter 3 to establish (42) and (43).

Section Assumption 4.1

1. *M is a d-dimensional continuous local $\mathbb{F}$-martingale.*
2. $E\left[\int_0^T \alpha_s^* \, d\langle M\rangle_s \, \alpha_s\right] < \infty.$
3. *$\widetilde{M} = M - \int d\langle M\rangle \, \mu^G$ is a d-dimensional continuous local $\mathbb{G}^\circ$-martingale.*
4. $$E\left[\int_0^t (\mu_s^G)^* \, d\langle M\rangle_s \, \mu_s^G\right] < \infty \qquad \textit{for all } t \in [0,T). \tag{44}$$
   (compare Section Assumption 3.2 and the remark at the end of section 3.2)
5. $$\frac{1}{p_t^G} = \mathcal{E}\left(-\int (\mu_s^G)^* \, d\widetilde{M}_s\right)_t \qquad , \qquad t \in [0,T). \tag{45}$$

Remarks:

1. Assumption (45) means that the orthogonal martingale $\widetilde{N}$ in (36) should vanish. This is clearly hard to check in a general incomplete market, but our results provide by Theorem 3.7 at least upper bounds for the insider's utility gain. In the classical *complete* market model with $\mathbb{F}$ generated by the underlying Brownian motion W, (45) follows from Corollary 2.10 by applying the martingale representation theorem to each p^ℓ. In particular, all examples in Pikovsky and Karatzas (1996) *without* constraints on the insider's strategies are special cases of our subsequent results.

2. In all subsequent explicit examples (Example 4.2 and subsection 4.3), Section Assumption 4.1 is satisfied. This can easily be shown by direct, but lengthy calculations.

4.1 The Distribution of G is Atomic

Suppose first that G takes values in a countable set U so that

$$\mathcal{G}_t = \bigcap_{\epsilon>0} \Big(\mathcal{F}_{t+\epsilon} \vee \sigma(\{G = \ell\}, \ell \in U)\Big) \qquad , \qquad t \in [0,T].$$

Recall that the *entropy* of G is defined by

$$H(G) := -\sum_{\ell\in U} P[G = \ell] \, \log P[G = \ell]. \tag{46}$$

Theorem 4.1 *Suppose that G is a discrete random variable such that $H(G) < \infty$ and Section Assumption 4.1 holds. Then:*

1. *The insider's additional expected logarithmic utility up to time $t \in [0,T]$ is given by*

$$E[a_t] = H(G) - H(G|\mathcal{F}_t) \tag{47}$$

   with

$$H(G|\mathcal{F}_t) := -E\left[\sum_{\ell \in U} P[G = \ell|\mathcal{F}_t] \log P[G = \ell|\mathcal{F}_t]\right] \quad , \qquad t \in [0,T] \tag{48}$$

   being the conditional entropy of G given $\mathcal{F}_t$.

2. *In particular, if G is $\mathcal{F}_T$-measurable, then the insider's terminal additional expected logarithmic utility is given by*

$$E[a_T] = H(G). \tag{49}$$

Proof: Without loss of generality, we may assume that $U = \mathbb{N}$.

1. By Example 2.7 and the remark after Theorem 2.2, $\widetilde{M}$ is a local $\mathbb{G}$-martingale. For any $t \in [0,T]$, part 4 of Theorem 3.5 therefore implies that the utility gain $E[a_t]$ gives indeed the insider's additional expected logarithmic utility up to time t.

2. Since the nonnegative process $(-P[G = i|\mathcal{F}_t] \log P[G = i|\mathcal{F}_t])_{t \in [0,T]}$ is an $\mathbb{F}$-supermartingale, the conditional entropy is nonnegative and decreasing in its second argument, and therefore

$$0 \leq -E\left[\sum_{i=1}^{\infty} P[G = i|\mathcal{F}_t] \log P[G = i|\mathcal{F}_t]\right] \leq H(G) < \infty \qquad \text{for } t \in [0,T]. \tag{50}$$

   Fix $t \in [0,T)$. By Example 2.7, we have for $i \in \mathbb{N}$ that

$$p_t^i = \frac{P[G = i|\mathcal{F}_t]}{P[G = i]},$$

   and therefore Theorem 3.7, Section Assumption 4.1 and conditioning on $\mathcal{F}_t$ yield

$$\begin{aligned} E[a_t] &= E[\log p_t^G] \\ &= E\left[\sum_{i=1}^{\infty} \log p_t^i \, P[G = i|\mathcal{F}_t]\right] \\ &= E\left[\sum_{i=1}^{\infty} P[G = i|\mathcal{F}_t] \log P[G = i|\mathcal{F}_t]\right] - \sum_{i=1}^{\infty} P[G = i] \log P[G = i], \end{aligned}$$

   which is well-defined according to (50) and yields (47) for $t < T$. For each $i \in \mathbb{N}$, $P[G = i|\mathcal{F}_t] \to P[G = i|\mathcal{F}_T]$ as $t \to T$ by martingale convergence and therefore

$$\lim_{t \to T} E\Big[P[G = i|\mathcal{F}_t] \log P[G = i|\mathcal{F}_t]\Big] = E\Big[P[G = i|\mathcal{F}_T] \log P[G = i|\mathcal{F}_T]\Big] \tag{51}$$

by dominated convergence, since $x \mapsto x \log x$ is bounded on $[0, 1]$. Moreover, (50) implies by Fubini's theorem that the series $-\sum_{i=1}^{\infty} E\Big[P[G = i|\mathcal{F}_t] \log P[G = i|\mathcal{F}_t]\Big]$ is absolutely convergent for each $t \in [0, T]$, and so (51) implies that

$$\lim_{t \to T} H(G|\mathcal{F}_t) = H(G|\mathcal{F}_T).$$

This completes the proof of (47).

3. If G is $\mathcal{F}_T$-measurable, then $P[G = i|\mathcal{F}_T] = \mathbf{I}_{\{G=i\}}$, and so the right-hand side in (51) becomes zero, since $0 \log 0 = 1 \log 1 = 0$. Thus, $H(G|\mathcal{F}_T) = 0$, completing the proof by (47). **q.e.d.**

Remark: Since $H(G|\mathcal{F}_t)$ measures the amount of uncertainty about the outcome of G if one has the information $\mathcal{F}_t$, we can interpret (47) as follows: for each $t \in [0, T]$, the utility gain of an insider up to time t equals the amount of uncertainty of the ordinary investor about G at time 0 minus the amount of uncertainty of the ordinary investor about G at time t and is therefore just the amount of certainty that the ordinary investor has gained about G by time t. Note also that the utility gain in (47) becomes zero if G is independent of $\mathcal{F}_t$ for all $t \in [0, T]$.

Example 4.2 Suppose that the insider's additional information in the classical *complete* market model consists of an interval-type information about the outcome of the external noise W, i.e., $G := \mathbf{I}_{\{W_T \in (a_1,b_1) \times \cdots \times (a_d,b_d)\}}$ with $a_i, b_i \in I\!R \cup \{-\infty, \infty\}$ and $a_i < b_i$ for $i = 1, \dots, d$. The insider's additional expected logarithmic utility is then by (49)

$$E[a_T] = -p \log p - (1 - p) \log(1 - p),$$

where $p = \prod_{i=1}^{d} p_i$ with $p_i := P[W_T^i \in (a_i, b_i)] = \Phi\Big(b_i/\sqrt{T}\Big) - \Phi\Big(a_i/\sqrt{T}\Big)$ for $i = 1, \dots, d$. In particular, if the insider has information about the outcome of only one noise term, i.e., $G := \mathbf{I}_{\{W_T^i \in (a_i,b_i)\}}$, his additional expected logarithmic utility is given by

$$E[a_T] = -p_i \log p_i - (1 - p_i) \log(1 - p_i).$$

This closed-form solution in particular answers a question by Pikovsky and Karatzas (1996) who conjectured that the additional expected utility is finite in this example.

We next consider the case when G has infinite entropy.

Theorem 4.3 *Suppose that G is a discrete random variable such that $H(G) = \infty$ and Section Assumption 4.1 holds. If G is $\mathcal{F}_T$-measurable, then $E[a_T] = \infty$, and thus the insider's additional expected logarithmic utility up to the terminal time T becomes infinite.*

Proof: Without loss of generality, assume that $U = I\!N$. For $n \in I\!N$, consider the random variable $G^n := \sum_{i=1}^{n} i\,\mathbf{I}_{\{G=i\}}$ and define the filtration $\mathbb{G}^n := (\mathcal{G}_t^n)_{t\in[0,T]}$ by $\mathcal{G}_t^n := \bigcap_{\epsilon>0} \Big(\mathcal{F}_{t+\epsilon} \vee \sigma(G^n)\Big)$. If $E[a_T^n]$ denotes the terminal utility gain corresponding to $\mathbb{G}^n$, then $\mathbb{G} \supseteq \mathbb{G}^n$ implies that

$$\text{expected additional utility in } \mathbb{G} \;\geq\; \text{expected additional utility in } \mathbb{G}^n \;\geq\; E[a_T^n] \tag{52}$$

by part 3 of Theorem 3.5. According to Theorem 4.1, we obtain for $n \in I\!N$

$$E[a_T^n] = H(G^n) = -\sum_{i=1}^{n} P[G=i]\,\log P[G=i].$$

But since $H(G) = \infty$, the right-hand side above diverges to ∞ as $n \to \infty$, and so the assertion follows from (52). **q.e.d.**

4.2 The Distribution of G is not Purely Atomic

Theorem 4.4 *Suppose G is $\mathcal{F}_T$-measurable with values in the Polish space $(U,\mathcal{U})$ and has a distribution which is not purely atomic and such that Section Assumption 4.1 holds. Then $E[a_T] = \infty$, and thus the insider's additional expected logarithmic utility up to the terminal time T becomes infinite.*

Proof: Choose $B \in \mathcal{U}$ such that B does not contain any atoms of G and such that $P[G \in B] = c > 0$. Then for each $n \in I\!N$, we can find a partition $(B_i^n)_{i=1,\dots,n}$ of B such that $P[G \in B_i^n] = \frac{c}{n}$ for $i = 1,\dots,n$. For each $n \in I\!N$, the random variable $G^n := \sum_{i=1}^{n} i\,\mathbf{I}_{\{G\in B_i^n\}}$ has entropy

$$H(G^n) = -\sum_{i=1}^{n} P[G \in B_i^n]\,\log P[G \in B_i^n] = c\log n - c\log c,$$

and since this goes to ∞ as $n \to \infty$, the same argument as for Theorem 4.3 completes the proof. **q.e.d.**

4.3 Terminal Information Distorted by Noise

In this subsection, we consider the classical *complete* market model as described after Section Assumption 3.2. Suppose that the insider's information about the outcome of W_T is distorted by some independent noise so that he knows the value of

$$G := \Big(\lambda_1 W_T^1 + (1-\lambda_1)\varepsilon_1, \dots, \lambda_d W_T^d + (1-\lambda_d)\varepsilon_d\Big)^*,$$

where for $i = 1,\dots,d$, the random variables ε_i are independent, independent of $\mathcal{F}_T$ and normally distributed with mean 0 and variance $\sigma_i^2 > 0$, and λ_i are numbers in

$[0,1]$, and not all $\lambda_i = 1$. For each $t \in [0,T]$, the conditional distribution of G given $\mathcal{F}_t$ is then multivariate normal with mean $m_t = (\lambda_1 W_t^1, \ldots, \lambda_d W_t^d)^*$ and variance $V_t = \mathrm{diag}\Big(\lambda_i^2(T-t) + (1-\lambda_i)^2\sigma_i^2\Big)_{i=1,\ldots,d}$, and its Radon-Nikodym derivative with respect to Lebesgue measure is given by

$$q_t^\ell = \prod_{i=1}^d \frac{1}{\sqrt{2\pi(\lambda_i^2(T-t) + (1-\lambda_i)^2\sigma_i^2)}} \exp\left(-\frac{(\ell_i - \lambda_i W_t^i)^2}{2(\lambda_i^2(T-t) + (1-\lambda_i)^2\sigma_i^2)}\right)$$

for $\ell \in I\!R^d$. By the remark following Theorem 2.2, $\widetilde{M}$ is therefore a local $\mathbb{G}$-martingale, and so part 4 of Theorem 3.7, combined with Section Assumption 4.1 and the comment following Definition 3.6, implies that the insider's additional expected logarithmic utility up to time $t < T$ is given by

$$E[a_t] = E[\log p_t^G] = E\left[\log \frac{q_t^G}{q_0^G}\right] = \frac{1}{2}\sum_{i=1}^d \log \frac{\lambda_i^2 T + (1-\lambda_i)^2\sigma_i^2}{\lambda_i^2(T-t) + (1-\lambda_i)^2\sigma_i^2}$$

and up to the terminal time T by

$$\begin{aligned} E[a_T] &= \lim_{t\to T} E[\log p_t^G] \\ &= \begin{cases} \frac{1}{2}\sum\limits_{i=1}^d \log \frac{\lambda_i^2 T + (1-\lambda_i)^2\sigma_i^2}{(1-\lambda_i)^2\sigma_i^2} & , \text{if } \lambda_i \in [0,1) \text{ for all } i = 1,\ldots,d \\ \infty & , \text{if } \lambda_i = 1 \text{ for at least one } i. \end{cases} \end{aligned} \quad (53)$$

This extends Theorem 3.3 of Pikovsky and Karatzas (1996) by giving a closed-form solution instead of bounds only. Furthermore, the quantity in (53) is decreasing in each σ_i and tends to ∞ if σ_i goes to 0 for at least one i, which is exactly what intuition suggests should happen.

Acknowledgement: We are grateful to Dirk Becherer, TU Berlin, for his interest in this work and helpful discussions. We also thank an unknown referee for his useful and very prompt comments.

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