On the Integrability Condition in the Multiplicative Ergodic Theorem for Stochastic Differential Equations

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October 13, 1997

Abstract

The multiplicative ergodic theorem is valid under an integrability condition on the linearized flow with respect to an invariant measure. We investigate the case were the flow is generated by a stochastic differential equation and give a criterion in terms of the vector fields and the (generally non-adapted) invariant measure assuring the validity of the integrability condition.

Key words and phrases: stochastic differential equations, random dynamical systems, stochastic flows, multiplicative ergodic theorem, integrability conditions.

1991 AMS subject classification: primary 60 H 10, 93 E 03; secondary 34 C 35, 34 F 05

1 Introduction

Smooth ergodic theory is based on Oseledets' fundamental Multiplicative Ergodic Theorem [18]. It provides us with a random substitute of linear algebra (spectral theory) and hence is at the basis of local theory of nonlinear deterministic and random dynamical systems under an invariant measure. See Arnold [2] for a survey, and the forthcoming monograph [3] for a comprehensive presentation.

More specifically (see [3, section 4.2]), let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be an ergodic metric dynamical system, and let $\varphi : \mathbb{R} \times \Omega \times M \to M$, $(t, \omega, x) \mapsto \varphi(t, \omega)x$ be a local C^1 random dynamical system (RDS) (or local C^1 cocycle) over θ on a Riemannian manifold M of dimension $d \geq 1$. Let further μ be an invariant measure of the skew product flow $\Theta(t)(\omega, x) := (\theta_t\omega, \varphi(t, \omega)x)$ with marginal $\mathbb{P}$ on Ω. Then the Multiplicative Ergodic Theorem holds provided the derivative flow $T\varphi$ satisfies the following two integrability conditions (IC):

$$\int_{\Omega\times M} \sup_{0\leq t\leq 1} \log^+ \|T\varphi(t, \omega, x)\|_{x,\varphi(t,\omega)x}\, \mu(d\omega, dx) < \infty, \tag{1.1}$$

$$\int_{\Omega\times M} \sup_{0\leq t\leq 1} \log^+ \|T\varphi(t, \omega, x)^{-1}\|_{\varphi(t,\omega)x,x}\, \mu(d\omega, dx) < \infty. \tag{1.2}$$

The possible explosion of φ in finite time is taken care of automatically since every invariant measure μ is supported by the invariant set $E \subset \Omega \times M$ of those (ω, x) for which $\varphi(\cdot, \omega)x$ never explodes (see [3, section 1.8]). If $E = \Omega \times M$, φ is said to be global.

We now suppose that our (local) cocycle is generated by a stochastic differential equation (SDE) on M, given by

$$dx_t = f_0(x_t)dt + \sum_{j=1}^{m} f_j(x_t) \circ dW_t^j =: \sum_{j=0}^{m} f_j(x_t) \circ dW_t^j, \quad x_0 = x \in M, \tag{1.3}$$

where for some $\delta \in (0, 1]$

$$f_0 \in \mathcal{C}^{1,\delta}, \quad f_1, \ldots, f_m \in \mathcal{C}^{2,\delta}, \tag{1.4}$$

and $\mathcal{C}^{k,\delta}$ is the space of functions which are k times continuously differentiable and whose k-th derivative is locally δ-Hölder continuous. Here time $\mathbb{T} = \mathbb{R}$ is two-sided, and $(\Omega, \mathcal{F}, \mathbb{P})$ is the two-sided canonical Wiener space. More precisely, $\Omega = \mathcal{C}(\mathbb{R}, \mathbb{R}^m)$ endowed with the compact-open topology, $\mathcal{F}$ is the Borel σ-algebra on Ω, $\mathbb{P}$ the Wiener measure on $\mathcal{F}$, and $W_t(\omega) = (W_t^1(\omega), \ldots, W_t^m(\omega)) := \omega(t)$ is the canonical Wiener process. Further, $\theta_t\omega := \omega(t + \cdot) - \omega(t)$ preserves the Wiener measure and is ergodic. Let $\mathcal{F}_s^t := \sigma(\omega(u) - \omega(v) : s \leq u, v \leq t)$, $-\infty \leq s \leq t \leq \infty$.

It is well-known (see Kunita [14] and Arnold [3, section 2.3]) that under (1.4), (1.3) generates a local C^1 RDS φ over θ. The subject of this paper is to find convenient sufficient conditions in terms of the vector fields f_j, $0 \leq j \leq m$, and the invariant measure μ which imply (1.1) and (1.2). This task is complicated by the fact that, if we disintegrate μ with respect to its marginal $\mathbb{P}$,

$$\mu(d\omega, dx) = \mu_\omega(dx)\mathbb{P}(d\omega),$$

the random measure $\omega \mapsto \mu_\omega$ on M generally depends on the whole history $\mathcal{F} = \mathcal{F}_{-\infty}^{\infty}$ of the Wiener process.

2 Particular cases

Random differential equations

It is informative to compare the SDE case with the simpler one in which the C^1 RDS φ is generated by the (pathwise) random differential equation

$$\dot{x}_t = f(\theta_t\omega, x_t), \quad x_0 = x \in M.$$

Then $T\varphi$ solves $\dot{v}_t = Tf(\theta_t\omega, v_t)$ which is equivalent to the pair consisting of the original equation (horizontal part) and the variational equation (vertical part)

$$v_t' = \nabla f(\theta_t\omega, \varphi(t,\omega)x)v_t,$$

where the prime denotes the absolute derivative and ∇f is the covariant derivative of the vector field f.

Writing the last equation in polar coordinates on the unit sphere bundle SM and using an elementary estimate we obtain (see [3, section 6.4])

$$\sup_{0\le t\le 1} \log^+ \|T\varphi(t,\omega,x)\| \le \log d + \int_0^1 \|\nabla f(\theta_t\omega, \varphi(t,\omega)x)\| dt. \tag{2.1}$$

The same estimate is obtained for $\|T\varphi(t,\omega,x)^{-1}\|$ using an analogous argument for $(T\varphi)^{*-1}$ which is generated by $v_t' = (-\nabla f)^* v_t$. Integrating (2.1) with respect to μ and taking into account the invariance of μ we immediately obtain that φ under μ satisfies the IC (1.1) and (1.2) provided

$$\|\nabla f(\cdot,\cdot)\| \in L^1(\mu).$$

For $M = \mathbb{R}^d$, $\nabla f(x)v = Df(x)v$, where $Df(x)$ is the Jacobian of f at x. In particular, a linear random differential equation $\dot{x}_t = A(\theta_t\omega)x_t$ in $\mathbb{R}^d$ generates a global linear RDS satisfying the IC for the invariant measure $\mu_\omega(dx) = \delta_0(dx)$ provided $A \in L^1(\mathbb{P})$.

Linear SDE

Consider the linear SDE

$$dx_t = \sum_{j=0}^{m} A_j x_t \circ dW_t^j, \quad x_0 = x \in \mathbb{R}^d. \tag{2.2}$$

It generates a global linear RDS Φ with invariant measure $\mu_\omega(dx) = \delta_0(dx)$. We claim that the IC

$$\alpha^{\pm} \in L^1(\mathbb{P}) \quad \text{for} \quad \alpha^{\pm}(\cdot) := \sup_{0\le t\le 1} \log^+ \|\Phi(t,\cdot)^{\pm 1}\|,$$

are automatically satisfied. Since $\Psi(t,\omega) := \Phi(t,\omega)^{*-1}$ satisfies the linear SDE $dy_t = \sum_{j=0}^m (-A_j^*) y_t \circ dW_t^j$ it suffices again to check the condition for α^+.

Using Khasminskii's method [12, section VI.6–10] of rewriting (2.2) in polar coordinates $r = |x| \in (0,\infty)$, $s = \frac{x}{|x|} \in S^{d-1}$ we obtain

$$dS_t = \sum_{j=0}^m h_j(S_t) \circ dW_t^j, \quad S_0 = s \in S^{d-1}, \tag{2.3}$$

and

$$dR_t = \sum_{j=0}^m q_j(S_t) R_t \circ dW_t^j = Q(S_t) R_t dt + \sum_{j=1}^m q_j(S_t) R_t \, dW_t^j, \quad R_0 = r \in (0,\infty), \tag{2.4}$$

where $h_j(s) := A_j s - \langle A_j s, s\rangle s$, $q_j(s) := \langle A_j s, s\rangle$ and

$$Q(s) := \langle A_0 s, s\rangle \pm \frac{1}{2} \sum_{j=1}^m \left(\langle A_j^2 s, s\rangle + \langle A_j s, A_j s\rangle - 2\langle A_j s, s\rangle^2 \right),$$

where "+" is valid for time $\mathbb{R}^+$ and "−" for time $\mathbb{R}^-$. Integrating (2.4) and taking into account that $|Q(s)| \le C$ on S^{d-1} we obtain by an elementary estimate (see [3, section 6.2])

$$\alpha^+ \le \log d + C + \max_i \sum_{j=1}^m \sup_{0\le t\le 1} \left| \int_0^t q_j(S_u(e_i))\, dW_u^j \right|.$$

where $(e_i)_{1\le i\le d}$ is the canonical basis of $\mathbb{R}^d$. By Burkholder's inequality for any $p > 0$ and $s \in S^{d-1}$

$$\begin{aligned} \mathbb{E}\left(\sup_{0\le t\le 1} \left| \int_0^t q_j(S_u(s))\, dW_u^j \right| \right)^{2p} &\le C_p \, \mathbb{E}\left(\int_0^1 q_j(S_u(s))^2 du \right)^p \\ &\le C_p \, \|A_j\|^{2p} < \infty, \end{aligned}$$

so that even $\alpha^+ \in L^p(\mathbb{P})$ for any $p > 0$.

SDE on a compact manifold

If M is a compact manifold, then under (1.4) the SDE (1.3) generates a *global* C^1 RDS φ (see again Kunita [14] and Arnold [3, section 2.3]). It was first proved by Carverhill [7, Appendix A.2] (for M a bounded open domain of $\mathbb{R}^d$) and Baxendale [6, Proposition 2.1] that φ automatically satisfies the IC for *any* μ. This follows from the stronger result

$$\mathbb{E}\left(\sup_{x\in M} \sup_{0\le t\le 1} \|T\varphi(t,\cdot,x)^{\pm 1}\| \right) < \infty.$$

The latter result was considerably improved by Kifer [13] who showed that for $f_0 \in \mathcal{C}^{k,\delta}$ and $f_1, \dots, f_m \in \mathcal{C}^{k+1,\delta}$ for some $k \geq 1$ and $0 < \delta \leq 1$ and for any $p \geq 1$ the C^k RDS φ generated by (1.3) satisfies

$$\mathbb{E}\left(\sup_{0 \leq t \leq 1} \|\varphi(t,\cdot)\|_k + \sup_{0 \leq t \leq 1} \|\varphi(t,\cdot)^{-1}\|_k \right)^p < \infty,$$

where $\|\cdot\|_k$ is the norm in $\mathcal{C}^k(M,M)$.

Invariant Markov measures

Consider the case $M = \mathbb{R}^d$. The SDE (1.3) in the Stratonovich as well as in the equivalent Itô form is

$$dx_t = \sum_{j=0}^m f_j(x_t) \circ dW_t^j = \hat{f}_0(x_t)dt + \sum_{j=1}^m f_j(x_t)\, dW_t^j, \quad x_0 = x \in \mathbb{R}^d, \qquad (2.5)$$

with

$$\hat{f}_0(x) := f_0(x) \pm \frac{1}{2} \sum_{j=1}^m D_{f_j} f_j(x),$$

where "+" holds for time $\mathbb{R}^+$, "−" for $\mathbb{R}^-$ and $D_f f := \sum_{i=1}^d f^i \frac{\partial}{\partial x_i} f$.

We now assume the stronger conditions

$$f_0 \in \mathcal{C}_b^{1,\delta}, \quad f_1, \dots, f_m \in \mathcal{C}_b^{2,\delta}, \quad \sum_{j=1}^m D_{f_j} f_j \in \mathcal{C}_b^{1,\delta} \qquad (2.6)$$

for some $0 < \delta \leq 1$, where $\mathcal{C}_b^{k,\delta}$ denotes the space of C^k vector fields which grow at most linearly in x, have bounded derivatives of order 1 up to k, and whose k-th derivatives are globally δ-Hölder continuous. Then also $\hat{f}_0 \in \mathcal{C}_b^{1,\delta}$, and under (2.6) the SDE (2.5) generates a global C^1 RDS φ [14, 3]. Further, the Jacobian $D\varphi$ of φ at x solves

$$dv_t = \sum_{j=0}^m Df_j(x_t)v_t \circ dW_t^j = D\hat{f}_0(x_t)v_t dt + \sum_{j=1}^m Df_j(x_t)v_t\, dW_t^j, \quad v_0 = v. \quad (2.7)$$

Using polar coordinates as in the linear case the SDE (2.7) is seen to be equivalent to the two SDE (2.3) and (2.4), where now in the expressions for h_j, q_j and Q, $A_j(x) = Df_j(x)$ is the Jacobian of f at x, and A_j^2 in Q is replaced with

$$B_j(x) = D(A_j(x)f_j(x)) = A_j(x)^2 + (D^2 f_j)(x) f_j(x),$$

the Jacobian of $g_j(x) = A_j(x)f_j(x)$. Hence $|q_j(x,s)| \leq \|A_j(x)\|$ and

$$|Q(x,s)| \leq \|A_0(x)\| + \frac{1}{2} \sum_{j=1}^m (\|B_j(x)\| + 3\|A_j(x)\|^2) =: Q_0(x), \ (x,s) \in M \times S^{d-1}.$$

Again it suffices to deal with the first IC $\alpha^+ \in L^1(\mu)$, where

$$\alpha^+(\cdot) := \sup_{0 \le t \le 1} \log^+ \|D\varphi(t,\cdot)\|.$$

Proceeding as in the linear case and denoting the marginal of μ on $\mathbb{R}^d$ by ρ and the integral with respect to a measure ν by $\mathbb{E}_\nu$, we obtain

$$\mathbb{E}_\mu \alpha^+ \le \log d + \mathbb{E}_\rho Q_0 + \max_i \sum_{j=1}^m \mathbb{E}_{\mathbb{P}} \mathbb{E}_{\mu_\omega} \sup_{0 \le t \le 1} \left| \int_0^t q_j(\varphi(u,\cdot)x, S_u(\cdot,x)e_i)\, dW_u^j \right|. \tag{2.8}$$

Due to our global assumptions (2.6) $Q_0(x) \le C$ on M, hence $\mathbb{E}_\rho Q_0 < \infty$ for any μ. The expectation of the stochastic integrals in (2.8) can in general not be estimated by means of classical stochastic analysis.

There is, however, one important particular case which permits the use of Burkholder's inequality, namely the case where μ is a Markov measure, i.e., where $\omega \mapsto \mu_\omega$ is measurable with respect to the past $\mathcal{F}_{-\infty}^0$ or with respect to the future $\mathcal{F}_0^\infty$ of the Wiener process W. For Markov measures with respect to the past, it has been show by Crauel [8, 9] that they are in a on-to-one correspondence with the stationary measures ρ of the Markov process generated by the SDE (1.3) for time $\mathbb{R}^+$, equivalently, with the solutions of the Fokker-Planck equation $L^*\rho = 0$, $L := f_0 + \frac{1}{2}\sum_{j=1}^m (f_j)^2$, the correspondence being given by

$$\mu_\omega \mapsto \rho = \mathbb{E}\mu_\cdot, \quad \rho \mapsto \mu_\omega = \lim_{t\to\infty} \varphi(-t,\omega)^{-1}\rho.$$

Further, by restriction of the RDS φ to one-sided time $\mathbb{T} = \mathbb{R}^+$ and to the σ-algebra $\mathcal{F}_0^\infty$, μ corresponds to the Θ-invariant product measure $\mu^+ = \mathbb{P} \times \rho$.

In this case for the stochastic integrals in (2.8) Burkholder's inequality for any $p > 0$ and $s \in S^{d-1}$ and then Jensen's inequality for $p \ge 1$ yield

$$\begin{aligned} & \mathbb{E}_\mu \left(\sup_{0 \le t \le 1} \left| \int_0^t q_j(\varphi(u,\cdot)x, S_u(\cdot,x)s) dW_u^j \right| \right)^{2p} \qquad (2.9) \\ \le\ & C_p\, \mathbb{E}_\rho\, \mathbb{E}_{\mathbb{P}} \left(\int_0^1 \|A_j(\varphi(u,\cdot)x)\|^2 du \right)^p \le C_p\, \mathbb{E}_\rho \|A_j(\cdot)\|^{2p}. \end{aligned}$$

By (2.6), $\|A_j(x)\| \le C$ on M, hence the right-hand side of (2.9) is finite for any $p \ge 1$ and any ρ. Consequently, even $\mathbb{E}_\mu(\alpha^+)^p < \infty$ for any $p > 0$.

The final result is that under the assumptions (2.6) assuring the generation of a global C^1 RDS φ by (2.5), the IC are automatically met for *any* Markov measure μ.

Suppose the coefficients only satisfy the conditions (1.4), but we happen to know that the local RDS φ possesses an invariant Markov measure μ with marginal ρ. Then, using for example $p = 1$ in (2.9), the estimates (2.8) and (2.9) show that the IC are met provided $\|Df_0\| \in L^1(\rho)$, $\|D((Df_j)f_j)\|$ and $\|Df_j\|^2 \in L^1(\rho)$ for $j = 1, \dots, m$.

We are, however, not able to extend these simple statements from Markov measures to arbitrary invariant measures.

3 A general sufficient condition

We work from now on in $M = \mathbb{R}^d$ and consider the SDE (2.5) and (2.7) under the assumptions

$$f_0 \in \mathcal{C}_b^{1,1}, \quad f_1, \dots, f_m \in \mathcal{C}_b^{2,\delta}, \quad \sum_{j=1}^m D_{f_j} f_j \in \mathcal{C}_b^{1,1}, \tag{3.1}$$

for some $0 < \delta \leq 1$, which are slighty stronger than (2.6). (2.5) hence generates a global C^1 RDS φ whose Jacobian $D\varphi$ solves the matrix version of (2.7) with initial value $D\varphi(0,\cdot) = I$, where I is the $d \times d$ identity matrix.

Let μ be any invariant measure. We will first obtain certain estimates for the moments of φ and $D\varphi$, employing the combination of Burkholder's inequality, Hölder's inequality and Gronwall's lemma. But contrary to the usual proofs we shall carefully keep track of how the constants appearing in the estimates depend on p, as we will later take averages with respect to p. For this reason we have to go into some details of otherwise well-known lines of reasoning.

The vector and matrix norms in finite-dimensional Euclidean spaces are denoted by $|\cdot|$, while $\|\cdot\|_p$ denotes the norm in $L^p(\mathbb{P})$. We also suppress the dependence on ω and write $\varphi(t,x) = \varphi(t,\cdot)x$. Universal constants appearing in the proofs will be denoted by c_1, c_2, ...

3.1. Proposition. *There exists $c > 0$ such that for any $p \geq 2$ and any $x, y \in \mathbb{R}^d$*

$$\mathbb{E}\left(\sup_{0\leq t\leq 1} |\varphi(t,x) - \varphi(t,y)|^p\right) \leq \exp(cp^2)|x-y|^p \tag{3.2}$$

and

$$\mathbb{E}\left(\sup_{0\leq t\leq 1} |D\varphi(t,\cdot,x) - D\varphi(t,\cdot,y)|^p\right) \leq \exp(cp^2)(|x-y| \wedge 1)^p. \tag{3.3}$$

Proof. (i) First recall from Barlow and Yor [5, page 207] that there exists a universal constant $c_1 > 0$ such that for any continuous martingale $(M_t)_{t\geq 0}$ with quadratic variation $(\langle M\rangle_t)_{t\geq 0}$ and any $p \geq 2$ we have

$$\| \sup_{0\leq t\leq 1} |M_t| \,\|_p \leq c_1 p^{1/2} \|\langle M\rangle_1^{1/2}\|_p. \tag{3.4}$$

We fix $p \geq 2$ and $x, y \in \mathbb{R}^d$ and put

$$f(t) := \| \sup_{0\leq s\leq t} |\varphi(s,x) - \varphi(s,y)| \,\|_p.$$

Then a combination of the global Lipschitz conditions for the coefficients of (2.5), inequality (3.4) for the martingale part as well as a simpler argument for the drift part of (2.5) involving just Jensen's and Hölder's inequalities yield (see Imkeller [10, page 253])

$$f(t) \leq \sqrt{2}|x-y| + c_2 p^{1/2} \left(\int_0^t f(s)^2 ds \right)^{1/2}.$$

Hence Gronwall's lemma implies

$$f(1) \leq \sqrt{2}|x-y| \exp(2c_2^2 p),$$

the p-th power of which gives (3.2) with $c = \frac{1}{2}\log 2 + 2c_2^2$.

(ii) For the proof of (3.3) we proceed in two steps. The first one consists in establishing a boundedness result for $D\varphi$. Let for fixed $x \in \mathbb{R}^d$

$$h(t) := \| \sup_{0 \leq s \leq t} |D\varphi(s,x)| \, \|_p.$$

Then the boundedness of $D\hat{f}_0$ and Df_j, $j = 1, \ldots, m$, gives by the same reasoning used already in part (i) of this proof [10]

$$h(t) \leq 1 + c_3 p^{1/2} \left(\int_0^t h(s)^2 ds \right)^{1/2},$$

whence, by Gronwall's lemma,

$$h(1) = \| \sup_{0 \leq t \leq 1} |D\varphi(t,x)| \, \|_p \leq \exp(c_4 p) \quad \text{for any } p \geq 2, \tag{3.5}$$

the upper bound being independent of x. (3.5) will be crucial for the second step in which we treat spatial increments of $D\varphi$.

We start with the following elementary inequality derived from (2.7). For $0 \leq t \leq 1$ we have

$$\begin{aligned}
&|D\varphi(t,x) - D\varphi(t,y)| \\
&\leq \; \left| \int_0^t (D\hat{f}_0(\varphi(s,x)) - D\hat{f}_0(\varphi(s,y))) D\varphi(s,x) ds \right| \\
&\quad + \left| \sum_{j=1}^m \int_0^t (Df_j(\varphi(s,x)) - Df_j(\varphi(s,y))) D\varphi(s,x) \, dW_s^j \right| \\
&\quad + \left| \int_0^t D\hat{f}_0(\varphi(s,y)) (D\varphi(s,x) - D\varphi(s,y)) ds \right| \\
&\quad + \left| \sum_{j=1}^m \int_0^t Df_j(\varphi(s,y)) (D\varphi(s,x) - D\varphi(s,y)) dW_s^j \right| \\
&=: \; T_1(t) + T_2(t) + T_3(t) + T_4(t).
\end{aligned}$$

For $0 \le t \le 1$, we introduce the notation

$$f_i(t) := \| \sup_{0\le s\le t} T_i(s)\|_p.$$

Assumption (3.1) implies that $D\hat{f}_0$ is globally Lipschitz continuous (it is here where we need the slightly stronger conditions), hence

$$|D\hat{f}_0(\varphi(s,x)) - D\hat{f}_0(\varphi(s,y))| \le c_5|\varphi(s,x) - \varphi(s,y)|.$$

This entails, using (3.2) and (3.5) to obtain the second inequality,

$$\begin{aligned} f_1(1) &\le c_5 p^{1/2} \| \sup_{0\le t\le 1} |\varphi(t,x) - \varphi(t,y)| \|_{2p} \, \| \sup_{0\le t\le 1} |D\varphi(t,x)| \, \|_{2p} \\ &\le \exp(c_6 p)|x-y|. \end{aligned}$$

Similarly, using the global Lipschitz continuity of Df_j and a combination of Hölder's and Burkholder's inequality,

$$f_2(1) \le \exp(c_7 p)|x-y|.$$

Finally, for f_3 and f_4 we use again a combination of Hölder's and Burkholder's inequality together with the estimates for f_1 and f_2. Putting

$$g(t) := \| \sup_{0\le s\le t} |D\varphi(s,x) - D\varphi(s,y)| \, \|_p,$$

we obtain

$$g(t) \le \exp(c_8 p)|x-y| + c_9 p^{1/2}\left(\int_0^t g(s)^2 ds\right)^{1/2}.$$

By Gronwall's lemma

$$g(1) \le \exp(c_{10} p)|x-y|.$$

This combined with (3.5) and the fact that $g(1) \le 2h(1)$ yields (3.3). □

We now consider expectations $\mathbb{E}\,F(X)$ of random variables X where F is a Young function (for basic information on Young functions and Young's inequality see Adams [1] or Meyer [16]).

3.2. Lemma. *(i) The function $F_c : \mathbb{R}^+ \to \mathbb{R}^+$ defined for $c > 0$ by*

$$F_c(x) := \int_1^\infty x^p \exp(-cp^2)dp$$

satisfies $F_c(0) = 0$, $F_c(x) > 0$ for $x > 0$ and has a positive first and second derivative for $x > 0$. Hence F_c is strictly increasing and convex, thus is a Young function. Further,

$$F_c(x) = \sqrt{\frac{\pi}{c}}\,\Phi(\frac{\log x}{\sqrt{2c}})\exp\left(\frac{(\log x)^2}{4c}\right),$$

where

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\,dt,$$

and

$$\exp\left(\frac{(\log x)^2}{4c}\right) \le F_c(x) \le \sqrt{\frac{\pi}{c}} \exp\left(\frac{(\log x)^2}{4c}\right). \tag{3.6}$$

(ii) The function $G_c : \mathbb{R}^+ \to \mathbb{R}^+$ defined for $c > 0$ by

$$G_c(x) := (x+1)\exp\left(c(\log(x+1))^2\right) - 1 \tag{3.7}$$

satisfies $G_c(0) = 0$, $G_c(x) > 0$ for $x > 0$ and has a positive first and second derivative. Hence G_c is strictly increasing and convex, thus is a Young function. Further, the inverse of G_c satisfies for any $c > 0$ and $x \ge 0$

$$\exp\left(-\frac{1}{2c} + (\frac{1}{4c^2} + \frac{1}{c}\log(x+1))^{\frac{1}{2}}\right) - 1 = G_c^{-1}(x) \le \exp\left((\frac{1}{c}\log(x+1))^{\frac{1}{2}}\right). \tag{3.8}$$

The proof is elementary. For (3.6) see [10, p. 257].

3.3. Proposition. *There exist constants $c > 0$ and $M > 0$ such that uniformly for all $x, y \in \mathbb{R}^d$*

$$\mathbb{E}\left(F_c\left(\sup_{0\le t\le 1} \frac{|D\varphi(t,x) - D\varphi(t,y)|}{|x-y| \wedge 1}\right)\right) \le M \tag{3.9}$$

and

$$\mathbb{E}\left(G_c\left(\sup_{0\le t\le 1} \frac{|D\varphi(t,x) - D\varphi(t,y)|}{|x-y| \wedge 1}\right)\right) \le M. \tag{3.10}$$

Proof. Let c' be the constant of (3.3) and let $c > c'$. Then the expression in (3.9) to be estimated has upper bound

$$M = \int_1^\infty \exp\left((c'-c)p^2\right) dp$$

which is finite.

As to (3.10), there clearly exist constants α_i, β_i, $i = 1, 2$, and $x_0 > 0$ such that for $x \ge x_0$

$$\alpha_1 \exp\left(\beta_1 (\log x)^2\right) \le G_c(x) \le \alpha_2 \exp\left(\beta_2 (\log x)^2\right).$$

Now apply Lemma 3.2(i) and (3.9). □

We now utilize an appropriate version of the Garsia-Rodemich-Rumsey lemma to obtain a rate of increase of $D\varphi(t,x)$ in the spatial variable x.

Assume that ν is a finite measure on $\mathcal{B}(\mathbb{R}^d)$ (we will choose a particular ν which suits our purposes best below) and consider

$$Y_c = \int_{\mathbb{R}^d \times \mathbb{R}^d} G_c \left(\sup_{0 \le t \le 1} \frac{|D\varphi(t,x) - D\varphi(t,y)|}{|x-y| \wedge 1} \right) \nu(dx)\nu(dy).$$

By Proposition 3.3 we can and will choose a constant $c > 0$ for which $\mathbb{E} Y_c < \infty$.

3.4. Proposition. *Let the metric d in $\mathbb{R}^d$ be defined by $d(x,y) := |x-y| \wedge 1$ and let $B_r^d(z) := \{u \in \mathbb{R}^d : d(z,u) \le r\}$. Then for any $x, y \in \mathbb{R}^d$*

$$\sup_{0 \le t \le 1} |D\varphi(t,x) - D\varphi(t,y)| \le 8 \sup_{z \in \{x,y\}} \int_0^{d(x,y)} G_c^{-1} \left(\frac{4Y_c}{\nu(B_r^d(z))^2} \right) dr,$$

Proof. The inequality follows from a slight modification of the proof of the version of the Garsia-Rodemich-Rumsey lemma given in [4, pp. 25–28]. In the path by path argument made there the Young function G_c just has to be replaced by G_c/Y_c. □

By constructing a particular *majorizing measure* ν we shall now derive the key asymptotic estimate for the spatial growth of $D\varphi$. Along with the ideas which led to Proposition 3.4 this estimate can be generalized to semimartingale flows of the Kunita type [14]. For this general class these arguments even give best asymptotic rates for the spatial growth of flows and their derivatives (see Imkeller and Scheutzow [11]), improving the results of Kunita [14, Exercise 4.6.9] and Mohammed and Scheutzow [17].

3.5. Theorem (Spatial Growth of Jacobian). *Let φ be the C^1 RDS generated by the SDE (2.5) under the conditions (3.1). Then there exist constants $c > 0$ and $b > 0$ such that the Jacobian $D\varphi$ of φ satisfies*

$$\mathbb{E}\, G_c \left(\sup_{r>0} \frac{\sup_{|x| \le r} \sup_{0 \le t \le 1} |D\varphi(t,x)|}{\exp \sqrt{b \log^+ r}} \right) < \infty\ ,$$

where G_c is defined by (3.7)

Proof. We define a majorizing measure ν (for the terminology see e. g. Ledoux and Talagrand [15]) by setting

$$\nu(\cdot) := \left(\sum_{n \in \mathbb{N}} \frac{1}{n^{d+1}} 1_{R_n} \right) \lambda(\cdot),$$

where λ is Lebesgue measure in $\mathbb{R}^d$, $R_n := B_n \setminus B_{n-1}$, $B_r := \{u \in \mathbb{R}^d : |u| \le r\}$. Then ν is finite since $\lambda(R_n) = c_1(n^d - (n-1)^d) \le c_1 n^{d-1}$.

We now apply Proposition 3.4 in the special case $y = 0$ and obtain

$$\sup_{0\le t\le 1} |D\varphi(t,x) - D\varphi(t,0)| \le c_2 \sup_{z\in\{x,0\}} \int_0^{d(x,0)} G_c^{-1}\left(\frac{c_3 Y_c}{\nu(B_r^d(z))^2}\right) dr,$$

We next estimate $\nu(B_r^d(z))$ for $z \in \{x, 0\}$ and $0 < r \le d(x,0)$. By the definition of ν it is obviously enough to consider $z = x$ and fix $n \in \mathbb{N}$ such that $n-1 \le |x| \le n$. Note also that $d(x,0) \le 1$ so that it suffices to consider $r \le 1$. Then $B_r^d(x) \subset B_{n+1}$ and therefore

$$\nu(B_r^d(x)) \ge \frac{c_4}{(n+1)^{d+1}} r^d .$$

Hence uniformly in $|x| \le n$

$$\begin{aligned}
&\sup_{z\in\{x,0\}} \int_0^{d(x,0)} G_c^{-1}\left(\frac{c_3 Y_c}{\nu(B_r^d(z))^2}\right) dr \\
\le\ & \int_0^1 G_c^{-1}\left(\frac{c_3 Y_c}{\nu(B_r^d(x))^2}\right) dr \\
\le\ & c_5 G_c^{-1}(Y_c) G_c^{-1}(n^{2(d+1)}) \int_0^1 G_c^{-1}\left(\frac{1}{r^{2d}}\right) dr \\
\le\ & c_6 G_c^{-1}(Y_c) \exp\sqrt{\frac{2(d+1)}{c}\log n} .
\end{aligned}$$

Take $c > 0$ such that $\mathbb{E} Y_c < \infty$ and $b = 2(d+1)/c$ to finish the proof. □

3.6. Corollary (Sufficient Condition for IC). *Let φ be the C^1 RDS generated by the SDE (2.5) under the conditions (3.1) and let μ be any (not necessarily invaraint) probability measure on $\Omega \times \mathbb{R}^d$ with marginal $\mathbb{P}$ on Ω. Denote the marginal of μ on $\mathbb{R}^d$ by ρ. If*

$$\mathbb{E}\int_{\mathbb{R}^d} \sqrt{\log^+ |x|}\, \mu_\cdot(dx) = \int_{\mathbb{R}^d} \sqrt{\log^+ |x|}\, \rho(dx) < \infty\,, \qquad (3.11)$$

then

$$\mathbb{E}\int_{\mathbb{R}^d} \sup_{0\le t\le 1} \log^+ |D\varphi(t,\cdot,x)^{\pm 1}| \mu_\cdot(dx) < \infty,$$

i.e. the IC of the Multiplicative Ergodic Theorem are satisfied.

Proof. It suffices to check the IC for $D\varphi$ (see Sect. 2). Let b and c be given by Theorem 3.5 and set

$$Z_b := \sup_{r>0} \frac{\sup_{|x|\le r} \sup_{0\le t\le 1} |D\varphi(t,x)|}{\exp\sqrt{b\log^+ r}} .$$

Then, with $R_n = B_n \setminus B_{n-1}$ as above, Theorem 3.5 yields

$$\begin{aligned}
&\int_{\mathbb{R}^d} \sup_{0\le t\le 1} \log^+ |D\varphi(t,\cdot,x)| \mu.(dx) \\
&\quad \le \sum_{n\in\mathbb{N}} \log^+ \sup_{x\in R_n} \sup_{0\le t\le 1} |D\varphi(t,x)| \mu.(R_n) \\
&\quad \le \sum_{n\in\mathbb{N}} \log^+ (Z_b \exp\sqrt{b\log n}) \mu.(R_n) \\
&\quad \le \log^+ Z_b + \sum_{n\in\mathbb{N}} \sqrt{b\log n}\, \mu.(R_n) .
\end{aligned}$$

By Theorem 3.5 $\log^+ Z_b$ is clearly integrable. Moreover, we have

$$\sum_{n\in\mathbb{N}} \sqrt{\log n}\, \mu.(R_n) \le \int_{\mathbb{R}^d} \sqrt{\log(|x|+1)}\, \mu.(dx) ,$$

and the right-hand side has finite expectation by assumption (3.11). □

3.7. Remark. Corollary 3.6 implies that for an SDE on a compact manifold satisfying the local versions of (3.1) the IC of the multiplicative ergodic theorem are automatically satisfied for *any* φ-invariant measure. Just embed the manifold isometrically into some $\mathbb{R}^d$ and extend the vector fields as usual to compactly supported vector fields in $\mathbb{R}^d$ using a tubular neighborhood of M. Since any μ is compactly supported on M condition (3.11) holds. ■

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