LETTER TO THE EDITOR

Lévy Flights: Transitions and Meta-Stability

Peter Imkeller and Ilya Pavlyukevich†
Institut für Mathematik, Humboldt-Universität zu Berlin, Rudower Chaussee 25, 12489 Berlin, Germany

Abstract. We consider Lévy flights of stability index $\alpha \in (0, 2)$ in a potential landscape in the limit of small noise parameter. We give a purely probabilistic description of the random dynamics based on a special decomposition of the driving Lévy processes into independent small jumps and compound Poisson parts. We prove that escape times from a potential well are exponentially distributed and their mean values increase as a power $\varepsilon^{-\alpha}$ of the noise intensity ε. This allows to obtain meta-stability results for a jump-diffusion in a double-well potential.

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1. Introduction

Dynamical systems subject to small random perturbations receive much attention both in the physical and mathematical literature. Most of the interesting questions relate to the problem of the first exit from a domain and the corresponding problem of transitions between domains of attraction of the underlying deterministic dynamical system and meta-stability. The properties of the random system are mainly determined by the nature of the noise. The study of perturbations by white Gaussian noise has the longest history (see e.g. [1, 2]), and richest bibliography. The standard mathematical reference on this subject is the book [3].

Recently non-Gaussian, in particular Lévy noises with heavy tails — Lévy flights (LFs) — have been introduced in many systems of sciences and economics. They are observed for instance in Greenland ice core measurements (see [4]), and thus used to model important qualitative features of paleoclimatic processes through low-dimensional dynamical systems. In biology Lévy flights are observed for example in the behavioural pattern of certain species such as albatrosses [5] or anchovies [6]. They are used to account for the uncertainties in price fluctuations in dynamical models of financial markets [7]. Lévy flights also naturally appear in particle evolutions along polymer chains [8, 9].

† To whom correspondence should be addressed (pavljuke@mathematik.hu-berlin.de)

In this paper we give a purely probabilistic description of small scale Lévy flights in an external potential, i.e. we investigate equations of motion of overdamped particles perturbed by small discontinuous noise processes with heavy tails. In the limit of small noise intensity we derive the exit law from a potential well, the analogue of the Kramers' law for Gaussian diffusions, and then obtain some meta-stability results. The rigorous proofs of the results formulated in this paper can be found in the forthcoming works [10, 11]. In the present paper we restrict ourselves to heuristic arguments and refrain from presenting the technical details.

2. Lévy Flights

Lévy flight is a synonym for symmetric stable Lévy process. Mathematically it describes a random Markov process $L = (L_t)_{t\geq 0}$ with independent stationary increments and marginals with symmetric stable laws of index $\alpha \in (0, 2)$. The Fourier transform of the marginal L_t, $t \geq 0$, has a very simple form,

$$\mathbf{E}\mathrm{e}^{i\lambda L_t} = \mathrm{e}^{-c(\alpha)t|\lambda|^\alpha}, \quad c(\alpha) = 2\int_0^\infty \frac{1-\cos y}{y^{1+\alpha}}\,dy. \tag{1}$$

In case $\alpha = 2$ we set $c(2) = \frac{1}{2}$, and (1) becomes the Fourier transform of a standard Brownian motion. However, due to the divergence $c(\alpha) \uparrow \infty$ as $\alpha \uparrow 2$, Brownian motion cannot be seen as a weak limit of LFs. The properties of the sample paths of L, in fact, are quite different for $\alpha = 2$ and $\alpha < 2$. Firstly, LFs are discontinuous (pure jump) processes whereas the Brownian motion has continuous paths. Secondly, Brownian motion has moments of all orders, whereas $\mathbf{E}|L_t|^\gamma < \infty$ iff $\gamma < \alpha$. One can also show that the tails of Lévy flights are heavy, i.e. $\mathbf{P}(L_t > u) \sim u^{-\alpha}$, $u \to \infty$, quite the opposite of the exponentially light Gaussian tails. Further, for $\alpha \in (0, 1)$, the path variation of Lévy flights is bounded on finite time intervals, and unbounded for $\alpha \in [1, 2)$.

Even if the form of the Fourier transform (1) is very simple, the marginals' density p can be expressed by elementary functions only in two cases, namely for

$$\alpha = 1, \text{ where } p(x) = \frac{1}{\pi}\frac{1}{x^2+1}, \text{ and } \alpha = 2, \text{ where } p(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}. \tag{2}$$

In the first case, L is called Cauchy process. More generally, for all $\alpha \in (0, 2)$, stable densities and distribution functions are known in terms of higher transcendental Meijer's G- and Fox's H-functions (see [12, Chapter 6]).

Although LFs are very well understood (see e.g. [13, 14] for a general theory), it is much more difficult to describe their behaviour in an external potential U (see e.g. [15, 16]). The dynamics is then given by a stochastic differential equation

$$X_t^\varepsilon = x - \int_0^t U'(X_{s-}^\varepsilon)\,ds + \varepsilon L_t, \quad , x \in \mathbb{R}, \quad t \geq 0, \tag{3}$$

where the positive parameter ε denotes the noise intensity. A frequently used approach to this problem consists in investigation of the corresponding Fokker-Planck equation which is a partial differential equation fractional in the spatial coordinate. This study is a difficult task, and analytically accessible solutions can be derived only for a few particular potentials and values of α (see [15, 17, 18]).

We study the process X^ε by probabilistic methods in the limit as the scale parameter $\varepsilon \to 0$. Thus, equation (3) becomes a natural generalisation of the Smoluchovski approximation of the Langevin equation for non-Gaussian stable noises. However, due to the heavy-tail nature of random perturbation, the limiting dynamics of X^ε differs drastically from its Gaussian counterpart.

Finally we stress that our approach uses neither special properties of LFs such as the scaling property, nor involves analysis of the Fokker-Planck equation, and thus can be generalised to a larger class of driving processes. Firstly, we can add to L a Brownian component and a constant drift considering a process $L_t + aB_t + bt$, $a \geq 0$, $b \in \mathbb{R}$. Secondly, it is enough to require that the heaviest tail of the Lévy measure ν is regularly varying with some negative index, i.e. $\mathbf{E}|L_t|^\gamma < \infty$ for some positive γ which is not necessarily smaller that 2.

3. Typical Behaviour of LFs in External Potentials

In this section we assume that the potential U has 'parabolic' form, i.e. $xU'(x) \geq 0$, $U'(x) = 0$ iff $x = 0$ and $U''(0) = M > 0$. We also impose the regularity condition $U(x) = |x|^{2+c}$, $x \to -\infty$. Under these assumptions, the deterministic dynamical system

$$X_t^0 = x - \int_0^t U'(X_s^0)\, ds \tag{4}$$

has a unique asymptotically stable attractor at the origin. Let $I = [-b, a]$ be a bounded or unbounded interval containing zero, $-\infty \leq -b < 0 < a < \infty$.

In this section we give a path-wise description of LFs in a potential U for small values of the scale parameter ε.

As a main tool of our analysis, we decompose LFs L into sums of ε-dependent small and large jump components. This can be done with the help of the Lévy-Khinchin formula for infinitely divisible distributions (see [14]). Indeed, the Fourier transform (1) can be represented in the following more complicated integral form

$$\mathbf{E}\mathrm{e}^{i\lambda L_t} = \exp\left\{ t \int_{\mathbb{R}\backslash\{0\}} \left[\mathrm{e}^{i\lambda y} - 1 - i\lambda y \mathbf{1}_{\{|y|\leq 1\}}(y)\right] \frac{dy}{|y|^{1+\alpha}} \right\}, \tag{5}$$

where the indicator function of a Borel set $A \subseteq \mathbb{R}$ is given by $\mathbf{1}_A(y) = 1$ if $y \in A$ and $\mathbf{1}_A(y) = 0$ otherwise. The most important ingredient of the representation (5) is the so called Lévy measure of the random process L given by

$$\nu(A) = \int_{A\backslash\{0\}} \frac{dy}{|y|^{1+\alpha}}, \quad A \text{ Borel set in } \mathbb{R}. \tag{6}$$

The Lévy measure controls the intensity and sizes of the jumps of the process. If we denote by $\Delta L_t = L_t - L_{t-}$ the jump size of L at time t, $t > 0$, and the number of jumps on the time interval $(0, t]$ belonging to the set A by

$$N(t, A) = \sharp\{s : (s, \Delta L_s) \in (0, t] \times A\}, \tag{7}$$

it turns out that $N(t, A)$ has a Poisson distribution with mean $t\nu(A)$ (which can possibly be infinite). For any $\alpha \in (0, 2)$, the Lévy measure of any neighbourhood of 0 is infinite, hence LFs make infinitely many very small jumps on any time interval. Moreover, the tails of the density $|y|^{1+\alpha}$ determine big jumps of LFs. Thus big jumps have finite mean for $\alpha \in (1, 2)$, and infinite mean for $\alpha \in (0, 1]$.

Let us now decompose the process L into the sum of two independent processes with relatively small and big jumps. We introduce two new Lévy measures by setting

$$\nu_\xi^\varepsilon(A) = \nu\left(A \cap \{x : |x| \le \varepsilon^{-1/2}\}\right), \tag{8}$$

$$\nu_\eta^\varepsilon(A) = \nu\left(A \cap \{x : |x| > \varepsilon^{-1/2}\}\right), \tag{9}$$

and two Lévy processes ξ^ε and η^ε with the corresponding Fourier transforms:

$$\mathbf{E}\mathrm{e}^{i\lambda\xi_t^\varepsilon} = \exp\left\{t \int_{\mathbb{R}\setminus\{0\}} \left[\mathrm{e}^{i\lambda y} - 1 - i\lambda y \mathbf{1}(|y| \le 1)\right] \nu_\xi^\varepsilon(dy)\right\}, \tag{10}$$

$$\mathbf{E}\mathrm{e}^{i\lambda\eta_t^\varepsilon} = \exp\left\{t \int_{\mathbb{R}\setminus\{0\}} \left[\mathrm{e}^{i\lambda y} - 1 - i\lambda y \mathbf{1}(|y| \le 1)\right] \nu_\eta^\varepsilon(dy)\right\}. \tag{11}$$

It is clear that $L_t = \xi_t^\varepsilon + \eta_t^\varepsilon$ since $\nu(A) = \nu_\xi^\varepsilon(A) + \nu_\eta^\varepsilon(A)$, and the processes ξ^ε and η^ε are independent. Let us investigate them in more detail.

First, since $\nu_\xi^\varepsilon(\mathbb{R}) = \infty$, the process ξ_t^ε makes infinitely many jumps on each time interval. Its jumps are, however, bounded by the threshold $\varepsilon^{-1/2}$. Thus ξ_t^ε has finite variance, and more generally moments of all orders.

On the contrary, the Lévy measure of the process η^ε is finite, and we note

$$\beta_\varepsilon = \nu_\eta^\varepsilon(\mathbb{R}) = \int_{-\infty}^{-\varepsilon^{-1/2}} \frac{dy}{|y|^{1+\alpha}} + \int_{\varepsilon^{-1/2}}^{\infty} \frac{dy}{y^{1+\alpha}} = 2\int_{\varepsilon^{-1/2}}^{\infty} \frac{dy}{y^{1+\alpha}} = \frac{2}{\alpha}\varepsilon^{\alpha/2}. \tag{12}$$

Hence, η^ε is a compound Poisson process with jumps of absolute value larger than $\varepsilon^{-1/2}$. Let τ_k^ε and W_k^ε, $k \ge 0$, be the jump arrival times and jump sizes under the convention $\tau_0^\varepsilon = W_0^\varepsilon = 0$. Then the inter-arrival times $T_k^\varepsilon = \tau_k^\varepsilon - \tau_{k-1}^\varepsilon$, $k \ge 1$, are independent and exponentially distributed with mean β_ε^{-1}, and the probability distribution function of W_k^ε is given by

$$\mathbf{P}(W_k^\varepsilon < u) = \frac{1}{\beta_\varepsilon}\int_{-\infty}^{u} \nu_\eta^\varepsilon(dy) = \frac{1}{\beta_\varepsilon}\int_{-\infty}^{u} \mathbf{1}_{\{|y| > \varepsilon^{-1/2}\}}(y)\frac{dy}{|y|^{1+\alpha}}. \tag{13}$$

Consider now the process X^ε given by equation (3). On the inter-arrival intervals $[\tau_{k-1}^\varepsilon, \tau_k^\varepsilon)$, $k \ge 1$, it is driven only by the process $\varepsilon\xi^\varepsilon$, and at the time instants τ_k^ε it makes a jump of the size $\varepsilon W_k^\varepsilon$. Recall that the jumps of $\varepsilon\xi^\varepsilon$ are bounded by $\sqrt{\varepsilon}$. Since the variance of $\varepsilon\xi^\varepsilon$ vanishes in the limit of small ε, the random trajectory X_t^ε should not deviate much from the deterministic trajectory X_t^0 of the underlying dynamical system on the intervals $[\tau_{k-1}^\varepsilon, \tau_k^\varepsilon)$. Indeed, in case of the bounded interval $[-b, a]$ the following estimate holds true:

$$\mathbf{P}\big(\sup_{t\in[0,T_1^\varepsilon)} |X_t^\varepsilon(x) - X_t^0(x)| \ge \varepsilon^\gamma\big) \le \exp(\varepsilon^{-r}), \quad \varepsilon \downarrow 0, \tag{14}$$

for some positive γ, r, and $x \in (-b, a)$. The rigorous proof of this inequality is a tedious task, so we just sketch the idea in the case when the process X^ε is an Ornstein-Uhlenbeck process. In this case, the potential function satisfies $U(x) = Mx^2/2$, and equations (3) and (4) have a closed form solution given by

$$X_t^\varepsilon(x) = x\mathrm{e}^{-Mt} + \varepsilon\left(\xi_t^\varepsilon - M\int_0^t \xi_{s-}^\varepsilon \mathrm{e}^{-M(t-s)}\,ds\right), \tag{15}$$

$$X_t^0(x) = x\mathrm{e}^{-Mt}. \tag{16}$$

Consequently, for any $t \geq 0$,

$$\sup_{s\in[0,t]} |X_t^\varepsilon(x) - X_t^0(x)| \leq 2 \sup_{s\in[0,t]} |\varepsilon\xi_t^\varepsilon|. \tag{17}$$

Then, from the independence of ξ^ε and T_1^ε and the reflection principle for symmetric Lévy processes we obtain

$$\begin{aligned}\mathbf{P}\left(\sup_{t\in[0,T_1^\varepsilon)} |X_t^\varepsilon(x) - X_t^0(x)| \geq \varepsilon^\gamma\right) &\leq \mathbf{P}\left(\sup_{t\in[0,T_1^\varepsilon)} |\varepsilon\xi_t^\varepsilon| \geq \frac{\varepsilon^\gamma}{2}\right)\\ &\leq 4\int_0^\infty \beta_\varepsilon \mathrm{e}^{-\beta_\varepsilon t}\mathbf{P}\left(\varepsilon\xi_t^\varepsilon \geq \frac{\varepsilon^\gamma}{2}\right)dt \\ &\leq \max_{t\in[0,\varepsilon^{-\alpha/2-\delta}]}\mathbf{P}\left(\varepsilon\xi_t^\varepsilon \geq \frac{\varepsilon^\gamma}{2}\right) + 4\int_0^{\varepsilon^{-\alpha/2-\delta}} \beta_\varepsilon \mathrm{e}^{-\beta_\varepsilon t}\,dt = \mathcal{O}(\exp(\varepsilon^{-r}))\end{aligned} \tag{18}$$

for some $\delta > 0$, where the exponential estimate for the probability in the last line follows from Chebyshev's inequality. For details see [10, 11].

Inequality (14) means that on the inter-arrival periods, the random trajectory X_t^ε follows the deterministic trajectory X_t^0 with probability close to 1. Due to the properties of the potential, for any starting point x, $X_t^0(x)$ converges to 0 as $t \to \infty$. Let us consider a ε^γ-neighbourhood of the origin and estimate the relaxation time $T(x,\varepsilon)$ that $X_t^0(x)$ needs to reach it, starting from $x \in (-b, a)$. Solving equation (4) results in

$$T(x,\varepsilon) \leq \max\left\{-\int_{-\infty}^{-\varepsilon^\gamma} \frac{dy}{U'(y)}, \int_{\varepsilon^\gamma}^a \frac{dy}{U'(y)}\right\} \leq R|\ln\varepsilon|, \quad \varepsilon \downarrow 0, \tag{19}$$

for some positive R. Here we have used that $|U'|$ increases faster than linearly at $-\infty$ so the integrals in the previous formula converge. It is of crucial importance for our argument to notice that the relaxation time has logarithmic order in ε and compares to the average time between jumps through

$$T(x,\varepsilon) \ll \mathbf{E}T_1^\varepsilon = \frac{1}{\beta_\varepsilon} = \frac{\alpha}{2\varepsilon^{\alpha/2}}, \quad \varepsilon \downarrow 0. \tag{20}$$

This implies that with probability close to 1, before the arrival time τ_k^ε the process X^ε has relaxed to a small $2\varepsilon^\gamma$-neighbourhood of 0.

Now we can describe the typical behaviour of the sample paths of $X^\varepsilon(x)$. Indeed, starting at $x \in (-b, a)$, $X^\varepsilon(x)$ follows the deterministic trajectory $X^0(x)$ until the first arrival time τ_1^ε. Due to the inequality (20), just before the big jump the process is located near the origin, i.e. $X_{\tau_1^\varepsilon-}^\varepsilon(x) \approx 0$. Consequently its new location is also known and given by

$$X_{\tau_1^\varepsilon}^\varepsilon = X_{\tau_1^\varepsilon-}^\varepsilon + \varepsilon W_1^\varepsilon \approx \varepsilon W_1^\varepsilon. \tag{21}$$

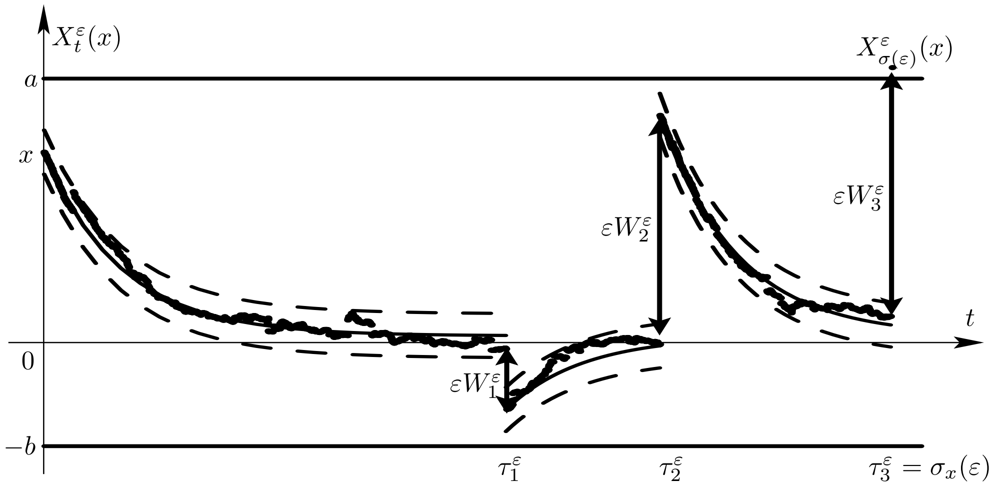

Figure 1. Predominant behaviour of Lévy Flights in external 'parabolic' potential.

From now on, X^ε follows the deterministic trajectory starting at $X^\varepsilon_{\tau_1^\varepsilon}$, and at the next jump time τ_2^ε it jumps to the neighbourhood of $\varepsilon W_2^\varepsilon$, etc. (see Figure 1).

Thus we can summarise the pathwise behaviour of X^ε as follows: $X_0^\varepsilon(x) = x$, with high probability $X^\varepsilon_{\tau_k^\varepsilon}(x) \approx \varepsilon W_k^\varepsilon$, $k \geq 1$, and on the intervals $[\tau_{k-1}^\varepsilon, \tau_k^\varepsilon)$, X^ε follows the deterministic trajectory X^0.

4. Kramers' Law for Lévy Flights

We next discuss the law and the mean value of the first exit time form the interval $I = [-b, a]$

$$\sigma_x(\varepsilon) = \inf\{t > 0 : X_t^\varepsilon(x) \notin I\}, \quad x \in (-b, a), \quad \varepsilon > 0. \tag{22}$$

This can easily be achieved now with the results of the previous Section.

Indeed, we note that X^ε can roughly leave $[-b, a]$ only at one of the time instants τ_k^ε while jumping by the distance $\varepsilon W_k^\varepsilon$ from a small neighbourhood of 0.

The probability to jump out of the interval $[-b, a]$ can be calculated explicitly from (13), to yield the formula

$$\mathbf{P}\left(\varepsilon W_1^\varepsilon \notin [-b, a]\right) = \frac{1}{\beta_\varepsilon} \left(\int_{-\infty}^{-\frac{b}{\varepsilon}} \frac{dy}{|y|^{1+\alpha}} + \int_{\frac{a}{\varepsilon}}^{\infty} \frac{dy}{y^{1+\alpha}} \right) = \frac{\varepsilon^\alpha}{\alpha \beta_\varepsilon} \left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha} \right]. \tag{23}$$

We can therefore calculate the mean value of $\sigma(\varepsilon)$ using the previous formula, the independence of jump sizes and the fact that $\tau_k^\varepsilon = T_1^\varepsilon + \cdots + T_k^\varepsilon$. We obtain

$$\begin{aligned}
\mathbf{E}_x \sigma(\varepsilon) &\approx \sum_{k=1}^\infty \mathbf{E}\tau_k^\varepsilon \cdot \mathbf{P}_x(\sigma(\varepsilon) = \tau_k^\varepsilon) \\
&\approx \sum_{k=1}^\infty k \cdot \mathbf{E}T_1^\varepsilon \cdot \mathbf{P}(\varepsilon W_1^\varepsilon \in I, \ldots, \varepsilon W_{k-1}^\varepsilon \in I, \varepsilon W_k^\varepsilon \notin I)
\end{aligned}$$

$$= \beta_\varepsilon \mathbf{P}(\varepsilon W_1^\varepsilon \notin I) \sum_{k=1}^\infty k(1-\mathbf{P}(\varepsilon W_1^\varepsilon \notin I))^{k-1}$$
$$= \frac{\beta_\varepsilon}{\mathbf{P}(\varepsilon W_1^\varepsilon \notin I)} = \frac{\alpha}{\varepsilon^\alpha}\left[\frac{1}{a^\alpha}+\frac{1}{b^\alpha}\right]^{-1}. \tag{24}$$

Similar arguments allow to estimate the law of the first exit time. Indeed, for all $k \geq 1$, τ_k^ε has a Gamma(β_ε, k) law with density at time t given by $\beta_\varepsilon \mathrm{e}^{-\beta_\varepsilon t}\frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!}$. Hence we may write for $u \geq 0$

$$\mathbf{P}_x(\sigma(\varepsilon) > u) \approx \sum_{k=1}^\infty \mathbf{P}(\tau_k^\varepsilon > u)\cdot \mathbf{P}_x(\sigma(\varepsilon) = \tau_k^\varepsilon)$$
$$\approx \sum_{k=1}^\infty \mathbf{P}(\tau_k^\varepsilon > u)\cdot \mathbf{P}(\varepsilon W_1^\varepsilon \in I, \dots, \varepsilon W_{k-1}^\varepsilon \in I, \varepsilon W_k^\varepsilon \notin I)$$
$$= \sum_{k=1}^\infty \int_u^\infty \beta_\varepsilon \mathrm{e}^{-\beta_\varepsilon t}\frac{(\beta_\varepsilon t)^{k-1}}{(k-1)!}\,dt \cdot (1-\mathbf{P}(\varepsilon W_1^\varepsilon \notin I))^{k-1}\cdot \mathbf{P}(\varepsilon W_1^\varepsilon \notin I)$$
$$= \beta_\varepsilon \mathbf{P}(\varepsilon W_1^\varepsilon \notin I)\int_u^\infty \mathrm{e}^{-\beta_\varepsilon t}\sum_{k=1}^\infty \frac{(\beta_\varepsilon t)^{k-1}(1-\mathbf{P}(\varepsilon W_1^\varepsilon \notin I))^{k-1}}{(k-1)!}\,dt$$
$$= \beta_\varepsilon \mathbf{P}(\varepsilon W_1^\varepsilon \notin I)\int_u^\infty \mathrm{e}^{-\beta_\varepsilon t}\mathrm{e}^{\beta_\varepsilon t(1-\mathbf{P}(\varepsilon W_1^\varepsilon \notin I))}\,dt$$
$$= \exp\left(-u\beta_\varepsilon \mathbf{P}(\varepsilon W_1^\varepsilon \notin I)\right) = \exp\left(-u\frac{\varepsilon^\alpha}{\alpha}\left[\frac{1}{a^\alpha}+\frac{1}{b^\alpha}\right]\right). \tag{25}$$

This can be paraphrased by saying that in the limit of small ε, the exit time $\sigma_x(\varepsilon)$ is exponentially distributed with mean described by (24).

P. Ditlevsen in [19] determined the rate of the mean value of $\sigma_x(\varepsilon)$ as a function of ε using some discrete time approximation of equation (3) and analysing the Fokker-Planck equation.

The exit problem from the potential well was also studied in [20] for LFs with $\alpha \in [1,2)$. The analytically derived asymptotic results in the Cauchy case $\alpha = 1$ are in agreement with our (24) whereas numerical estimates for $\alpha \in (1,2)$ seem to be not very conclusive yet.

It is instructive to compare the results just obtained with their well-known counterparts for diffusions driven by Brownian motions of small intensity ε. Together with (3) consider the diffusion $\hat{X}^\varepsilon$ which solves the stochastic differential equation

$$\hat{X}_t^\varepsilon = x - \int_0^t U'(\hat{X}_s^\varepsilon)\,ds + \varepsilon W_t, \tag{26}$$

where W is a standard one-dimensional Brownian motion, and U is the same potential as in (3). For the diffusion $\hat{X}^\varepsilon$ we define the first exit time from the interval I by

$$\hat{\sigma}_x(\varepsilon) = \inf\{t \geq 0 \,:\, \hat{X}_t^\varepsilon(x) \notin [-b,a]\}, \quad x \in (-b,a). \tag{27}$$

Then the following statements hold for $\hat{\sigma}(\varepsilon)$ in the limit of small ε.

1. The first exit time $\hat{\sigma}_x(\varepsilon)$ is exponentially large in ε^{-2}. To state the law more precisely, assume that $U(a) < U(-b)$. Then for any $\delta > 0$, $x \in (-b, a)$, according to [3]:

$$\mathbf{P}_x(\mathrm{e}^{(2U(a)-\delta)/\varepsilon^2} < \hat{\sigma}(\varepsilon) < \mathrm{e}^{(2U(a)+\delta)/\varepsilon^2}) \to 1 \quad \text{as} \quad \varepsilon \to 0. \tag{28}$$

Moreover, $\varepsilon^2 \ln \mathbf{E}_x \hat{\sigma}(\varepsilon) \to 2U(a)$.

The mean value of the first exit time in the small noise limit (Kramers' law) can be calculated more explicitly by (see [2, 21])

$$\mathbf{E}_x \hat{\sigma}(\varepsilon) \approx \frac{\varepsilon\sqrt{\pi}}{U'(a)\sqrt{U''(0)}} \mathrm{e}^{2U(a)/\varepsilon^2}. \tag{29}$$

For its understanding note that the boundary points a and $-b$ are non-characteristic, i.e. $U'(a), U'(-b) \neq 0$. This leads to a somewhat different formulation of Kramers' law compared with the formula in the original paper [2].

2. The normalised first exit time is exponentially distributed [22, 23, 24]: for $u \geq 0$ we have

$$\mathbf{P}_x\left(\frac{\hat{\sigma}(\varepsilon)}{\mathbf{E}_x\hat{\sigma}(\varepsilon)} > u\right) \to \mathrm{e}^{-u} \quad \text{as} \quad \varepsilon \to 0, \tag{30}$$

uniformly in x on compact subsets of $(-b, a)$.

As we see, $\hat{\sigma}(\varepsilon)$ and $\sigma(\varepsilon)$ have essentially different orders of growth as $\varepsilon \to 0$. The exit times of the processes driven by α-stable noise are much shorter because of the presence of large jumps which occur with probability polynomially small in ε. To leave the interval, the diffusion $\hat{X}^\varepsilon$ has to overcome a potential barrier of height either $U(-b)$ or $U(a)$. So in the case considered here, $\hat{X}^\varepsilon_{\hat{\sigma}(\varepsilon)} = a$ with an overwhelming probability. The diffusion has to 'climb' in the potential landscape. This also explains why the pre-factor in (29) depends on geometric properties of U such as the slope at the exit point and the curvature at the local minimum, the place where the diffusion spends most of its time before exiting.

The process X^ε on the contrary uses the possibility to exit the interval at one large jump. This is the reason why the asymptotic exit time depends mainly on the distance between the stable point 0 and the interval's boundaries. The potential's geometry does not play a big role for the low order approximations of the exit time $\sigma(\varepsilon)$. Although it is important for the proof, it does not appear in the pre-factors of the mean exit time in (24) and remains hidden in the error terms.

5. Meta-stable Behaviour

Assume now that the potential U has two wells with minima located in $-p$ and q, and a saddle point at the origin, $-p < 0 < q$. We continue to assume that all extreme points are non-degenerate, and U increases at infinity of the order $|x|^{2+c}$ for some positive c. For example, one can consider a standard quartic potential $U(x) = \frac{x^4}{4} + (p-q)\frac{x^3}{3} - pq\frac{x^2}{2}$.

It is clear that for small values of ε the process X^ε spends most of its time in small neighbourhoods of the potential's local minima jumping between the wells at random times. Since we have only two wells, the transition time is at the same time the exit time from one of the wells.

In the previous section we have studied exit times for a well with non-characteristic boundaries. The situation in the present Section is a bit more complicated. This is due to the presence of the saddle point at which the force vanishes, and a simple comparison of the random dynamics of X^ε and X^0 as before is not possible.

However we can reduce the exit problem to the one solved before by excluding some small, say ε^γ-neighbourhood of the saddle point, and considering the exit from the domains $(-\infty, -\varepsilon^\gamma]$ and $[\varepsilon^\gamma, \infty)$. The dynamics in these domains is not essentially different from the typical behaviour described in Section 3. We only need to give an estimate for the relaxation time $T(x,\varepsilon)$ which takes into account the singularity at the origin. Thus, for instance for the left well we obtain

$$T(x,\varepsilon) \leq \max\left\{ -\int_{-\infty}^{-p-\varepsilon^\gamma} \frac{dy}{U'(y)}, \int_{-p+\varepsilon^\gamma}^{-\varepsilon^\gamma} \frac{dy}{U'(y)} \right\} \leq R_1 |\ln \varepsilon|, \quad \varepsilon \downarrow 0. \quad (31)$$

The relaxation time has again just a logarithmic order due to the non-degeneracy of the potential's extreme points.

We further notice that the probabilities to jump out of the domain $(-\infty, -\varepsilon^\gamma]$ and into the domain $[\varepsilon^\gamma, \infty)$ are equal up to terms of higher order to the expression

$$\mathbf{P}(\varepsilon W_1^\varepsilon > p - \varepsilon^\gamma) \approx \mathbf{P}(\varepsilon W_1^\varepsilon > p + \varepsilon^\gamma) \approx \frac{\varepsilon^{\alpha/2}}{2p^\alpha}, \quad (32)$$

whereas the probability to jump into the ε^γ-neighbourhood of the saddle point is negligible and given by

$$\mathbf{P}(\varepsilon W_1^\varepsilon + p \in [-\varepsilon^\gamma, \varepsilon^\gamma]) \propto \varepsilon^{\alpha/2+\gamma} \ll \varepsilon^{\alpha/2}. \quad (33)$$

(This explains why we should work with ε-dependent neighbourhoods.)

Finally, we conclude that the transition times from the left to the right well resp. vice versa have mean values

$$\mathbf{E}\tau_{pq}(\varepsilon) \approx \frac{\alpha p^\alpha}{\varepsilon^\alpha} \quad \text{and} \quad \mathbf{E}\tau_{qp}(\varepsilon) \approx \frac{\alpha q^\alpha}{\varepsilon^\alpha}, \quad \varepsilon \downarrow 0, \quad (34)$$

and are asymptotically exponentially distributed. Thus, the main features of the process X^ε in the small noise limit are retained by a Markov jump process, and on the time scale $\varepsilon^{-\alpha}$ we obtain the following convergence in the sense of finite dimensional distributions:

$$X^\varepsilon_{t/\varepsilon^\alpha}(x) \to Y_t, \quad t > 0, \quad \varepsilon \downarrow 0, \quad (35)$$

where Y is a Markov process on the state space $\{-p, q\}$ with the following matrix as infinitesimal generator

$$\frac{1}{\alpha}\begin{pmatrix} -p^{-\alpha} & p^{-\alpha} \\ q^{-\alpha} & -q^{-\alpha} \end{pmatrix} \quad \text{and} \quad Y_0 = \begin{cases} -p, & \text{if } x < 0, \\ q, & \text{if } x > 0. \end{cases} \quad (36)$$

Again, let us compare the result obtained with its Gaussian counterpart. Here we refer to [25], where this problem was first studied.

Let us again consider a Gaussian diffusion $\hat{X}^\varepsilon$ which solves equation (26). Since it is well known that in the Gaussian case the height of the potential barriers plays a crucial role, we assume that $U(0) = 0$, $U(-p) = -H$, $U(q) = -h$ and $0 < h < H$, i.e. the left well is deeper. Then due to Kramers' law, the system has two different intrinsic time scales, given by the mean exit times from the wells (compare with (34)):

$$\mathbf{E}\hat{\tau}_{pq}(\varepsilon) \approx \frac{2\pi}{\sqrt{U''(-p)|U''(0)|}}\mathrm{e}^{2H/\varepsilon^2} \quad \text{and} \quad \mathbf{E}\hat{\tau}_{qp}(\varepsilon) \approx \frac{2\pi}{\sqrt{U''(-q)|U''(0)|}}\mathrm{e}^{2h/\varepsilon^2}, \quad \varepsilon \downarrow 0. \tag{37}$$

Exponentially different Kramers' times lead to the following meta-stable behaviour of $\hat{X}^\varepsilon$:

$$\hat{X}^\varepsilon_{t\lambda^\varepsilon}(x) \to \hat{Y}_t, \quad \varepsilon \downarrow 0, \tag{38}$$

in the sense of finite dimensional distributions, where λ^ε is such that $\lambda^\varepsilon/\mathbf{E}\hat{\tau}_{qp}(\varepsilon) \to 1$, and $\hat{Y}$ is a Markov process on $\{-p, q\}$ with the infinitesimal matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \hat{Y}_0 = \begin{cases} -p, & \text{if } x < 0, \\ q, & \text{if } x > 0. \end{cases} \tag{39}$$

As we see, the main difference between LFs and Gaussian dynamics consists not only in different intrinsic time scales — polynomial vs. exponential, — but also in qualitatively different limiting behaviour. In the heavy-tail case, the states of the limiting process are recurrent, whereas in the Gaussian case, the minimum of the deepest well is absorbing.

6. Conclusion

We determine the probability law and the mean value of escape times from a potential well for all values of the stability index $\alpha \in (0, 2)$ in the limit of small noise. Escape times have exponential distribution, and their averages increase as $\varepsilon^{-\alpha}$ with pre-factors depending on α and the distance between the potential's local extrema.

In the case of a double-well potential, we determine a new time scale on which the Lévy-driven diffusion converges to a two-state Markov process with some non-trivial generator.

Our methods are purely probabilistic. They also work for all Lévy noises with heavy (regularly varying) tails of any index.

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