Large deviations for Hilbert space valued Wiener processes: a sequence space approach

Andreas Andresen, Peter Imkeller, Nicolas Perkowski
Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
12489 Berlin
Germany

October 19, 2011

Abstract

Ciesielski's isomorphism between the space of α-Hölder continuous functions and the space of bounded sequences is used to give an alternative proof of the large deviation principle for Wiener processes with values in Hilbert space.

Introduction

The large deviation principle (LDP) for Brownian motion β on $[0,1]$ - contained in Schilder's theorem - describes the exponential decay of the probabilities with which $\sqrt{\varepsilon}\beta$ takes values in closed or open subsets of the path space of continuous functions in which the trajectories of β live. The path space is equipped with the topology generated by the uniform norm. The decay is dominated by a rate function capturing the 'energy' $\frac{1}{2}\int_0^1(\dot{f}(t))^2dt$ of functions f on the Cameron-Martin space for which a square integrable derivative exists. A version of Schilder's theorem for a Q-Wiener processes W taking values in a separable Hilbert space H is well known (see Da Prato and Zabczyk (1992)). Here Q is a self adjoint positive trace class operator on H. If $(\lambda_i)_{i\geq 0}$ are its summable eigenvalues with respect to an eigenbasis $(e_k)_{k\geq 0}$ in H, W may be represented with respect to a sequence of one dimensional Wiener processes $(\beta_k)_{k\geq 0}$ by $W=\sum_{k=0}^\infty \lambda_k\beta_k\, e_k$. The LDP in this framework can be derived by means of techniques of reproducing kernel Hilbert spaces (see Da Prato and Zabczyk (1992)). The rate function is then given by an analogous energy functional for which $\dot{f}^2$ is replaced by $\|Q^{-\frac{1}{2}}\dot{F}\|^2$ for continuous functions F possessing square integrable derivatives $\dot{F}$ on $[0,1]$.

Schilder's theorem for β may for instance be derived via approximation of β by random walks from LDP principles for discrete processes (see Dembo and Zeitouni (1998)). Baldi and Roynette Baldi and Roynette (1992) give a very elegant alternative proof of Schilder's theorem, the starting point of which is a Fourier decomposition of β by a complete orthonormal system (CONS) in $L^2([0,1])$. The rate function for β is then simply calculated by the rate functions of one-dimensional Gaussian unit variables. In this approach, the LDP is first proved for balls of the topology, and then generalized by means of exponential tightness to open and closed sets of the topology. As a special feature of the approach, Schilder's theorem is obtained in a stricter sense on all spaces of Hölder continuous functions of order $\alpha<\frac{1}{2}$. This enhancement results quite naturally from a characterization of the Hölder topologies on function spaces by appropriate infinite sequence spaces (see Ciesielski (1960)). Representing the one-dimensional Brownian motions β_k for instance by the CONS of Haar functions on $[0,1]$, we obtain a description of the Hilbert space valued Wiener process W in which a double sequence of independent standard normal variables describes randomness. Starting with this observation, in this paper we extend the direct proof of Schilder's theorem by Baldi and Roynette Baldi and Roynette (1992) to Q-Wiener spaces W with values on H. On the way, we also retrieve the enhancement of the LDP to spaces of Hölder continuous functions on $[0,1]$ of order $\alpha < \frac{1}{2}$.

In Section 1 we first give a generalization of Ciesielski's isomorphism of spaces of Hölder continuous functions and sequence spaces to functions with values on Hilbert spaces. We briefly recall the basic notions of Gaussian measures and Wiener processes on Hilbert spaces. Using Ciesielski's isomorphism we give a Schauder representation of Wiener processes with values in H. Additionally we give a short overview of concepts and results from the theory of LDP needed in the derivation of Schilder's theorem for W. In the main Section 2 the alternative proof of the LDP for W is given. We first introduce a new norm on the space of Hölder continuous functions $C_\alpha([0,1],H)$ with values in H which is motivated by the sequence space representation in Ciesielski's isomorphism, and generates a coarser topology. We adapt the description of the rate function to the Schauder series setting, and then prove the LDP for a basis of the coarser topology using Ciesielski's isomorphism. We finally establish the last ingredient, the crucial property of exponential tightness, by construction of appropriate compact sets in sequence space.

1 Preliminaries

In this section we collect some ingredients needed for the proof of a large deviations principle for Hilbert space valued Wiener processes. We first prove Ciesielski's theorem for Hilbert space valued functions which translates properties of functions into properties of the sequences of their Fourier coefficients with respect to complete orthonormal systems in $L^2([0,1])$. We summarize some basic properties of Wiener processes W with values in a separable Hilbert space H. We then discuss Fourier decompositions of W, prove that its trajectories lie almost surely in $C^0_\alpha([0,1],H)$ and describe its image under the Ciesielski isomorphism. We will always denote by H a separable Hilbert space equipped with a symmetric inner product $\langle\cdot,\cdot\rangle$ that induces the norm $\|\cdot\|_H$ and a countable complete orthonormal system (CONS) (e_k) $k\in\mathbb{N}$.

1.1 Ciesielski's isomorphism

The **Haar functions** $(\chi_n, n\geq 0)$ are defined as $\chi_0\equiv 1$,

$$\chi_{2^k+l}(t) := \begin{cases} \sqrt{2^k}, & \frac{2l}{2^{k+1}} \leq t \leq \frac{2l+1}{2^{k+1}}, \\ -\sqrt{2^k}, & \frac{2l+1}{2^{k+1}} \leq t \leq \frac{2l+2}{2^{k+1}}, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

The Haar functions form a CONS of $L^2([0,1],dx)$. Note that because of their wavelet structure, the integral $\int_{[0,1]}\chi_n df$ is well-defined for all functions f. For $n=2^k+l$ where $k\in\mathbb{N}$ and $0\leq l\leq 2^k-1$ we have $\int_{[0,1]}\chi_n dF = \sqrt{2^k}[2F(\frac{2l+1}{2^{k+1}}) - F(\frac{2l+2}{2^{k+1}}) - F(\frac{2l}{2^{k+1}})]$, and it does not matter whether F is a real or Hilbert space valued function.

The primitives of the Haar functions are called **Schauder functions**, and they are given by

$$\phi_n(t) = \int_0^t \chi_n(s)ds,\ t\in[0,1],\ n\geq 0.$$

Slightly abusing notation, we denote the α-Hölder seminorms on $C_\alpha([0,1];H)$ and on $C_\alpha([0,1];\mathbb{R})$ by the same symbols

$$\|F\|_\alpha := \sup_{0\leq s<t\leq 1}\frac{\|F(t)-F(s)\|_H}{|t-s|^\alpha}, \quad F\in C_\alpha([0,1];H),$$
$$\|f\|_\alpha := \sup_{0\leq s<t\leq 1}\frac{|f(t)-f(s)|}{|t-s|^\alpha}, \quad f\in C_\alpha([0,1];\mathbb{R})$$

$C_\alpha([0,1];H)$ is of course the space of all functions $F:[0,1]\to H$ such that $\|F\|_\alpha<\infty$, and similarly for $C_\alpha([0,1];\mathbb{R})$. We also denote the supremum norm on $C([0,1];H)$ and $C([0,1];R)$ by the same symbol $\|\cdot\|_\infty$.

Denote in the sequel for an H-valued function F its orthogonal component with respect to e_k by $F_k=\langle F,e_k\rangle, k\geq 0$. Further denote by P_k resp. R_k the orthogonal projectors on $\mathrm{span}(e_1,\cdots,e_k)$ resp. its orthogonal complement, $k\geq 0$. For every $F\in C_\alpha([0,1];H)$, every $k\geq 0, s,t\in[0,1]$ we have

$$|\langle F(t),e_k\rangle-\langle F(s),e_k\rangle|\leq\|F(t)-F(s)\|_H$$

More generally, for any $k\geq 0, s,t\in[0,1]$ we have

$$\|P_kF(t)-P_kF(s)\|_H\leq\|F(t)-F(s)\|_H,\quad \|R_kF(t)-R_kF(s)\|_H\leq\|F(t)-F(s)\|_H.$$

Our approach starts with the observation that we may decompose functions $F\in C_\alpha([0,1];H)$ by double series with respect to the system $(\phi_n\, e_k: n,k\geq 0)$.

Lemma 1. *Let $\alpha\in(0,1)$ and $F\in C_\alpha([0,1];H)$. Then we have*

$$F=\sum_n\int_{[0,1]}\chi_n dF\phi_n=\sum_{n=0}^\infty\sum_{k=0}^\infty\int_{[0,1]}\chi_n dF_k e_k\phi_n$$

with convergence in the uniform norm on $C([0,1];H)$.

Proof. For the real valued functions $F_k, k\geq 0$, the representation

$$F_k=\sum_{n=0}^\infty\int_{[0,1]}\chi_n dF_k\,\phi_n$$

is well known from Ciesielski (1960). Therefore we may write for $F\in C_\alpha([0,1];H)$

$$\begin{aligned}F&=\sum_{k=0}^\infty F_k e_k\\&=\sum_{k=0}^\infty e_k\sum_{n=0}^\infty\int_{[0,1]}\chi_n dF_k\phi_n\\&=\sum_{n=0}^\infty\sum_{k=0}^\infty\int_{[0,1]}\chi_n dF_k e_k\phi_n\\&=\sum_{n=0}^\infty\int_{[0,1]}\chi_n dF\phi_n.\end{aligned}$$

To justify the exchange in the order of summation and the convergence in the uniform norm, we have to show

$$\lim_{N,m\to\infty}\left\|\sum_{n\geq N}\int_{[0,1]}\chi_n dR_mF\phi_n\right\|_\infty=0.$$

For this purpose, note first that by definition of the Haar system for any $n,m\geq 0, n=2^k+l$, where $0\leq l\leq 2^k-1$

$$\begin{aligned}\left\|\int_{[0,1]}\chi_n dR_mF\right\|_H&=\sqrt{2^k}\left\|2R_mF\left(\frac{2l+1}{2^{k+1}}\right)-R_mF\left(\frac{2l+2}{2^{k+1}}\right)-R_mF\left(\frac{2l}{2^{k+1}}\right)\right\|_H\\&\leq 2\|R_mF\|_\alpha 2^{-\alpha(k+1)}2^{\frac{1}{2}k}\\&=\|R_mF\|_\alpha 2^{-\alpha(k+1)+\frac{1}{2}k+1}.\end{aligned}$$

Therefore and for $K \geq 0$ such that $2^K \leq N \leq 2^{K+1}$, using the fact that $\phi_{2^k+l}, 0 \leq l \leq 2^k - 1$ have disjoint support and that $\|\phi_{2^k+l}\|_\infty \leq 2^{-\frac{k}{2}-1}$, we obtain

$$\begin{aligned}
\left\| \sum_{n \geq N} \int_{[0,1]} \chi_n dR_m F \phi_n \right\|_\infty &\leq \sum_{k \geq K} \left\| \sum_{0 \leq l \leq 2^k - 1} \int_{[0,1]} \chi_{2^k+l} dF \phi_{2^k+l} \right\|_\infty \\
&\leq \sum_{k \geq K} \sup_{0 \leq l \leq 2^k - 1} \left\| \int_{[0,1]} \chi_{2^k+l} dR_m F \right\|_\infty 2^{-\frac{k}{2}-1} \\
&\leq \sum_{k \geq K} \|R_m F\|_\alpha 2^{-\alpha(k+1)} \\
&\leq \|R_m F\|_\alpha \sum_{k \geq K} (2^\alpha)^{-k} \xrightarrow[K,m \to \infty]{} 0.
\end{aligned}$$

Here we use $\|R_m F\|_\alpha \leq \|F\|_\alpha < \infty$ for all $m \geq 0$, the fact that $\lim_{m\to\infty} R_m F(t) = 0$ for any $t \in [0,1]$, and dominated convergence to obtain $\lim_{m\to\infty} \|R_m F\|_\alpha = 0$. ☐

A closer inspection of the coefficients in the decomposition of Lemma 1 leads us to the following isomorphism, described by Ciesielski (1960) in the 1-dimensional case. To formulate it, denote by $\mathcal{C}_0^H$ the space of H-valued sequences $(\eta_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \|\eta_n\|_H = 0$. If we equip $\mathcal{C}_0^H$ with the supremum norm (using again the symbol $\|\cdot\|_\infty$), it becomes a Banach space.

Theorem 1 (Ciesielski's isomorphism for Hilbert spaces)**.** *Let $0 < \alpha < 1$. Let (χ_n) denote the Haar functions, and (ϕ_n) denote the Schauder functions. Let for $0 \leq n = 2^k + l \geq 0$, where $0 \leq l \leq 2^k - 1$*

$$c_0(\alpha) := 1, \quad c_n(\alpha) := 2^{k(\alpha - 1/2) + \alpha - 1}.$$

Define

$$T_\alpha^H : C_\alpha^0([0,1]; H) \to \mathcal{C}_0^H \qquad F \mapsto \left(c_n(\alpha) \int_{[0,1]} \chi_n dF \right)_{n \in \mathbb{N}}$$

Then T_α^H is continuous and bijective, its operator norm is 1, and its inverse is given by

$$(T_\alpha^H)^{-1} : \mathcal{C}_0^H \to C_\alpha^0([0,1]; H), \quad (\eta_n) \mapsto \sum_{n=0}^\infty \frac{\eta_n}{c_n(\alpha)} \phi_n,$$

The norm of $(T_\alpha^H)^{-1}$ is bounded by

$$\|(T_\alpha^H)^{-1}\| \leq \frac{2}{(2^\alpha - 1)(2^{1-\alpha} - 1)}.$$

Proof. Observe that for $n \in \mathbb{N}$ with $n = 2^k + l$, $0 \leq l \leq 2^k - 1$

$$\begin{aligned}
&\left\| \int_{[0,1]} \chi_n dF \right\|_H \\
&= \sqrt{2^k} \left\| 2F\left(\frac{2l+1}{2^{k+1}}\right) - F\left(\frac{2l+2}{2^{k+1}}\right) - F\left(\frac{2l}{2^{k+1}}\right) \right\|_H \\
&\leq \frac{1}{2c_\alpha(n)} \left(\frac{\|F(\frac{2l+2}{2^{k+1}}) - F(\frac{2l+1}{2^{k+1}})\|_H}{2^{-\alpha(k+1)}} + \frac{\|F(\frac{2l+1}{2^{k+1}}) - F(\frac{2l}{2^{k+1}})\|_H}{2^{-\alpha(k+1)}} \right) \\
&\leq \frac{1}{c_\alpha(n)} \sup_{t,s\in[0,1],\ |t-s| \leq 2^{-k-1}} \frac{\|F(t) - F(s)\|_H}{|t-s|^\alpha} \\
&\leq \frac{1}{c_\alpha(n)} \|F\|_\alpha.
\end{aligned}$$

This gives the desired bound on the norm. Moreover, since $F \in C^0_\alpha([0,1],H)$ we have

$$\lim_{n\to\infty} c_\alpha(n) \left\| \int_{[0,1]} \chi_n dF \right\|_H \leq \lim_{n\to\infty} \sup_{t,s\in[0,1],\ |t-s|\leq 2^{-k-1}} \frac{\|F(t)-F(s)\|_H}{|t-s|^\alpha} = 0.$$

Thus the range of T^H_α is indeed contained in $\mathcal{C}^H_0$. Taking $F:[0,1]\to H$ with $F(s)=se_1$ for $s\in[0,1]$ we find that $T^H_\alpha(F)=(e_1,0,0,...)$, thus $\|F\|_\alpha = \|T^H_\alpha(F)\|_\infty$. Therefore $\|T^H_\alpha\|=1$. Clearly T^H_α is injective.

To see that T^H_α is bijective and that the inverse is bounded as claimed, define

$$A: \mathcal{C}^H_0 \to C^0_\alpha([0,1];H), \quad (\eta_n) \mapsto \sum_{n=0}^\infty \frac{\eta_n}{c_n(\alpha)}\phi_n.$$

Now a straightforward calculation using the orthogonality of the $(\chi_n)_{n\geq 0}$ gives for any $(\eta_n)_{n\geq 0}\subset \mathcal{C}^H_0$

$$\begin{aligned} T^H_\alpha \circ A((\eta_n)_{n\geq 0}) &= T^H_\alpha\left(\sum_{n=0}^\infty \frac{\eta_n}{c_n(\alpha)}\phi_n\right) \\ &= \left(\sum_{n,m=0}^\infty \eta_n \int_{[0,1]} \chi_m d\phi_n\right)_{m\in\mathbb{N}} \\ &= \left(\sum_{n,m=0}^\infty \eta_n \int \chi_n(t)\chi_m(t)dt\right)_{m\in\mathbb{N}} \\ &= (\eta_m)_{m\geq 0}. \end{aligned}$$

Consequently we can infer that $A=(T^H_\alpha)^{-1}$.

We still have to show that $(T^H_\alpha)^{-1}$ satisfies the claimed norm inequality and maps every sequence $(\eta_n)_{n\geq 0}\in\mathcal{C}^H_0$ to an element of $C^0_\alpha([0,1],H)$. For this purpose let $(\eta_n)_{n\geq 0}\in \mathcal{C}^H_0$, set $F=(T^H_\alpha)^{-1}((\eta_n))$ and let $s,t\in[0,1]$ be given. Then we have

$$\|F(t)-F(s)\|_H \leq \|(\eta_n)_{n\geq 0}\|_\infty \left(|t-s| + \sum_{k=0}^\infty \sum_{l=0}^{2^k-1} \frac{|\phi_{2^k+l}(t)-\phi_{2^k+l}(s)|}{c_{2^k}(\alpha)}\right).$$

The term in brackets on the right hand side is exactly the one appearing in the real valued case (Ciesielski (1960)). Consequently we have the same bound, given by

$$\|(T^H_\alpha)^{-1}\| \leq \frac{1}{(2^\alpha-1)(2^{\alpha-1}-1)}.$$

A more careful estimation yields

$$\|F(t)-F(s)\|_H \leq [\|\eta_0\||t-s| + \left(\sum_{k=0}^\infty \sum_{l=0}^{2^k-1} \frac{1}{c_{2^k}(\alpha)} \|\eta_{2^k+l}\||\phi_{2^k+l}(t)-\phi_{2^k+l}(s)|\right).$$

This is the same expression as in the real valued case. Its well known treatment implies

$$\lim_{|t-s|\to 0} \frac{\|F(t)-F(s)\|_H}{|t-s|^\alpha} = 0.$$

This finishes the proof. □

1.2 Wiener processes on Hilbert spaces

We recall some basic concepts of Gaussian random variables and Wiener processes with values in a separable Hilbert space H. Especially we will derive a Fourier sequence decomposition of Wiener processes. Our presentation follows Da Prato and Zabczyk (1992).

Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $m \in H$ and $Q : H \to H$ a positive self adjoint operator. An H-valued random variable X such that for every $h \in H$

$$E[\exp(i\langle h, X\rangle)] = \exp\left(i\langle h, m\rangle - \frac{1}{2}\langle Qh, h\rangle\right).$$

is called Gaussian with covariance operator Q and mean $m \in H$. We denote the law of X by $\mathcal{N}(m, Q)$.

By Proposition 2.15 of Da Prato and Zabczyk (1992), Q has to be a positive, self-adjoint trace class operator, i.e. a bounded operator from H to H that satisfies

1. $\langle Qx, x\rangle \geq 0$ for every $x \in H$,
2. $\langle Qx, x\rangle = \langle x, Qx\rangle$ for every $x \in H$,
3. $\sum_{k=0}^{\infty} \langle Qe_k, e_k\rangle < \infty$ for every CONS $(e_k)_{k\geq 0}$.

If Q is a positive, self-adjoint trace class operator on H, then there exists a CONS $(e_k)_{k\geq 0}$ such that $Qe_k = \lambda_k e_k$, where $\lambda_k \geq 0$ for all k and $\sum_{k=0}^{\infty} \lambda_k < \infty$. Note that for such a Q, an operator $Q^{1/2}$ can be defined by setting $Q^{1/2}e_k := \sqrt{\lambda_k}e_k, k \in \mathbb{N}_0$. Then $Q^{1/2}Q^{1/2} = Q$.

Definition. Let Q be a positive, self-adjoint trace class operator on H. A Q-Wiener process $(W(t) : t \in [0, 1])$ is a stochastic process with values in H such that

1. $W(0) = 0$,
2. W has continuous trajectories,
3. W has independent increments,
4. $\mathcal{L}(W(t) - W(s)) = \mathcal{N}(0, (t - s)Q)$

In this case $(W(t_1), \ldots, W(t_n))$ is H^n-valued Gaussian for all $t_1, \ldots, t_n \in [0, 1]$. By Proposition 4.2 of Da Prato and Zabczyk (1992) we know that such a process exists for every positive, self-adjoint trace class operator Q on H. To get the Fourier decomposition of a Q-Wiener process along the Schauder basis we use a different standard characterization.

Lemma 2. *A stochastic process Z on $(H, \mathcal{B}(H))$ is a Q-Wiener process iff*

- $Z_0 = 0$ $\mathbb{P}$-*a.s.,*
- *Z has continuous trajectories,*
- *cov*$(\langle v, Z_t\rangle\langle w, Z_s\rangle) = (t \wedge s)\langle v, Qw\rangle$ $\forall v, w \in H$, $\forall 0 \leq s \leq t < \infty$,
- $\forall (v_1, ..., v_n) \in H^n$ $(\langle v_1, Z\rangle, ..., \langle v_n, Z\rangle)$ *is a* $\mathbb{R}^n$*-valued Gaussian process.*

Independent Gaussian random variables with values in a Hilbert space asymptotically allow the following bounds.

Lemma 3. *Let $Z_n \sim \mathcal{N}(0, Q)$, $n \in \mathbb{N}$, be iid. Then there exists an a.s. finite real valued random variable C such that*

$$\|Z_n\|_H \leq C\sqrt{\log n}\ \mathbb{P}\ a.s..$$

Proof. By using the exponential integrability of $\lambda\|Z_n\|_H^2$ for small enough λ and Markov's inequality, we obtain that there exist $\lambda, c \in \mathbb{R}_+$ such that for any $a > 0$

$$\mathbb{P}(\|Z\|_H > a) \leq ce^{-\lambda a^2}.$$

Thus for $\alpha > 1$ and n big enough

$$\mathbb{P}\left(\|Z_n\|_H \geq \sqrt{\lambda^{-1}\alpha \log n}\right) \leq cn^{-\alpha}.$$

We set $A_n = \left\{\|Z_n\|_H \geq \sqrt{\lambda^{-1}\alpha \log n}\right\}$ and have

$$\sum_{n=0}^{\infty} \mathbb{P}(A_n) < \infty.$$

Hence the lemma of Borel-Cantelli gives, that $\mathbb{P}(\limsup_n A_n) = 0$, i.e. $\mathbb{P} - a.s.$ for almost all $n \in \mathbb{N}$ we have $\|Z_n\|_H \leq \sqrt{\lambda^{-1}\alpha \log n}$. In other words

$$C := \sup_{n\geq 0} \frac{\|Z_n\|_H}{\sqrt{\log n}} < \infty \ \mathbb{P} - a.s.$$

□

Using Lemma 3 and the characterization of Q-Wiener processes of Lemma 2, we now obtain its Schauder decomposition which can be seen as a Gaussian version of Lemma 1.

Proposition 1. *Let $\alpha \in (0, 1/2)$, let $(\phi_n)_{n\geq 0}$ be the Schauder functions and $(Z_n)_{n\geq 0}$ a sequence of independent, $\mathcal{N}(0, Q)$-distributed Gaussian variables, where Q is a positive self adjoint trace class operator on H. The series defined process*

$$W_t = \sum_{n=0}^{\infty} \phi_n(t) Z_n, \quad t \in [0, 1],$$

converges $\mathbb{P}$-a.s. with respect to the $\|\cdot\|_\alpha$-norm on $[0, 1]$ and is an H-valued Q-Wiener process.

Proof. We have to show that the process defined by the series satisfies the conditions given in Lemma 2. The first and the two last conditions concerning the covariance structure and Gaussianity of scalar products have standard verifications. Let us just argue for absolute and $\|\cdot\|_\alpha$-convergence of the series, thus proving Hölder-continuity of the trajectories.

Since T_α^H is an isomorphism and since any single term of the series is even Lipschitz-continuous, it suffices to show that

$$\left(T_\alpha^H\left(\sum_{n=0}^{m} \phi_n Z_n\right) : m \in \mathbb{N}\right)$$

is a Cauchy sequence in $\mathcal{C}_0^H$. Let us first calculate the image of term N under T_α^H. We have

$$(T_\alpha^H \phi_n Z_n)_N = 1_{\{n=N\}} c_N(\alpha) Z_N.$$

Therefore for $m_1, m_2 \geq 0, m_1 \leq m_2$

$$\sum_{n=m_1}^{m_2} (T_\alpha^H \phi_n Z_n)_N = 1_{\{m_1 \leq N \leq m_2\}} c_N(\alpha) Z_N = \left(T_\alpha^H\left(\sum_{n=m_1}^{m_2} \phi_n Z_n\right)\right)_N.$$

So if we can prove that $c_N(\alpha) Z_N$ a.s. converges to 0 in H as $N \to \infty$, the proof is complete. But this follows immediately from Lemma 3: $c_N(\alpha)$ decays exponentially fast, and $\|Z_N\|_H \leq C\sqrt{\log N}$. □

In particular we showed that for $\alpha < 1/2$ W a.s. takes its trajectories in

$$C^0_\alpha([0,1];H) := \left\{ F : [0,1] \to H, F(0) = 0, \lim_{\delta\to 0} \sup_{\substack{t\neq s,\\ |t-s|<\delta}} \frac{\|F(t)-F(s)\|_H}{|t-s|^\alpha} = 0 \right\}$$

By Lipschitz continuity of the scalar product, we also have $\langle F, e_k\rangle \in C^0_\alpha([0,1];\mathbb{R})$: Since P_k and R_k are orthogonal projectors and therefore Lipschitz continuous, we obtain that for $F \in C^0_\alpha([0,1];H)$

$$\sup_{k\geq 0} \|\langle F, e_k\rangle\|_\alpha \leq \|F\|_\alpha.$$

We also saw that $T^H_\alpha(W)$ is well defined almost surely. As a special case this is also true for the real valued Brownian motion. We have by Proposition 1

$$T^H_\alpha(W) = (c_n(\alpha)Z_n)$$

where $(Z_n)_{n\geq 0}$ is a sequence of i.i.d. $\mathcal{N}(0,Q)-$variables.

Plainly, the representation of the preceding Lemma can be used to prove the representation formula for Q-Wiener processes by scalar Brownian motions according to Da Prato and Zabczyk (1992), Theorem 4.3.

Proposition 2. *Let W be a Q-Wiener process. Then*

$$W(t) = \sum_{k=0}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k,\ t \in [0,1],$$

where the series on the right hand side $\mathbb{P}$-a.s. converges uniformly on $[0,1]$, and $(\beta_k)_{k\geq 0}$ is a sequence of independent real valued Brownian motions.

Proof. Using arguments as in the proof of Theorem 1 and Lemma 3 to justify changes in the order of summation we get

$$W = \sum_{n=0}^{\infty} \phi_n Z_n = \sum_{k\geq 0}\sum_{n\geq 0} \phi_n \langle Z_n, e_k\rangle e_k = \sum_{k\geq 0} \sqrt{\lambda_k} \sum_{n\geq 0} \phi_n N_{n,k} e_k = \sum_{k\geq 0} \sqrt{\lambda_k}\beta_k e_k,$$

where the equivalences are $\mathbb{P}$-a.s. and $(N_{n,k})_{n,k\geq 0}$, $(\beta_k)_{k\geq 0}$ are real valued iid $\mathcal{N}(0,1)$ random variables resp. Brownian motions. For the last step we applied Proposition 1 for the one-dimensional case. □

1.3 Large deviations

Let us recall some basic notions of the theory of large deviations that will suffice to prove the large deviation principle for Hilbert space valued Wiener processes. We follow Dembo and Zeitouni (1998). Let X be a topological Hausdorff space. Denote its Borel σ-algebra by $\mathcal{B}$.

Definition (Rate function)**.** A function $I : X \to [0,\infty]$ is called a rate function if it is lower semi-continuous, i.e. if for every $C \geq 0$ the set

$$\Psi_I(C) := \{x \in X : I(x) \leq C\}$$

is closed. It is called a good rate function, if $\Psi_I(C)$ is compact. For $A \in \mathcal{B}$ we define $I(A) := \inf_{x\in A} I(x)$.

Definition (Large deviation principle)**.** Let I be a rate function. A family of probability measures $(\mu_\varepsilon)_{\varepsilon>0}$ on $(X, \mathcal{B})$ is said to satisfy the large deviation principle (LDP) with rate function I if for any closed set $F \subset X$ and any open set $G \subset X$ we have

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(F) \le -I(F) \text{ and}$$
$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G) \ge -I(G).$$

Definition (Exponential tightness)**.** A family of probability measures $(\mu_\varepsilon)_{\varepsilon>0}$ is said to be exponentially tight if for every $a > 0$ there exists a compact set $K_a \subset X$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(K_a^c) < -a.$$

In our approach to Schilder's Theorem for Hilbert space valued Wiener processes we shall mainly use the following proposition which basically states that the rate function has to be known for elements of a sub-basis of the topology.

Proposition 3. *Let $\mathcal{G}_0$ be a collection of open sets in the topology of X such that for every open set $G \subset X$ and for every $x \in G$ there exists $G_0 \in \mathcal{G}_0$ such that $x \in G_0 \subset G$. Let I be a rate function and let $(\mu_\varepsilon)_{\varepsilon>0}$ be an exponentially tight family of probability measures. Assume that for every $G \in \mathcal{G}_0$ we have*

$$-\inf_{x\in G} I(x) = \lim_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G).$$

Then I is a good rate function, and $(\mu_\varepsilon)_\varepsilon$ satisfies an LDP with rate function I.

Proof. Let us first establish the lower bound. In fact, let G be an open set. Choose $x \in G$, and a basis set G_0 such that $x \in G_0 \subset G$. Then evidently

$$\liminf_{\varepsilon\to 0} \varepsilon \ln \mu_\varepsilon(G) \ge \liminf_{\varepsilon\to 0} \varepsilon \ln \mu_\varepsilon(G_0) = -\inf_{y\in G_0} I(y) \ge -I(x).$$

Now the lower bound follows readily by taking the sup of $-I(x), x \in G$, on the right hand side, the left hand side not depending on x.

For the upper bound, fix a compact subset K of X. For $\delta > 0$ denote

$$I^\delta(x) = (I(x) - \delta) \wedge \frac{1}{\delta}, \quad x \in X.$$

For any $x \in K$, use the lower semicontinuity of I, more precisely that $\{y \in X : I(y) > I^\delta(x)\}$ is open to choose a set $G_x \in \mathcal{G}_0$ such that

$$-I^\delta(x) \ge \limsup_{\varepsilon\to 0} \varepsilon \ln \mu_\varepsilon(G_x).$$

Use compactness of K to extract from the open cover $K \subset \cup_{x\in K} G_x$ a finite subcover $K \subset \cup_{i=1}^n G_{x_i}$. Then with a standard argument we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \mu_\varepsilon(K) \le \max_{1\le i\le n} \limsup_{\varepsilon\to 0} \varepsilon \ln \mu_\varepsilon(G_{x_i}) \le -\min_{1\le i\le n} I^\delta(x_i) \le -\inf_{x\in K} I^\delta(x).$$

Now let $\delta \to 0$. Finally use exponential tightness to show that I is a good rate function (see Dembo and Zeitouni (1998), Section 4.1). □

The following propositions show how large deviation principles are transferred between different topologies on a space, or via continuous maps to other topological spaces.

Proposition 4 (Contraction principle). *Let X and Y be topological Hausdorff spaces, and let $I : X \to [0,\infty]$ be a good rate function. Let $f : X \to Y$ be a continuous mapping. Then*

$$I' : Y \to [0,\infty], I'(y) = \inf\{I(x) : f(x) = y\}$$

is a good rate function, and if $(\mu_\varepsilon)_{\varepsilon>0}$ satisfies an LDP with rate function I on X, then $(\mu_\varepsilon \circ f^{-1})_{\varepsilon>0}$ satisfies an LDP with rate function I' on Y.

Proposition 5. *Let $(\mu_\varepsilon)_{\varepsilon>0}$ be an exponentially tight family of probability measures on $(X, \mathcal{B}_{\tau_2})$ where $\mathcal{B}_{\tau_2}$ are the Borel sets of τ_2. Assume (μ_ε) satisfies an LDP with rate function I with respect to some Hausdorff topology τ_1 on X which is coarser than τ_2, i.e. $\tau_2 \subset \tau_1$. Then $(\mu_\varepsilon)_{\varepsilon>0}$ satisfies the LDP with respect to τ_2, with good rate function I.*

The main idea of our sequence space approach to Schilder's Theorem for Hilbert space valued Wiener processes will just extend the following large deviation principle for a standard normal variable with values in $\mathbb{R}$ to sequences of i.i.d. variables of this kind.

Proposition 6. *Let Z be a standard normal variable with values in $\mathbb{R}$,*

$$I : \mathbb{R} \to [0,\infty),\ x \mapsto \frac{x^2}{2},$$

and for Borel sets B in $\mathbb{R}$ let $\mu_\varepsilon(B) := \mathbb{P}(\sqrt{\varepsilon}Z \in B)$. Then $(\mu_\varepsilon)_{\varepsilon>0}$ satisfies a LDP with good rate function I.

2 Large Deviations for Hilbert Space Valued Wiener Processes

Ciesielski's isomorphism and the Schauder representation of Brownian motion yield a very elegant and simple method of proving large deviation principles for the Brownian motion. This was first noticed by Baldi and Roynette (1992) who gave an alternative proof of Schilder's theorem based on this isomorphism. We follow their approach and extend it to Wiener processes with values on Hilbert spaces. In this entire section we always assume $0 < \alpha < 1/2$. By further decomposing the orthogonal 1-dimensional Brownian motions in the representation of an H-valued Wiener process by its Fourier coefficients with respect to the Schauder functions, we describe it by double sequences of real-valued normal variables.

2.1 Appropriate norms

We work with new norms on the spaces of α-Hölder continuous functions given by

$$\|F\|_\alpha' := \|T_\alpha^H F\|_\infty = \sup_{k,n} \left| c_n(\alpha) \int_{[0,1]} \chi_n(s) d\langle F, e_k\rangle(s) \right|, F \in C_\alpha^0([0,1]; H),$$

$$\|f\|_\alpha' := \|T_\alpha f\|_\infty = \sup_{n} \left| c_n(\alpha) \int_{[0,1]} \chi_n(s) df(s) \right|, f \in C_\alpha^0([0,1]; \mathbb{R}).$$

Since T_α^H is one-to-one, $\|.\|_\alpha'$ is indeed a norm. Also, we have $\|.\|_\alpha' \le \|.\|_\alpha$. Hence the topology generated by $\|.\|_\alpha'$ is coarser than the usual topology on $C_\alpha^0([0,1], H)$.

Balls with respect to the new norms $U_\alpha^\delta(F) := \{G \in C_\alpha^0([0,1]; H) : \|G - F\|_\alpha' < \delta\}$ for $F \in C_\alpha^0([0,1]; H), \delta > 0$, have a simpler form for our reasoning, since the condition that for $\delta > 0$ a function $G \in C_\alpha^0([0,1], H)$ lies in $U_\alpha^\delta(F)$ translates into the countable set of one-dimensional conditions $|\langle T_\alpha^H(F)_n - T_\alpha^H(G)_n, e_k\rangle| < \delta$ for all $n, k \ge 0$. This will facilitate the proof of the LDP for the basis of open balls of the topology generated by $\|.\|_\alpha'$. We will first prove the LDP in the topologies generated by these norms and then transfer the result to the finer sequence space topologies using Proposition 5, and finally to the original function space using Ciesielski's isomorphism and Proposition 4.

2.2 The rate function

Recall that Q is supposed to be a positive self-adjoint trace-class operator on H. Let $H_0 := (Q^{1/2}H, \|\cdot\|_0)$, equipped with the inner product

$$\langle x, y\rangle_{H_0} := \langle Q^{-1/2}x, Q^{-1/2}y\rangle_H,$$

that induces the norm $\|\cdot\|_0$ on H_0. We define the Cameron-Martin space of the Q-Wiener process W by

$$\mathcal{H} := \left\{ F \in C([0,1]; H) : F(\cdot) = \int_0^\cdot U(s)ds \text{ with } U \in L^2([0,1]; H_0) \right\}.$$

Here $L^2([0,1]; H_0)$ is the space of measurable functions U from $[0,1]$ to H_0 such that $\int_0^1 \|U\|_{H_0}^2 dx < \infty$. Define the function I via

$$\begin{aligned} I &: C([0,1]; H) \to [0, \infty] \\ F &\mapsto \inf \left\{ \frac{1}{2} \int_0^1 \|U(s)\|_{H_0}^2 ds : U \in L^2([0,1]; H_0), F(\cdot) = \int_0^\cdot U(s)ds \right\} \end{aligned}$$

where by convention $\inf \emptyset = \infty$. In the following we will denote any restriction of I to a subspace of $C([0,1]; H)$ (e.g. to $(C_\alpha([0,1]; H))$ by I as well. We will use the structure of H to simplify our problem. It allows us to compute the rate function I from the rate function of the one dimensional Brownian by the following Lemma.

Lemma 4. *Let $\tilde{I} : C([0,1]; \mathbb{R})$ be the rate function of the Brownian motion, i.e.*

$$\tilde{I}(f) := \begin{cases} \int_0^1 |\dot{f}(s)|^2 ds, & f(\cdot) = \int_0^\cdot \dot{f}(s)ds \text{ for a square integrable function } \dot{f}, \\ \infty, & \text{otherwise.} \end{cases}$$

Let $(\lambda_k)_{k \geq 0}$ be the sequence of eigenvalues of Q. Then for all $F \in C([0,1]; H)$ we have

$$I(F) = \sum_{k=0}^\infty \frac{1}{\lambda_k} \tilde{I}(\langle F, e_k\rangle).$$

where we convene that $c/0 = \infty$ for $c > 0$ and $0/0 = 0$.

Proof. Let $F \in C([0,1]; H)$.

1. First assume $I(F) < \infty$. Then there exists $U \in L^2([0,1]; H_0)$ such that $F = \int_0^\cdot U(s)ds$ and thus $\langle F, e_k\rangle = \int_0^\cdot \langle U(s), e_k\rangle ds$ for $k \geq 0$. Consequently we have by monotone convergence

$$\begin{aligned} \frac{1}{2}\int_0^1 \|U(s)\|_{H_0}^2 ds &= \frac{1}{2}\int_0^1 \left\| \sum_{k=0}^\infty \langle U(s), e_k\rangle e_k \right\|_{H_0} ds \\ &= \frac{1}{2}\int_0^1 \sum_{k=0}^\infty \langle U(s), e_k\rangle^2 \langle Q^{-\frac{1}{2}}e_k, Q^{-\frac{1}{2}}e_k\rangle ds \\ &= \frac{1}{2}\int_0^1 \sum_{k=0}^\infty \frac{1}{\lambda_k}\langle U(s), e_k\rangle^2 ds \\ &= \frac{1}{2}\sum_{k=0}^\infty \int_0^1 \frac{1}{\lambda_k}\langle U(s), e_k\rangle^2 ds \\ &= \sum_{k=0}^\infty \frac{1}{\lambda_k}\tilde{I}(\langle F, e_k\rangle). \end{aligned}$$

   The last expression does not depend on the choice of U. Hence we get that $I(F) < \infty$ implies $I(F) = \sum_{k=0}^\infty \frac{1}{\lambda_k}\tilde{I}(\langle F, e_k\rangle)$.

2. Conversely assume $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} \tilde{I}(\langle F, e_k \rangle) < \infty$. Since $\tilde{I}(\langle F, e_k \rangle) < \infty$ for all $k \geq 0$, we know that there exists a sequence $(U_k)_{k \geq 0}$ of square-integrable real-valued functions such that $\langle F, e_k \rangle = \int_0^{\cdot} U_k(s) ds$. Further, those functions U_k satisfy by monotone convergence

$$\int_0^1 \sum_{k=0}^{\infty} \frac{1}{\lambda_k} |U_k(s)|^2 ds = \sum_{k=0}^{\infty} \frac{1}{\lambda_k} \int_0^1 |U_k(s)|^2 ds = \sum_{k=0}^{\infty} \frac{2}{\lambda_k} \tilde{I}(\langle F, e_k \rangle) < \infty.$$

So if we define $U(s) := \sum_{k=0}^{\infty} U_k(s) e_k, s \in [0,1]$, then $U \in L^2([0,1]; H_0)$. This follows from

$$U \in L^2([0,1]; H_0) \text{ iff } \int_0^1 \|U(s)\|_{H_0}^2 ds = \int_0^1 \sum_{k=0}^{\infty} \frac{1}{\lambda_k} |U_k(s)|^2 ds < \infty.$$

Finally we obtain by dominated convergence ($\|F(t)\|_H < \infty$)

$$F(t) = \sum_{k=0}^{\infty} \langle F(t), e_k \rangle e_k = \sum_{k=0}^{\infty} e_k \int_0^t U_k(s) ds = \int_0^t U(s) ds,$$

such that

$$I(F) \leq \frac{1}{2} \int_0^1 \|U(s)\|_{H_0}^2 ds = \frac{1}{2} \int_0^1 \sum_{k=0}^{\infty} \frac{1}{\lambda_k} |U_k(s)|^2 ds < \infty.$$

Combining the two steps we obtain $I(F) < \infty$ iff $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} \tilde{I}(\langle F, e_k \rangle) < \infty$ and in this case

$$I(F) = \sum_{k=0}^{\infty} \frac{1}{\lambda_k} \tilde{I}(\langle F, e_k \rangle).$$

This completes the proof. □

Lemma 4 allows us to show that I is a rate function.

Lemma 5. *I is a rate function on $(C_\alpha^0([0,1]; H), \|.\|'_\alpha)$.*

Proof. For a constant $C \geq 0$ we have to prove that if $(F_n)_{n \geq 0} \subset \Psi_I(C) \cap C_\alpha^0([0,1]; H)$ converges in $C_\alpha^0([0,1]; H)$ to F, then F is also in $\Psi_I(C)$.

It was observed in Baldi and Roynette (1992) that $\tilde{I}$ is a rate function for the $\|.\|'_\alpha$-topology on $C_0^\alpha([0,1|; \mathbb{R})$. By our assumption we know that for every $k \in \mathbb{N}$, $(\langle F_n, e_k \rangle)_{n \geq 0}$ converges in $(C_0^\alpha([0,1|; \mathbb{R}), \|.\|'_\alpha)$ to $\langle F, e_k \rangle$. Therefore

$$\tilde{I}(\langle F, e_k \rangle) \leq \liminf_{n \to \infty} \tilde{I}(\langle F_n, e_k \rangle),$$

so by Lemma 4 and by Fatou's lemma

$$\begin{aligned} C &\geq \liminf_{n \to \infty} I(F_n) = \liminf_{n \to \infty} \sum_{k=0}^{\infty} \frac{1}{\lambda_k} \tilde{I}(\langle F_n, e_k \rangle) \geq \sum_{k=0}^{\infty} \frac{1}{\lambda_k} \liminf_{n \to \infty} \tilde{I}(\langle F_n, e_k \rangle) \\ &\geq \sum_{k=0}^{\infty} \frac{1}{\lambda_k} \tilde{I}(\langle F, e_k \rangle) = I(F). \end{aligned}$$

Hence $F \in \Psi_I(C)$. □

2.3 LDP for a sub-basis of the coarse topology

To show that the Q-Wiener process $(W(t) : t \in [0,1])$ satisfies a LDP on $(C_\alpha([0,1];H), \|.\|_\alpha)$ with good rate function I as defined in the last section we now show that the LDP holds for open balls in our coarse topology induced by $\|.\|_\alpha'$. The proof is an extension of the version of Baldi and Roynette (1992) for the real valued Wiener process.

For $\varepsilon > 0$ denote by μ_ε the law of $\sqrt{\varepsilon}W$, i.e. $\mu_\varepsilon(A) = \mathbb{P}(\sqrt{\varepsilon}W \in A)$, $A \in \mathcal{B}(H)$.

Lemma 6. *For every $\delta > 0$ and every $F \in C_\alpha^0([0,1];H)$ we have*

$$\lim_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(U_\alpha^\delta(F)) = -\inf_{G \in U_\alpha^\delta(F)} I(G).$$

Proof. 1. Write $T_\alpha^H F = (\sum_{k=0}^\infty F_{n,k} e_k)_{n \in \mathbb{N}}$. Then $\sqrt{\varepsilon}W$ is in $U_\alpha^\delta(F)$ if and only if

$$\sup_{k,n \geq 0} \left| \sqrt{\varepsilon} c_n(\alpha) \int_0^1 \chi_n d\langle W, e_k \rangle - F_{k,n} \right| < \delta.$$

Now for $k \geq 0$ we recall $\langle W, e_k \rangle = \sqrt{\lambda_k}\beta_k$, where $(\beta_k)_{k \geq 0}$ is a sequence of independent standard Brownian motions. Therefore for $n, k \geq 0$

$$\left| \int_0^1 \chi_n d\langle W, e_k \rangle \right| = \left| \sqrt{\lambda_k} Z_{k,n} \right|,$$

where $(Z_{k,n})_{k,n \geq 0}$ is a double sequence of independent standard normal variables. Therefore by independence

$$\begin{aligned} \mu_\varepsilon(U_\alpha^\delta(F)) &= \mathbb{P}\left(\bigcap_{k,n \in \mathbb{N}_0} \left| c_n(\alpha) \sqrt{\varepsilon \lambda_k} Z_{k,n} - F_{k,n} \right| < \delta \right) \\ &= \prod_{k=0}^\infty \prod_{n=0}^\infty \mathbb{P}\left(c_n(\alpha) \sqrt{\varepsilon \lambda_k} Z_{k,n} \in (F_{k,n} - \delta, F_{k,n} + \delta) \right). \end{aligned}$$

To abbreviate, we introduce the notation

$$\mathbb{P}_{k,n}(\varepsilon) = \mathbb{P}\left(c_n(\alpha) \sqrt{\varepsilon \lambda_k} Z_{k,n} \in (F_{k,n} - \delta, F_{k,n} + \delta) \right), \ \varepsilon > 0, n, k \in \mathbb{N}_0.$$

For every $k \geq 0$ we split $\mathbb{N}_0$ into subsets Λ_i^k, $i = 1,2,3,4$, for each of which we will calculate $\prod_{k=0}^\infty \prod_{n \in \Lambda_i^k} \mathbb{P}_{n,k}(\varepsilon)$ separately. Let

$$\begin{aligned} \Lambda_1^k &= \{n \geq 0 : 0 \notin [F_{k,n} - \delta, F_{k,n} + \delta]\} \\ \Lambda_2^k &= \{n \geq 0 : F_{k,n} = \pm\delta\} \\ \Lambda_3^k &= \{n \geq 0 : [-\delta/2, \delta/2] \subset [F_{k,n} - \delta, F_{k,n} + \delta]\} \\ \Lambda_4^k &= (\Lambda_1^k \cup \Lambda_2^k \cup \Lambda_3^k)^c. \end{aligned}$$

By applying Ciesielski's isomorphism to the real-valued functions $\langle F, e_k \rangle$, we see that for every fixed k, Λ_3^k contains nearly all n. Since $(T_\alpha^H F)_n$ converges to zero in H, in particular $\sup_{k \geq 0} |F_{k,n}|$ converges to zero as $n \to \infty$. But for every fixed n, $(F_{k,n})_k$ is in l^2 and therefore converges to zero. This shows that for large enough k we must have $\Lambda_3^k = \mathbb{N}_0$, and therefore $\cup_k (\Lambda_3^k)^c$ is finite.

2. First we examine $\prod_{k=0}^\infty \prod_{n \in \Lambda_3^k} \mathbb{P}_{k,n}(\varepsilon)$. Note that for $n \in \Lambda_3^k$ we have

$$[-\delta/2, \delta/2] \subset [F_{k,n} - \delta, F_{k,n} + \delta],$$

and therefore

$$\prod_{k=0}^{\infty}\prod_{n\in\Lambda_3^k}\mathbb{P}_{k,n}(\varepsilon)\geq\prod_{k=0}^{\infty}\prod_{n\in\Lambda_3^k}\mathbb{P}\left(Z_{k,n}\in\left(-\frac{\delta}{2c_n(\alpha)\sqrt{\varepsilon\lambda_k}},\frac{\delta}{2c_n(\alpha)\sqrt{\varepsilon\lambda_k}}\right)\right)$$
$$=\prod_{k=0}^{\infty}\prod_{n\in\Lambda_3^k}\left(1-\sqrt{\frac{2}{\pi}}\int_{\delta/(2c_n(\alpha)\sqrt{\varepsilon\lambda_k})}^{\infty}e^{-u^2/2}du\right).$$

For $a>1$ we have $\int_a^\infty e^{-x^2/2}dx\leq e^{-a^2/2}$. Thus for small enough ε:

$$\prod_{k=0}^{\infty}\prod_{n\in\Lambda_3^k}\mathbb{P}_{k,n}(\varepsilon)\geq\prod_{k=0}^{\infty}\prod_{n\in\Lambda_3^k}\left(1-\sqrt{\frac{2}{\pi}}\exp\left(-\frac{\delta^2}{8c_n^2(\alpha)\varepsilon\lambda_k}\right)\right).$$

This amount will tend to 1 if and only if its logarithm tends to 0 as $\varepsilon\to 0$. Since $\log(1-x)\leq -x$ for $x\in(0,1)$, it suffices to prove that

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\sum_{n\geq 0}\exp\left(-\frac{\delta^2}{8c_n^2(\alpha)\varepsilon\lambda_k}\right)=0. \tag{2}$$

This is true by dominated convergence, because $c_n(\alpha)=2^{n(\alpha-1/2)+\alpha-1}$, and since $(\lambda_k)\in l_1$.

We will make this more precise. First observe that for $a>0$

$$e^{-a}\leq\frac{1}{a}e^{-1}$$
$$\text{if }\log(a)-a\leq -1.$$

For $k,n\geq 0$ we write $\eta_{n,k}=\frac{\delta^2}{8c_n^2(\alpha)\varepsilon\lambda_k}$. Clearly there exists a finite set $T\subset\mathbb{N}_0^2$ such that $\log(\eta_{n,k})-\eta_{n,k}\leq -1$ for all $(n,k)\in T^c$. We set $C=\sum_{(n,k)\in T}e^{-\eta_{n,k}}$ and get

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\exp\left(-\frac{\delta^2}{8c_n^2(\alpha)\varepsilon\lambda_k}\right)=C+\sum_{(n,k)\in T^c}^{\infty}e^{-\eta_{n,k}}$$
$$\leq C+\sum_{(n,k)\in T^c}^{\infty}\frac{1}{\eta_{n,k}}e^{-1}$$
$$\leq C+\frac{8\varepsilon e^{-1}}{\delta^2}\sum_{k\geq 0}\lambda_k\sum_{n\geq 0}c_n(\alpha)^2<\infty.$$

3. Since $\cup_{k\geq 0}\Lambda_4^k$ is finite, and since for every n in Λ_4^k the interval $(\mathcal{F}_{k,n}-\delta,\mathcal{F}_{k,n}+\delta)$ contains a small neighborhood of 0, we have

$$\lim_{\varepsilon\to 0}\prod_{k=0}^{\infty}\prod_{n\in\Lambda_4^k}\mathbb{P}_{k,n}(\varepsilon)=1. \tag{3}$$

4. Again because $\cup_{k\geq 0}\Lambda_2^k$ is finite, we obtain from its definition that

$$\lim_{\varepsilon\to 0}\prod_{k=0}^{\infty}\prod_{n\in\Lambda_2^k}\mathbb{P}_{k,n}(\varepsilon)=2^{-|\cup_k\Lambda_2^k|}. \tag{4}$$

5. Finally we calculate $\lim_{\varepsilon\to 0}\prod_{k=0}^{\infty}\prod_{n\in\Lambda_1^k}\mathbb{P}_{k,n}(\varepsilon)$. For given k,n define

$$\bar{F}_{k,n}=\begin{cases} F_{k,n}-\delta, & F_{k,n}>\delta,\\ F_{k,n}+\delta, & F_{k,n}<-\delta.\end{cases}$$

We know that $Z_{k,n}$ is standard normal, so that by Proposition 6 for $n\in\Lambda_1^k$

$$\lim_{\varepsilon\to 0}\varepsilon\log\mathbb{P}^0_{k,n}(\varepsilon)=-\frac{\bar{F}^2_{k,n}}{2c_n^2(\alpha)\lambda_k},$$

and therefore again by the finiteness of $\cup_k\Lambda_1^k$

$$\lim_{\varepsilon\to 0}\varepsilon\log\prod_{k=0}^{\infty}\prod_{n\in\Lambda_1^k}\mathbb{P}^0_{k,n}(\varepsilon)=-\sum_{k=0}^{\infty}\sum_{n\in\Lambda_1^k}\frac{\bar{F}^2_{k,n}}{2c_n^2(\alpha)\lambda_k}. \tag{5}$$

6. Combining (2) - (5) we obtain

$$\lim_{\varepsilon\to 0}\varepsilon\log\mu_\varepsilon(U_\alpha^\delta(F))=-\sum_{k=0}^{\infty}\frac{1}{\lambda_k}\sum_{n\in\Lambda_1^k}\frac{\bar{F}^2_{k,n}}{2c_n^2(\alpha)}.$$

So if we manage to show

$$-\sum_{k=0}^{\infty}\frac{1}{\lambda_k}\sum_{n\in\Lambda_1^k}\frac{\bar{F}^2_{k,n}}{2c_n^2(\alpha)}=-\inf_{G\in U_\alpha^\delta(F)}I(G),$$

the proof is complete. By Ciesielski's isomorphism, every $G\in C^0_\alpha([0,1];H)$ has the representation

$$G=\sum_{k=0}^{\infty}e_k\sum_{n=0}^{\infty}\frac{G_{k,n}}{c_n(\alpha)}\phi_n.$$

Its derivative fullfills (if it exists) for any $k\geq 0$

$$\langle\dot{G},e_k\rangle=\sum_{n=0}^{\infty}\frac{G_{k,n}}{c_n(\alpha)}\chi_n.$$

Since the Haar functions $(\chi_n)_{n\geq 0}$ are a CONS for $L^2([0,1])$, we see that $\tilde{I}(\langle G,e_k\rangle)<\infty$ if and only if $(G_{k,n}/c_n(\alpha))\in l_2$, and in this case

$$\tilde{I}(\langle G,e_k\rangle)=\frac{1}{2}\int_0^1\langle\dot{G}(s),e_k\rangle^2ds=\sum_{n=0}^{\infty}\frac{G^2_{k,n}}{2c_n^2(\alpha)}.$$

So we finally obtain with Lemma 4 the desired equality

$$\begin{aligned}\inf_{G\in U_\alpha^\delta(F)}I(G)&=\inf_{G\in U_\alpha^\delta(F)}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}\tilde{I}(\langle G,e_k\rangle)=\inf_{G\in U_\alpha^\delta(F)}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}\sum_{n=0}^{\infty}\frac{G^2_{k,n}}{2c_n^2(\alpha)}\\&=\sum_{k=0}^{\infty}\frac{1}{\lambda_k}\sum_{n\in\Lambda_1^k}\frac{\bar{F}^2_{k,n}}{2c_n^2(\alpha)}.\end{aligned}$$

□

2.4 Exponential tightness

The final ingredient needed in the proof of the LDP for Hilbert space valued Wiener processes is exponential tightness. It will be established in two steps. The first step claims exponential tightness for the family of laws of $\sqrt{\varepsilon}Z, \varepsilon > 0$, where Z is an H-valued $\mathcal{N}(0, Q)$-variable.

Lemma 7. *Let $\varepsilon > 0$ let $\nu_\varepsilon = \mathbb{P} \circ (\sqrt{\varepsilon}Z)^{-1}$ for a centered Gaussian random variable Z with values in the separable Hilbert space H and covariance operator Q. Then $(\nu_\varepsilon)_{\varepsilon \in (0,1]}$ is exponentially tight. More precisely for every $a > 0$ there exists a compact subset K_a of H, such that*

$$\nu_\varepsilon(K_a^c) \leq e^{-a/\varepsilon}$$

Proof. We know that for a sequence $(b_k)_{k \geq 0}$ converging to 0, the operator $T_{(b_k)} := \sum_{k=0}^{\infty} b_k \langle \cdot, e_k \rangle e_k$ is compact. That is, for bounded sets $A \subset H$ the set $T_{(b_k)}(A)$ is precompact in H. Since H is complete, this means that $cl(T_{(b_k)}(A))$ is compact. Let $a' > 0$ to be specified later. Denote by $B(0, \sqrt{a'}) \subset H$ the ball of radius $\sqrt{a'}$ in H. We will show that there exists a zero sequence $(b_k)_{k \geq 0}$, such that the compact set $K_{a'} = cl(T_{(b_k)}(B(0, \sqrt{a'})))$ satisfies for all $\varepsilon \in (0, 1]$

$$\mathbb{P}(\sqrt{\varepsilon}Z \in (K_{a'})^c) \leq ce^{-a'/\varepsilon}. \tag{6}$$

with a constant $c > 0$ that does not depend on a'. Thus for given a, we can choose $a' > a$ such that for every $\varepsilon \in (0, 1]$

$$c \leq e^{(a'-a)/\varepsilon}$$

and therefore the proof is complete once we proved (6).

Since Z is Gaussian, $e^{\lambda \|Z\|_H}$ is integrable for small λ, and we can apply Markov's inequality to obtain constants $\lambda(Q), c(Q) > 0$ such that $\mathbb{P}(\|Z\|_H \geq \sqrt{a'}) \leq c(Q)e^{-\lambda(Q)a'}$.

Note that if $(\lambda_k)_{k \geq 0} \in l^1$, we can always find a sequence $(c_k)_{k \geq 0}$ such that $\lim_{k \to \infty} c_k = \infty$ and $\sum_{k \geq 0} c_k \lambda_k < \infty$. For $\beta > 0$ that will be specified later, we set $b_k = \sqrt{\frac{\beta}{c_k}}$ for all $k \geq 0$. We can define $(T_{(b_k)})^{-1} = \sum_{k=0}^{\infty} \frac{1}{b_k} \langle \cdot, e_k \rangle e_k$. This gives

$$\begin{aligned}
\mathbb{P}(\sqrt{\varepsilon}Z \in (K_{a'})^c) &\leq \mathbb{P}(\sqrt{\varepsilon}(T_{(b_k)})^{-1}(Z) \notin B(0, \sqrt{a'})) \\
&= \mathbb{P}(\|(T_{(b_k)})^{-1}(Z)\|_H^2 \geq \frac{a'}{\varepsilon}) \\
&= \mathbb{P}\left(\sum_{k=0}^{\infty} c_k |\langle Z, e_k \rangle|^2 \geq \frac{\beta a'}{\varepsilon}\right) \\
&= \mathbb{P}\left(\|\tilde{Z}\|_H \geq \sqrt{\frac{\beta a'}{\varepsilon}}\right),
\end{aligned}$$

where $\tilde{Z}$ is a centered Gaussian random variable with trace class covariance operator

$$\tilde{Q} = \sum_{k=0}^{\infty} c_k \lambda_k \langle \cdot, e_k \rangle e_k.$$

Consequently we obtain

$$\mathbb{P}(\sqrt{\varepsilon}Z \in (K_{a'})^c) \leq c(\tilde{Q})e^{-\frac{\lambda(\tilde{Q})\beta a'}{\varepsilon}}$$

Choosing $\beta = \frac{1}{\lambda(\tilde{Q})}$ proves the claim (6). $\square$

With the help of Lemma 7 we are now in a position to prove exponential tightness for the family $(\mu_\varepsilon)_{\varepsilon\in(0,1]}$.

Lemma 8. *$(\mu_\varepsilon)_{\varepsilon\in(0,1]}$ is an exponentially tight family of probability measures on $(C^0_\alpha([0,1];H),\ \|.\|_\alpha)$.*

Proof. Let $a>0$. We will construct a suitable set of the form

$$\tilde{K}^a = \prod_{n=0}^{\infty} K_n^a$$

such that

$$\limsup_{l\to\infty} \varepsilon_l \log \mu_{\varepsilon_l}\left[\left(\left(T_\alpha^H\right)^{-1}\tilde{K}^a\right)^c\right] \le -a.$$

Here each K_n^a is a compact subset of H, such that the diameter of K_n^a tends to 0 as n tends to ∞. Then $\tilde{K}^a$ will be sequentially compact in $\mathcal{C}_0^H$ by a diagonal sequence argument. Since $\mathcal{C}_0^H$ is a metric space, $\tilde{K}^a$ will be compact. As we saw in Theorem 1, $(T_\alpha^H)^{-1}$ is continuous, so that then $K^a := (T_\alpha^H)^{-1}(\tilde{K}^a)$ is compact in $(C^0_\alpha([0,1],H),\|\cdot\|_\alpha)$.

Let $\nu_\varepsilon = \mathbb{P}\circ(\sqrt{\varepsilon}Z)^{-1}$ for a random variable Z on H with $Z\sim\mathcal{N}(0,Q)$. By Lemma 7, we can find a sequence of compact sets $(K_n^a)_{n\in\mathbb{N}}\subset H$ such that for all $\varepsilon\in(0,1]$:

$$\nu_\varepsilon((K_n^a)^c) \le \exp\left(\frac{-(n+1)a}{\varepsilon}\right).$$

To guarantee that the diameter of the K_n^a converges to zero, denoting by $\overline{B}(0,d)$ the closed ball of radius d around 0, we set

$$\tilde{K^a} := \prod_{n=0}^{\infty} c_n(\alpha)\left(\overline{B}\left(0,\sqrt{\frac{a(n+1)}{\lambda}}\right)\cap K_n^a\right).$$

Since $c_n(\alpha)\sqrt{a(n+1)/\lambda}\to 0$ as $n\to\infty$, this is a compact set in $\mathcal{C}_0^H$. Thus $K^a := (T_\alpha^H)^{-1}(\tilde{K^a})$ is compact in $(C^0_\alpha([0,1],H),\|\cdot\|_\alpha)$.

Remember that by Lemma 1 we have $W=\sum_{n=0}^\infty \phi_n Z_n$, where $(Z_n)_{n\ge0}$ is an i.i.d. sequence of $\mathcal{N}(0,Q)-$ variables. This implies $T_\alpha^H(W) = (c_n(\alpha)Z_n)_{n\ge0}$ and thus for any $\varepsilon\in(0,1]$

$$\begin{aligned}
\mu_\varepsilon((K^a)^c) &= \mathbb{P}\left[\cup_{n\in\mathbb{N}_0}\left\{c_n(\alpha)\sqrt{\varepsilon}Z_n \in \left(c_n(\alpha)\left(B\left(0,\sqrt{\frac{a(n+1)}{\lambda}}\right)\cap K_n^a\right)\right)^c\right\}\right]\\
&\le \sum_{n=0}^{\infty}\left(\nu_\varepsilon((K_n^a)^c) + \mathbb{P}\left(\|Z_n\|\ge\sqrt{\frac{a(n+1)}{\varepsilon\lambda}}\right)\right)\\
&\le \sum_{n=0}^{\infty}\left(e^{\frac{-(n+1)a}{\varepsilon}} + ce^{\frac{-a(n+1)}{\varepsilon}}\right)\\
&= (1+c)\frac{e^{\frac{-a}{\varepsilon}}}{1-e^{\frac{-a}{\varepsilon}}}.
\end{aligned}$$

So we have

$$\limsup_{\varepsilon\to0} \varepsilon\log\mu_\varepsilon((K^a)^c) \le -a,$$

□

We now combine the arguments given so far to obtain an LDP in the Hölder spaces.

Lemma 9. *$(\mu_\varepsilon)_{\varepsilon\in(0,1]}$ satisfies an LDP on $(C^0_\alpha([0,1];H),\|.\|_\alpha)$ with good rate function I.*

Proof. We know $\|.\|'_\alpha \leq \|.\|_\alpha$. Therefore the $\|.\|'_\alpha$-topology is coarser, which in turn implies that every compact set in the $\|.\|_\alpha$-topology is also a compact set in the $\|.\|'_\alpha$-topology. From Lemma 8 we thus obtain that $(\mu_\varepsilon)_{\varepsilon\in(0,1]}$ is also exponentially tight on $(C^0_\alpha([0,1];H),\|.\|^0_\alpha)$.

Proposition 3 implies that $(\mu_\varepsilon)_{\varepsilon\in(0,1]}$ satisfies an LDP with good rate function I on $(C^0_\alpha([0,1];H),\|.\|^0_\alpha)$.

Finally we obtain from Proposition 5 and from Lemma 8 that $(\mu_\varepsilon)_{\varepsilon\in(0,1]}$ satisfies an LDP with good rate function I on $(C^0_\alpha([0,1];H),\|.\|_\alpha)$. □

We may now extend the LDP from $(C^0_\alpha([0,1];H),\|.\|_\alpha)$ to $(C_\alpha([0,1];H),\|.\|_\alpha)$. This is an immediate consequence of the contraction principle (Proposition 4), since the inclusion map from $C^0_\alpha([0,1];H)$ to $C_\alpha([0,1];H)$ is continuous. Similarly we can transfer the LDP from $C^0_\alpha([0,1];H)$ to $C([0,1];H)$, the space of continuous functions on $[0,1]$ with values in H, equipped with the uniform norm.

Theorem 2. *Let $(W(t) : t\in[0,1])$ be a Q-Wiener process and let for $\varepsilon\in(0,1]$ μ_ε be the law of $\sqrt{\varepsilon}W$. Then $(\mu_\varepsilon)_{\varepsilon\in(0,1]}$ satisfies an LDP on $(C([0,1];H),\|.\|_\infty)$ with rate function I.*

Proof. First we can transfer the LDP from $(C^0_\alpha([0,1];H),\|.\|_\alpha)$ to $(C^0_\alpha([0,1];H),\ \|.\|_\infty)$. This is because on $C^0_\alpha([0,1];H)$, $\|.\|_\infty\leq\|.\|_\alpha$, whence the $\|.\|_\infty$-topology is coarser. Therefore I is a good rate function for the $\|.\|_\infty$-topology as well, and $(\mu_\varepsilon)_{\varepsilon\in(0,1]}$ satisfies an LDP on $(C^0_\alpha([0,1];H),\ \|.\|_\infty)$ with good rate function I.

The inclusion map from $(C^0_\alpha([0,1];H),\ \|.\|_\infty)$ to $(C([0,1];H),\ \|.\|_\infty)$ is continuous, so that an application of the contraction principle (Propisition 4) finishes the proof. □

References

P. Baldi and B. Roynette. Some exact equivalents for the brownian motion in hölder norm. *Probab. Theory Relat. Fields*, 93:457–484, 1992.

Z. Ciesielski. On the isomorphisms of the spaces h_α and m. *Bull. Acad. Pol. Sci.*, 8:217–222, 1960.

G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Cambridge University Press, 1992.

A. Dembo and O. Zeitouni. *Large Deviation Techniques and Applications*. Springer-Verlag, 1998.