

Limit theorems for p -variations of solutions of SDEs driven by additive non-Gaussian stable Lévy noise

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November 19, 2008[†]

Abstract

In this paper we study the asymptotic properties of the power variations of stochastic processes of the type $X = Y + L$, where L is an α -stable Lévy process, and Y a perturbation which satisfies some mild Lipschitz continuity assumptions. We establish local functional limit theorems for the power variation processes of X . In case X is a solution of a stochastic differential equation driven by L , these limit theorems provide estimators of the stability index α . They are applicable for instance to model fitting problems for paleo-climatic temperature time series taken from the Greenland ice core.

Keywords: power variation; stable Lévy process; tightness; Skorokhod topology; stability index; model selection; estimation; Kolmogorov-Smirnov statistics; paleo-climatic time series; Greenland ice core data.

MSC 2000: primary 60G52, 60F17; secondary 60H10, 62F10, 62M10, 86A40.

1 Introduction

Stochastic differential equations are being used for quite a while as meso-scopic models for natural phenomena. In one of their simplest variants, they consist of deterministic differential equations perturbed by a noise term. The subclass in which the noise is Gaussian arises for instance from microscopic models described by coupled systems of deterministic differential equations on different time scales, in the limit of infinite fast scale, as the fluctuations of the slow scale component around its averaged version are considered. With a view in particular towards the mathematical interpretation of financial time series, the theory for stochastic differential equations the noise term of which is given by general (discontinuous and non-Gaussian) *semimartingales* has received considerable attention during the recent years.

It is reasonable on a quite general level to model real data by stochastic differential equations. Usually neither their deterministic nor their noise component can be deduced from first principles for instance from microscopic models, but may be selected by statistical inference or *model fit* from the time series they are supposed to interpret. The central question of the *model selection problem* that motivated this paper asks for the best choice of the noise term.

More formally, suppose we wish to model a real time series by the dynamics $X = (X_t)_{t \geq 0}$ of a real valued process of the type

$$X_t = x + \int_0^t f(s, X_s) ds + \eta_t, \quad t \geq 0, \quad \text{(1.1)} \quad \boxed{\text{eq:1}}$$

where the process η is the noise component. Then the problem of a model fit consists in the choice of a drift term f and a noise term η , so that the solution of (1.1) is in the best possible agreement with the data of the given time series.

The example of (1.1) which inspired us most was investigated in the papers [3, 4], where P. Ditlevsen used (1.1) as a model fit for temperature data (yearly averages) obtained from the Greenland ice core describing aspects of the evolution of the Earth's climate during the last Ice Age, extending over about 100,000 years, in particular the catastrophic warmings and coolings in the Northern hemisphere, the so-called Dansgaard-Oeschger events [2]. The drift term was chosen as a gradient of a double well potential (climatic quasi-potential), the local minima corresponding to the cold and warm meta-stable climate states. In order to find a good fit for the noise component, P. Ditlevsen performed a histogram analysis for the residuals of the ice core time series, the temperature increments measured between adjacent data points, i.e. years. He conjectured that the noise

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may contain a strong α -stable component with $\alpha \approx 1.75$, and plotted an estimate for the drift term assuming the stationarity of the solution.

In this paper we resume Ditlevsen's model selection problem for the fit of the noise component from the perspective of a new testing method to be developed. Following Ditlevsen, we work under the model assumption that the noise η has an α -stable Lévy component (symmetric or skewed). We search for a test statistics discriminating well between different α , and capable of testing for the right one. We shall show that this job is well done by the *equidistant p -variation* - in the sequel called *p -variation* - of the process X defined for all $p > 0$ as

$$V_p^n(X)_t := \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n X|^p \quad (1.2)$$

where $\Delta_i^n X := X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$ for $1 \leq i \leq n, n \in \mathbf{N}$. We first observe that for those values of p relevant for our analysis the main contribution to the p -variation of X comes from the α -stable component of the noise. We next prove local limit theorems which hold under very mild assumptions on the drift term f and allow to determine the stability index α asymptotically. We finally use these limit theorems in Section 3 below to analyze the real data from the Greenland ice core with our methods, and come to an estimate $\alpha \approx 0.7$, surprisingly quite different from the one obtained by Ditlevsen [4], [3].

The paper is organized as follows. In section 2 we set the stage for stating our main results, which are applied to the ice core data in section 3. The remaining sections are devoted to the proof of our main functional limit theorems, starting with the convergence of the finite dimensional laws in section 4, continuing with the proof of the tightness of the laws in the Skorokhod topology in section 5, and ending with the robustness proof of the convergence with respect to adding terms of finite variation in section 6.

ACKNOWLEDGEMENTS: P.I. and I.P. thank DFG SFB 555 for financial support. C.H. thanks DFG IRTG for financial support. C.H. is grateful to R. Schilling for his valuable comments. The authors thank P. Ditlevsen for providing the ice-core data.

2 Object of study and main results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space. We assume that the filtration satisfies the usual hypotheses in the sense of [8], i.e. it is right continuous, and \mathcal{F}_0 contains all the \mathbf{P} -null sets of \mathcal{F} .

For $\alpha \in]0, 2]$ let $L = (L_t)_{t \geq 0}$ be an α -stable Lévy process, i.e. a process with right continuous trajectories possessing left side limits (rcll) and stationary independent increments whose marginal laws satisfy

$$\ln \mathbf{E} e^{i\lambda L_t} = \begin{cases} -tC^\alpha |\lambda|^\alpha \left(1 - i\beta \operatorname{sgn}(\lambda) \tan \frac{\pi\alpha}{2}\right), & \alpha \neq 1, \\ -tC |\lambda| \left(1 - i\beta \frac{2}{\pi} \operatorname{sgn}(\lambda) \log |\lambda|\right), & \alpha = 1, \end{cases} \quad t \geq 0, \quad (2.1) \quad \text{eq:L}$$

$C > 0$ being the scale parameter and $\beta \in [-1, 1]$ the skewness. We adopt the standard notation from [9] and write $L_1 \sim S_\alpha(C, \beta, 0)$.

We also make use of the Lévy–Khinchin formula for L which takes the following form in our case (see [5, Chapter XVII.3] for details):

$$\ln \mathbf{E} e^{i\lambda L_1} = \begin{cases} C_F \int_{\mathbb{R} \setminus \{0\}} (e^{i\lambda x} - 1 - i\lambda x \mathbb{I}_{\{|x| < 1\}}) \left[\frac{1 - \beta}{2} \mathbb{I}_{\{x < 0\}} + \frac{1 + \beta}{2} \mathbb{I}_{\{x > 0\}} \right] \frac{dx}{|x|^{1+\alpha}}, & \alpha \neq 1, \\ C_F \int_{\mathbb{R} \setminus \{0\}} (e^{i\lambda x} - 1 - i\lambda \sin x) \left[\frac{1 - \beta}{2} \mathbb{I}_{\{x < 0\}} + \frac{1 + \beta}{2} \mathbb{I}_{\{x > 0\}} \right] \frac{dx}{|x|^{1+\alpha}}, & \alpha = 1, \end{cases} \quad (2.2)$$

where C_F denotes the scale parameter in Feller's notation and equals

$$C_F = \begin{cases} C^\alpha \left(\cos \left(\frac{\pi\alpha}{2} \right) \Gamma(\alpha) \right)^{-1}, & \alpha \neq 1, \\ C \frac{2}{\pi}, & \alpha = 1. \end{cases} \quad (2.3)$$

Recall that a totally asymmetric process L with $\beta = 1$ ($\beta = -1$) is called spectrally positive (negative). A spectrally positive α -stable process with $\alpha \in (0, 1)$ has a.s. increasing sample paths and is called *subordinator*.

In the sequel, ' \xrightarrow{D} ' denotes convergence in the Skorokhod topology, ' \xrightarrow{d} ' denotes convergence of finite-dimensional (marginal) distributions, and ' $\xrightarrow{u.c.p.}$ ' stands for uniform convergence on compacts in probability.

The main results of this paper are presented in the following three theorems. The first theorem deals with the asymptotic behavior of the p -variation for a stable Lévy process itself. As we will see later, this behavior does not change under perturbations by stochastic processes that satisfy some mild conditions.

Theorem 2.1 Let $(L_t)_{t \geq 0}$ be an α -stable Lévy process with $L_1 \sim S_\alpha(C, \beta, 0)$. If $p > \alpha/2$ then

$$(V_p^n(L)_t - nt B_n(\alpha, p))_{t \geq 0} \xrightarrow{\mathcal{D}} (L'_t)_{t \geq 0} \quad \text{as } n \rightarrow \infty, \quad (2.4)$$

where $L'_1 \sim S_{\alpha/p}(C', 1, 0)$ with the scale

$$C' = \begin{cases} C^p \left(\frac{\cos(\frac{\pi\alpha}{2p})\Gamma(1 - \frac{\alpha}{p})}{\cos(\frac{\pi\alpha}{2})\Gamma(1 - \alpha)} \right)^{p/\alpha}, & \alpha \neq p, \\ C, & \alpha = p. \end{cases} \quad (2.5) \quad \text{eq:C}$$

The normalizing sequence $(B_n(\alpha, p))_{n \geq 1}$ is deterministic and given by

$$B_n(\alpha, p) = \begin{cases} n^{-p/\alpha} \mathbf{E}|L_1|^p, & p \in (\alpha/2, \alpha), \\ \mathbf{E} \sin(n^{-1}|L_1|^\alpha), & p = \alpha, \\ 0, & p > \alpha. \end{cases} \quad (2.6) \quad \text{eq:B}$$

We remark that the skewness parameter β of L does not influence the convergence of $V_p^n(L)_t$ and does not appear in the limiting process since the p -variation depends only on the absolute values of the increments of L . Moreover, for $p > \alpha$ the limiting process L' is a subordinator.

We next perturb L by some other process Y . We impose no restrictions on dependence properties of Y and L . The only conditions on Y concern the behavior of its p -variation. We formulate two theorems, the first for $p \in (\alpha/2, 1) \cup (\alpha, \infty)$, and the second for $p \in (1, \alpha]$.

Theorem 2.2 Let $(L_t)_{t \geq 0}$ be an α -stable stochastic process, with $L_1 \sim S_\alpha(C, \beta, 0)$ and $(Y_t)_{t \geq 0}$ be another stochastic process that satisfies

$$V_p^n(Y) \xrightarrow{u.c.p.} 0, \quad n \rightarrow \infty, \quad (2.7)$$

for some $p \in (\alpha/2, 1) \cup (\alpha, \infty)$. Then

$$(V_p^n(L + Y)_t - nt B_n(\alpha, p))_{t \geq 0} \xrightarrow{\mathcal{D}} (L'_t)_{t \geq 0} \quad \text{as } n \rightarrow \infty, \quad (2.8)$$

with L' and $(B_n(\alpha, p))_{n \geq 1}$ defined in [\(2.5\)](#) and [\(2.6\)](#).

The methods used to prove [Theorem 2.2](#) do not work for $p \in (1, \alpha]$, and in this case we have to impose stronger conditions on the process Y .

Theorem 2.3 Let $(L_t)_{t \geq 0}$ be an α -stable stochastic process with $L_1 \sim S_\alpha(C, \beta, 0)$, $\alpha \in (1, 2)$ and let $(Y_t)_{t \geq 0}$ be another stochastic process. Let $p \in (1, \alpha]$ and $T > 0$. If Y is such that for every $\delta > 0$ there exists $K(\delta) > 0$ that satisfies

$$\mathbf{P}(|Y_s(\omega) - Y_t(\omega)| \leq K(\delta)|s - t| \text{ for all } s, t \in [0, T]) \geq 1 - \delta, \quad (2.9)$$

the process Y does not contribute to the limit of $V_p^n(L + Y)$, i.e.

$$(V_p^n(L + Y)_t - nt B_n(\alpha, p))_{0 \leq t \leq T} \xrightarrow{\mathcal{D}} (L'_t)_{0 \leq t \leq T}, \quad n \rightarrow \infty \quad (2.10)$$

with L' and $(B_n(\alpha, p))_{n \geq 1}$ defined in [\(2.5\)](#) and [\(2.6\)](#).

To be able to study models of the type [\(1.1\)](#) we formulate the following corollary of [Theorems 2.3](#) and [2.2](#) which takes into account that Lebesgue integral processes are absolutely continuous w.r.t. the time variable and thus qualify as small process perturbations in the sense of the Theorems.

Corollary 2.1 Let $(L_t)_{t \geq 0}$ be an α -stable stochastic process, with $L_1 \sim S_\alpha(C, \beta, 0)$, and let $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be a locally bounded function such that for some $x \in \mathbb{R}$ and $T > 0$ the unique strong solution for

$$X_t = x + \int_0^t f(s, X_s) ds + L_t \quad (2.11) \quad \text{eq:dif_X}$$

exists on the time interval $[0, T]$. Then for $p > \alpha/2$ we have

$$(V_p^n(X)_t - nt B_n(\alpha, p))_{0 \leq t \leq T} \xrightarrow{\mathcal{D}} (L'_t)_{0 \leq t \leq T} \quad (2.12)$$

as $n \rightarrow \infty$, with L' and $(B_n(\alpha, p))_{n \geq 1}$ defined in [\(2.5\)](#) and [\(2.6\)](#).

The functional convergence of power variations of symmetric stable Lévy processes to stable processes was first studied by Greenwood in [?], where more general *non-equidistant* power variations were considered, and in particular for $p > \alpha$ the convergence to subordinators was proved. Further, more general results on power variations of Lévy processes are obtained by Greenwood and Fristedt in [?]. In [?] and [?], Jacod proves convergence results for p -variations of general Lévy processes and semimartingales. In particular, several laws of large numbers and central limit theorems are established. Our results are different from Jacod's because we consider processes possessing no second moments so that only the generalized central limit theorem can apply. Moreover, we consider in addition convergence of perturbed processes. Corcuera, Nualart and Woerner in [1] consider p -variations of a (perturbed) integrated α -stable process of the type $X = Y + \int_0^\cdot u_{s-} dL_s$ with some cadlag adapted process u . For $u = 1$, our setting results. [1] contains a law of large numbers for $0 < p < \alpha$ and a functional central limit theorem for $0 < p \leq \alpha/2$. However, very restrictive conditions on possible perturbation processes Y are imposed, so that the results are not applicable to processes of the type (1.1).

3 Applications to real data

In this section we illustrate our convergence results and show how they can be applied to the estimation of the stability index α . We emphasize that the conclusions obtained are somewhat heuristic. Additional work has to be done to provide more precise statistical properties of the p -variation processes as estimators for the stability index, and to describe the decision rule of the testing procedure along with its quality features.

We first work with simulated data. Assume they are realizations of the SDE (2.11) where L is a stable process with unknown stability index α . From the data set we extract m samples with n data points each by taking adjacent non-overlapping groups of n consecutive points. This way we get the samples $(X^{(i)})_{1 \leq i \leq m}$ with $X^{(i)} = (X_{\frac{1}{n}}^{(i)}, X_{\frac{2}{n}}^{(i)}, \dots, X_1^{(i)})$, $1 \leq i \leq m$. Along each sample we calculate the p -variation

$$V_p^n(X^{(i)})_1 = \sum_{j=1}^n |\Delta_j^n X^{(i)}|^p, \quad 1 \leq i \leq m, \quad (3.1)$$

where the parameter p takes values in some appropriately chosen interval $[p_1, p_2]$.

Due to Corollary 2.1, the random variables $V_p^n(X^{(i)})_1$ converge to the stable random variable L'_1 , possessing stability index α/p . In order to estimate α , we compare the law of $V_p^n(X^{(i)})_1$ with some known stable reference law. Thus for each p we calculate the empirical distribution function of $(V_p^n(X^{(i)})_1)_{1 \leq i \leq m}$ given by

$$G_{p,n}(x) := \frac{1}{m} \sum_{i=1}^m \mathbb{I}_{(-\infty, x]}(V_p^n(X^{(i)})_1), \quad x \in \mathbb{R}, \quad (3.2)$$

and consider for convenience the reference $\frac{1}{2}$ -subordinator with scale parameter $C' > 0$ whose probability distribution function $F_{1/2, C'}$ can be explicitly calculated by

$$F_{1/2, C'}(x) = \sqrt{\frac{C'}{2\pi}} \int_0^x \frac{e^{-C'/2y}}{y^{3/2}} dy, \quad x \geq 0. \quad (3.3)$$

This will be a candidate for the limiting law of the random variable L'_1 . Since the scale parameter C' of L'_1 is not known, we numerically calculate the following distance of the Kolmogorov–Smirnov type:

$$D_n(C', p) = \min_{p \in [p_1, p_2], C' \in [c_1, c_2]} \sup_{x \geq 0} |G_{p,n}(x) - F_{1/2, C'}(x)| \quad (3.4)$$

where $0 < p_1 < p_2$ and $0 < c_1 < c_2$ are to be chosen appropriately. Assume $D_n(C', p)$ attains a unique minimum at $C' = c^*$ and $p = p^*$. Since $V_p^n(X)_1$ converges to a $\frac{1}{2}$ -subordinator if and only if $p = 2\alpha$, we immediately obtain the estimator $\alpha^* = p^*/2$ for the stability index of the driving process L . From the equations (2.5) we also get the estimator c^* for the scale parameter of L_1 .

In Fig. 1 we plot the section $p \mapsto D(c^*, p)$ of D for the simulated data from equation (2.11) with $f(\cdot, x) = \cos x$, $x \in \mathbb{R}$. Here $L_1 \sim S()$ and we see a well pronounced minimum corresponding to the value $p =$

We next study the real ice core data, analyzed earlier by Ditlevsen in [3, 4]. We analogously extract samples and use the p -variation process and the Kolmogorov–Smirnov distance to test the time series of the log-calcium signal of the ice core data over the time period from approximately 90 150 to 10 150 years before present, and its part corresponding to 55 150 to 15 150 years before present.

The empirical distance $D(C', p)$ for the real data equally exhibits a unique global minimum in the (C', p) -domain, which yields the estimate $\alpha^* \approx 0.75$. It is striking that our estimate differs from Ditlevsen's by a quantity very close to 1.

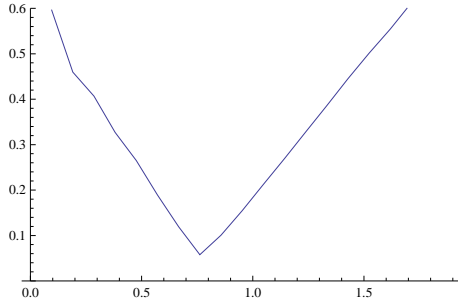


Figure 1: $D(c^*, p)$ simulated data, L is a 0.75-stable Lévy process, $n = m = 200$ fig:kss-s

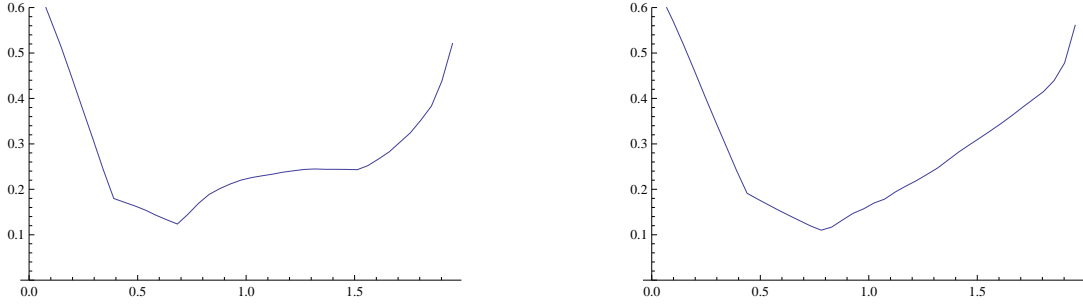


Figure 2: $D(c^*, p)$ for the whole ice-core data set, $n = m = 282$ (l.) and for its part from 55 150 to 15 150 years b.p., $n = m = 200$ (r.).

4 Convergence of the finite dimensional laws of $V_p^n(L)_t$

fig:kss-ia

To prove the convergence of the marginal distributions we use the following theorem which is a direct result of the well known generalized central limit theorem for i.i.d. random variables with infinite variance (e.g. see Theorem 3 in Feller [5, Chapter XVII.5]).

Proposition 4.1 *Let $(\eta_i)_{i \geq 1}$ be a sequence of non-negative i.i.d. random variables with a regularly varying tail such that*

$$\mathbf{P}(\eta_1 > x) \approx C \frac{2-\alpha}{\alpha} x^{-\alpha/p} \quad \text{as } x \rightarrow +\infty. \quad (4.1)$$

for some $\alpha \in (0, 2)$, $p > \alpha/2$ and $C > 0$. Then for any $t > 0$ we have

$$\left(\frac{t}{n}\right)^{p/\alpha} \sum_{i=1}^n \eta_i - b_{t,n}(\alpha, p) \xrightarrow{d} t^{p/\alpha} Z, \quad \text{as } n \rightarrow \infty, \quad (4.2)$$

with

$$b_{t,n}(\alpha, p) = \begin{cases} n \left(\frac{t}{n}\right)^{p/\alpha} \mathbf{E}\eta_1, & p \in (\alpha/2, \alpha) \\ n \mathbf{E} \sin\left(\frac{t\eta_1}{n}\right), & p = \alpha, \\ 0, & p > \alpha. \end{cases} \quad (4.3)$$

where $Z \sim S_{\alpha/p}(C', 1, 0)$ with C' as defined in (2.5).

Let L be an α -stable Lévy process as defined in (2.1) and let $p > \alpha/2$. To study the finite dimensional distributions of $V_p^n(L)_t$ we note that due to the independence of increments of L it suffices to establish the convergence of marginal laws for a fixed $t > 0$. Further, the stationarity and independence of increments of L and the self-similarity property $L_t \stackrel{d}{=} t^{1/\alpha} L_1$ implies that

$$V_p^n(L)_t = \sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n L|^p \stackrel{d}{=} \sum_{i=1}^{\lfloor nt \rfloor} \frac{|\Delta_i^1 L|^p}{n^{p/\alpha}}, \quad (4.4) \quad \text{eq:a}$$

with $\Delta_i^1 L \stackrel{d}{=} L_1 \sim S_\alpha(C, \beta, 0)$ being i.i.d. random variables. It is easy to see that the random variables $|\Delta_i^1 L|^p$ have a distribution function with a regularly varying tail, namely

$$\mathbf{P}(|\Delta_i^1 L|^p > x) = \mathbf{P}(|L_1| > x^{1/p}) \approx C_F \frac{2-\alpha}{\alpha} x^{-\alpha/p} \quad \text{as } x \rightarrow +\infty, \quad (4.5)$$

and thus we can apply Proposition 4.1 to the sum (4.4). Taking into account that $nt/[nt] \rightarrow 1$ as $n \rightarrow \infty$, we obtain convergence of the finite dimensional laws in Theorem 2.1.

5 Tightness of the laws

5.1 Aldous' criterion

To establish the tightness of the sequence $(V_p^n(L))_{n \geq 1}$ is more complicated. Although the idea of the proof is based on an application of Aldous' criterion, the technical details depend strongly on the relationship between p and α . We first formulate a version of Aldous' criterion for tightness that is applicable in our case.

On the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we define filtrations $\mathcal{G}^n = (\mathcal{G}_t^n)_{t \geq 0}$ generated by the power variation process $V_p^n(L)$, i.e.

$$\mathcal{G}_t^n = \sigma(V_p^n(L)_s : s \leq t) = \sigma(|L_{\frac{k}{n}}| : 0 \leq k \leq [nt]). \quad (5.1)$$

Let \mathcal{T}_N^n be the set of \mathcal{G}^n -stopping times that are bounded by $N > 0$. Then the sequence $V_p^n(L)$ is tight if and only if the following two conditions hold (see p. 350 and p. 356 in [6]):

1. for all $\varepsilon > 0$ and $N > 0$ there are $n_0 \geq 1$ and $K > 0$, so that for all $n \geq n_0$ we have

$$\mathbf{P}(\sup_{t \leq N} V_p^n(L)_t \geq K) \leq \varepsilon; \quad (5.2) \quad \text{eq:b}$$

2. for all $N \geq 0$ and $\varepsilon > 0$ we have

$$\lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{S, T \in \mathcal{T}_N^n, \\ S \leq T \leq S + \theta}} \mathbf{P}(|V_p^n(L)_T - V_p^n(L)_S| \geq \varepsilon) = 0. \quad (5.3) \quad \text{eq:c}$$

Condition (5.2) is obviously satisfied due to the monotonicity of the p -variation process and convergence of its marginal distributions to a stable law.

To deal with condition (5.3) we note that since $(V_p^n(L))_{t \geq 0}$ has piecewise constant paths, the filtrations \mathcal{G}^n increase only at discrete time instants $t = k/n$, $k \geq 0$, i.e.

$$\mathcal{G}_t^n = \mathcal{G}_{\frac{k}{n}}^n \text{ for } \frac{k}{n} \leq t < \frac{k+1}{n}. \quad (5.4)$$

Thus instead of \mathcal{T}_N^n we can consider in (5.3) a family of \mathcal{G}^n -stopping times with finitely many values

$$\mathcal{S}_N^n := \{S \in \mathcal{T}_N^n : S \text{ takes values in } \{0, \frac{1}{n}, \dots, N\}\}. \quad (5.5)$$

5.2 Tightness for $p > \alpha$

The case $p > \alpha$ is simple because the compensating sequence $B_{n,t}(\alpha, p)$ vanishes. Thus we can use the monotonicity of the process $V_p^n(L)_t$.

Let $N \geq 1$ and $\varepsilon > 0$ be fixed. We use the stationarity and independence of increments of L and convergence the marginal distributions of $V_p^n(L)$ to L' to obtain the following limit:

$$\begin{aligned} & \lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{S, T \in \mathcal{S}_N^n, \\ S \leq T \leq S + \theta}} \mathbf{P}(|V_p^n(L)_T - V_p^n(L)_S| \geq \varepsilon) \\ &= \lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{S, T \in \mathcal{S}_N^n, \\ S \leq T \leq S + \theta}} \mathbf{P}\left(\sum_{i=nS+1}^{nT} |\Delta_i^n L|^p \geq \varepsilon\right) \\ &\leq \lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{S \in \mathcal{S}_N^n} \mathbf{P}\left(\sum_{i=nS+1}^{nS+[n\theta]} |\Delta_i^n L|^p \geq \varepsilon\right) \\ &= \lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{S \in \mathcal{S}_N^n} \sum_{j=0}^{nN} \mathbf{P}\left(\sum_{i=j+1}^{j+[n\theta]} |\Delta_i^n L|^p \geq \varepsilon, S = \frac{j}{n}\right) \\ &= \lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbf{P}\left(\sum_{i=1}^{[n\theta]} |\Delta_i^n L|^p \geq \varepsilon\right) \\ &= \lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \mathbf{P}(V_p^n(L)_\theta \geq \varepsilon) \\ &= \lim_{\theta \downarrow 0} \mathbf{P}(L'_\theta \geq \varepsilon) = 0. \end{aligned} \quad (5.6)$$

5.3 Tightness for $\frac{\alpha}{2} < p < \alpha$

The case $p < \alpha$ has to be treated differently since the process

$$X_t^n = V_p^n(L)_t - [nt]n^{-p/\alpha}\mathbf{E}|L_1|^p \quad (5.7)$$

need not be monotone.

Let $\theta > 0$. As in the preceding section we use the independence and stationarity of increments of L to obtain the estimate

$$\begin{aligned} \sup_{\substack{S, T \in \mathcal{S}_N^n, \\ S \leq T \leq S + \theta}} \mathbf{P}(|X_T^n - X_S^n| \geq \varepsilon) &\leq \sup_{S \in \mathcal{S}_N^n} \mathbf{P}\left(\max_{1 \leq k \leq [n\theta]} \left| \sum_{i=nS+1}^{nS+k} \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \varepsilon\right) \\ &\leq \mathbf{P}\left(\max_{1 \leq k \leq [n\theta]} \left| \sum_{i=1}^k \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \varepsilon\right) \\ &\leq \max_{1 \leq k \leq [n\theta]} 3\mathbf{P}\left(\left| \sum_{i=1}^k \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{3}\right), \end{aligned} \quad (5.8) \quad \boxed{\text{eq:h}}$$

where the last inequality follows from Lemma 20.2 in [\[10\]](#).^{Sato-99} Now we have to show that the probabilities in the latter inequality converge to zero uniformly in k as $n \rightarrow \infty$ and $\theta \downarrow 0$.

For any $\delta > 0$ we can choose $\theta_0 > 0$ and $n_0 \geq 1$ such that

$$\mathbf{P}\left(\left| \sum_{i=1}^{[n\theta_0]} \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{6}\right) \leq \frac{\delta}{6} \quad (5.9) \quad \boxed{\text{eq:d}}$$

for all $n \geq \frac{n_0}{2}$. Without loss of generality we can assume that $\theta_0 < 1$. We will show now that

$$\mathbf{P}\left(\left| \sum_{i=1}^{[n\theta]} \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{3}\right) \leq \frac{\delta}{3} \quad (5.10) \quad \boxed{\text{eq:e}}$$

holds for all $\theta < \theta_0$ and $n \geq n_0$. Indeed let us fix $\theta < \theta_0$ and assume first that $\theta < \theta_0/2$. With the help of the triangle inequality and the estimate [\(5.9\)](#) we can conclude that for $n \geq n_0$

$$\begin{aligned} &\mathbf{P}\left(\left| \sum_{i=1}^{[n\theta]} \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{3}\right) \\ &\leq \mathbf{P}\left(\left| \sum_{i=1}^{[n\theta_0]} \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{6}\right) + \mathbf{P}\left(\left| \sum_{i=1+[n\theta]}^{[n\theta_0]} \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{6}\right) \\ &\leq \frac{\delta}{6} + \mathbf{P}\left(\left| \sum_{i=1}^{[n\theta_0]-[n\theta]} \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{6}\right). \end{aligned} \quad (5.11)$$

Recalling that $\theta_0 - \theta > \theta_0/2$ and $\theta_0 < 1$ we find $n > n' \geq \frac{n_0}{2}$ such that $[n'\theta_0] = [n\theta_0] - [n\theta]$. Thus employing the self-similarity of stable processes and [\(5.9\)](#) we have

$$\begin{aligned} &\mathbf{P}\left(\left| \sum_{i=1}^{[n\theta_0]-[n\theta]} \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{6}\right) \\ &= \mathbf{P}\left(\left| \sum_{i=1}^{[n'\theta_0]} \left(|\Delta_i^{n'} L|^p - (n')^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{6}\right) \\ &= \mathbf{P}\left(\left(\frac{n'}{n}\right)^{p/\alpha} \left| \sum_{i=1}^{[n'\theta_0]} \left(|\Delta_i^{n'} L|^p - (n')^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{6}\right) \\ &\leq \mathbf{P}\left(\left| \sum_{i=1}^{[n'\theta_0]} \left(|\Delta_i^{n'} L|^p - (n')^{-p/\alpha} \mathbf{E}|L_1|^p \right) \right| \geq \frac{\varepsilon}{6}\right) \leq \frac{\delta}{6}. \end{aligned} \quad (5.12)$$

So the inequality (5.10) is proved for $\theta < \theta_0/2$ and n big enough. Further, if $\theta_0/2 \leq \theta < \theta_0$ the proof is even easier because for $n \geq n_0$ we can directly find n' with the same properties as above such that

$$\begin{aligned} \mathbf{P}\left(\left|\sum_{i=1}^{[n\theta]} \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p\right)\right| \geq \frac{\varepsilon}{3}\right) &= \mathbf{P}\left(\left|\sum_{i=1}^{[n'\theta_0]} \left(|\Delta_i^{n'} L|^p - n'^{-p/\alpha} \mathbf{E}|L_1|^p\right)\right| \geq \frac{\varepsilon}{3}\right) \\ &\leq \mathbf{P}\left(\left|\sum_{i=1}^{[n'\theta_0]} \left(|\Delta_i^{n'} L|^p - (n')^{-p/\alpha} \mathbf{E}|L_1|^p\right)\right| \geq \frac{\varepsilon}{3}\right) \leq \mathbf{P}\left(\left|\sum_{i=1}^{[n'\theta_0]} \left(|\Delta_i^{n'} L|^p - (n')^{-p/\alpha} \mathbf{E}|L_1|^p\right)\right| \geq \frac{\varepsilon}{6}\right) < \frac{\delta}{3}. \end{aligned} \quad (5.13) \quad \boxed{\text{eq:f}}$$

Thus we have demonstrated that for any $\delta > 0$ and $\varepsilon > 0$ there exist $\theta_0 > 0$ and $n_0 \geq 1$ such that

$$3\mathbf{P}\left(\left|\sum_{i=1}^{[n\theta]} \left(|\Delta_i^n L|^p - n^{-p/\alpha} \mathbf{E}|L_1|^p\right)\right| \geq \frac{\varepsilon}{3}\right) \leq \delta, \quad n \geq n_0, \quad \theta < \theta_0, \quad (5.14)$$

which together with (5.8) implies the second condition of Aldous' criterion, namely

$$\lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{S, T \in \mathcal{S}_N^n, \\ S \leq T \leq S + \theta}} \mathbf{P}(|X_T^n - X_S^n| \geq \varepsilon) \leq \delta. \quad (5.15)$$

But δ was arbitrary.

5.4 Tightness for $p = \alpha$

For this part of the proof we need two lemmas that will enable us to control moments of stable processes.

Lemma 5.1 *Let L be an α -stable Lévy process and let $k \geq 1$. Then there are positive numbers $C_{\alpha, k}$, $K_{\alpha, k}$ and N_α such that for $y \geq N_\alpha$ we have*

$$K_{\alpha, k} y^{(k-1)\alpha} \leq \mathbf{E}(|L_1|^{k\alpha} \mathbb{I}_{\{|L_1| \leq y\}}) \leq C_{\alpha, k} y^{(k-1)\alpha} \quad (5.16)$$

if $k > 1$ and

$$K_{\alpha, 1} \ln y \leq \mathbf{E}(|L_1|^\alpha \mathbb{I}_{\{|L_1| \leq y\}}) \leq C_{\alpha, 1} \ln y. \quad (5.17)$$

In particular there is $C_\alpha > 0$ such that $\mathbf{P}(|L_1| \geq y) \leq C_\alpha y^{-\alpha}$ for $y \geq N_\alpha$.

Proof: We first prove the estimates from above. Let F denote the probability distribution function of L_1 . Then there are positive constants c_α and n_α such that for $x \geq n_\alpha$

$$1 - F(x) \leq \frac{c_\alpha}{x^\alpha} \quad \text{and} \quad F(-x) \leq \frac{c_\alpha}{x^\alpha}. \quad (5.18) \quad \boxed{\text{eq:p}}$$

Integration by parts yields for $y \geq n_\alpha$

$$\begin{aligned} \mathbf{E}\left(|L_1|^{k\alpha} \mathbb{I}_{\{|L_1| \leq y\}}\right) &= \int_{-y}^y |x|^{k\alpha} dF(x) \\ &= -x^{k\alpha} F(-x) \Big|_0^y + \int_0^y k\alpha x^{k\alpha-1} F(-x) dx \\ &\quad - x^{k\alpha} (1 - F(x)) \Big|_0^y + \int_0^y k\alpha x^{k\alpha-1} (1 - F(x)) dx \\ &\leq 2 \int_0^{n_\alpha} k\alpha x^{k\alpha-1} dx + 2 \int_{n_\alpha}^y k\alpha x^{k\alpha-1} \frac{c_\alpha}{x^\alpha} dx \\ &= 2n_\alpha^{k\alpha} + 2c_\alpha \int_{n_\alpha}^y k\alpha x^{(k-1)\alpha-1} dx \\ &= 2n_\alpha^{k\alpha} + 2c_\alpha \begin{cases} \frac{k}{k-1} (y^{(k-1)\alpha} - n_\alpha^{(k-1)\alpha}), & k > 1, \\ k(\ln y - \ln n_\alpha), & k = 1, \end{cases} \\ &\leq 2n_\alpha^{k\alpha} + 2c_\alpha \begin{cases} \frac{k}{k-1} y^{(k-1)\alpha}, & k > 1, \\ k \ln y, & k = 1. \end{cases} \end{aligned} \quad (5.19)$$

Now we can choose constants $C_{\alpha, k}$ and N_α such that the estimates from above are satisfied. The estimates from below follow analogously from the inequalities

$$1 - F(x) \geq \frac{k_\alpha}{x^\alpha} \quad \text{and} \quad F(-x) \geq \frac{k_\alpha}{x^\alpha} \quad (5.20)$$

for some $k_\alpha > 0$ and x big enough. ■

:ineq_cos

Lemma 5.2 *Let L be an α -stable Lévy process. For any $a, b > 0$ there exist $C_{\alpha,n}(a, b)$ and $N_\alpha(a, b)$ such that for $\lambda \in [0, 2/a]$ and $n \geq N_\alpha(a, b)$ we have*

$$\left| \mathbf{E} \exp \left(i\lambda \frac{|L_1|^\alpha}{n} \right) - \exp \left(i\lambda \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \right| \leq C_{\alpha,n}(a, b) \quad (5.21) \quad \text{eq:g}$$

with $C_{\alpha,n}(a, b)$ satisfying

$$\lim_{b \rightarrow 0} \lim_{n \rightarrow \infty} C_{\alpha,n}(a, b)nb = 0 \quad \text{and} \quad \lim_{a \rightarrow \infty} \lim_{n \rightarrow \infty} C_{\alpha,n}(a, b)n = 0. \quad (5.22) \quad \text{eq:k}$$

Proof: We split the left-hand side of (5.21) into the real and the imaginary part to obtain the simple estimate

$$\begin{aligned} & \left| \mathbf{E} \exp \left(i\lambda \frac{|L_1|^\alpha}{n} \right) - \exp \left(i\lambda \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \right| \\ & \leq \left| \mathbf{E} \cos \left(\frac{\lambda |L_1|^\alpha}{n} \right) - \cos \left(\lambda \mathbf{E} \sin \frac{|L_1|^\alpha}{n} \right) \right| + \left| \mathbf{E} \sin \left(\frac{\lambda |L_1|^\alpha}{n} \right) - \sin \left(\lambda \mathbf{E} \sin \frac{|L_1|^\alpha}{n} \right) \right|. \end{aligned} \quad (5.23)$$

Throughout the proof we shall use the following elementary inequalities:

$$\begin{aligned} x - \frac{x^3}{6} & \leq \sin x \leq x, \quad x \geq 0, \\ \cos x & \geq 1 - \frac{x^2}{2}, \quad x \in \mathbb{R}. \end{aligned} \quad (5.24)$$

Let $a, b > 0$, $\lambda \in [0, 2/a]$ and denote $m_n = m_n(a, b) := na^{1/4}b^{1/2}$.

We first estimate the real part. To this end, we apply Lemma 5.1 with \tilde{n} big enough so that $m_n^{1/\alpha} \geq N_\alpha$ or $n \geq N_\alpha^\alpha a^{-1/4}b^{-1/2}$, to obtain

$$\begin{aligned} 1 & \geq \mathbf{E} \cos \left(\frac{\lambda |L_1|^\alpha}{n} \right) \\ & \geq \mathbf{E} \cos \left(\frac{\lambda |L_1|^\alpha}{n} \mathbb{I}_{\{|L_1| \leq m_n^{1/\alpha}\}} \right) - \mathbf{P}(|L_1| \geq m_n^{1/\alpha}) \\ & \geq \mathbf{E} \left(\left(1 - \frac{\lambda^2 |L_1|^{2\alpha}}{2n^2} \right) \mathbb{I}_{\{|L_1| \leq m_n^{1/\alpha}\}} \right) - \mathbf{P}(|L_1| \geq m_n^{1/\alpha}) \\ & = 1 - \mathbf{E} \left(\frac{\lambda^2 |L_1|^{2\alpha}}{2n^2} \mathbb{I}_{\{|L_1| \leq m_n^{1/\alpha}\}} \right) - 2\mathbf{P}(|L_1| \geq m_n^{1/\alpha}) \\ & \geq 1 - \frac{C_{\alpha,2}\lambda^2 m_n}{2n^2} - \frac{2C_\alpha}{m_n} \\ & \geq 1 - \frac{2C_{\alpha,2}b^{1/2}}{na^{7/4}} - \frac{2C_\alpha}{na^{1/4}b^{1/2}} =: 1 - C_{\alpha,n}^{(1)}(a, b). \end{aligned} \quad (5.25)$$

Analogously we get an estimate for the second summand:

$$\begin{aligned} 1 & \geq \cos \left(\lambda \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \geq 1 - \frac{\lambda^2}{2} \left(\mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right)^2 \\ & \geq 1 - \frac{\lambda^2}{2} \left(\mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \mathbb{I}_{\{|L_1| \leq m_n^{1/\alpha}\}} \right) + \mathbf{P}(|L_1| \geq m_n^{1/\alpha}) \right)^2 \\ & \geq 1 - \frac{\lambda^2}{2} \left(\mathbf{E} \left(\frac{|L_1|^\alpha}{n} \mathbb{I}_{\{|L_1| \leq m_n^{1/\alpha}\}} \right) + \mathbf{P}(|L_1| \geq m_n^{1/\alpha}) \right)^2 \\ & \geq 1 - \frac{2}{a^2} \left(\frac{C_{\alpha,1} \ln m_n}{n\alpha} + \frac{C_\alpha}{m_n} \right)^2 \\ & = 1 - \frac{2}{a^2} \left(\frac{C_{\alpha,1} \ln(na^{1/4}b^{1/2})}{n\alpha} + \frac{C_\alpha}{na^{1/4}b^{1/2}} \right)^2 =: 1 - C_{\alpha,n}^{(2)}(a, b). \end{aligned} \quad (5.26)$$

Summarizing we have

$$\left| \mathbf{E} \cos \left(\frac{\lambda |L_1|^\alpha}{n} \right) - \cos \left(\lambda \mathbf{E} \sin \frac{|L_1|^\alpha}{n} \right) \right| \leq C_{\alpha,n}^{(1)}(a, b) + C_{\alpha,n}^{(2)}(a, b). \quad (5.27) \quad \text{eq:j}$$

We now estimate the imaginary part. With analogous arguments we obtain

$$\begin{aligned}
& \mathbf{E} \sin \left(\frac{\lambda |L_1|^\alpha}{n} \right) - \sin \left(\lambda \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \\
& \leq \mathbf{E} \sin \left(\frac{\lambda |L_1|^\alpha}{n} \right) - \lambda \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) + \frac{\lambda^3}{6} \left(\mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right)^3 \\
& \leq \mathbf{E} \left(\left(\sin \left(\frac{\lambda |L_1|^\alpha}{n} \right) - \lambda \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \mathbb{I}_{\{|L_1| \leq m_n^{1/\alpha}\}} \right) + (1 + \lambda) \mathbf{P}(|L_1| \geq m_n^{1/\alpha}) \\
& \quad + \frac{4}{3a^3} \left(\frac{C_{\alpha,1} \ln(na^{1/4}b^{1/2})}{n\alpha} + \frac{C_\alpha}{na^{1/4}b^{1/2}} \right)^3 \\
& \leq \mathbf{E} \left(\left(\frac{\lambda |L_1|^\alpha}{n} - \frac{\lambda |L_1|^\alpha}{n} + \frac{\lambda |L_1|^{3\alpha}}{6n^3} \right) \mathbb{I}_{\{|L_1| \leq m_n^{1/\alpha}\}} \right) + \frac{(1 + 2/a)C_\alpha}{m_n} \\
& \quad + \frac{4}{3a^3} \left(\frac{C_{\alpha,1} \ln(na^{1/4}b^{1/2})}{n\alpha} + \frac{C_\alpha}{na^{1/4}b^{1/2}} \right)^3 \\
& \leq \frac{C_{\alpha,3}b}{3na^{1/2}} + \frac{(1 + 2/a)C_\alpha}{na^{1/4}b^{1/2}} + \frac{4}{3a^3} \left(\frac{C_{\alpha,1} \ln(na^{1/4}b^{1/2})}{n\alpha} + \frac{C_\alpha}{na^{1/4}b^{1/2}} \right)^3 =: C_{\alpha,n}^{(3)}(a, b).
\end{aligned} \tag{5.28}$$

The lower bound is obtained similarly, and reads

$$\begin{aligned}
& \mathbf{E} \sin \left(\frac{\lambda |L_1|^\alpha}{n} \right) - \sin \left(\lambda \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \\
& \geq \mathbf{E} \sin \left(\frac{\lambda |L_1|^\alpha}{n} \right) - \lambda \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \\
& \geq \mathbf{E} \left(\left(\sin \frac{\lambda |L_1|^\alpha}{n} - \lambda \sin \frac{|L_1|^\alpha}{n} \right) \mathbb{I}_{\{|L_1| \leq m_n^{1/\alpha}\}} \right) - (1 + \lambda) \mathbf{P}(|L_1| \geq m_n^{1/\alpha}) \\
& \geq \mathbf{E} \left(\left(\frac{\lambda |L_1|^\alpha}{n} - \frac{\lambda^3 |L_1|^{3\alpha}}{6n^3} - \frac{\lambda |L_1|^\alpha}{n} \right) \mathbb{I}_{\{|L_1| \leq m_n^{1/\alpha}\}} \right) - \frac{(1 + \frac{2}{a})C_\alpha}{m_n} \\
& \geq -\frac{4C_{\alpha,3}b}{3a^{5/2}n} - \frac{(1 + 2/a)C_\alpha}{na^{1/4}b^{1/2}} =: C_{\alpha,n}^{(4)}(a, b).
\end{aligned} \tag{5.29}$$

Combining these estimates with (5.27) and denoting $C_{\alpha,n}(a, b) := C_{\alpha,n}^{(1)}(a, b) + C_{\alpha,n}^{(2)}(a, b) + C_{\alpha,n}^{(3)}(a, b) + C_{\alpha,n}^{(4)}(a, b)$ we obtain inequality (5.21). Property (5.22) of the limits is straightforward. \blacksquare

Now we can show tightness for $p = \alpha$. We use the inequality analogous to (5.8) which states that for any $\varepsilon > 0$

$$\begin{aligned}
& \sup_{\substack{S, T \in \mathcal{S}_n^{\alpha}, \\ S \leq T \leq S + \theta}} \mathbf{P} \left(\left| \sum_{i=nS+1}^{nT} \left(|\Delta_n L_i|^\alpha - \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \right| \geq \varepsilon \right) \\
& \leq \max_{1 \leq k \leq \lfloor n\theta \rfloor} 3\mathbf{P} \left(\left| \sum_{i=1}^k \left(|\Delta_n L_i|^\alpha - \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \right| \geq \frac{\varepsilon}{3} \right).
\end{aligned} \tag{5.30}$$

Therefore we have to show that

$$\lim_{\theta \downarrow 0} \limsup_{n \rightarrow \infty} \max_{1 \leq k \leq \lfloor n\theta \rfloor} \mathbf{P}(|X_n^k| \geq \varepsilon) = 0 \tag{5.31}$$

with

$$X_n^k = \sum_{i=1}^k \left(|\Delta_n L_i|^\alpha - \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right), \quad k, n \geq 1. \tag{5.32}$$

The truncation inequality (see e.g. Theorem 5.1 in [Kallenberg-02](#) [7]) provides an upper bound for the tail of X_n^k :

$$\mathbf{P}(|X_n^k| \geq \varepsilon) \leq \frac{\varepsilon}{2} \int_{-2/\varepsilon}^{2/\varepsilon} \left(1 - \mathbf{E} e^{i\lambda X_n^k} \right) d\lambda \leq 2 \sup_{0 \leq \lambda \leq 2/\varepsilon} \left(1 - \Re \mathbf{E} e^{i\lambda X_n^k} \right), \tag{5.33} \quad \boxed{\text{eq:1}}$$

where $\Re c$ denotes the real part of a complex number c . Since X_n^k is a sum of i.i.d. random variables, its characteristic function can be factorized and we get for $\lambda \geq 0$

$$\begin{aligned} \mathbf{E}e^{i\lambda X_n^k} &= \left[\mathbf{E} \exp \left(i\lambda \left(\frac{|L_1|^\alpha}{n} - \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \right) \right]^k \\ &= \exp \left[k \ln \mathbf{E} \exp \left(i\lambda \left(\frac{|L_1|^\alpha}{n} - \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \right) \right] \\ &= \exp \left[k \ln \frac{\mathbf{E} \exp(i\lambda \frac{|L_1|^\alpha}{n})}{\exp(i\lambda \mathbf{E} \sin(\frac{|L_1|^\alpha}{n}))} \right] \\ &= \exp \left[k \ln \left(1 + \frac{\mathbf{E} \exp(i\lambda \frac{|L_1|^\alpha}{n}) - \exp(i\lambda \mathbf{E} \sin(\frac{|L_1|^\alpha}{n}))}{\exp(i\lambda \mathbf{E} \sin(\frac{|L_1|^\alpha}{n}))} \right) \right]. \end{aligned} \quad (5.34)$$

Denote

$$\psi_{n,\lambda} := \frac{\mathbf{E} \exp(i\lambda \frac{|L_1|^\alpha}{n}) - \exp(i\lambda \mathbf{E} \sin(\frac{|L_1|^\alpha}{n}))}{\exp(i\lambda \mathbf{E} \sin(\frac{|L_1|^\alpha}{n}))}, \quad (5.35)$$

and note that due to Lemma [5.2](#) for any $\varepsilon, \theta > 0$ we can estimate

$$|\psi_{n,\lambda}| = \left| \mathbf{E} \exp \left(i\lambda \frac{|L_1|^\alpha}{n} \right) - \exp \left(i\lambda \mathbf{E} \sin \left(\frac{|L_1|^\alpha}{n} \right) \right) \right| \leq C_{\alpha,n}(\varepsilon, \theta), \quad (5.36)$$

with $\lim_{\theta \rightarrow 0} \lim_{n \rightarrow \infty} C_{\alpha,n}(\varepsilon, \theta)\theta n = 0$.

Let $\theta > 0$ be fixed and let $n_0(\alpha, \varepsilon, \theta)$ be such that $|\psi_{n,\lambda}| \leq 1$ for $n \geq n_0(\alpha, \varepsilon, \theta)$. Recalling the elementary approximations $\ln(1+z) = z + p(z)$ and $e^z = 1 + z + q(z)$ with $|p(z)|, |q(z)| \leq |z|^2$, $|z| \leq 1$, and the estimate [\(5.33\)](#) we get

$$\begin{aligned} \mathbf{P}(|X_n^k| \geq \varepsilon) &\leq 2 \sup_{\lambda \in [0, 2/\varepsilon]} (1 - \Re(\exp[k \ln(1 + \psi_{n,\lambda})])) \\ &= 2 \sup_{\lambda \in [0, 2/\varepsilon]} \left(-\Re \left(k\psi_{n,\lambda} + kp(\psi_{n,\lambda}) + q(k\psi_{n,\lambda} + kp(\psi_{n,\lambda})) \right) \right) \\ &\leq 2 \sup_{\lambda \in [0, 2/\varepsilon]} \left(k|\psi_{n,\lambda}| + k|p(\psi_{n,\lambda})| + |q(k\psi_{n,\lambda} + kp(\psi_{n,\lambda}))| \right). \end{aligned} \quad (5.37)$$

For $n \geq \max(n_0, N_\alpha(\varepsilon, \theta))$ with $N_\alpha(\varepsilon, \theta)$ as defined in Lemma [5.2](#) we already know that

$$k|\psi_{n,\lambda}| + k|p(\psi_{n,\lambda})| \leq \theta(n+1)|\psi_{n,\lambda}| + \theta(n+1)|\psi_{n,\lambda}|^2 \leq 2\theta(n+1)C_{\alpha,n}(\varepsilon, \theta). \quad (5.38)$$

Applying Lemma [5.2](#) we conclude that for $\theta < \theta_0$, $n \geq \max(n_0, N_\alpha(\varepsilon, \theta_0))$ and any $\varepsilon > 0$

$$\lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} \max_{1 \leq k \leq [n\theta]} \mathbf{P}(|X_n^k| \geq \varepsilon) = 0, \quad (5.39)$$

so that condition [\(5.3\)](#) of Aldous' criterion holds.

To prove condition [\(5.2\)](#) we proceed in exactly the same way as for [\(5.3\)](#), taking advantage of the fact that for fixed $N \geq 1$ the equation

$$\lim_{K \rightarrow \infty} \lim_{n \rightarrow \infty} C_{\alpha,n}(K, N)Nn = 0 \quad (5.40)$$

holds due to Lemma [5.2](#).

6 Generalization to sums of processes

We finally discuss the situation of Theorem [2.3](#), where besides the Lévy process L another process Y is given. To see that $V_p^n(L)$ and $V_p^n(L+Y)$ have equivalent asymptotic behavior we apply Lemma VI.3.31 in [\[6\]](#). Under the conditions of Theorems [2.2](#) and [2.3](#) it is enough to show that $V_p^n(L+Y) - V_p^n(L) \xrightarrow{u.c.p.} 0$ as $n \rightarrow \infty$.

6.1 Equivalence for $p \leq 1$

Let L be an α -stable Lévy process, $p \in (\alpha/2, 1]$ and let Y be such that for $V_p^n(Y) \xrightarrow{u.c.p.} 0$, as $n \rightarrow \infty$. Note that due to the monotonicity properties of $V_p^n(Y)$, the latter convergence condition is equivalent to $V_p^n(Y)_N \xrightarrow{\mathbf{P}} 0$ for any $N \in \mathbb{N}$. Then a simple application of the triangle inequality yields the proof. In fact, for any $N \in \mathbb{N}$ we have

$$\begin{aligned} \sup_{0 \leq t \leq N} |V_p^n(L+Y)_t - V_p^n(L)_t| &\leq \sum_{i=1}^{nN} \left| |\Delta_i^n(L+Y)|^p - |\Delta_i^n L|^p \right| \\ &\leq \sum_{i=1}^{nN} |\Delta_i^n(L+Y) - \Delta_i^n L|^p = V_p^n(Y)_N \xrightarrow{\mathbf{P}} 0, \quad n \rightarrow \infty. \end{aligned} \quad (6.1)$$

6.2 Equivalence for $p > \alpha$

Assume again that $V_p^n(Y)_t \xrightarrow{\mathbf{P}} 0$, $n \rightarrow \infty$, for any $t \geq 0$. Denote $m := [p]$, so that $p = m + q$ with $q \in [0, 1)$. Then for any $N \in \mathbb{N}$ we have

$$\begin{aligned}
\sup_{t \leq N} |V_p^n(L+Y)_t - V_p(L)_t| &= \sup_{t \leq N} \left| \sum_{i=1}^{[nt]} |\Delta_i^n(L+Y)|^{m+q} - |\Delta_n L_i|^{m+q} \right| \\
&= \sup_{t \leq N} \left| \sum_{i=1}^{[nt]} |\Delta_i^n(L+Y)|^q \sum_{k=0}^m \binom{m}{k} |\Delta_n L_i|^k |\Delta_i^n Y|^{m-k} - |\Delta_i^n L|^{m+q} \right| \\
&\leq \sum_{k=0}^{m-1} \binom{m}{k} \sum_{i=1}^{nN} \left| |\Delta_i^n(L+Y)|^q |\Delta_i^n L|^k |\Delta_i^n Y|^{m-k} \right| \\
&\quad + \sum_{i=1}^{nN} \left| |\Delta_i^n(L+Y)|^q |\Delta_i^n L|^m - |\Delta_i^n L|^{m+q} \right| \\
&\leq \sum_{k=0}^{m-1} \binom{m}{k} \sum_{i=1}^{nN} |\Delta_i^n L|^{k+q} |\Delta_i^n Y|^{m-k} + \sum_{k=0}^{m-1} \binom{m}{k} \sum_{i=1}^{nN} |\Delta_i^n L|^k |\Delta_i^n Y|^{m-k+q} \\
&\quad + \sum_{i=1}^{nN} |\Delta_i^n L|^m \left| |\Delta_i^n(L+Y)|^q - |\Delta_i^n L|^q \right| \\
&\leq \sum_{k=0}^{m-1} \binom{m}{k} \sum_{i=1}^{nN} |\Delta_i^n L|^{k+q} |\Delta_i^n Y|^{m-k} + \sum_{k=0}^{m-1} \binom{m}{k} \sum_{i=1}^{nN} |\Delta_i^n L|^k |\Delta_i^n Y|^{m-k+q} \\
&\quad + \mathbb{I}_{\{q>0\}} \sum_{i=1}^{nN} |\Delta_i^n L|^m |\Delta_i^n Y|^q.
\end{aligned} \tag{6.2}$$

The right-hand side of the latter inequality is essentially a sum of $2m+1$ terms of the type $\sum_{i=1}^{nN} |\Delta_i^n L|^k |\Delta_i^n Y|^{p-k}$, where $k \in [0, p] \cap \mathbb{N}$.

Applying Hölder's inequality we get

$$\sum_{i=1}^{[nt]} |\Delta_n L_i|^k |\Delta_n Y_i|^{p-k} \leq \left(\sum_{i=1}^{[nt]} |\Delta_n L_i|^p \right)^{k/p} \left(\sum_{i=1}^{[nt]} |\Delta_n Y_i|^p \right)^{(p-k)/p}. \tag{6.3}$$

The first factor in the latter formula converges in probability to a finite limit, since $p > \alpha$. The second factor converges to 0 in probability due to the assumption on Y , and Theorem 2.2 is proven.

6.3 Equivalence for $\alpha \in (1, 2)$, $p \in (1, \alpha]$

The main technical difficulty of this case arises from the fact that the p -variation of L for $p < \alpha$ does not exist. In particular, the events when increments of the stable process L become very large have to be considered carefully. For $T > 0$ and some $c \geq 1$ which will be specified later define the following sets:

$$\begin{aligned}
J_c^n(\omega) &:= \{i \in [0, [nT]] : |\Delta_i^n L(\omega)| > c\}, \\
A_c^n(j) &:= \{\omega \in \Omega : |J_c^n(\omega)| = j\}, \quad j = 0, \dots, [nT].
\end{aligned} \tag{6.4}$$

The set J_c^n contains the time instants (in the scale $\frac{1}{n}$), where the increments of the process L are 'large', i. e. exceed c . The set $A_c^n(j)$ describes the event that the number of large increments equals j .

Let $\delta > 0, \varepsilon > 0$. According to the conditions of Theorem (2.3), let $K' := K(\frac{\delta}{2}) > 0$, and set

$$B = \{\omega : |Y_s(\omega) - Y_t(\omega)| \leq K'|s-t| \text{ for all } s, t \in [0, T]\}, \tag{6.5}$$

so that $\mathbf{P}(B) \geq 1 - \frac{\delta}{2}$.

We estimate

$$\begin{aligned}
& \mathbf{P}\left(\sup_{0 \leq t \leq T} |V_p^n(L+Y)_t - V_p^n(L)_t| > \varepsilon\right) \\
& \leq \mathbf{P}\left(\left\{\sum_{i=1}^{[nT]} \left||\Delta_i^n(L+Y)|^p - |\Delta_i^n L|^p\right| > \varepsilon\right\} \cap B\right) + \mathbf{P}(\bar{B}) \\
& \leq \sum_{j=0}^{[nT]} \mathbf{P}\left(\left\{\sum_{i=1}^{[nT]} \left||\Delta_i^n(L+Y)|^p - |\Delta_i^n L|^p\right| > \varepsilon\right\} \cap A_c^n(j) \cap B\right) + \frac{\delta}{2} \\
& \leq \sum_{j=0}^{[nT]} \mathbf{P}\left(\left\{\sum_{i \notin J_c^n} \left||\Delta_i^n(L+Y)|^p - |\Delta_i^n L|^p\right| > \frac{\varepsilon}{2}\right\} \cap A_c^n(j) \cap B\right) \\
& + \sum_{j=0}^{[nT]} \mathbf{P}\left(\left\{\sum_{i \in J_c^n} \left||\Delta_i^n(L+Y)|^p - |\Delta_i^n L|^p\right| > \frac{\varepsilon}{2}\right\} \cap A_c^n(j) \cap B\right) + \frac{\delta}{2} \\
& =: D^{(1)}(n, c) + D^{(2)}(n, c) + \frac{\delta}{2}.
\end{aligned} \tag{6.6}$$

In the following two steps we show that for appropriately chosen $c = c(\varepsilon) > 0$ and n big enough, $D^{(1)}(n, c) = 0$ and $D^{(2)}(n, c) < \delta/2$. This will finish the proof.

Step 1. To estimate $D^{(1)}(n, c)$, let $\omega \in B$. Using the elementary inequality $|a^p - b^p| \leq \max\{a, b\}^{p-1} p|a - b|$ which holds for $a, b \geq 0$ and $p \geq 1$ we estimate

$$\begin{aligned}
\sum_{i \notin J_c^n(\omega)} \left||\Delta_i^n(L(\omega) + Y(\omega))|^p - |\Delta_i^n L(\omega)|^p\right| & \leq \sum_{i \notin J_c^n(\omega)} p\left(c + \frac{K'}{n}\right)^{p-1} \left||\Delta_i^n(L(\omega) + Y(\omega))| - |\Delta_i^n L(\omega)|\right| \\
& \leq \sum_{i \notin J_c^n(\omega)} p\left(c + \frac{K'}{n}\right)^{p-1} |\Delta_i^n Y| \\
& \leq \sum_{i \notin J_c^n(\omega)} p\left(c + \frac{K'}{n}\right)^{p-1} \frac{K'}{n} \\
& \leq p\left(c + \frac{K'}{n}\right)^{p-1} K'T,
\end{aligned} \tag{6.7}$$

where we have used that the ‘small’ increments of L are bounded by c . So we have

$$\begin{aligned}
D^{(1)}(n, c) & = \sum_{j=0}^{[nT]} \mathbf{P}\left(\left\{\left|\sum_{i \notin J_c^n} |\Delta_i^n L + \Delta_i^n Y|^p - |\Delta_i^n L|^p\right| > \frac{\varepsilon}{2}\right\} \cap A_c^n(j) \cap B\right) \\
& \leq \sum_{j=0}^{[nT]} \mathbf{P}\left(\left\{p\left(c + \frac{K'}{n}\right)^{p-1} K'T > \frac{\varepsilon}{2}\right\} \cap A_c^n(j) \cap B\right) \\
& \leq \mathbf{P}\left(p\left(c + \frac{K'}{n}\right)^{p-1} K'T > \frac{\varepsilon}{2}\right) = 0
\end{aligned} \tag{6.8}$$

for all $n \geq n_1$ with

$$n_1 := \left\lceil 2K' \left(\frac{2pK'T}{\varepsilon}\right)^{\frac{1}{p-1}} \right\rceil \quad \text{and} \quad c = c(\varepsilon) := \frac{1}{4} \left(\frac{\varepsilon}{2pK'T}\right)^{\frac{1}{p-1}}. \tag{6.9}$$

Step 2. To estimate $D^{(2)}(n, c)$, let $n \geq \frac{K'}{c}$ so that for $\omega \in B$ and $i \in J_c^n(\omega)$ we have

$$\left|\frac{\Delta_i^n Y(\omega)}{\Delta_i^n L(\omega)}\right| \leq \frac{K'}{nc} \leq 1. \tag{6.10}$$

By means of the elementary inequality $(1 + |x|)^p - 1 \leq 3|x|$ which holds for $1 \leq p \leq 2$ and $|x| \leq 1$ this implies

that

$$\begin{aligned}
& \sum_{i \in J_c^n(\omega)} \left| |\Delta_i^n(L(\omega) + Y(\omega))|^p - |\Delta_i^n L(\omega)|^p \right| \\
& \leq \sum_{i \in J_c^n(\omega)} |\Delta_i^n L(\omega)|^p \left[\left(1 + \left| \frac{\Delta_i^n Y(\omega)}{\Delta_i^n L(\omega)} \right| \right)^p - 1 \right] \\
& \leq \sum_{i \in J_c^n(\omega)} |\Delta_i^n L(\omega)|^p 3 \left| \frac{\Delta_i^n Y(\omega)}{\Delta_i^n L(\omega)} \right| \\
& \leq \sum_{i \in J_c^n(\omega)} |\Delta_i^n L(\omega)|^p \frac{3K'}{nc}.
\end{aligned} \tag{6.11}$$

This in turn immediately yields

$$D^{(2)}(n, c) \leq \sum_{j=1}^{[nT]} \mathbf{P} \left(A_c^n(j) \cap \left\{ \sum_{i \in J_c^n} |\Delta_i^n L|^p \frac{3K'}{nc} > \frac{\varepsilon}{2} \right\} \right) \leq \sum_{j=1}^{[nT]} \mathbf{P} \left(A_c^n(j) \cap \bigcup_{i \in J_c^n} \left\{ |\Delta_i^n L|^p > \frac{\varepsilon nc}{6jK'} \right\} \right). \tag{6.12} \quad \boxed{\text{eq:m}}$$

Since all $\Delta_i^n L$, $i = 1, \dots, [nT]$, are i.i.d., and only j of them exceed the threshold c , we can estimate the probability for this event precisely. Indeed, denoting

$$p_n := \mathbf{P}(|\Delta_1^n L| > c) = \mathbf{P}(|L_1| > cn^{1/\alpha}), \tag{6.13}$$

we continue the estimates in [\(6.12\)](#) to get

$$\begin{aligned}
D^{(2)}(n, c) & \leq \sum_{j=1}^{[nT]} \binom{[nT]}{j} \mathbf{P} \left(\left\{ \{|\Delta_i^n L| > c, i = 1, \dots, j\} \cap \{|\Delta_i^n L| \leq c, i = j+1, \dots, [nT]\} \right\} \cap \bigcup_{i=1}^j \left\{ |\Delta_i^n L|^p > \frac{\varepsilon nc}{6jK'} \right\} \right) \\
& = \sum_{j=1}^{[nT]} \binom{[nT]}{j} j \mathbf{P} \left(\left\{ \{|\Delta_i^n L| > c, i = 1, \dots, j\} \cap \{|\Delta_i^n L| \leq c, i = j+1, \dots, [nT]\} \right\} \cap \left\{ |\Delta_1^n L|^p > \frac{\varepsilon nc}{6jK'} \right\} \right) \\
& \leq \sum_{j=1}^{[nT]} \binom{[nT]}{j} j p_n^{j-1} (1-p_n)^{[nT]-j} \mathbf{P} \left(|\Delta_1^n L|^p > \frac{\varepsilon nc}{6jK'} \right).
\end{aligned} \tag{6.14} \quad \boxed{\text{eq:n}}$$

With help of the inequalities [\(5.18\)](#), we obtain the estimates

$$p_n \leq \frac{\tilde{c}}{c^\alpha n} \tag{6.15} \quad \boxed{\text{eq:pp}}$$

and

$$\mathbf{P} \left(|\Delta_1^n L|^p > \frac{\varepsilon nc}{6jK'} \right) = \mathbf{P} \left(|L_1| > \left(\frac{\varepsilon nc}{6jK'} \right)^{1/p} n^{1/\alpha} \right) \leq \frac{\tilde{c}}{n} \left(\frac{\varepsilon nc}{6jK'} \right)^{-\alpha/p} \tag{6.16} \quad \boxed{\text{eq:o}}$$

holding for some $\tilde{c} > 0$, for all $n \geq n_2 \geq \frac{K'}{c}$ and $1 \leq j \leq [nT]$. Combining [\(6.14\)](#) and [\(6.16\)](#), denoting the constant pre-factor by C , and recalling that $\frac{\alpha}{p} \leq 2$, we obtain for $n \geq n_2$ that

$$D^{(2)}(n, c) \leq \frac{C}{p_n n^{1+\alpha/p}} \sum_{j=1}^{[nT]} \binom{[nT]}{j} j^{1+\alpha/p} p_n^j (1-p_n)^{[nT]-j} \leq \frac{C}{p_n n^{1+\alpha/p}} \sum_{j=1}^{[nT]} \binom{[nT]}{j} j^3 p_n^j (1-p_n)^{[nT]-j}. \tag{6.17}$$

The sum in the previous formula represents the third moment of a binomial distribution, and thus can be calculated explicitly. By means of the asymptotic inequality [\(5.18\)](#) we get

$$\begin{aligned}
D^{(2)}(n, c) & \leq C \frac{(nT-2)(nT-1)nT p_n^3 + 3(nT-1)nT p_n^2 + nT p_n}{p_n n^{1+\alpha/p}} \\
& \leq \frac{CT}{n^{\alpha/p}} ((nT p_n)^2 + 3nT p_n + 1).
\end{aligned} \tag{6.18}$$

Now choose $n \geq n_3 \geq n_2$ big enough to ensure that this expression is smaller than $\frac{\delta}{2}$. This completes the proof of Theorem [2.3](#). thm: add tip

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