An explicit description of the Lyapunov exponents of the noisy damped harmonic oscillator

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Abstract

The Lyapunov exponents of the linearization

$$\ddot{x} \ = \ -x + 2\beta\dot{x} + \sigma\xi_t x$$

of a noisy Duffing-van der Pol oscillator are key quantities in the investigation of the stochastic Hopf bifurcation of this system. Considering the white noise case we derive a simple equation exhibiting them explicitly as functions of the fourth moment of the invariant measure of an associated diffusion with drift given by a potential function and additive noise, and, consequently, in terms of hypergeometric functions. This representation leads to different kinds of complete and explicit asymptotic expansions, as well as a rather complete account of global properties of the Lyapunov exponents as functions of β and σ.

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Introduction

The deterministic Duffing-van der Pol oscillator

$$\ddot{x} \;=\; -x + 2\beta\dot{x} - x^3 - x^2\dot{x}$$

exhibits a Hopf bifurcation near a value of the bifurcation parameter β (damping) at which its eigenvalues are purely imaginary. Adding white noise to the system results in splitting the degeneracy. Two different Lyapunov exponents appear and the "stochastic Hopf bifurcation scenario" emerges. Its pattern has been observed in computer simulations by Schenk-Hoppé [12].

As long as the largest of the two exponents as a function of the bifurcation parameter is negative, 0 as a fixed point of the motion is stable. Stability is lost as soon as the top exponent λ_1 crosses the axis and two regimes of different qualitative behavior emerge which depends on the sign of the second Lyapunov exponent λ_2.

Though looking rather appealing in the simulations, only a few of the sketched features have been understood mathematically. See Schenk-Hoppé [12] for an account of this.

Basic to an understanding of this bifurcation scenario are global and local properties of the Lyapunov exponents as functions of the damping and noise parameters which figure in the linearization at 0 of this system. Passing to the coordinates $v = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$, the linearization at 0 of the noisy Duffing-van der

Pol oscillator is given by

$$dv_t \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 2\beta \end{pmatrix} v_t \, dt + \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix} v_t \circ dW_t, \qquad (1)$$

in which $(W_t)_{t\in\mathbb{R}}$ is a two-sided one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. The key to investigate their properties are the Furstenberg-Khasminskii formulas in which the Lyapunov exponents are represented as spatial means in terms of the invariant measures of the projection of (1) onto the unit circle. The infinitesimal generator of this diffusion was asymptotically expanded in σ (for $\sigma \to 0$) as well as the invariant measure to obtain the first terms in an expansion of the exponents as functions of σ in Arnold, Papanicolaou, Wihstutz [5] and later in a more general setting in Pardoux, Wihstutz [10]. In a related paper by Arnold, Eizenberg, Wihstutz [2] a corresponding large noise asymptotics is made in a still more general setting by means of large deviations methods, without, however, yielding more concrete results for the system considered here.

Our aim in writing this paper is to collect as many results on local and global properties of the Lyapunov exponents as functions of noise and damping as possible to support a better understanding of the stochastic Hopf bifurcation than one is able to get from the knowledge of the first term in their expansions for small or large noise. To achieve this, we start again with the Furstenberg-Khasminskii formulas. The key to our analysis is a very simple explicit integral representation of the exponents λ_1 and λ_2 given in section 1 by

$$\begin{aligned} \lambda_1(\beta,\sigma) &= \beta + \frac{\sigma^2}{4} C(\beta,\sigma) \\ \lambda_2(\beta,\sigma) &= \beta - \frac{\sigma^2}{4} C(\beta,\sigma) \end{aligned}$$

where $C(\beta,\sigma)$ is the fourth moment of the invariant measure of a linear diffusion given by

$$dx_t \;=\; -U'(x_t)dt + \sigma dW_t,$$

with potential drift

$$U(x) \;=\; \frac{x^6}{12} - (2\beta^2 - 1)x^2, \quad x \in \mathbb{R}$$

(Theorem 1). In Theorem 2 this representation is seen to lead to an explicit description of λ_1, λ_2 in terms of hypergeometric functions.

In section 2, Theorem 1 is further exploited. In Theorem 3 we give an explicit expansion of λ_1 and λ_2 in Laurent series near $\sigma = \infty$. In Theorem 4, for the case $|\beta| < 1$ the expansion near $\sigma = 0$ starting with the terms exhibited by Pardoux, Wihstutz [10] is completely and explicitly derived form Theorem 1. As a by-product of Theorem 3, it is seen not to converge anywhere except at $\sigma = 0$. In case $|\beta| > 1$, a similar expansion involves some ideas related to large deviations since in this case the invariant measure degenerates as $\sigma \to 0$. The expansion is given in Theorem 5. A 3D plot of $\lambda_1(\beta, \sigma)$ exhibiting the global features of the function is given based on the formula of Theorem 1 again.

Finally, in Theorem 6 of section 3 we show some global properties of λ_1 and λ_2 as functions of the damping parameter β. For σ fixed, λ_1 is increasing on $\mathbb{R}_+$, λ_2 increasing on $\mathbb{R}_-$. Like the function for the deterministic case $\sigma = 0$, λ_1 may decrease on $\mathbb{R}_-$, λ_2 on $\mathbb{R}_+$. For $|\beta| < 1$ fixed, λ_1 is increasing in σ. Finally we show that for each $\sigma > 0$ there is exactly one root of λ_1 as a function of β.

1 Explicit formulas for the Lyapunov exponents

For the convenience of the non specialist reader we shortly recall some basic facts about Lyapunov exponents and the Furstenberg-Khasminskii formula. See also [4] or [1] for an overview of this subject. Consider a linear Stratonovich differential equation

$$dX_t \;=\; A_0 X_t dt + \sum_{j=1}^{m} A_j X_t \circ dW_t^j, \quad X_0 = x \in \mathbb{R}^d. \tag{2}$$

where $A_0, \ldots, A_m \in \mathbb{R}^{d\times d}$ and $W^1, \ldots, W^m$ are independent Wiener processes with tow sided time. Let $(\Phi_t x)_{t\in\mathbb{R}}$ denote the solution of (2) with initial condition $X_0 = x$. By the multiplicate ergodic theorem there exists a random splitting $\mathbb{R}^d = E_1(\omega) \oplus \ldots \oplus E_p(\omega)$ and (deterministic) Lyapunov exponents $\lambda_1 > \ldots > \lambda_p$ such that

$$x \in E_i(\omega) \setminus \{0\} \iff \lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi_t(\omega)x\| \;=\; \lambda_i.$$

For $x \neq 0$ we may decompose the solution of (2) into its radial and angular part by setting $r_t = |X_t|$, $s_t = \frac{X_t}{r_t}$. Using Ito's formula we see that $(s_t)_{t\in\mathbb{R}}$ is a diffusion on the unit sphere S^{d-1} which is generated by the SDE

$$ds_t \;=\; g_{A_0}(s_t)\; dt + \sum_{j=1}^{m} g_{A_j}(s_t) \circ dW_t^j,$$

where we set

$$g_A(s) \;=\; As - \langle s, As\rangle \qquad (A \in \mathbb{R}^{d\times d}, s \in S^{d-1}).$$

For the radial part $(r_t)_{t\in\mathbb{R}}$ we obtain

$$\begin{aligned} r_t &= r_0\; \exp\left(\int_0^t \langle s_\tau, A_0 s_\tau\rangle\; d\tau + \sum_{j=1}^{m}\int_0^t \langle s_\tau, A_j s_\tau\rangle \circ dW_\tau^j\right) \\ &= r_0\; \exp\left(\int_0^t [h_{A_0}(s_\tau) + \sum_{j=1}^{m} k_{A_j}(s_\tau)]\; d\tau + \sum_{j=1}^{m}\int_0^t \langle s_\tau, A_j s_\tau\rangle\; dW_\tau^j\right), \end{aligned}$$

where

$$\begin{aligned} h_A(s) &= \langle s, As\rangle, \\ k_A(s) &= \frac{1}{2}\langle (A+A^\star)s, As\rangle - \langle s, As\rangle^2, \qquad A \in \mathbb{R}^{d\times d}, s \in S^{d-1}. \end{aligned}$$

Since the Ito integrals in the formula for r_t are of order $\sqrt{t}$ we have for large t:

$$\frac{1}{t}\log r_t \;\sim\; \frac{1}{t}\int_0^t [h_{A_0}(s_\tau) + \sum_{j=1}^{m} k_{A_j}(s_\tau)]\; d\tau.$$

Suppose now that the following hypoellipticity condition is fulfilled:

$$\dim \mathcal{L}(g_{A_0}, \ldots, g_{A_m})(s) \;=\; d-1 \quad \text{for all } s \in S^{d-1}, \tag{3}$$

where $\mathcal{L}(g_{A_0}, \ldots, g_{A_m})(s)$ denotes the Lie algebra generated by the these vector fields at the point s. Then it is known that the Oseledets spaces E_i possess smooth densities (see [7]). Hence x possesses almost surely a nonvanishing component in E_1 and we have

$$\lambda_1 \;=\; \lim_{t\to\infty}\frac{1}{t}\log r_t \quad \text{a.s.}$$

Furthermore it is known that under the hypoellipticity condition (3) the diffusion $(s_t)_{t\in\mathbb{R}}$ possesses a unique invariant smooth distribution whose density p solves the adjoint Fokker Planck equation

$$L^\star p \;=\; 0,$$

where

$$L \;=\; g_{A_0} + \frac{1}{2}\sum_{j=1}^{m} g_{A_j}^2$$

denotes the generator of the diffusion (s_t) and $L^\star$ is the formal adjoint of L. So the ordinary ergodic theorem tells us that

$$\lambda_1 \;=\; \lim_{t\to\infty}\frac{1}{t}\log r_t \;=\; \int_{S^{d-1}} [h_{A_0}(s) + \sum_{j=1}^{m} k_{A_j}(s)]\; p(s)ds. \tag{4}$$

This is the Furstenberg Khasminskii formula for λ_1. By inverting time one obtains a similar formula for the lowest Lyapunov exponent λ_p. Finally one has the trace formula

$$\sum_{i=1}^{p} d_i\lambda_i \;=\; \mathrm{trace}(A_0), \tag{5}$$

where $d_1,\ldots,d_p$ denote the (deterministic) dimensions of the Oseledets spaces. Given the fact that in our case we are in a two dimensional setting, the invariant measure of the angular part of the adjoint Fokker-Planck equation is easily computed. Calculations of this type were executed for example in Nishioka [9], Guo [16]. In order to simplify the integral in (4) we use a decomposition of q by splitting off a term belonging to the range of the infinitesimal generator, to obtain a very simple explicit representation of the Lyapunov exponents in terms of different types of hypergeometric functions.

In the sequel denote $A = \begin{pmatrix} 0 & 1 \\ -1 & 2\beta \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}$ for $\beta, \sigma \in \mathbb{R}$. The linearization of the Duffing-van der Pol oscillator we consider at the fixed point 0 is given by

$$dv_t \;=\; Av_t dt + Bv_t \circ dW_t. \tag{6}$$

The Furstenberg-Khasminskii formula for the top Lyapunov exponent is given by

$$\lambda_1 \;=\; \int_{S^1} [h_A(s) + k_B(s)]\; \rho(ds), \tag{7}$$

where ρ is the invariant measure with respect to the angular component of (6). Note that is sufficient to compute only the top Lyapunov exponent λ_1 since the Lyapunov exponents λ_1, λ_2 are related by the trace formula

$$\lambda_1 + \lambda_2 \;=\; \mathrm{trace}(A) \;=\; 2\beta.$$

In our particular case, the functionals become

$$\begin{aligned} h_A(s) &= 2\beta s_2^2, \quad g_B(s) \;=\; \begin{pmatrix} -\sigma s_1^2 s_2 \\ \sigma s_1^3 \end{pmatrix}, \quad g_A(s) \;=\; \begin{pmatrix} s_2 - 2\beta s_1 s_2^2 \\ -s_1 + 2\beta s_2 s_1^2 \end{pmatrix}, \\ k_B(s) &= \frac{1}{2}\sigma^2 s_1^2 - \sigma^2 s_1^2 s_2^2 \\ &= \frac{1}{2}\sigma^2(s_1^4 - s_1^2 s_2^2), \qquad s \in S^1. \end{aligned}$$

In order to further compute λ_1 we introduce coordinates on S^1 by setting

$$s \;=\; \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \phi \in [-\frac{\pi}{2}, \frac{3}{2}\pi[.$$

In these coordinates, we have new functionals

$$\bar{h}_A(\phi) \;=\; 2\beta \sin^2\phi, \tag{8}$$

$$\bar{g}_B(\phi) \;=\; \begin{pmatrix} -\sigma\cos^2\phi\sin\phi \\ \sigma\cos^3\phi \end{pmatrix} \quad \bar{g}_A(\phi) \;=\; \begin{pmatrix} \sin\phi - 2\beta\cos\phi\sin^2\phi \\ -\cos\phi + 2\beta\sin\phi\cos^2\phi \end{pmatrix} \tag{9}$$

$$\bar{k}_B(\phi) \;=\; \frac{1}{2}\sigma^2(\cos^4\phi - \sin^2\phi\cos^2\phi), \qquad \phi \in [-\frac{\pi}{2}, \frac{3}{2}\pi[, \tag{10}$$

and the Furstenberg-Khasminskii formula

$$\lambda_1 \;=\; \int_{-\pi/2}^{\pi/2} (2\beta\sin^2\phi + \frac{1}{2}\sigma^2[\cos^4\phi - \sin^2\phi\cos^2\phi])\; \bar{\rho}(d\phi),$$

where $\bar{\rho}$ denotes the corresponding image measure of ρ under the mapping $\phi \longmapsto \binom{\cos\phi}{\sin\phi}$. In deriving (1) note that (7) is actually a spatial mean on P^1, the one-dimensional projective space, which is expressed in (8) – (10) by the fact that the functionals have period π. To determine the invariant measure $\bar{\rho}$ we have to solve the adjoint Fokker-Planck equation for the angular part of (6). To do this, we pass to the coordinate $\phi \in [-\pi/2, \pi/2[$ as well. We obtain the stochastic differential equation

$$d\phi_t \;=\; -\frac{1}{\sin\phi_t} ds_1(t) \;=\; (-1 + 2\beta \sin\phi_t \cos\phi_t)dt + \sigma \cos^2\phi_t \circ dW_t \qquad (11)$$

with periodic boundary conditions at $-\pi/2, \pi/2$.

The infinitesimal generator of the corresponding diffusion on $[-\pi/2, \pi/2[$ is given by

$$Lf(\phi) \;=\; b(\phi) + \frac{1}{2}cc'(\phi)f'(\phi) + \frac{1}{2}c^2(\phi)f''(\phi), \qquad (12)$$

where

$$\begin{aligned} b(\phi) &= -1 + 2\beta\cos\phi\sin\phi, \\ c(\phi) &= \sigma\cos^2\phi, \end{aligned}$$

$\phi \in [-\pi/2, \pi/2[,\ f \in \mathcal{C}^2([-\pi/2, \pi/2[)$ with periodic boundary conditions. In terms of the trigonometric functions we have

$$Lf(\phi) \;=\; ([-1+2\beta\sin\phi\cos\phi] - \sigma^2\sin\phi\cos^3\phi)f'(\phi) + \frac{1}{2}\sigma^2\cos^4\phi f''(\phi). \quad (13)$$

Since for the vector fields generating the diffusion (11) on $[-\pi/2, \pi/2[$ Hörmander's hypoellipticity condition is fulfilled, the invariant measure $\bar{\rho}$ possesses a $\mathcal{C}^\infty$-density p. To determine p, we then have to solve the ordinary differential equation with periodic boundary conditions on $[-\pi/2, \pi/2[$ given by

$$-(\frac{1}{2}c^2p)' + (b + \frac{1}{2}cc')p \;=\; \alpha$$

with a suitable constant α, to be determined from the boundary conditions and norming. This leads to

$$p' \;=\; \left(\frac{2b}{c^2} - \frac{c'}{c}\right)p + \frac{2\alpha}{c^2}. \qquad (14)$$

By variation of constants the solutions of (14) is easily seen to be given by

$$p(\phi) \;=\; G(\phi)\frac{\displaystyle\int_{-\pi/2}^{\phi} \frac{2}{c^2 G(\theta)}\; d\theta}{\displaystyle\int_{-\pi/2}^{\pi/2} G(\phi) \int_{-\pi/2}^{\phi} \frac{2}{c^2 G(\theta)}\; d\theta\; d\phi},$$

where

$$G(\phi) \;=\; \frac{c(0)}{c(\phi)}\; \exp\left(-\frac{2}{\sigma^2}[\tan\phi + \frac{\tan^3\phi}{3} - \beta\tan^2\phi]\right),$$

$\phi \in [-\pi/2, \pi/2[$ (see also Nishioka [9]).

So, finally, by putting

$$q(\phi) \;=\; 2\beta\sin^2\phi + \frac{1}{2}\sigma^2[\cos^4\phi - \sin^2\phi\cos^2\phi], \quad \phi \in [-\pi/2, \pi/2],$$

we obtain the formula

$$\lambda_1 \;=\; \int_{-\pi/2}^{\pi/2} q(\phi)p(\phi)\; d\phi.$$

We shall next considerably simplify the preceding formula by splitting off q a term contained in $rg(L)$. This leads to the following formula, in which we set $\gamma = \frac{\sigma^2}{2}$, as in the rest of the paper.

Theorem 1 *Set*

$$\gamma \;=\; \frac{\sigma^2}{2}.$$

Then we have

$$\lambda_1 \;=\; \int_{-\pi/2}^{\pi/2} \tan\phi\; p(\phi)\; d\phi \;=\; \beta + \frac{\gamma}{2}\int_0^{\infty} v\; q(v)\; dv,$$

where

$$q(v) \;=\; \frac{\displaystyle\frac{1}{\sqrt{v}}\exp(-v(1-\beta^2) - \frac{v^3\gamma^2}{12})}{\displaystyle\int_0^{\infty} \frac{1}{\sqrt{u}}\exp(-u(1-\beta^2) - \frac{u^3\gamma^2}{12})\; du}.$$

Proof: Consider

$$\begin{aligned} f(\phi) &= -\ln\cos\phi, \quad \phi \in]-\pi/2, \pi/2[, \\ f(-\pi/2) &= \infty. \end{aligned}$$

Then formally

$$\begin{aligned} Lf(\phi) &= (-1 + 2\beta\sin\phi\cos\phi - \sigma^2\cos^3\phi\sin\phi)\tan\phi + \frac{\sigma^2}{2}\cos^4\phi\frac{1}{\cos^2\phi} \\ &= -\tan\phi + q(\phi), \tan\phi - q_-(\phi), \quad \phi \in [-\pi/2, \pi/2[, \end{aligned}$$

hence

$$\begin{aligned} \lambda_1 &= \int_{-\pi/2}^{\pi/2} q(\phi)\ p(\phi)\ d\phi \\ &= \int_{-\pi/2}^{\pi/2} \tan\phi\ p(\phi)\ d\phi. \end{aligned} \tag{15}$$

Though formal, this calculation can be made precise by approximating f suitably by $\mathcal{C}^\infty$-functions. Next, we substitute $\tan\phi = s,\ \tan\theta = t$ to obtain for the numerator of (15) the expression

$$\int_{-\infty}^{\infty} s \int_{-\infty}^{s} \exp\big(-[s + s^3/3 - \beta s^2 - t - t^3/3 + \beta t^2]\big)\ dt\ ds.$$

We replace t with $u + s$. Then the expression in the exponent simplifies to give

$$\begin{aligned} &[s + s^3/3 - \beta s^2 - (u+s) - (u+s)^3/3 + \beta(u+s)^2)] \\ &= -u\left(s + \frac{u - 2\beta}{2}\right)^2 - \frac{u^3}{12} - u(1-\beta^2). \end{aligned}$$

Hence the numerator of (15) becomes

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{0}^{\infty} s\ \exp\left(-\Big[u(s - \frac{u+2\beta}{2})^2 + \frac{u^3}{12} + u(1-\beta^2)\Big]\right)\ du\ ds \\ &= \sqrt{2\pi\cdot\frac{2}{\gamma}}\int_0^\infty \frac{1}{\sqrt{u}}\ \left(\frac{u+2\beta}{2}\right)\ \exp\left(-\frac{1}{\gamma}\Big[\frac{u^3}{12} + u(1-\beta^2)\Big]\right)\ du. \end{aligned} \tag{16}$$

A simpler calculation for the denominator combines with (16) to give

$$\lambda_1 = \frac{\displaystyle\int_0^\infty \frac{1}{\sqrt{u}} \left(\frac{u+2\beta}{2}\right) \exp\left(-\frac{1}{\gamma}\left[\frac{u^3}{12}+u(1-\beta)^2\right]\right) du}{\displaystyle\int_0^\infty \frac{1}{\sqrt{u}} \exp\left(-\frac{1}{\gamma}\left[\frac{u^3}{12}+u(1-\beta)^2\right]\right) du} \tag{17}$$

$$= \beta + \frac{\gamma}{2}\int_0^\infty v\, q(v)\, dv,$$

where the last line follows after substituting $v = \frac{u}{\gamma}$. □

The integral representation of the top Lyapunov exponent λ_1 obtained in Theorem 1 is simple enough to allow a representation in terms of hypergeometric functions. Since these functions are implemented in many numerical subroutine libraries we were able to give a 3D plot of the function $(\beta,\gamma) \longmapsto \lambda_1(\beta,\gamma)$, using the built-in function `HypergeometricPFQ` of Mathematica (see Figure 1).

For $k, l \in \mathbb{N}$, $a_1, \ldots, a_k, b_1, \ldots, b_l \in \mathbb{R} \setminus \mathbb{Z}^-$ we recall the hypergeometric function

$${}_kF_l(a_1,\ldots,a_k;b_1,\ldots,b_l;x) = \sum_{n=0}^\infty \frac{(a_1)_n \ldots (a_k)_n}{(b_1)_n \ldots (b_l)_n} \frac{x^n}{n!}, \quad x \in I,$$

where $(a)_n = \prod_{i=0}^{n-1}(a+i)$, $n \geq 0$, and $I \subset \mathbb{R}$ is an interval centered at 0. We do not discuss their radius of convergence, but remark that the types we shall be using here converge on $\mathbb{R}$. In this notation, Theorem 1 yields the following explicit formula, in which Γ is Euler's Gamma function.

Theorem 2 *Assume* $\gamma = \frac{\sigma^2}{2} > 0$ *and let* $\alpha = \frac{4(\beta^2-1)^3}{9\gamma^2}$. *Then for* $|\beta| \neq 1$, *setting*

$$\begin{aligned} G(\beta,\gamma) \quad = \quad & \Big[\ \gamma^{4/3}\ 2\cdot 3^{1/3}\ \Gamma(\tfrac{1}{2})^2\ {}_1F_2(\tfrac{1}{2};\tfrac{1}{3},\tfrac{2}{3};\alpha) \\ & -\gamma^{2/3}(\beta^2-1)\cdot 2\cdot 3^{1/6}\ \Gamma(-\tfrac{1}{3})\Gamma(\tfrac{2}{3})\ {}_1F_2(\tfrac{5}{6};\tfrac{2}{3},\tfrac{4}{3};\alpha) \\ & +(\beta^2-1)^2\ 2^{1/3}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{6})\ {}_1F_2(\tfrac{5}{6};\tfrac{2}{3},\tfrac{4}{3};\alpha)\ \Big] \Big/ \\ & \Big[\ \gamma^{4/3}\ 2^{1/3}\ \Gamma(\tfrac{1}{6})\Gamma(\tfrac{1}{2})\ {}_1F_2(\tfrac{1}{6};\tfrac{1}{3},\tfrac{2}{3};\alpha) \\ & +\gamma^{2/3}(\beta^2-1)\cdot 2\cdot 3^{1/3}\ \Gamma(\tfrac{1}{2})^2{}_1F_2(\tfrac{1}{2};\tfrac{2}{3},\tfrac{4}{3};\alpha) \\ & +(\beta^2-1)^2\ 6^{2/3}\Gamma(\tfrac{5}{6})\Gamma(\tfrac{1}{2})\ {}_1F_2(\tfrac{5}{6};\tfrac{4}{3},\tfrac{5}{3};\alpha)\ \Big], \end{aligned}$$

we have

$$\begin{aligned} \lambda_1 &= \beta + \frac{\gamma^{1/3}}{2} G(\beta,\gamma), \\ \lambda_2 &= \beta - \frac{\gamma^{1/3}}{2} G(\beta,\gamma). \end{aligned}$$

For $|\beta| = 1$ *we have*

$$\begin{aligned} \lambda_1 &= \beta + \frac{\gamma^{1/3}}{2}\cdot 12^{1/3}\ \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})}, \\ \lambda_2 &= \beta - \frac{\gamma^{1/3}}{2}\cdot 12^{1/3}\ \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})}. \end{aligned}$$

Proof: Let us give some arguments for the numerator function. We have by dominated convergence

$$\begin{aligned} & \int_0^\infty \sqrt{v}\ \exp\left(-v((1-\beta^2) - \frac{v^3\gamma^2}{12}\right)\ dv \qquad (18) \\ &= \frac{1}{3}\left(\frac{12}{\gamma^2}\right)^{1/2} \sum_{n=0}^{\infty}\left((\beta^2-1)\cdot\frac{12^{1/3}}{\gamma^{2/3}}\right)^n \frac{1}{n!}\int_0^\infty v^{n/3-1/2}\ \exp(-v)\ dv \\ &= \frac{1}{3}\left(\frac{12}{\gamma^2}\right)^{1/2} \sum_{n=0}^{\infty}\left((\beta^2-1)\cdot\frac{12^{1/3}}{\gamma^{2/3}}\right)^n \frac{1}{n!}\ \Gamma\left(\frac{n}{3}+\frac{1}{2}\right). \end{aligned}$$

For $|\beta| = 1$ this and a similar formula for the denominator yield the result.

For $|\beta| \neq 1$, $j = 0, 1, 2$ observe that the terms of order $n = j \pmod 3$ yield the jth term in the numerator of $G(\beta, \gamma)$, modulo a common multiplicative constant. A similar argument for the denominator gives the result in this case. □

2 Asymptotic expansions of the Lyapunov exponents as functions of the noise parameter

Theorem 2 will imply in particular that the Lyapunov exponents as functions of $z = \gamma^{1/3}$ have Laurent series with just one term of negative order 1 at $z = \infty$. We shall confirm this claim first and give the expansion at $z = \infty$. As a consequence, λ_1 and λ_2 cannot be analytic at $z = 0$. Hence, if there is an asymptotic expansion at $\gamma = 0$, it cannot converge at any point except $\gamma = 0$. Nonetheless, expansions of this type exist and posses some interest for example for the theory of bifurcations of the noisy Duffing-van der Pol oscillator (see Arnold [1], Arnold, Namachchivaya, Schenk-Hoppé [3]). Their first terms have been calculated in a more general setting by Auslender, Mil'stejn [6] (see also Kozin, Promodrou [8]), Arnold, Papanicolaou, Wihstutz [5] and Pardoux, Wihstutz [10]. In these papers a method was used which is based on a formal expansion of the infinitesimal generator and the density of the invariant measure, without taking into account the latter explicitly. We shall show now, how Theorems 1 and 2 yield any term of the asymptotic expansions at $\gamma = 0$ rather easily. The arguments given here are straightforward in case $|\beta| \leq 1$, and involve an elementary consideration in the spirit of large deviations in case $|\beta| > 1$.

We state the following Lemma giving an explicit solution for a recursive equation.

Lemma 1 *Let $(a_n)_{n \in \mathbb{N}_0}$, $(b_n)_{n \in \mathbb{N}_0}$ be sequences of real numbers. Suppose that $(c_n)_{n \in \mathbb{N}_0}$ is recursively defined by*

$$\sum_{k=0}^{n} c_k \, a_{n-k} \;=\; b_n, \quad n \geq 0,$$

and that $a_0 \neq 0$. *Then we have*

$$\begin{aligned} c_0 &= \frac{b_0}{a_0}, \\ c_n &= \sum_{m=1}^{n} \frac{(-1)^{m+1}}{a_0^m} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\ldots+l_m=n}} a_{l_1}\ldots a_{l_{m-1}} \left(b_{l_m} - \frac{b_0}{a_0} a_{l_m} \right) \quad (n \geq 1). \end{aligned}$$

Proof: By passing to $\frac{a_n}{a_0}$, $n \geq 0$, respectively $\frac{b_n}{a_0}$, $n \geq 0$, we may assume $a_0 = 1$. Now the formula can be verified by elementary algebraic operations. □

One can now use Lemma 1 to get an analytic expansion of the Lyapunov exponents at ∞.

Theorem 3 *Let* $\gamma > 0$. *Then we have*

$$\lambda_{1/2} = \beta \pm \frac{12^{1/3}}{2}\gamma^{1/3} \sum_{n=0}^{\infty} \left((\beta^2 - 1) \cdot \frac{12^{1/3}}{\gamma^{2/3}} \right)^n c_n,$$

where

$$\begin{aligned} c_0 &= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})}, \\ c_n &= \sum_{m=1}^{n} \frac{(-1)^{m+1}}{\Gamma(\frac{1}{6})^m} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\ldots+l_m=n}} \prod_{i=1}^{m-1} \frac{\Gamma(\frac{l_i}{3}+\frac{1}{6})}{l_i!} \left[\frac{\Gamma(\frac{l_m}{3}+\frac{1}{2})}{l_m!} - \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})} \right] \\ & (n \geq 1). \end{aligned}$$

Proof: Starting with the expansions

$$\begin{aligned} & \int_0^\infty \sqrt{v}\ \exp\left(-v((1-\beta^2) \cdot \frac{v^3\gamma^2}{12} \right)\ dv \\ &= \frac{1}{3} \left(\frac{12}{\gamma^2} \right)^{1/2} \sum_{n=0}^{\infty} \left((\beta^2 - 1) \cdot \frac{12^{1/3}}{\gamma^{2/3}} \right)^n \frac{1}{n!}\ \Gamma\left(\frac{n}{3} + \frac{1}{2} \right), \end{aligned}$$

$$\begin{aligned} & \int_0^\infty \frac{1}{\sqrt{v}}\ \exp\left(-v((1-\beta^2) - \frac{v^3\gamma^2}{12} \right)\ dv \\ &= \frac{1}{3} \left(\frac{12}{\gamma^2} \right)^{1/6} \sum_{n=0}^{\infty} \left((\beta^2 - 1) \cdot \frac{12^{1/3}}{\gamma^{2/3}} \right)^n \frac{1}{n!}\ \Gamma\left(\frac{n}{3} + \frac{1}{6} \right), \end{aligned}$$

which were derived in the proof of Theorem 2, we take the quotient in order to describe λ_1, λ_2. To find the coefficients of the (absolutely convergent) quotient series, we then obviously have to apply Lemma 1 with $a_n = \frac{\Gamma(\frac{n}{3}+\frac{1}{6})}{n!}$, $b_n = \frac{\Gamma(\frac{n}{3}+\frac{1}{2})}{n!}$, $n \geq 0$. This immediately yields our formula. □

Corollary 1 *The asymptotic expansions of λ_1, λ_2 in $\gamma = \frac{\sigma^2}{2}$ at $\gamma = 0$ do not converge at any point γ except $\gamma = 0$.*

Proof: Theorem 3 states that λ_1 and λ_2 posses Laurent series in $\gamma^{1/3}$ at ∞ with just one term of negative order 1. Hence 0 is an essential singularity for the functions in $\gamma^{1/3}$, hence also in γ. □

Let us now deal with the asymptotic expansions in γ near $\gamma = 0$. We first treat the case $|\beta| < 1$ in which Theorem 2 yields a simple result.

Theorem 4 *Suppose $|\beta| < 1$. Let*

$$\begin{aligned} c_0 &= 2, \\ c_n &= 12 \sum_{m=1}^{n} (-1)^{m+1} \sum_{\substack{l_1, \ldots, l_m \geq 1 \\ l_1 + \ldots + l_m = n}} \prod_{i=1}^{m} \frac{(6l_i)!}{(3l_i)! l_i!} \cdot l_m \quad (n \geq 1). \end{aligned}$$

Then the formal (but not convergent) asymptotic expansion of λ_1, λ_2 in $\gamma = \frac{\sigma^2}{2}$ at $\gamma = 0$ is given by

$$\lambda_{1/2} = \beta \pm \frac{\gamma}{8(1-\beta^2)} \sum_{n=0}^{\infty} c_n \left(\frac{\gamma^2}{3 \cdot 2^8 (\beta^2 - 1)^3} \right)^n$$

Proof: We start with the formula of Theorem 1. By expanding $\exp(-\frac{v^3\gamma^2}{12})$ this time, up to order n, we obtain the formula

$$\begin{aligned} &\int_0^\infty \sqrt{v}\ \exp\left(-v(1-\beta^2) - \frac{v^3\gamma^2}{12}\right)\ dv \qquad (19) \\ &= \frac{1}{\sqrt{1-\beta^2}} \int_0^\infty \sqrt{u}\ \exp\left(-u - u^3 \frac{\gamma^2}{12(1-\beta^2)^3}\right)\ du \\ &= \frac{1}{\sqrt{1-\beta^2}^3} \sum_{k=0}^{n} \left(\frac{\gamma^2}{12(\beta^2-1)^3}\right)^k \cdot \frac{1}{k!} \int_0^\infty u^{3k+1/2} \mathrm{e}^{-u}\ du + o(\gamma^{2n+1}) \\ &= \frac{1}{\sqrt{1-\beta^2}^3} \sum_{k=0}^{n} \left(\frac{\gamma^2}{12(\beta^2-1)^3}\right)^k \cdot \frac{1}{k!}\ \Gamma(3k+3/2) + o(\gamma^{2n+1}). \end{aligned}$$

The companion expansion for the denominator is given by

$$\int_0^\infty \frac{1}{\sqrt{v}} \, \exp\left(-v(1-\beta^2) - \frac{v^3\gamma^2}{12}\right) \, dv \tag{20}$$
$$= \frac{1}{\sqrt{1-\beta^2}} \, \sum_{k=0}^{n} \left(\frac{\gamma^2}{12(\beta^2-1)^3}\right)^k \cdot \frac{1}{k!} \, \Gamma(3k+1/2) + o(\gamma^{2n+1}).$$

Now

$$\Gamma\left(3n+\frac{1}{2}\right) = \frac{(6n)!}{(3n)!2^{6n}} \sqrt{\pi}, \quad n \geq 0.$$

Hence (19) and (20) may be written in the alternative form

$$\frac{\sqrt{\pi}}{\sqrt{1-\beta^2}^3} \, \sum_{k=0}^{n} \left(\frac{\gamma^2}{3\cdot 2^8(\beta^2-1)^3}\right)^k \frac{(6k)!}{(3k)!k!} + o(\gamma^{2n+1}). \tag{21}$$

$$\frac{\sqrt{\pi}}{4\sqrt{1-\beta^2}} \, \sum_{k=0}^{n} \left(\frac{\gamma^2}{3\cdot 3^8(\beta^2-1)^3}\right)^k \frac{(6k+2)!}{(3k+1)!k!} + o(\gamma^{2n+1}). \tag{22}$$

Hence we have to apply Lemma 1 with

$$a_n = \frac{(6n)!}{(3n)!n!}, \quad b_n = \frac{(6n+2)!}{(3n+1)!n!}, \quad n \geq 0.$$

This yields

$$\begin{aligned} c_0 &= 2 \quad \text{and} \\ c_n &= \sum_{m=1}^{n} (-1)^{m+1} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\ldots+l_m=n}} \prod_{i=1}^{m} \frac{(6l_i)!}{(3l_i)!l_i!} \cdot [2(6l_m+1)-2] \\ &= 12 \sum_{m=1}^{n} (-1)^{m+1} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\ldots+l_m=n}} \prod_{i=1}^{m} \frac{(6l_i)!}{(3l_i)!l_i!} \cdot l_m \qquad (n \geq 1). \end{aligned}$$

This is the claimed formula. □

Remarks: 1. The divergence of the series associated with λ_1, λ_2 is exhibited by showing the first few terms in the expansion already. We have

$$\lambda_{1/2} \;=\; \beta \pm \frac{\gamma}{4(1-\beta^2)} \mp \frac{15\gamma^3}{64(1-\beta^2)^4} \pm \frac{1695\gamma^5}{1024(1-\beta^2)^7} \mp \frac{59025\gamma^7}{2084(1-\beta^2)^{10}} + O(\gamma^9).$$

The first two terms in this expansion were given by Pardoux, Wihstutz [10] (see also Arnold [1]).

2. The formula for λ_1, λ_2 in case $|\beta| = 1$ given in Theorem 2 indicates that there is no asymptotic expansion at $\gamma = 0$ in this case. Let us now turn to the case $|\beta| > 1$. Here, as $\gamma \to 0$, both numerator and denominator of the formula of Theorem 1 explode. Hence we need another type of argument.

Theorem 5 *Assume $|\beta| > 1$. Then the formal (but not convergent) asymptotic expansion of λ_1, λ_2 in $\gamma = \frac{\sigma^2}{2}$ at $\gamma = 0$ is given by*

$$\lambda_{1/2} \;=\; \beta \pm \sum_{n=0}^{\infty} c_n (\beta^2 - 1)^{-(3n-1)/2} \, \gamma^n, \tag{23}$$

where the c_n can be computed as follows:
There exist a $\mathcal{C}^\infty$ function ϕ defined on a neighborhood of $x_0 = \sqrt{2}$ whose derivatives at x_0 can be computed via the recursion

$$\phi(x_0) \;=\; \sqrt{2}, \quad \phi'(x_0) \;=\; \frac{1}{2\sqrt{2}}, \quad \phi'' \;=\; -\frac{4(4+6\phi^2+\phi^4)}{3\phi^3(2+\phi^2)^3} \tag{24}$$

and $\mathcal{C}^\infty$ functions $f, g : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$f^{(n)}(0) \;=\; \frac{1}{2^n}(\phi^2\phi')^{(2n)}(x_0), \quad g^{(n)}(0) \;=\; \frac{1}{2^n}\phi^{(2n+1)}(x_0) \tag{25}$$

such that

$$c_n \;=\; \frac{1}{2n!} \left(\frac{d}{dt}\right)^n \frac{f(t)}{g(t)}\bigg|_{t=0}, \qquad n \geq 0.$$

Proof: For $\gamma > 0$ we have

$$\lambda_{1/2} \;=\; \beta \pm \frac{\gamma}{2} \frac{\displaystyle\int_0^\infty \sqrt{v}\mathrm{e}^{-(\frac{v^3\gamma^2}{12} - v(\beta^2-1))}\, dv}{\displaystyle\int_0^\infty \frac{1}{\sqrt{v}}\mathrm{e}^{-(\frac{v^3\gamma^2}{12} - v(\beta^2-1))}\, dv}$$

(Theorem 1). Substituting $v = x^2 \frac{\sqrt{\beta^2-1}}{\gamma}$ we get

$$\begin{aligned} \lambda_{1/2} &= \beta \pm \frac{1}{2}(\beta^2-1)^{1/2} \frac{\displaystyle\int_0^\infty x^2 \mathrm{e}^{-\frac{1}{\gamma}(\beta^2-1)^{3/2}(\frac{x^6}{12}-x^2)}\, dx}{\displaystyle\int_0^\infty \mathrm{e}^{-\frac{1}{\gamma}(\beta^2-1)^{3/2}(\frac{x^6}{12}-x^2)}\, dx} \\ &= \beta \pm \frac{1}{2}(\beta^2-1)^{1/2} \frac{\displaystyle\int_0^\infty x^2 \mathrm{e}^{-\frac{1}{t}H(x)}\, dx}{\displaystyle\int_0^\infty \mathrm{e}^{-\frac{1}{t}H(x)}\, dx}, \end{aligned}$$

where $t = \frac{\gamma}{(\beta^2-1)^{3/2}}$ and

$$H(x) = \frac{x^6}{12} - x^2 + \frac{4}{3}.$$

For $\gamma = 0$ (deterministic case) we have of course $\lambda_{1/2} = \beta \pm \sqrt{\beta^2-1}$ (the eigenvalues of A). So, setting

$$\begin{aligned} \rho_t(dx) &= \frac{1}{Z_t}\mathrm{e}^{-\frac{1}{t}H(x)}dx \quad \text{where } Z_t = \int_0^\infty \mathrm{e}^{-\frac{1}{t}H(x)}\, dx \text{ for } \gamma > 0, \\ \rho_0 &= \delta_{\sqrt{2}}, \end{aligned}$$

we have

$$\lambda_{1/2} = \beta \pm \frac{1}{2}\sqrt{\beta^2-1}\int_{\mathbb{R}^+} x^2\, \rho_t(dx), \quad \gamma \geq 0.$$

So we have to study the behavior of $\int_{\mathbb{R}^+} x^2 \rho_t(dx)$ for $t \to 0$. The potential H has a global minimum at $x_0 = \sqrt{2}$ where $H(x_0) = H'(x_0) = 0$ and $H''(x_0) > 0$. Therefore we can find a neighborhood U of x_0, $\epsilon > 0$, and a strictly increasing $\mathcal{C}^\infty$ function $\phi :]x_0-\epsilon, x_0+\epsilon[\to U$, such that

$$H \circ \phi(x) = \frac{(x-x_0)^2}{2}, \quad x \in]x_0-\epsilon, x_0+\epsilon[.$$

This is seen as follows. Since H has a root of order two at $x = x_0$, H can be written as $H(x) = (x-x_0)^2 G(x)$ where G is a polynomial with $G(x_0) > 0$. Now define ϕ by $\phi^{-1}(x) = x_0 + (x-x_0)\sqrt{2G(x)}$ in some neighborhood of x_0.

Using the coordinate transformation $x = \phi(y)$ on U, we have for $t > 0$

$$\begin{aligned}\int_{\mathbb{R}^+} x^2 \ \rho_t(dx) &= \frac{\displaystyle\int_{x_0-\epsilon}^{x_0+\epsilon} \phi(x)^2 \phi'(x) \mathrm{e}^{-\frac{(x-x_0)^2}{2t}} \ dx + \int_{\mathbb{R}^+ \setminus U} x^2 \mathrm{e}^{-\frac{1}{t}H(x)} \ dx,}{\displaystyle\int_{x_0-\epsilon}^{x_0+\epsilon} \phi'(x) \mathrm{e}^{-\frac{(x-x_0)^2}{2t}} \ dx + \int_{\mathbb{R}^+ \setminus U} \mathrm{e}^{-\frac{1}{t}H(x)} \ dx} \\ &= \frac{\mathbb{E}(\phi(B_t)^2 \phi'(B_t)) + r_1(t)}{\mathbb{E}(\phi'(B_t)) + r_2(t)},\end{aligned}$$

where $(B_t)_{t \geq 0}$ is a Brownian motion starting at x_0 and the remainder terms r_1, r_2 are given by

$$\begin{aligned} r_1(t) &= \frac{1}{\sqrt{2\pi t}} \left(\int_{\mathbb{R}^+ \setminus \ ]x_0-\epsilon, x_0+\epsilon[} \phi(x)^2 \phi'(x) \mathrm{e}^{-\frac{(x-x_0)^2}{2t}} \ dx + \int_{\mathbb{R}^+ \setminus U} x^2 \mathrm{e}^{-\frac{1}{t}H(x)} \ dx \right), \\ r_2(t) &= \frac{1}{\sqrt{2\pi t}} \left(\int_{\mathbb{R}^+ \setminus \ ]x_0-\epsilon, x_0+\epsilon[} \phi'(x) \mathrm{e}^{-\frac{(x-x_0)^2}{2t}} \ dx + \int_{\mathbb{R}^+ \setminus U} \mathrm{e}^{-\frac{1}{t}H(x)} \ dx \right). \end{aligned}$$

Now it is easy to see that r_1, r_2 are infinitely flat at $t = 0$ (i.e. r_1, r_2 and all their derivatives vanish for $t \to 0$). This shows that $t \longmapsto \int_{\mathbb{R}^+} x^2 \rho_t(dx)$ is $\mathcal{C}^\infty$, the derivatives at $t = 0$ being given by

$$\left(\frac{d}{dt}\right)^n \int_{\mathbb{R}_+} x^2 \ \rho_t(dx) \Big|_{t=0} = \left(\frac{d}{dt}\right)^n \frac{f(t)}{g(t)} \Big|_{t=0}$$

with

$$f(t) \ = \ \mathbb{E}(\phi(B_t)^2 \phi'(B_t)), \quad g(t) \ = \ \mathbb{E}(\phi'(B_t)).$$

Using Ito's formula it is easy to see that the derivatives of f, g at $t = 0$ are given by (25). From the definition of H and ϕ it follows that the derivatives of ϕ at x_0 are given by the recursion (24). So, noting that $\frac{d}{d\gamma} = (\beta^2 - 1)^{-3/2} \frac{d}{dt}$, (23) follows easily. □

3 Global properties of the Lyapunov exponents as functions of the noise and damping parameters

In this section we shall collect a few results about the global behavior of λ_1 and λ_2 as functions of the noise parameter σ and the damping parameter

β. They again will turn out to be be consequences of the simple formulas of Theorem 1. The global behavior of λ_1 is illustrated in figure 4 which was obtained using the built-in Mathematica function `FindRoot`.

For simplicity of exposition we shall from now on interpret the integral term in Theorem 1 as an expectation of a nonnegative random variable V with density

$$q(v) \;=\; \frac{\frac{1}{\sqrt{v}}\exp\left(-v(1-\beta^2)-\frac{v^3\gamma^2}{12}\right)}{\int_0^\infty \frac{1}{\sqrt{u}}\exp\left(-u(1-\beta^2)-\frac{u^3\gamma^2}{12}\right)\,du}, \qquad v>0$$

(recall that $\gamma=\frac{\sigma^2}{2}$). Then Theorem 1 may be paraphrased by stating

$$\lambda_1 \;=\; \beta+\frac{\gamma}{2}\mathbb{E}(V), \tag{26}$$

$$\lambda_2 \;=\; \beta-\frac{\gamma}{2}\mathbb{E}(V). \tag{27}$$

Since the density q of V does not depend on the sign of β, we have obviously the equality

$$\lambda_2(\beta,\gamma) \;=\; -\lambda_1(-\beta,\gamma).$$

For this reason we will formulate the global results only for the top Lyapunov exponent λ_1.

Theorem 6

i) Let $\gamma>0$ fixed. Then $\beta\longmapsto\lambda_1(\beta,\gamma)$ is increasing on $\mathbb{R}_+$.

ii) If $|\beta|\le 1$ the function $\gamma\longmapsto\lambda_1(\beta,\gamma)$ is increasing.

iii) There exists a smooth strictly decreasing function $f:\mathbb{R}_-\longrightarrow\mathbb{R}_+$ with $\lim_{\beta\to-\infty}f(\beta)=\infty$, such that $\{(\beta,\gamma)\in\mathbb{R}\times\mathbb{R}_+\mid\lambda_1(\beta,\gamma)=0\}=\mathrm{graph}(f)$. In particular, for each $\gamma\ge 0$ the function $\beta\longmapsto\lambda_1(\beta,\gamma)$ possesses a unique root.

Proof: Let $\delta=\frac{\gamma}{2}=\frac{\sigma^2}{4}$ and

$$m_k \;=\; \mathbb{E}(V^k) \;=\; \int_0^\infty v^k q(v)\,dv,$$

denote the k-th moment of V $(k \geq 0)$. Since we may always interchange the order of differentiation and integration, we obtain

$$\begin{aligned}
\frac{\partial m_1}{\partial \beta} &= 2\beta(m_2 - m_1^2), \\
\frac{\partial m_1}{\partial \delta} &= \frac{2}{3}\delta(m_1 m_3 - m_4).
\end{aligned}$$

Furthermore integration by parts leads to the recursion

$$m_{k+3} = \frac{1}{\delta^2}\Big[(k+\frac{1}{2})m_k - (1-\beta^2)m_{k+1}\Big] \quad (k \geq 0).$$

Putting all together, we arrive at

$$\begin{aligned}
\lambda_1 &= \beta + \delta m_1, \\
\frac{\partial \lambda_1}{\partial \beta} &= 1 + 2\beta\delta(m_2 - m_1^2), \\
\frac{\partial \lambda_1}{\partial \delta} &= \frac{1}{3}m_1 + \frac{2}{3}(1-\beta^2)(m_2 - m_1^2).
\end{aligned}$$

Since $m_2 - m_1^2 \geq 0$ for all $\beta \in \mathbb{R}, \gamma > 0$, i) and ii) are proved.

In order to prove iii) we need some more estimates for the partial derivatives $\frac{\partial \lambda_1}{\partial \beta}$ and $\frac{\partial \lambda_1}{\partial \delta}$ on the set $\{\lambda_1 = 0\}$.
Since we know already that λ_1 has no root for $\beta > 0$ we will from now on assume that $\beta \leq 0$.
Let $\beta < 0, \gamma > 0$ and assume that $\lambda_1(\beta, \gamma) = 0$. Then, using the above calculations, we obtain that

$$m_1 = \frac{|\beta|}{\delta}, \quad m_3 = \frac{1}{\delta^2}\left[\frac{1}{2} + (\beta^2 - 1)\frac{|\beta|}{\delta}\right].$$

So the Cauchy-Schwarz inequality yields the estimate

$$m_2 \leq (m_1 m_3)^{\frac{1}{2}} = \frac{|\beta|^{\frac{1}{2}}}{\delta^{\frac{3}{2}}}\left[\frac{1}{2} + (\beta^2 - 1)\frac{|\beta|}{\delta}\right]^{\frac{1}{2}}.$$

Hence

$$\begin{aligned}
\frac{\partial \lambda_1}{\partial \beta} &= 1 + 2|\beta|\delta m_1^2 - 2|\beta|\delta m_2 \\
&\geq 1 + 2\frac{|\beta|^3}{\delta} - 2\frac{|\beta|^{\frac{3}{2}}}{\delta^{\frac{1}{2}}}\left[\frac{1}{2} + (\beta^2 - 1)\frac{|\beta|}{\delta}\right]^{\frac{1}{2}} \geq 0,
\end{aligned}$$

since

$$\left(1+2\frac{|\beta|^3}{\delta}\right)^2 \;>\; 4\frac{|\beta|^3}{\delta}\left[\frac{1}{2}+(\beta^2-1)\frac{|\beta|}{\delta}\right].$$

In a similar manner one sees that $\frac{\partial\lambda_1}{\partial\delta} > 0$ if $\beta < -1$ (in case $|\beta| \le 1$ this is already known).

Finally we note for later use that for $\beta < -1$

$$\frac{\partial\lambda_1}{\partial\delta} \;=\; \frac{1}{3|\beta|\delta}\left[1+(\beta^2-1)\frac{\partial\lambda_1}{\partial\beta}\right] \;\ge\; \frac{\beta^2-1}{3|\beta|\delta}\,\frac{\partial\lambda_1}{\partial\beta}.$$

Now we are able to prove that $\lambda_1 = 0$ is the graph of some strictly decreasing function $f : \mathbb{R}_- \longrightarrow \mathbb{R}_+$.
Let Γ be a component of $\{\lambda_1 = 0\} \cap]-\infty, 0[\times]0,\infty[$. Since $\frac{\partial\lambda_1}{\partial\beta}, \frac{\partial\lambda_1}{\partial\delta} > 0$ on Γ we see that Γ must be the graph of a strictly decreasing function $f : I \longrightarrow]0,\infty[$, were I is an open subinterval of $]-\infty, 0[$. (Note that I must be open, since Γ is an embedded sub-manifold of $]-\infty,0[\times]0,\infty[$.)
Hence, if $\inf I > -\infty$, then

$$\delta_\star \;=\; \lim_{\beta\to\inf I} f(\beta) = \infty.$$

(Otherwise we had $\lambda_1(\inf I, \delta_\star) \;=\; 0$ (by continuity) hence $(\inf I, \delta_\star) \in \Gamma$, since Γ is a component of $\{\lambda_1 = 0\}\cap]-\infty,0[\times]0,\infty[$. But this cannot be the case since I is open.)
By similar arguments we see that Γ must lead into the axes. But the origin is the only point on the axes where $\lambda_1 = 0$ (just remember that $\lambda_1(\beta,\delta)$ is explicitly known for $\delta = 0$ and $\delta \longmapsto \lambda_1(0,\delta)$ is strictly increasing). Hence we have $\sup I = 0$ and $\lim_{\beta\to 0} f(\beta) = 0$. This also shows that $\{\lambda_1 = 0\}\cap]-\infty,0[\times]0,\infty[$ has at most one component, since $\delta \longmapsto \lambda_1(\beta,\delta)$ is strictly increasing for $|\beta| \le 1$.
Now, since we know from the explicit formula of Theorem 2 that λ_1 has a root for $\beta = -1$, we can conclude that $\{\lambda_1 = 0\}$ has exactly one component, which is the graph of a smooth strictly decreasing function $f :]\beta_\star, 0] \longrightarrow \mathbb{R}_+$ where $\beta_\star < -1$, and if $\beta_\star > -\infty$, then $\lim_{\beta\to\beta_\star} f(\beta) = \infty$.
But f cannot explode for finite β since we have for $\beta < -1$ and $\delta = f(\beta)$

$$|f'(\beta)| \;=\; \frac{\frac{\partial\lambda_1}{\partial\beta}}{\frac{\partial\lambda_1}{\partial\delta}} \;\le\; \frac{3|\beta|\delta}{\beta^2-1} \;=\; \frac{3|\beta|}{\beta^2-1}\,f(\beta).$$

So it follows that $\beta_\star = -\infty$.

Finally we have to show that $\lim_{\beta\to-\infty} f(\beta) = \infty$.
Fix $\beta < -1$ and let $\rho_t(dx)$ be as in the proof of theorem 5. Using the notation of the proof of theorem 5, after some lengthy calculations we see that

$$\frac{d}{dt}\int_0^\infty x^2\rho_t(dx)\bigg|_{t=0} \;=\; \frac{f'(0)g(0)-f(0)g'(0)}{g(0)^2} \;<\; 0.$$

Since one sees easily that $\int_0 x^2\rho_t(dx) \to \infty$ for $t \to \infty$, the function $t \longmapsto \int x^2\rho_t(dx)$ possesses local minima. Let t_0 be the smallest local minimum of this function. Then $\gamma_0 = t_0(\beta^2-1)^{3/2}$ is the smallest local minimum of the function $\gamma \longmapsto \lambda_1(\beta,\gamma)$.
Since on the set $\{\lambda_1 = 0\}$ we have $\frac{\partial\lambda_1}{\partial\gamma} > 0$, we see that the smallest local minimum γ_0 must be is smaller than the unique root of the function $\gamma \longmapsto \lambda_1(\beta,\gamma)$.
Hence the line $\{\lambda_1 = 0\}$ must lie above the minimum line $\gamma = t_0(\beta^2-1)^{3/2}$, i.e.

$$\lim_{\beta\to-\infty} f(\beta) \;\geq\; \lim_{\beta\to-\infty} t_0(\beta^2-1)^{\frac{3}{2}} \;=\; \infty.$$

□

Remark: Numerical evidence shows that the function $t \longmapsto \int x^2\rho_t(dx)$ possesses a unique global minimum $t_0 > 0$ (numerically $t_0 = 1.69461$), and that this function is decreasing on $[0,t_0[$ and increasing on $]t_0,\infty[$. This means that for $|\beta| > 1$ the function $\gamma \longmapsto \lambda_1(\beta,\gamma)$ is increasing on $[0, t_0(\beta^2-1)^{3/2}[$ and decreasing on $]t_0(\beta^2-1)^{3/2},\infty[$. (For $|\beta| < 1$ we already know that $\gamma \longmapsto \lambda_1(\beta,\gamma)$ is increasing on $\mathbb{R}^+$.)

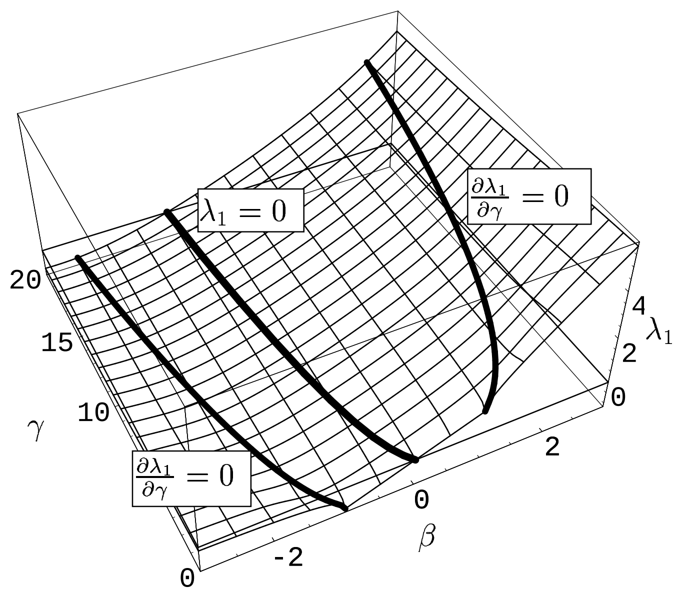

Figure 1: λ_1 as a function of β and γ

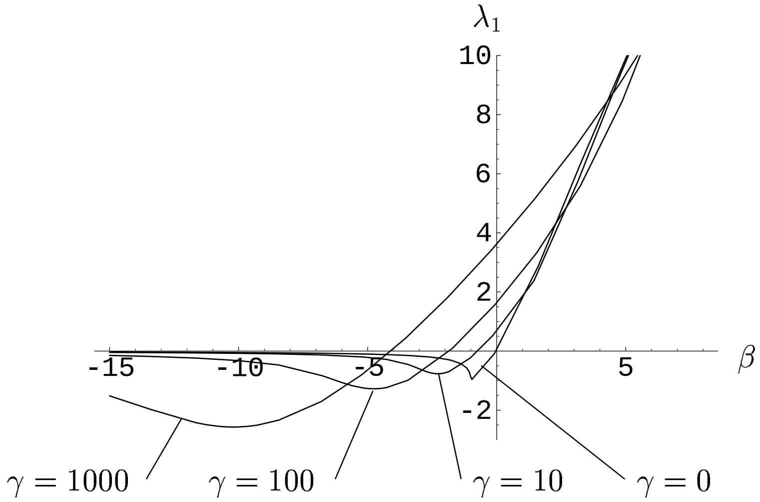

Figure 2: λ_1 as a function of β for fixed γ

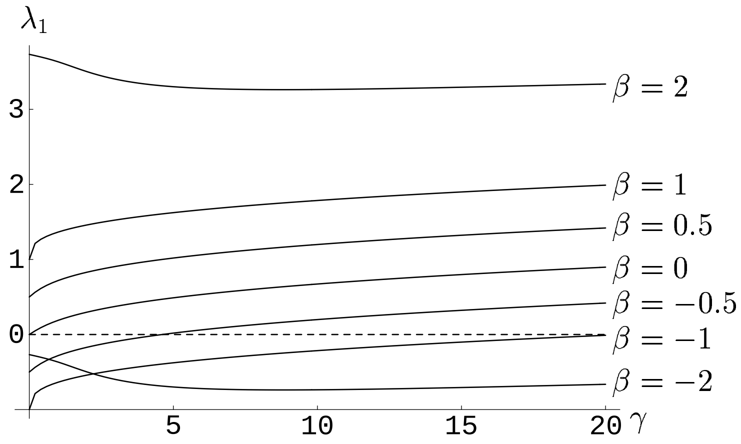

Figure 3: λ_1 as a function of γ for fixed β

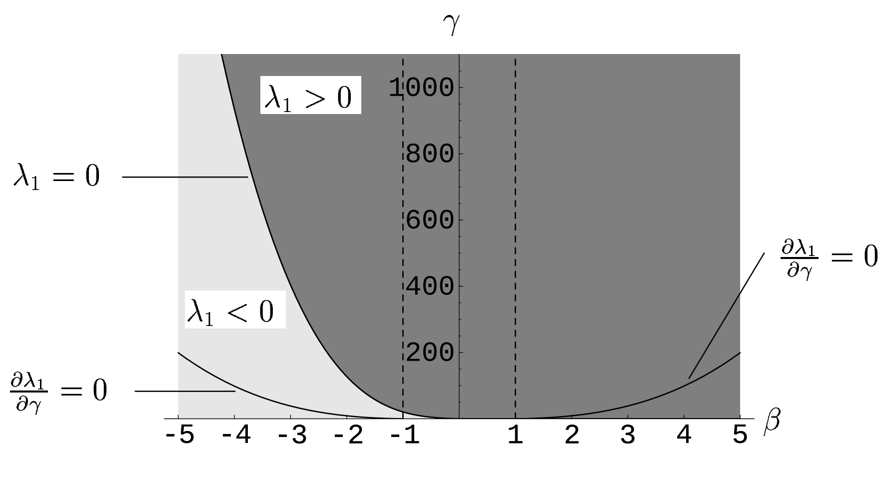

Figure 4: Roots of λ_1

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