Some formulas for Lyapunov exponents and rotation numbers in two dimensions and the stability of the harmonic oscillator and the inverted pendulum

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Abstract

Lyapunov exponents and rotation numbers of linear two dimensional stochastic differential equations are described by variants of Furstenberg-Khasminskii formulas exhibiting the interaction of drift and diffusion in terms of Lie brackets of their projections into projective space. In case of one diffusion matrix of sheer type and general drift, the formulas simplify to expressions containing the moments of one dimensional diffusions of potential type.

Applications are given to the following systems perturbed by white noise: the harmonic oscillator and the inverted pendulum linearized in its unstable equilibrium position. Their Lyapunov exponents and rotation numbers are explicited in terms of hypergeometric functions, and are asymptotically expanded into series as functions of the noise parameter. A complete account of the stability diagrams of the systems is given. Lines of change of stability and of maximal stability are described in the planes spanned by the damping and noise resp. restoring force and noise parameters. The area in the planes where stabilization by noise for the inverted pendulum takes place is investigated.

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Introduction

Lyapunov exponents and rotation numbers are among the most important invariants of dynamical systems perturbed by noise. Lyapunov exponents describe the asymptotic exponential growth rate of the trajectories of the random dynamical system, while rotation numbers give their asymptotic rate of rotation, in more than two dimensions with respect to a moving plane. In the Furstenberg-Khasminskii formulas, these numbers appear as spatial mean values via ergodic theory (see Furstenberg [14], Khasminskii [19], and Arnold [3] for an exposition of the theory and more references).

In this paper, we shall start from the Furstenberg-Khasminskii representation, to give simple formulas for Lyapunov exponents and rotation numbers of linear stochastic differential equations in two dimensions. They exhibit the Lyapunov exponents as perturbations of the trace of the drift matrix. The perturbation is expressed in terms of Lie brackets linking the projections of the drift and diffusion vector fields into projective space, integrated with respect to the invariant measures of the forward and backward infinitesimal generators on projective space. They become particularly simple in case of just one diffusion matrix. In case the diffusion matrix is of sheer type, we can further simplify the formulas thus obtained, and express the exponents in terms of the moments of the invariant measure of a simple one-dimensional diffusion of gradient type, this way extending the work begun in [15] for the noisy damped harmonic oscillator to general drift. The typical formulas exhibited link the eigenvalues of the drift matrix with both Lyapunov exponents and rotation numbers of the system perturbed by noise by expressions which are rather easy to investigate. They not only allow to recover the well known results on their asymptotic expansions for small and large noise, but even give access to a complete description of their global behaviour over the whole range of the systems' parameters. The resulting complete stability diagrams we are able to give are new, and have no counterpart in the existing literature on stochastic systems. In fact, the Furstenberg-Khasminskii approach to describe ergodic invariants has been followed in an immense number of papers, partly for real and partly for white noise. Here is a selective, by no means complete list of references: Ariarathnam, Xie [1], Arnold, Crauel Wihstutz [5], Arnold, Eizenberg, Wihstutz [6], Arnold, Papanicolaou, Wihstutz [9], Kozin, Prodromou [18], Namachchivaya, Van Roessel, Doyle [23], Pardoux, Wihstutz [26], [27], Pinsky [28], [29], Pinsky, Wihstutz [31], [32], Wedig [40], [41], Wihstutz [42].

The detailed study of stability diagrams and possibilities of stabilization by noise on the basis of the simple formulas given is done for two particular simple mechanical systems: the harmonic oscillator and the inverted pendulum linearized at its unstable equilibrium position. Thus the equation we investigate is the second order linear stochastic differential equation

$$\ddot{y} - 2\beta\dot{y} + \alpha\, y + \sigma\, \dot{W}\, y = 0,$$

with a one-dimensional Brownian motion W, noise parameter σ, damping parameter β, and strength of the restoring force α. Using the canonical passage to a two dimensional system of the first order, i. e. setting

$$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix},$$

we obtain the linear stochastic differential equation

$$dx_t = \begin{bmatrix} 0 & 1 \\ -\alpha & 2\beta \end{bmatrix} x_t dt + \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix} x_t \circ dW_t.$$

Positive α corresponds to the usual harmonic oscillator, while negative α describes the case of the inverted pendulum. The *Lyapunov exponents* $\lambda_1 \geq \lambda_2$ of the system are given as the (a.s.) exponential growth rates of the vector x_t as $t \to \infty$, i. e. they are the possible values of $\lim_{t\to\infty} \frac{1}{t} \ln ||x_t||$. Due to Birkhoff's ergodic theorem, they are constant. We interpret *stability* of our linear system as a synomym for a negative leading exponent λ_1, since it gives the minimal possible rate on an exponential scale with which both individual trajectories are attracted to 0, as well as two point motions are mutually attracted to each other. If x is given in polar coordinates (r, ϕ), the *rotation number* ρ of the system, another ergodic constant, is the asymptotic rate of rotation of x, in other words the asymptotic average of the integral increments of the angular process ϕ_t, formally $\rho = \lim_{t\to\infty} \frac{1}{t} \int_0^t d\phi_s$. For two dimensional linear deterministic systems governed by an autonomous matrix A, Lyapunov exponent and rotation number are just given by the real resp. imaginary parts of the eigenvalues of A.

Setting $\gamma = \sigma^2/2$, the top exponent and the rotation number become functions $\lambda_1(\alpha, \beta, \gamma)$ and $\rho(\alpha, \beta, \gamma)$ of the three parameters. First of all, our formulas allow to expand the two invariants asymptotically in γ near $\gamma = \infty$ (where they are analytic) and near $\gamma = 0$ (where the expansions consequently fail to be absolutely convergent). We can also express the exponents entirely in terms of three types of hypergeometric functions. Asymptotic expansions and explicit descriptions are then used to study analytically and plot the global behavior of λ_1 and ρ as functions of two parameters, while one is kept fixed. In particular we find and describe their *null lines*, i.e. the lines where their stability behavior changes from contraction to explosion or vice versa, and their *minimal lines*, i.e. lines on which maximal stability with respect to the noise parameter is attained. If a system parameter changes so that in the given two dimensional parameter space the top Lyapunov exponent increases, the system *loses stability*, i. e. one and two point motions are attracted at slower rates, if it decreases, the system *gains stability*, i. e. they are attracted at ever faster rates. If the exponent crosses a null line becoming positive, the asymptotic behaviour of both the one and two point motion changes qualitatively: instead of being attracted by 0 they are repelled from 0, both at an exponential rate determined by the constant Lyapunov exponent. On the minimal line, the exponential growth rate of one and two point motions is minimal. In particular, as long as this line runs in the domain of negative top exponent, it gives the maximal contraction rate. These lines will be represented as one-dimensional graphs in the $(\alpha, \gamma)-$ resp. $(\beta, \gamma)-$planes, while the respective third parameter is kept fixed.

For the harmonic oscillator, the picture is given in the $(\beta, \gamma)-$plane, for its interest in the question of the stochastic degeneration of the deterministic Hopf bifurcation the system undergoes at $\beta = 0$. We see that for $\beta > 0$, the system remains unstable for all noise intensities. For $\beta < -1$, the function $\gamma \mapsto \lambda_1(\alpha, \beta, \gamma)$ first gains stability, then crosses a minimal line with a simple equation given by

$$\beta \mapsto c_0 (\beta^2 - 1)^{\frac{3}{2}},$$

with a numerically known constant c_0, then monotonically loses stability, thereby crossing the null line.

The picture we obtain for the linear inverted pendulum is particularly interesting. For negative β, i.e. normal damping, there is a critical value $\alpha_0 < 0$ of the restoring force parameter such that for α between α_0 and 0 the function $\gamma \mapsto \lambda_1(\alpha, \beta, \gamma)$ crosses a null line twice, while crossing a minimal line between the two changes of stability. For $\alpha < \alpha_0$, the system is and remains unstable for all noise intensities. So while the restoring force is above the critical α_0, which is a constant multiple of β^2, stabilization of the inverted pendulum by noise is possible, if the noise intensity is kept within the limits of a well defined interval.

To explain what happens in more intuitive terms, let us fix a restoring force between 0 and the critical α_0. Then for noise turned off ($\gamma = 0$), the mathematical pendulum falls if it leaves its equilibrium position, despite the presence of damping. The top exponent is positive, and the system unstable. Now, as noise is turned on, while its intensity γ is still below a threshold value described by the corresponding crossing of the null line, noise is not strong enough to counterbalance the gravitation, and the pendulum still falls, just at an ever decreasing exponential rate. If the noise strength crosses the null line, the fast oscillating noise pushes strong enough, and often enough into the direction opposite to the gravitational force, so that trajectories of the pendulum are attracted to the equilibrium point. So at this null line, increasing the noise stabilizes the system. Increasing the noise intensity further will lead to still improving the counterbalancing effect, until at the minimal line this effect reaches its optimum. After this, the roughness of the erratic noise trajectories takes over. At the second crossing of the null line, the random pushing is strong enough to wipe out the balancing effect. The mathematical pendulum is pushed so hard into the direction opposite to the gravitational force, that it overshoots, i. e. it is pushed very rapidly into a position in which the gravitational force is in the same direction as the push. In this situation, even if the random push is initially opposite to the force, the effect of the latter is actually enhanced. Thus at the second null line crossing, by the overshooting effect, noise destabilizes. Increasing the noise strength further adds to this destabilizing effect. We should, at this point, hasten to emphasize, that we are of course discussing the mathematical, i. e. linear, version of the physical inverted pendulum. Still, the behaviour of the trajectories of the nonlinear pendulum can be accounted for, at least locally, in this situation, and our results give some new insight as well. As long as we are in the stability region just described, the general theory of local linearization and invariant manifolds for random dynamical systems (see Arnold [3], Chap. 7) implies that there is a random neighborhood of the equilibrium point 0, in which the trajectories of the nonlinear systems show stable behaviour, with Lyapunov exponents close to the ones of the linearization: there is a local *stable manifold*.

Of course, if damping is increased, the stability region generally becomes bigger. In particular, stabilization happens at lower values of the noise intensity, and destabilization at higher values. At the same time, the critical gravitational strength below which stabilization is no more possible, decreases, and our formula tells that it is a multiple of β^2.

Even if the white noise perturbation of our systems is replaced by a closely related real noise, complete stability diagrams are not available. It is therefore, at the moment, hard to say, whether the feature of stabilization followed by destabilization within the boundaries of a critical interval for α is a more general phenomenon. In particular, it can not be said if there is a critical interval at all. Studies of the asymptotic expansion for small noise of the Lyapunov exponents in this case (see Wihstutz [42]) seem to indicate, that there might be no critical lower bound for α.

Stabilization and destabilization by nonrandom types of forcing of Hamiltonian or dissipative systems such as our damped harmonic oscillator or inverted pendulum have been the subject of a huge number of studies, many of which date back to the first decades after 1900 (for example van der Pol [39], Stephenson [38]).We mention just a few topics most related to the topics of our investigation. Dissipative as well as non-dissipative harmonic oscillators with periodic or almost periodic noise are described in the well known Matthieu equations. Klotter [17] for example gives a broad treatment of Matthieu and related differential equations. The prototype of the systems studied is given by

$$\ddot{y} + (\lambda + \gamma \cos t)\, y = 0.$$

By using Floquet's theory, one can see that for certain areas of the (λ, γ)-plane the biggest Floquet exponent is positive (*exponential instability*) while for different areas it is zero (*marginal instability*). This way one obtains a nice stability map in the (λ, γ)-plane, the so-called *Ince-Strutt map*. For example, one can read from this map that if λ is fixed, increasing γ from 0 to ∞ gives an infinite number of successive changes of stability.

While for such systems with slow periodic forcing the analysis in the existing literature is focused on bifurcation studies, and stability diagrams such as the one just sketched may be available, this is no more the case for deterministic systems with fast periodic forcing which are more closely related to our white noise forcing. For example, Bellman, Bentsman, Meerkov [12] present an analysis of nonlinear systems with fast periodic or almost periodic forcing. The method of averaging over the fast oscillation plays an important role in studies of this type. Various stabilization and destabilization effects are known. But neither in this deterministic setting nor for related stochastic forcing by e. g. real noise stability diagrams for the whole parameter region have been given (see Mitchell [21]). For an overview of results for random forcing see Wihstutz [42], see also Kao, Wihstutz [16].

This paper does not intend to create new methods of investigation of dynamical systems perturbed by noise, which go essentially beyond the formulas of Furstenberg-Khasminskii. It presents a rather simple computational trick leading to the very simple formulas for Lyapunov exponents and rotation numbers in formulas (23) and (27). As simple as this trick may seem, its effect is rather convincing. It yields the first complete description of the global stability diagram of the well known and well studied systems considered. Our approach essentially improves the results of numerous studies of asymptotic expansions of Lyapunov exponents and rotation numbers for two-dimensional systems for small and large noise. In many papers the first coefficients of the expansions are derived, in the limit of small and large noise. Again we give a selective list of references, which may be completed by looking to the bibliographies of Arnold [3] or Wihstutz [42]: Ariarathnam, Xie [1], [2], Arnold, Crauel Wihstutz [5], Arnold, Eizenberg, Wihstutz [6], Arnold, Oeljeklaus, Pardoux [8], Arnold, Papanicolaou, Wihstutz [9], Auslender, Milstein [11], Wu, Guo [43], Kozin, Prodromou [18], Khasminskii, Moshchuk [20], Namachchivaya, Van Roessel, Doyle [23], Namachchivaya, Van Roessel, Talwar [24], Nishioka [25], Pardoux, Wihstutz [26], [27], Pinsky [28], Pinsky, Wihstutz [30], [31], [32], Wedig [40], [41].

The organization of the paper is as follows. In section 1, we derive the basic formulas expressing Lyapunov exponents and rotation numbers by variants of the Furstenberg-Khasminskii formulas in which the interaction between the drift and diffusion vector fields is expressed by their Lie brackets (Theorems 1.1 and 1.2 and their Corollaries).

Section 2 is devoted to specializing the formulas of section 1 to the setting of two matrices. It is here where the crucial simplification of the formulas for the exponents is derived. They simplify to moment equations of invariant measures of one-dimensional equations of potential type (Theorems 2.1 - 2.2 and their corollary).

In the third section we give explicit expressions of the Lyapunov exponents and rotation numbers of the harmonic oscillator and the inverted pendulum in terms of hypergeometric functions and derive asymptotic expansions at zero and infinite noise intensity (Theorems 3.1 - 3.4 and 3.7 - 3.10). In the main Theorems 3.5 and 3.6 we give a complete picture of the stability diagrams of the harmonic oscillator and the inverted pendulum, and in particular the possibilities for the inverted pendulum to be stabilized by noise.

Finally, we plot several stability diagrams exhibiting null lines, lines of maximal stability, as well as the areas of stability and instability, for a sample of different, but fixed, values of the damping parameter β, which indicate the shape of the surfaces of change of stability and maximal stability in the three dimensional parameter space. Due to the restriction $\alpha = 1$ in [15], the plots of [15] do not exhibit the most interesting parts we found in the present paper: the area between $\alpha = 0$ and the critical α for which both stabilization and destabilization happen, as the noise intensity increases. The analytical treatment of this main part of the global stability diagram in Theorems 3.5 and 3.6 therefore needed arguments essentially different from the ones used in [15].

Preliminaries and notation

Our basic probability space is the $m-$dimensional canonical Wiener space $(\Omega, \mathbf{F}, P)$, enlarged such as to carry an $m-$dimensional *Wiener process* indexed by $\mathbf{R}$. More precisely, $\Omega = C(\mathbf{R}, \mathbf{R}^m)$ is the set of continuous functions on $\mathbf{R}$ with values in $\mathbf{R}^m$, $\mathbf{F}$ the $\sigma-$algebra of Borel sets with respect to uniform convergence on compacts of $\mathbf{R}$, P the probability measure on $\mathbf{F}$ for which the *canonical Wiener process* $W_t = (W_t^1, ..., W_t^m), t \in \mathbf{R}$, satisfies that both $(W_t)_{t\geq 0}$ and $(W_{-t})_{t\geq 0}$ are usual $m-$dimensional Brownian motions. The natural filtration $\{\mathbf{F}_s^t = \sigma(W_u - W_v : s \leq u, v \leq t) : \mathbf{R} \ni s \leq t \in \mathbf{R}\}$ of W is assumed to be completed by the $P-$completion of $\mathbf{F}$. For $t \in \mathbf{R}$, let $\theta_t : \Omega \to \Omega, \omega \mapsto \omega(t + \cdot) - \omega(t)$, the *shift* on Ω by t. It is well known that θ_t preserves

Wiener measure P for any $t \in \mathbf{R}$ and is even ergodic for $t \neq 0$. Hence $(\Omega, \mathbf{F}, P, (\theta_t)_{t\in\mathbf{R}})$ is an ergodic *metric dynamical system* (see Arnold [3]). As usual, we use a "$\circ$" to denote Stratonovich integrals with respect to Wiener process.

For a random vector X, we denote by P_X the law of X with respect to P. Lie brackets of vector fields are denotes as usual by $[.,.]$, scalar products in $\mathbf{R}^m$ by the brackets $\langle .,. \rangle$.

For the convenience of the reader, we briefly sketch the concepts of Lyapunov exponents and rotation numbers for two-dimensional systems perturbed by white noise. See Arnold [3] for a complete account of these subjects. Let $A_0, \cdots, A_m$ be 2×2 matrices, and consider the linear Stratonivich stochastic differential equation

$$dx_t = A_0\, x_t\, dt + \sum_{i=1}^m A_i\, x_t \circ dW_t^i. \tag{1}$$

To exclude the trivial case, assume that at least one of $A_1, \cdots, A_m$ is non-zero. Let $(\Phi_t)_{t\in\mathbf{R}}$ denote the fundamental solution of (1). Then the multiplicative ergodic theorem due to Oseledets states that solution trajectories $(\Phi_t x)_{t\in\mathbf{R}}$ for $x \in \mathbf{R}^2$ can have at most two deterministic exponential growth rates. We give a more precise statement for the more interesting case of two exponents, the other case being much simpler.

There exists a random splitting $\mathbf{R}^2 = E_1(\omega) \oplus E_2(\omega)$ and real numbers $\lambda_1 > \lambda_2$ (the *Lyapunov exponents*) such that for P-a. e. $\omega \in \Omega, i = 1, 2$ we have

$$0 \neq x \in E_1(\omega) \iff \lim_{t\to\pm\infty} \frac{1}{t} \ln\, ||\Phi_t(\omega)\, x|| = \lambda_i. \tag{2}$$

To describe the Lyapunov exponents of the system as spatial averages, and also define the rotation number, we decompose the equation in the usual way into its radial and angular components $r_t = |x_t|, s_t = \frac{x_t}{r_t}$ (see for example Arnold, Oeljeklaus, Pardoux [8]). Using Itô's formula, radial and angular part are seen to fulfill the stochastic differential equations

$$dr_t = \overline{q}_0(s_t)\, r_t\, dt + \sum_{i=1}^m \overline{q}_i(s_t)\, r_t \circ dW_t^i, \tag{3}$$

$$ds_t = \overline{h}_0(s_t)\, dt + \sum_{i=1}^m \overline{h}_i(s_t) \circ dW_t^i. \tag{4}$$

The vector fields figuring in this decomposition are given by

$$\overline{q}_i(s) = \langle s, A_i\, s \rangle, \quad \overline{h}_i(s) = A_i\, s - \overline{q}_i(s)\, s, \quad s \in S^1, \quad 0 \leq i \leq m.$$

Since the angular motion is in fact a motion on one-dimensional projective space, we may work with the coordinate ϕ on $[-\frac{\pi}{2}, \frac{\pi}{2}[$ by setting $s = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$. This has been done for example in Pardoux, Wihstutz [27] as well. If we denote

$$A_i = \begin{bmatrix} a_{11}^i & a_{21}^i \\ a_{12}^i & a_{22}^i \end{bmatrix}, \quad 0 \leq i \leq m,$$

our vector fields are given in the ϕ-coordinates by the formulas

$$\begin{aligned} q_i(\phi) &= a^i_{11}\cos^2\phi + a^i_{22}\sin^2\phi + (a^i_{12}+a^i_{21})\cos\phi\ \sin\phi, \\ h_i(\phi) &= (a^i_{22}-a^i_{11})\sin\phi\ \cos\phi - a^i_{21}\sin^2\phi + a^i_{12}\cos^2\phi, \end{aligned}$$

$0 \le i \le m$, $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}[$. Hence the angular part of (3) is given by the stochastic differential equation

$$d\phi_t = h_0(\phi_t)\,dt + \sum_{i=1}^m h_i(\phi_t)\circ dW^i_t, \tag{5}$$

whereas the radial part (4) becomes

$$dr_t = q_0(\phi_t)\,r_t\,dt + \sum_{i=1}^m q_i(\phi_t)\,r_t\circ dW^i_t. \tag{6}$$

The radial equation being linear, its solution can be readily given by

$$\begin{aligned} r_t &= r_0\,\exp(\int_0^t q_0(\phi_u)\,du + \sum_{i=1}^m q_i(\phi_u)\ \circ dW^i_u) \\ &= r_0\,\exp(\int_0^t [q_0(\phi_u) + \frac{1}{2}\sum_{i=1}^m q_i' h_i(\phi_u)]\,du + \sum_{i=1}^m q_i(\phi_u)\,dW^i_u), \end{aligned}$$

where we have used the transformation of Stratonovich integrals into Itô integrals. Now let Γ_1, Γ_2 denote the random angles giving the position of the Oseledets spaces E_1, E_2 on projective space. Then, due to the simple representation of the radial part, the *Lyapunov exponents* given by (2) can be expressed in the form

$$\psi \in \Gamma_i(\omega) \iff \lim_{t\to\pm\infty}\frac{1}{t}\int_0^t [q_0(\phi_u(\omega,\psi)) + \frac{1}{2}\sum_{i=1}^m q_i' h_i(\phi_u(\omega,\psi))]\,du = \lambda_i, \tag{7}$$

if $\phi_t(.,\psi), t\in\mathbf{R}$, is the solution of (5) starting in ψ.

It is most convenient to define the rotation number in the notation now established. For a more complete account of the subject, in particular the definition of rotation numbers in dimension higher than 2, where they have to be given with respect to moving reference planes, see Arnold [3]. Intuitively, it expresses the asymptotic average rotation of the system described by the angular motion of equation (5). More formally, the corresponding generalization of the the multiplicative ergodic theorem states that there exists a real number (the *rotation number*) ρ such that

$$\rho = \lim_{t\to\pm\infty}\frac{1}{t}\int_0^t d\phi_u.$$

(5) thus yields the description analogous to (7)

$$\rho = \lim_{t\to\pm\infty}\int_0^t [h_0(\phi_u(.,\psi))\,dt + \frac{1}{2}\sum_{i=1}^m h_i' h_i(\phi_u(.,\psi))]du \tag{8}$$

for any initial value ψ. Note that ρ is an integral average of angular increments, and thus *not* confined to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}[$. The Lyapunov exponents of a linear random dynamical system can be considered as analogues of the real parts of the eigenvalues of the matrix inducing a nonautonomous linear ordinary differential equation, the rotation number analogues of their imaginary part. The term *rotation number* used for deterministic monotone maps of the unit circle is a correct analogue of the term we use here. The direct analogy has been investigated in the work of Ruffino ([34]), where the random dynamical system studied consists of products of random monotone maps of the unit circle. It is found that a version of the Oseledets multiplicative ergodic theorem holds, and that the concept of rotation is consistent with the classical one for deterministic systems. For the continuous parameter setting the analogy is briefly discussed in Ruffino [33].

We shall assume that the angular motion of the system is non-degenerate. More precisely, we suppose in the following that

(**H**) the Lie algebra generated by $h_0, \cdots, h_m$ is 1-dimensional throughout $[-\frac{\pi}{2}, \frac{\pi}{2}[$.

Since we assumed that not all of the matrices $A_1, \cdots, A_m$ are zero, (H) is equivalent to the usual hypoellipticity condition of Hörmander (see Arnold, Oeljeklaus, Pardoux [8], p. 216). In this case it is well known that the adjoint Fokker-Planck equation forward and backward in time associated with the angular motion (5) possesses unique solutions among the probability measures on the unit circle with a $C^\infty-$density, which describe the laws of the angular positions Γ_1, Γ_2 of the Oseledets spaces E_1, E_2 of the system. If the initial state ψ is distributed according to one of these invariant measures, the angular motion becomes stationary, and an application of Birkhoff's ergodic theorem transforms the temporal averages of (7) and (8) into spatial averages over $[-\frac{\pi}{2}, \frac{\pi}{2}[$ taken with the invariant measures, thus giving the well known *Furstenberg-Khasminskii formulas*. In fact, if the initial state is deterministic, due to the smoothness of the law of Γ_1, we will observe a. s. only the top exponent λ_1.

The infinitesimal generator of the forward (+) and backward (-) equation of the angular motion governs the respective forward and backward Fokker-Planck equations and is given by

$$L^{\pm} f = [\pm h_0 + \frac{1}{2}\sum_{i=1}^m h_i\, h_i']\, f' + \frac{1}{2}\sum_{i=1}^m h_i^2\, f'',$$

for sufficiently smooth f on $[-\frac{\pi}{2}, \frac{\pi}{2}[$. Denote by $p_\pm$ the smooth densities of the probability solutions of the adjoint Fokker-Planck equations, i, e.

$$(L^\pm)^*\, p^\pm = 0,$$

and denote the quantities appearing in (7), (8) by

$$Q_\pm = \pm q_0 + \frac{1}{2}\sum_{i=1}^m q_i'\, h_i \quad R = h_0 + \frac{1}{2}\sum_{i=1}^m h_i\, h_i'.$$

Then the Lyapunov exponents λ_1, λ_2 and the rotation number ρ of (1) are expressed by the following spatial averages

$$\lambda_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Q_+(\phi)\, p_+(\phi)\, d\phi, \tag{9}$$

$$\lambda_2 = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Q_-(\phi)\, p_-(\phi)\, d\phi, \tag{10}$$

and

$$\rho = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R(\phi)\, d\phi. \tag{11}$$

1 The basic formulas

The following relationship between the $q-$ and $h-$vector fields will be crucial in the derivation of our formulas for the Lyapunov exponents and the rotation number of the system under consideration.

Lemma 1.1 *Let* $A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$, *and*

$$\begin{aligned} q(\phi) &= a_{11}\cos^2\phi + a_{22}\sin^2\phi + (a_{12}+a_{21})\cos\phi\ \sin\phi, \\ h(\phi) &= (a_{22}-a_{11})\sin\phi\ \cos\phi - a_{21}\sin^2\phi + a_{12}\cos^2\phi, \end{aligned}$$

$\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}[$. *Then we have*

$$\begin{aligned} h' &= -2\,q + (a_{11}+a_{22}), \\ q' &= 2\,h + (a_{21}-a_{12}), \\ h'' &= -4\,h + 2(a_{12}-a_{21}). \end{aligned}$$

Proof:
We use the elementary relations for trigonometric functions

$$\begin{aligned} \sin^2\phi &= \frac{1}{2}(1-\cos 2\phi), \\ \cos^2\phi &= \frac{1}{2}(1+\cos 2\phi), \\ \sin\phi\ \cos\phi &= \frac{1}{2}\ \sin 2\phi, \end{aligned}$$

to write

$$q(\phi) = \frac{1}{2}[(a_{11}+a_{22}) + (a_{11}-a_{22})\cos 2\phi + (a_{12}+a_{21})\sin 2\phi], \tag{12}$$

$$h(\phi) = \frac{1}{2}[-(a_{21}-a_{12}) + (a_{22}-a_{11})\sin 2\phi + (a_{12}+a_{21})\cos 2\phi]. \tag{13}$$

Differentiating the functions in (12), the second one twice, obviously leads to the claimed equations. □

The first one of our basic representations of Lyapunov exponents will now be derived with the aid of the preceding Lemma. We split off $Q_\pm$ a component in the range of the infinitesimal generator to obtain formulas in which the interaction of the different vector fields of the angular motion is exhibited in terms of their Lie brackets.

We shall denote vectors of functions $(h_1,\cdots,h_m)$ by h and use $\langle .,.\rangle$ as well as symbol for the scalar product in $\mathbf{R}^m$. The Lie bracket is denoted as usually by $[.,.]$. For vectors of vector fields $(g_1,\cdots,g_n),(h_1,\cdots,h_m)$ we also write $[g,h]$ for the $n\times m-$matrix $[g_i,h_j]_{1\le i\le n,1\le j\le m}$.

Theorem 1.1 *We have*

$$\begin{aligned}\lambda_{1/2} &= \frac{trA_0}{2}+\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\langle h,[h_0,h]\rangle}{\langle h,h\rangle}(\phi)\,p_\pm(\phi)\,d\phi \\ &\quad\pm\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\langle h',h'\rangle\langle h,h\rangle-\langle h',h\rangle^2}{\langle h,h\rangle}(\phi)\,p_\pm(\phi)\,d\phi \\ &= \frac{trA_0}{2}+\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\langle h,[h_0,h]\rangle}{\langle h,h\rangle}(\phi)\,p_\pm(\phi)\,d\phi \\ &\quad\pm\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\langle h',[h,h]h\rangle}{\langle h,h\rangle}(\phi)\,p_\pm(\phi)\,d\phi.\end{aligned}$$

Proof:
Let $f=\frac{1}{2}\ln\langle h,h\rangle$. The formulas we are about to give are not completely rigorous, since integrability of the functions involved is not everywhere clear. We remark that they can be made completely rigorous by first choosing $\epsilon>0$ and working with $f_\epsilon=\frac{1}{2}\ln\langle h,h\rangle+\epsilon$ instead of f and letting $\epsilon\to 0$ in the end. We have

$$\begin{aligned}L^\pm f &= [\pm h_0+\frac{1}{2}\langle h,h'\rangle]\frac{\langle h,h'\rangle}{\langle h,h\rangle}+\frac{1}{2}\langle h,h\rangle\,[\frac{\langle h',h'\rangle+\langle h,h''\rangle}{\langle h,h\rangle}-2\frac{\langle h',h\rangle^2}{\langle h,h\rangle^2}] \qquad (14)\\ &= \pm h_0\frac{\langle h,h'\rangle}{\langle h,h\rangle}-\frac{1}{2}\frac{\langle h,h'\rangle^2}{\langle h,h\rangle}+\frac{1}{2}[\langle h',h'\rangle+\langle h,h''\rangle] \\ &= \pm h_0\frac{\langle h,h'\rangle}{\langle h,h\rangle}-\langle h,q'\rangle+\frac{1}{2}\frac{\langle h',h'\rangle\langle h,h\rangle-\langle h',h\rangle^2}{\langle h,h\rangle} \\ &= \pm h_0\frac{\langle h,h'\rangle}{\langle h,h\rangle}-2Q_\pm\pm 2q_0+\frac{1}{2}\frac{\langle h',h'\rangle\langle h,h\rangle-\langle h',h\rangle^2}{\langle h,h\rangle} \\ &= \pm h_0\frac{\langle h,h'\rangle}{\langle h,h\rangle}\mp h_0'-2Q_\pm\pm \mathrm{tr}A_0+\frac{1}{2}\frac{\langle h',h'\rangle\langle h,h\rangle-\langle h',h\rangle^2}{\langle h,h\rangle} \\ &= \pm\frac{\langle h,[h_0,h]\rangle}{\langle h,h\rangle}-2Q_\pm\pm \mathrm{tr}A_0+\frac{1}{2}\frac{\langle h',h'\rangle\langle h,h\rangle-\langle h',h\rangle^2}{\langle h,h\rangle}.\end{aligned}$$

Here we have used Lemma 1.1 twice. Since by definition $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}L^\pm f\,p_\pm(\phi)\,d\phi=0$, the representation formula for $\lambda_{1/2}$ is a direct consequence of (14) and the simple equation

$$\langle h',h'\rangle\langle h,h\rangle-\langle h',h\rangle^2=\langle h',[h,h]h\rangle.$$

□

Corollary 1.1 *If $m=1$, we have*

$$\lambda_{1/2}=\frac{trA_0}{2}+\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{[h_0,h_1]}{h_1}(\phi)\,p_\pm(\phi)\,d\phi.$$

Proof:
The second integral term in the representation of Theorem 1.1 vanishes if $m = 1$. $\square$

The trace term in the decomposition of the Lyapunov exponents according to Theorem 1.1 can be seen as the contribution of the drift vector field alone. The first integral term exhibits the angular interactions of the drift with the diffusion matrices, whereas the second one describes the interactions of the diffusion vector fields with each other. Note that this term is always positive, due to the inequality of Cauchy-Schwarz.

We shall next give some formulas linking the Lyapunov exponents with the rotation number.

Theorem 1.2 *We have*

$$\begin{aligned}
\lambda_{1/2} &= \frac{trA_0}{2} \pm \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0^2}{\langle h,h\rangle} p_\pm(\phi)\,d\phi - \rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{\langle h,h\rangle}(\phi)\,d\phi \\
&\pm\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\langle h',h'\rangle\langle h,h\rangle - \langle h',h\rangle^2}{\langle h,h\rangle}(\phi)\,p_\pm(\phi)\,d\phi \\
&= \frac{trA_0}{2} \pm \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0^2}{\langle h,h\rangle} p_\pm(\phi)\,d\phi - \rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{\langle h,h\rangle}(\phi)\,d\phi \\
&\pm\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\langle h',[h,h]h\rangle}{\langle h,h\rangle}(\phi)\,p_\pm(\phi)\,d\phi.
\end{aligned}$$

Proof:
We argue for λ_1. Note first that integration by parts and periodicity of the functions involved imply

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\langle h,[h_0,h]\rangle}{\langle h,h\rangle}(\phi)\,p_+(\phi)\,d\phi &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [h_0\frac{\langle h,h'\rangle}{\langle h,h\rangle} - h_0'](\phi)\,p_+(\phi)\,d\phi \qquad (15)\\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h_0\,[\frac{\langle h,h'\rangle}{\langle h,h\rangle}\,p_+ + p_+'](\phi)\,d\phi.
\end{aligned}$$

Now as a consequence of the Fokker-Planck equation there is a constant c to be determined later such that

$$[h_0 + \frac{1}{2}\langle h,h'\rangle]\,p_+ - \frac{1}{2}\,(\langle h,h\rangle\,p_+)' = c, \qquad (16)$$

equivalently

$$h_0\,p_+ - \frac{1}{2}\,[\langle h,h'\rangle\,p_+ + \langle h,h\rangle\,p_+'] = c,$$

which implies

$$\frac{2h_0}{\langle h,h\rangle} - \frac{2c}{\langle h,h\rangle} = \frac{\langle h,h'\rangle}{\langle h,h\rangle}p_+ + p_+'.$$

Using this in (15) yields

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\langle h,[h_0,h]\rangle}{\langle h,h\rangle}(\phi)\,p_+(\phi)\,d\phi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\,\frac{h_0^2}{\langle h,h\rangle}p_\pm(\phi)\,d\phi - 2\,c\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{\langle h,h\rangle}(\phi)\,d\phi. \qquad (17)$$

It remains to determine c. For this purpose, we use (16) to get

$$\rho = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R(\phi)\, p_+(\phi) d\phi = c + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\langle h, h\rangle\, p_+)'(\phi)\, d\phi = c,$$

where the last equation uses the periodicity of the functions again. Hence (17) together with Theorem 1.1 lead to the formula asserted for λ_1. The one for λ_2 is obtained by using L^- instead of L^+ in (16). This completes the proof. □

The well known trace formula for Lyapunov exponents can finally be applied to produce the following entirely symmetric representation.

Corollary 1.2 *We have*

$$\begin{aligned}\lambda_{1/2} &= \frac{trA_0}{2} \pm \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0^2}{\langle h, h\rangle} p_+(\phi)\, d\phi \pm \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\langle h', [h,h] h\rangle}{\langle h, h\rangle}(\phi)\, p_+(\phi)\, d\phi \\ &\mp \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{\langle h, h\rangle} d\phi.\end{aligned}$$

Proof:
Just use $\lambda_1 + \lambda_2 = \mathrm{tr}(A_0)$ in Theorem 1.2. □

Corollary 1.2 gives us yet another formula expressing the spectral gap $\lambda_1 - \lambda_2$ as a function of ρ.

Corollary 1.3 *We have*

$$\lambda_1 - \lambda_2 = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\frac{h_0^2}{\langle h, h\rangle} + \frac{1}{2}\frac{\langle h', [h,h]\, h\rangle}{\langle h, h\rangle}](\phi)\, p_+(\phi) d\phi - 2\,\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{\langle h, h\rangle}(\phi)\, d\phi.$$

In particular,

$$\lambda_1 - \lambda_2 \geq -2\,\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{\langle h, h\rangle}(\phi)\, d\phi.$$

Proof:
Just note that the first expression in the formula for the gap, which follows directly from the preceding corollary, is nonnegative. □

In case $m = 1$ our formulas simplify to the following.

Corollary 1.4 *Let $m = 1$. Then*

$$\begin{aligned}\lambda_{1/2} &= \frac{trA_0}{2} \pm \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0^2}{h_1^2}(\phi)\, p_\pm(\phi)\, d\phi - \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{h_1^2}(\phi)\, d\phi \\ &= \frac{trA_0}{2} \pm \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0^2}{h_1^2}(\phi)\, p_+(\phi)\, d\phi \mp \rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{h_1^2}(\phi)\, d\phi,\end{aligned}$$

and

$$\lambda_1 - \lambda_2 = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0^2}{h_1^2}(\phi)\, p_+(\phi) d\phi - 2\,\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{h_1^2}(\phi)\, d\phi.$$

In particular,

$$\lambda_1 - \lambda_2 \geq -2\,\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{h_1^2}(\phi)\, d\phi.$$

A final consequence of Theorem 1.2 is the following representation of the rotation number.

Corollary 1.5 *We have*

$$\rho = \frac{1}{2}\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\frac{h_0^2}{\langle h,h\rangle} + \frac{1}{2}\frac{\langle h',[h,h]\,h\rangle}{\langle h,h\rangle}](\phi)\,[p_+(\phi) - p_-(\phi)]\, d\phi}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{\langle h,h\rangle}(\phi)\, d\phi},$$

and in case $m = 1$

$$\rho = \frac{1}{2}\frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0^2}{h_1^2}(\phi)[p_+(\phi) - p_-(\phi)]\, d\phi}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h_0}{h_1^2}(\phi)\, d\phi}.$$

2 The case of two matrices

As applications of some of the formulas of the preceding section we now give some explicit formulas for Lyapunov exponents and rotation numbers in the case $m = 1$. Indeed, in one of three possible types of situations, among which we find the very relevant one of the noisy damped harmonic oscillator, which is treated in full detail in Imkeller, Lederer [15], we obtain a simple integral formula which links the eigenvalues of the drift matrix, and the Lyapunov exponents and rotation number of the stochastic system.

We start with recalling that a non-singular coordinate transformation does not affect the Lyapunov exponents of the system. Now there are three possible types of Jordan normal forms of A_1 given by the following formulas

$$\text{type I:} \quad A_1 = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix},$$

$$\text{type II:} \quad A_1 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix},$$

$$\text{type III:} \quad A_1 = \begin{bmatrix} \alpha & 0 \\ 1 & \alpha \end{bmatrix},$$

where $\alpha, \beta \in \mathbf{R}$, and $\beta \geq \alpha$ for type I, $\alpha^2 + \beta^2 = 1$ for type II. We shall generalize the analysis of [15], to discuss the type III system thoroughly. We may omit the superscript "0" in the entries of A_0. Our angular vector fields then become

$$h_0(\phi) \quad = \quad (a_{22} - a_{11}) \sin\phi \cos\phi - a_{21} \sin^2\phi + a_{12} \cos^2\phi$$

$$\begin{aligned} &= \frac{1}{2}[-(a_{21}-a_{12})+(a_{22}-a_{11})\,\sin 2\phi+(a_{12}+a_{21})\,\cos 2\phi], \\ h_1(\phi) &= \cos^2\phi = \frac{1}{2}[1+\cos 2\phi]. \end{aligned}$$

Hence the Lie bracket of h_0 and h_1 is given by

$$[h_0,h_1](\phi) = -(a_{22}-a_{11})\,cos^2\phi + 2\,a_{21}\,\sin\phi\,\cos\phi, \tag{18}$$

$\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}[$. So in case $a_{21}=0$ we obtain

$$[h_0,h_1] = -(a_{22}-a_{11})\,h_1,$$

and since $h_0(\frac{\pi}{2}) = h_1(\frac{\pi}{2}) = 0$, this is a degenerate case which does not fulfill (H). So it will be excluded from further considerations. Solving the Fokker-Planck equation in the given case is classical (see for example Kozin, Prodromou [18], or Nishioka [25]). The differential equation with periodic boundary conditions is this:

$$\begin{aligned} p_+'(\phi) &= (\frac{2h_0}{h_1^2} - \frac{h_1'}{h_1})(\phi)\,p_+(\phi) - \frac{2c}{h_1^2}(\phi) \\ &= \frac{1}{\cos^2\phi}\{[2(a_{22}-a_{11})\,\tan\phi - 2a_{21}\,\tan^2\phi + 2\,a_{12}]\,p_+(\phi) - \frac{h_1'}{h_1}\,p_+(\phi) - \frac{2c}{\cos^2\phi}\}, \end{aligned}$$

where $c=\rho$, the rotation number of the system, as remarked in the preceding section. The homogeneous part of this equation has the solution

$$p_0(\phi) = \exp((a_{22}-a_{11})\,\tan^2\phi - \frac{2}{3}\,a_{21}\,\tan^3\phi + 2a_{12}\,\tan\phi)\,\frac{1}{h_1(\phi)}.$$

So by variation of constants the equation is seen to be solved by the probability densities

$$p_+(\phi) = \frac{1}{d_+}\int_{-\frac{\pi}{2}}^{\phi} p_0(\phi)\,p_0^{-1}(\theta)\,\frac{1}{h_1^2}(\theta)\,d\theta, \text{ if } a_{21}>0, \tag{19}$$

$$p_+(\phi) = \frac{1}{d_-}\int_{\phi}^{-\frac{\pi}{2}} p_0(\phi)\,p_0^{-1}(\theta)\,\frac{1}{h_1^2}(\theta)\,d\theta, \text{ if } a_{21}<0, \tag{20}$$

where $d_\pm$ are the respective norming constants. Substituting the formulas obtained into Corollary 1.1 gives the following expression for λ_1:

$$\begin{aligned} \lambda_1 &= \frac{\mathrm{tr}A_0}{2} + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[(a_{11}-a_{22}) + 2\,a_{21}\,\tan\phi]\,p_+(\phi)\,d\phi \qquad (21) \\ &= a_{11} + a_{21}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\phi\,p_+(\phi)\,d\phi. \end{aligned}$$

One might ask at this place what the advantage of an application of Corollary 1.1 as opposed to a direct application of the Furstenberg-Khasminskii formulas (9), (10), (11) is. (21) describes λ_1 with an integral of the tan-function against the density of an invariant measure which can as well be given in terms of this trigonnometric function. A direct appeal to (9) would yield a rather complicated integral in terms of polynomials in the trigonometric functions sin and cos . In fact, the main feature of the direct calculation presented in [15] and giving the essential simplification leading from (9) to (21), has now been put in a general framework and produced the main formulas of section 1, which we could call *geometric versions* of the formulas of Furstenberg-Khasminskii.

It remains to further evaluate the integral in (21). To do this, we have to distinguish the cases $a_{21} > 0$ and $a_{21} < 0$. Let us concentrate on the former. For the latter, analogous arguments use (20) instead of (19).

Let us use, just as in [15], the coordinate transformation $s = \tan\phi, u = \tan\theta$. After another switch of signs of u, this gives

$$\begin{aligned} d_+ \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\phi\, p_+(\phi)\, d\phi \\ &= \int_{-\infty}^{\infty} s \int_{-\infty}^{s} \exp((a_{22}-a_{11})(s^2-u^2) - \frac{2}{3}a_{21}(s^3-u^3) + 2\,a_{12}(s-u))duds \\ = \quad \int_{-\infty}^{\infty} s \int_0^{\infty} \exp(-2va_{21}\,s^2 + [2a_{21}v^2 + 2v(a_{22}-a_{11})]s \\ &\qquad -(a_{22}-a_{11})v^2 - \frac{2}{3}a_{21}\,v^3 + 2\,a_{12}v)dvds. \end{aligned}$$

Now we can write

$$\begin{aligned} &2v\,a_{21}\,s^2 - [2a_{21}v^2 + 2v(a_{22}-a_{11})]s \\ &= 2va_{21}\,(s^2 - (v + \frac{(a_{22}-a_{11})}{a_{21}})s) \\ &= 2va_{21}\,(s - \frac{1}{2}(v + \frac{(a_{22}-a_{11})}{a_{21}}))^2 - (v + \frac{(a_{22}-a_{11})}{a_{21}})^2\,\frac{v\,a_{21}}{2}. \end{aligned}$$

Moreover, note that

$$4a_{21}a_{12} + (a_{22}-a_{11})^2 = (\mathrm{tr}A_0)^2 - 4\det A_0,$$

which, by the well known formula for the eigenvalues of A_0, say μ_1, μ_2, stating

$$\mu_{1/2} = \frac{\mathrm{tr}A_0}{2} \pm \frac{1}{2}\sqrt{(\mathrm{tr}A_0)^2 - 4\det A_0},$$

may also be described by $\mu_1 - \mu_2$. Therefore

$$\begin{aligned} &\int_{-\infty}^{\infty} s \int_0^{\infty} \exp(-2va_{21}\,s^2 + [2a_{21}v^2 + 2v(a_{22}-a_{11})]s \\ &\qquad -(a_{22}-a_{11})v^2 - \frac{2}{3}a_{21}\,v^3 + 2\,a_{12}v)dvds \\ &= \frac{1}{2}\int_0^{\infty}\int_{-\infty}^{\infty} \exp(-2va_{21}\,s^2)ds\ [v + \frac{a_{22}-a_{11}}{a_{21}}]\ \exp(-\frac{1}{6}a_{21}v^3 + \frac{v}{2a_{21}}(\mu_1-\mu_2)^2)\,dv. \end{aligned}$$

In the same way we see

$$d_+ = \int_0^{\infty}\int_{-\infty}^{\infty} \exp(-2va_{21}\,s^2)ds\ \exp(-\frac{1}{6}a_{21}v^3 + \frac{v}{2a_{21}}(\mu_1-\mu_2)^2)\,dv. \qquad (22)$$

So, after an evaluation of the integrals in s, (21) is seen to lead to

$$\lambda_1 = \frac{\mu_1+\mu_2}{2} + \frac{1}{2}\, a_{21}\, \frac{\int_0^\infty \sqrt{v}\exp(-\frac{1}{6}a_{21}\, v^3 + \frac{v}{2a_{21}}(\mu_1-\mu_2)^2)\, dv}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(-\frac{1}{6}a_{21}\, v^3 + \frac{v}{2a_{21}}(\mu_1-\mu_2)^2)\, dv}. \tag{23}$$

A careful inspection of the completely analogous arguments in case $a_{21} < 0$ finally leads to the following main result.

Theorem 2.1 *Suppose A_1 is of Jordan type III, i.e. $A_1 = \begin{bmatrix} \alpha & 0 \\ 1 & \alpha \end{bmatrix}$, for some $\alpha \in \mathbf{R}$. Assume further that A_0 is such that $a_{21} \neq 0$ and possesses eigenvalues μ_1, μ_2 with $\Re\mu_1 > \Re\mu_2$. Then the Lyapunov exponents λ_1, λ_2 of (1) satisfy*

$$\lambda_{1/2} = \frac{\mu_1+\mu_2}{2} + \frac{1}{2}\, |a_{21}|\, \frac{\int_0^\infty \sqrt{v}\exp(-\frac{1}{6}|a_{21}|\, v^3 + \frac{v}{2|a_{21}|}(\mu_1-\mu_2)^2)\, dv}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(-\frac{1}{6}|a_{21}|\, v^3 + \frac{v}{2|a_{21}|}(\mu_1-\mu_2)^2)\, dv}. \tag{24}$$

On the basis of Theorem 2.1, global properties of $\lambda_{1/2}$ as functions of $\mu_{1/2}$ may now be discussed as in [15].

We finally include the rotation number into the discussion of the case under consideration. Recall the constant $\rho = c$ from the differential equation for p_+, and compare with the norming constants $d_\pm$ to obtain

$$\rho = -\frac{1}{2d_+}, \quad \text{if}\, a_{21} > 0, \tag{25}$$

$$\rho = \frac{1}{2d_-}, \quad \text{if}\, a_{21} < 0. \tag{26}$$

So, the formulas obtained for the norming constants in the derivation of Theorem 2.1 lead to the following formulas.

Theorem 2.2 *Suppose A_1 is of Jordan type III, i.e. $A_1 = \begin{bmatrix} \alpha & 0 \\ 1 & \alpha \end{bmatrix}$, for some $\alpha \in \mathbf{R}$. Assume further that A_0 is such that $a_{21} \neq 0$ and possesses eigenvalues μ_1, μ_2 with $\Re\mu_1 > \Re\mu_2$. Then the rotation number of (1) satisfies*

$$\rho = -sgn(a_{21})\sqrt{\frac{|a_{21}|}{2\pi}}\, \frac{1}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(-\frac{1}{6}|a_{21}|\, v^3 + \frac{v}{2|a_{21}|}(\mu_1-\mu_2)^2)\, dv}. \tag{27}$$

Theorem 2.2 finally allows us to give an explicit formula linking Lyapunov exponents, rotation number and eigenvalues of the drift matrix.

Corollary 2.1 *Let A_0, A_1 be as in Theorem 2.2. Then we have the following relationship between the eigenvalues μ_1, μ_2 of A_0, the Lyapunov exponents λ_1, λ_2 and the rotation number ρ of (1)*

$$\lambda_{1/2} = \frac{\mu_1+\mu_2}{2} \mp \frac{1}{2}\, sgn(a_{21})\sqrt{\frac{|a_{21}|}{2\pi}}\; \rho \int_0^\infty \sqrt{v}\exp(-\frac{1}{6}|a_{21}|\, v^3 + \frac{v}{2|a_{21}|}(\mu_1-\mu_2)^2)\, dv,$$

and

$$\lambda_1 - \lambda_2 = -\frac{1}{2}\, sgn(a_{21})\sqrt{2\pi|a_{21}|}\; \rho \int_0^\infty \sqrt{v}\exp(-\frac{1}{6}|a_{21}|\, v^3 + \frac{v}{2|a_{21}|}(\mu_1-\mu_2)^2)\, dv.$$

3 Applications: the damped harmonic oscillator and the inverted pendulum

The physical system perturbed by white noise $\dot{W}$ we consider is given by the following second order stochastic differential equation

$$\ddot{y} - 2\beta\,\dot{y} + \alpha\,y + \sigma\,y\,\dot{W} = 0.$$

Here β is the damping constant, α the parameter controlling the strength of the restoring force. Positive α corresponds to the noisy damped harmonic oscillator, negative α appears in the case of the inverted pendulum. Introducing two-dimensional coordinates in the usual way $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, and setting

$$A_0 = \begin{bmatrix} 0 & 1 \\ -\alpha & 2\beta \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ \sigma & 0 \end{bmatrix},$$

we obtain the 2-dimensional stochastic differential equation

$$dx_t = A_0\,x_t\,dt + A_1\,x_t\,\circ dW_t, \tag{28}$$

where W is a one-dimensional Brownian motion. Using the explicit formulas of the preceding section and results from [15], we shall derive exact asymptotic expansions of Lyapunov exponents and rotation numbers. These and moment equations will then yield a complete description of global properties of the Lyapunov exponents and rotation numbers as functions of the three parameters α, β, σ. In particular, our results give a rather precise description of the stability diagram of the inverted pendulum and the damped harmonic oscillator.

In case $\alpha > 0$ we shall complement the results of [15]. We shall consider the biggest Lyapunov exponent λ_1, and show that for fixed α, the roots of the function are located on exactly one strictly decreasing line which is based at the point $\beta = 0, \sigma = 0$. Hence for negative β, the case of normal damping, there is exactly one noise value at which the stability of the system is lost. Below this value, the system is stable. In fact, maximal stability is achieved on another strictly decreasing line, which has a relatively simple equation, and is based in the point $\beta = -\sqrt{\alpha}, \sigma = 0$. For positive β, the non-physical case of negative damping, the system is always unstable.

In case $\alpha < 0$ the results are more interesting. Here we have to argue for the case of normal damping, i.e. for $\beta < 0$. We see that there is again a strictly decreasing line of maximal stability with the same simple equation as in the previous case, which, this time, is based at the point $\alpha = \beta^2, \sigma = 0$. The *null line*, at which stability changes, is more complex this time. It is based at the point $\alpha = 0, \sigma = 0$. Its slope at this point is given by $-2|\beta|$. There is a value α_0 depending on β, such that stability as a function of noise strength changes twice for $\alpha \in]\alpha_0, 0[$, first from instability to stability, and at some higher value of σ back to instability again. Hence in this regime, noise first stabilizes the system, and increasing it further eventually destabilizes it again. If $\alpha < \alpha_0$, the system is always unstable.

In the last subsection, we shall investigate the rotation numbers of the system. We give exact asymptotic expansions at $\sigma = \infty$ and $\sigma = 0$. In case $\alpha > \beta^2$, we shall in particular see that the rotation number, which is 0 for noise turned off, is infinitely flat near the deterministic boundary. In particular, only after an exponential renormalization, it admits a formal asymptotic expansion.

Let us first transform (28) into a system fitting exactly into the framework of the preceding section. This is done by scaling the Wiener process. Indeed, if $\sigma \neq 0$, then we use the fact that $\tilde{W}_t = \sigma W_{\frac{t}{\sigma^2}}, t \in \mathbf{R}$, is a Wiener process. We obtain an integral equation in terms of $\tilde{W}$ of the following form

$$x_t = x_0 + \int_0^{t\sigma^2} B_0 \, x_{\frac{u}{\sigma^2}} \frac{1}{\sigma^2} \, du + \int_0^{t\sigma^2} B_1 \, x_{\frac{u}{\sigma^2}} \circ d\tilde{W}_u, \tag{29}$$

where

$$B_0 = \begin{bmatrix} 0 & \frac{1}{\sigma^2} \\ -\frac{\alpha}{\sigma^2} & \frac{2\beta}{\sigma^2} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Setting $y_t = x_{\frac{t}{\sigma^2}}, t \in \mathbf{R}$, we obtain the stochastic differential equation

$$dy_t = B_0 \, y_t \, dt + B_1 \, y_t \circ d\tilde{W}_t. \tag{30}$$

The Lyapunov exponents λ_i of (28) and $\tilde{\lambda}_i$ of (30) are obviously related by the equation

$$\lambda_i = \sigma^2 \, \tilde{\lambda}_i, \quad i = 1, 2. \tag{31}$$

The eigenvalues of B_0 are given by

$$\tilde{\mu}_{1/2} = \frac{1}{\sigma^2}(\beta \pm \sqrt{\beta^2 - \alpha}),$$

hence Theorem 2.1 gives the formula

$$\begin{aligned} \tilde{\lambda}_{1/2} &= \frac{\beta}{\sigma^2} \pm \frac{1}{2\sigma^2} \frac{\int_0^\infty \sqrt{v} \exp(-\frac{1}{6\sigma^2} v^3 + 2v \frac{\beta^2 - \alpha}{\sigma^2}) dv}{\int_0^\infty \frac{1}{\sqrt{v}} \exp(-\frac{1}{6\sigma^2} v^3 + 2v \frac{\beta^2 - \alpha}{\sigma^2}) dv} \\ &= \frac{\beta}{\sigma^2} \pm \frac{1}{2} \frac{\int_0^\infty \sqrt{v} \exp(-\frac{1}{12} v^3 \gamma^2 + v(\beta^2 - \alpha)) dv}{\int_0^\infty \frac{1}{\sqrt{v}} \exp(-\frac{1}{12} v^3 \gamma^2 + v(\beta^2 - \alpha)) dv}, \end{aligned}$$

where $\gamma = \frac{\sigma^2}{2}$. Setting

$$q(\alpha, \beta, \gamma, v) = \frac{\frac{1}{\sqrt{v}} \exp(-\frac{1}{12} v^3 \gamma^2 + v(\beta^2 - \alpha))}{\int_0^\infty \frac{1}{\sqrt{u}} \exp(-\frac{1}{12} u^3 \gamma^2 + u(\beta^2 - \alpha)) du}, \quad \alpha, \beta \in \mathbf{R}, \gamma > 0, v \geq 0,$$

we therefore obtain the equation

$$\lambda_{1/2} = \beta \pm \frac{\gamma}{2} \int_0^\infty v \, q(\alpha, \beta, \gamma, v) \, dv. \tag{32}$$

In case $\alpha > 0$, if we rescale the parameters σ, β, we may regain formulas for Lyapunov exponents with the normalization $\alpha = 1$. Indeed, set

$$\hat{\beta} = \frac{\beta}{\sqrt{\alpha}}, \quad \hat{\gamma} = \frac{\gamma}{\alpha^{\frac{3}{2}}}, \tag{33}$$

to get

$$\lambda_{1/2} = \sqrt{\alpha}\, \hat{\lambda}_{1/2}, \tag{34}$$

where

$$\hat{\lambda}_{1/2} = \hat{\beta} \pm \frac{\hat{\gamma}}{2} \int_0^\infty v\, q(1, \hat{\beta}, \hat{\gamma}, v)\, dv$$

are the Lyapunov exponents belonging to the parameters $\hat{\alpha} = -1, \hat{\beta}, \hat{\gamma}$, which were extensively investigated in [15]. We first state a few results on asymptotic expansions of the Lyapunov exponents, before we discuss their global properties as functions of the three parameters separately for the cases $\alpha > 0$ of the harmonic oscillator, and $\alpha < 0$ of the inverted pendulum. As (32) indicates, we may subsequently confine our attention mostly to the top exponent λ_1.

3.1 Asymptotic expansions of the Lyapunov exponents

Following [15], with only slight modifications we can first give an explicit description of the top exponent in terms of hypergeometric functions. For $k, l \in \mathbf{N}$, $a_1, \ldots, a_k, b_1, \ldots, b_l \in \mathbf{R} \setminus \mathbf{Z}^-$ we first recall the following type of hypergeometric functions

$$_kF_l(a_1, \ldots, a_k; b_1, \ldots, b_l; x) \;=\; \sum_{n=0}^\infty \frac{(a_1)_n \ldots (a_k)_n}{(b_1)_n \ldots (b_l)_n} \frac{x^n}{n!}, \quad x \in I,$$

where $(a)_n = \prod_{i=0}^{n-1}(a+i)$, $n \geq 0$, and $I \subset \mathbf{R}$ is an interval centered at 0. We do not discuss their radius of convergence, but remark that the types we shall be using here converge on $\mathbf{R}$. In these terms, (32) and (33) allow us to apply Theorem 2 of [15], and we obtain the following result.

Theorem 3.1 *Assume $\gamma = \frac{\sigma^2}{2} > 0$ and define $\delta = \frac{4(\beta^2-\alpha)^3}{9\gamma^2}$. Then for $|\beta| \neq \sqrt{\alpha}$, setting*

$$\begin{aligned} G(\alpha,\beta,\gamma) \;=\; & \Big[\ \gamma^{4/3}\ 2\cdot 3^{1/3}\ \Gamma(\tfrac{1}{2})^2\ {}_1F_2(\tfrac{1}{2}; \tfrac{1}{3}, \tfrac{2}{3}; \delta) \\ & - \gamma^{2/3}(\beta^2-\alpha)\cdot 2\cdot 3^{1/6}\ \Gamma(-\tfrac{1}{3})\Gamma(\tfrac{2}{3})\ {}_1F_2(\tfrac{5}{6}; \tfrac{2}{3}, \tfrac{4}{3}; \delta) \\ & + (\beta^2-\alpha)^2\ 2^{1/3}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{6})\ {}_1F_2(\tfrac{7}{6}; \tfrac{4}{3}, \tfrac{5}{3}; \delta)\ \Big] \Big/ \\ & \Big[\ \gamma^{4/3}\ 2^{1/3}\ \Gamma(\tfrac{1}{6})\Gamma(\tfrac{1}{2})\ {}_1F_2(\tfrac{1}{6}; \tfrac{1}{3}, \tfrac{2}{3}; \delta) \\ & + \gamma^{2/3}(\beta^2-\alpha)\cdot 2\cdot 3^{1/3}\ \Gamma(\tfrac{1}{2})^2 {}_1F_2(\tfrac{1}{2}; \tfrac{2}{3}, \tfrac{4}{3}; \delta) \\ & + (\beta^2-\alpha)^2\ 6^{2/3}\Gamma(\tfrac{5}{6})\Gamma(\tfrac{1}{2})\ {}_1F_2(\tfrac{5}{6}; \tfrac{4}{3}, \tfrac{5}{3}; \delta)\ \Big], \end{aligned}$$

we have

$$\lambda_1 \;=\; \beta + \frac{\gamma^{1/3}}{2} G(\alpha,\beta,\gamma),$$

$$\lambda_2 \;=\; \beta - \frac{\gamma^{1/3}}{2} G(\alpha,\beta,\gamma).$$

For $|\beta| = \sqrt{\alpha}$ *we have*

$$\begin{aligned}\lambda_1 &= \beta + \frac{\gamma^{1/3}}{2} \cdot 12^{1/3}\, \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})},\\ \lambda_2 &= \beta - \frac{\gamma^{1/3}}{2} \cdot 12^{1/3}\, \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})}.\end{aligned}$$

For α, β arbitrary, we can give a series expansion of the exponents which is in fact a Laurent series at $\gamma = \infty$ in $\gamma^{\frac{1}{3}}$. This is the obvious extension of Theorem 3 of [15].

Theorem 3.2 *Let* $\gamma > 0$. *Then we have*

$$\lambda_{1/2} \;=\; \beta \pm \frac{12^{1/3}}{2} \gamma^{1/3} \sum_{n=0}^{\infty} \left((\beta^2 - \alpha) \cdot \frac{12^{1/3}}{\gamma^{2/3}} \right)^n c_n,$$

where

$$\begin{aligned} c_0 &= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})},\\ c_n &= \sum_{m=1}^{n} \frac{(-1)^{m+1}}{\Gamma(\frac{1}{6})^m} \sum_{\substack{l_1, \ldots, l_m \geq 1 \\ l_1 + \ldots + l_m = n}} \prod_{i=1}^{m-1} \frac{\Gamma(\frac{l_i}{3} + \frac{1}{6})}{l_i!} \left[\frac{\Gamma(\frac{l_m}{3} + \frac{1}{2})}{l_m!} - \frac{\Gamma(\frac{1}{2}}{\Gamma(\frac{1}{6})} \right]\\ &\quad (n \geq 1). \end{aligned}$$

Remark:
For $\alpha = \beta = 0$ Theorem 3.2 gives the formula

$$\lambda_{1/2} = \frac{12^{1/3}}{2} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})} \gamma^{1/3}.$$

An asymptotic expansion near $\gamma = 0$ of the form

$$\lambda_{1/2} = \frac{12^{1/3}}{2} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})} \gamma^{1/3} [a_0(\alpha,\beta) + a_1(\alpha,\beta)\,\gamma + a_2(\alpha,\beta)\,\gamma^2 + \cdots] \tag{35}$$

does not hold, however. This is indicated by Theorem 3.2, and follows precisely from the representation

$$\lambda_{1/2} = \frac{12^{1/3}}{2} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{6})} \gamma^{1/3} \frac{\int_0^\infty \sqrt{v}\, \exp(\frac{12^{\frac{1}{3}}(\beta^2 - \alpha)}{\gamma^{\frac{2}{3}}} v - v^3)\, dv}{\int_0^\infty \frac{1}{\sqrt{v}} \exp(\frac{12^{\frac{1}{3}}(\beta^2 - \alpha)}{\gamma^{\frac{2}{3}}} v - v^3)\, dv}. \tag{36}$$

In case α, β are such that $\alpha < \beta^2$, the quotient of integrals in (36) diverges along a sequence of parameters γ_n for which $n = \frac{12^{\frac{1}{3}}(\beta^2-\alpha)}{\gamma_n^{\frac{2}{3}}}, n \in \mathbf{N}$, since

$$\frac{\int_0^\infty \sqrt{v}\exp(nv - v^3)\,dv}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(nv - v^3)\,dv} = n\,\frac{\int_0^\infty \sqrt{v}\exp(n^{\frac{2}{3}}[v - v^3])\,dv}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(n^{\frac{2}{3}}[v - v^3])\,dv}$$

and the latter quotient of integrals converges to a nontrivial finite limit as $n \to \infty$, as is shown in the proof of Theorem 5 of [15], or Theorem 3.4 below. In case $\alpha > \beta^2$, the correct asymptotic expansion is given by the following Theorem, which also clearly shows that (35) cannot hold. It is the obvious extension of Theorem 4 of [15] to the case of general restoring force.

Theorem 3.3 *Suppose $|\beta| < \sqrt{\alpha}$. Let*

$$\begin{aligned} c_0 &= 2, \\ c_n &= 12\sum_{m=1}^{n}(-1)^{m+1} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\ldots+l_m = n}} \prod_{i=1}^{m} \frac{(6l_i)!}{(3l_i)!l_i!}\cdot l_m \quad (n \geq 1). \end{aligned}$$

Then the formal (but not convergent) asymptotic expansion of λ_1, λ_2 in $\gamma = \frac{\sigma^2}{2}$ at $\gamma = 0$ is given by

$$\lambda_{1/2} = \beta \pm \frac{\gamma}{8(\alpha - \beta^2)} \sum_{n=0}^{\infty} c_n \left(\frac{\gamma^2}{3\cdot 2^8(\beta^2-\alpha)^3}\right)^n$$

Our final asymptotic expansion is for the case $|\beta| > \sqrt{\alpha}$, and extends Theorem 5 of [15].

Theorem 3.4 *Assume $|\beta| > \sqrt{\alpha}$. Then the formal (but not convergent) asymptotic expansion of λ_1, λ_2 in $\gamma = \frac{\sigma^2}{2}$ at $\gamma = 0$ is given by*

$$\lambda_{1/2} = \beta \pm \sum_{n=0}^{\infty} c_n(\beta^2-\alpha)^{-(3n-1)/2}\,\gamma^n, \tag{37}$$

where the c_n can be computed as follows:
There exist a $\mathcal{C}^\infty$ function ϕ defined on a neighborhood of $x_0 = \sqrt{2}$ whose derivatives at x_0 can be computed via the recursion

$$\phi(x_0) = \sqrt{2}, \quad \phi'(x_0) = \frac{1}{2\sqrt{2}}, \quad \phi'' = -\frac{4(4+6\phi^2+\phi^4)}{3\phi^3(2+\phi^2)^3} \tag{38}$$

and $\mathcal{C}^\infty$ functions $f, g : \mathbf{R}^+ \to \mathbf{R}$ satisfying

$$f^{(n)}(0) = \frac{1}{2^n}(\phi^2\phi')^{(2n)}(x_0), \quad g^{(n)}(0) = \frac{1}{2^n}\phi^{(2n+1)}(x_0) \tag{39}$$

such that

$$c_n = \frac{1}{2\;n!}\left(\frac{d}{dt}\right)^n \frac{f(t)}{g(t)}\bigg|_{t=0}, \quad n \geq 0.$$

3.2 Global properties of the top Lyapunov exponent of the harmonic oscillator and inverted pendulum as a function of noise, damping and restoring force

Let us now concentrate on $\lambda_1 = \lambda_1(\alpha,\beta,\gamma)$ and study its global behavior as the parameters vary. We shall state our results in two Theorems, the first one for the case of positive restoring force, the second one for the inverted pendulum. In both Theorems, we will pay special attention to the eventual change of stability of the described system, which appears as the *null line* is crossed. In the result for the harmonic oscillator, we shall formulate the null line in the $(\beta,\gamma)-$plane, as $\alpha > 0$ is fixed.

Theorem 3.5 *i) For $\gamma > 0, \alpha > 0$ the function $\beta \mapsto \lambda_1(\alpha,\beta,\gamma)$ is increasing on $\mathbf{R}_+$, and we have* $\lim_{\beta\to-\infty} \lambda_1(\alpha,\beta,\gamma) = 0$, $\lim_{\beta\to\infty} \lambda_1(\alpha,\beta,\gamma) = \infty$.

ii) If $|\beta| \le \sqrt{\alpha}$ the function $\gamma \mapsto \lambda_1(\alpha,\beta,\gamma)$ is increasing, and we have $\lim_{\gamma\to\infty} \lambda_1(\alpha,\beta,\gamma) = \infty$, $\lim_{\gamma\to 0} \lambda_1(\alpha,\beta,\gamma) = \beta + \sqrt{\beta^2-\alpha}$ *if $|\beta| \le \sqrt{\alpha}$ and 2β, if $|\beta| \ge \sqrt{\alpha}$.*

iii) For $\beta \in \mathbf{R}, \gamma \ge 0$ the function $\alpha \mapsto \lambda_1(\alpha,\beta,\gamma)$ is decreasing and we have $\lim_{\alpha\to\infty} \lambda_1(\alpha,\beta,\gamma) = \beta$.

iv) For each $\alpha > 0$ there exists a smooth strictly decreasing function $f : \mathbf{R}_- \to \mathbf{R}_+$ with $\lim_{\beta\to-\infty} f(\beta) = \infty$, $\lim_{\beta\to 0} f(\beta) = 0$, *such that* $\{(\beta,\gamma) \in \mathbf{R}\times\mathbf{R}_+ \mid \lambda_1(\alpha,\beta,\gamma) = 0\} = \mathrm{graph}(f)$. *In particular, for each $\alpha \ge 0, \gamma \ge 0$ the function $\beta \mapsto \lambda_1(\alpha,\beta,\gamma)$ possesses a unique root.*

Proof:

i), ii), and iv) are immediate from Theorem 6 of [15] and the rescaling properties of the Lyapunov exponents for $\alpha > 0$ given by (33) and (34). To show iii), note first that we may confine our attention to $\gamma > 0$. Define the moments of the law with density q by

$$m_k = \int_0^\infty v^k\, q(\alpha,\beta,\gamma,v)\, dv,$$

$k \ge 0$. Then $\lambda_1(\alpha,\beta,\gamma) = \beta + \frac{\gamma}{2} m_1$, and therefore

$$\frac{\partial\lambda_1}{\partial\alpha} = \frac{\partial m_1}{\partial\alpha} = -\frac{\gamma}{2}\,(m_2 - m_1^2) < 0.$$

Hence λ_1 as a function of α is decreasing. Now recall the equation

$$\lambda_1(\alpha,\beta,\gamma) = \beta + \frac{\gamma}{2\alpha}\int_0^\infty v\, q(1,\hat\beta,\hat\gamma,v)\, dv,$$

with $\hat\beta = \frac{\beta}{\sqrt{\alpha}}, \hat\gamma = \frac{\gamma}{\alpha^{\frac{3}{2}}}$. But

$$\begin{aligned}\int_0^\infty v\, q(1,\hat\beta,\hat\gamma,v)\, dv &= \frac{\int_0^\infty \sqrt{v}\exp(-\frac{1}{12}v^3\frac{\gamma^2}{\alpha^3} + v(\frac{\beta^2}{\alpha}-1))du}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(-\frac{1}{12}v^3\frac{\gamma^2}{\alpha^3} + v(\frac{\beta^2}{\alpha}-1))du} \\ &\to \frac{\int_0^\infty \sqrt{v}\exp(-v)\, dv}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(-v)\, dv} = \frac{1}{2},\end{aligned}$$

as $\alpha \to \infty$. This implies the limiting equation claimed. □

We now treat the case of the inverted pendulum. In this case, it will be more informative to describe the null line in the $(\alpha, \gamma)-$plane, for fixed β. In this setting, it will clearly exhibit the range of possible restoring forces for which stabilization by noise takes place. In fact, as far as stabilization is concerned, the preceding Theorem already tells us that the range where $\beta > 0$ is not very interesting. We know that $\alpha \mapsto \lambda_1(\alpha, \beta, \gamma)$ is decreasing for any β and γ, and its limit as $\alpha \to \infty$ equals β. Hence for $\beta > 0$ the system is unstable, and remains unstable with increasing noise, whatever strength α of the restoring force is chosen.

Theorem 3.6 *i) For $\gamma > 0, \alpha \leq 0$ the function $\beta \mapsto \lambda_1(\alpha, \beta, \gamma)$ is increasing on $\mathbf{R}_+$, and we have $\lim_{\beta\to\infty} \lambda_1(\alpha, \beta, \gamma) = \infty$.*

ii) For $\beta \in \mathbf{R}, \gamma > 0$ the function $\alpha \mapsto \lambda_1(\alpha, \beta, \gamma)$ is decreasing, and we have $\lim_{\alpha\to-\infty} \lambda_1(\alpha, \beta, \gamma) = \infty$.

iii) For each $\beta \leq 0$ there exists a smooth function $f : \mathbf{R}_+ \to \mathbf{R}$ with $\lim_{\gamma\to 0} f(\gamma) = 0$, $\lim_{\gamma\to\infty} f(\gamma) = \infty$, and such that $\{(\gamma, \alpha) \in \mathbf{R}_+ \times \mathbf{R} \mid \lambda_1(\alpha, \beta, \gamma) = 0\} = \mathrm{graph}(f)$. f has a local maximum $f(0) = 0$, and a global minimum the value of which is given by $\beta^2(1 - \frac{4}{c^2})$, if c is the minimal value of $\frac{\gamma}{\sqrt{\beta^2-\alpha}} m_1$ as α, β, γ vary over their respective domains. Then for each $\alpha < \beta^2(1 - \frac{4}{c^2})$, the function $\gamma \mapsto \lambda_1(\alpha, \beta, \gamma)$ possesses no roots, for $\alpha \in]\beta^2(1 - \frac{4}{c^2}), 0]$, it possesses two roots, and changes sign from positive to negative at the first one, and from negative to positive at the second one, for $\alpha > 0$ it has one root, where it changes sign from negative to positive. The slope of f at $\gamma = 0$ is given by $-\frac{1}{2|\beta|}$.

Proof:
i) can be proved analogously as the first statement of the preceding Theorem.

To prove ii), according to the preceding Theorem, it remains to show that $\lim_{\alpha\to-\infty} \lambda_1(\alpha, \beta, \gamma) = \infty$. To show this, recall

$$\lambda_1(\alpha, \beta, \gamma) = \beta + \frac{\gamma}{2} m_1, \quad m_1 = \frac{\sqrt{\beta^2 - \alpha}}{\gamma} n_1,$$

where

$$n_1 = \frac{\int_0^\infty \sqrt{v} \exp(-\frac{1}{t}[\frac{v^3}{12} - v])\, dv}{\int_0^\infty \frac{1}{\sqrt{v}} \exp(-\frac{1}{t}[\frac{v^3}{12} - v])\, dv}, \quad t = \frac{\gamma}{(\beta^2 - \alpha)^{\frac{3}{2}}} \tag{40}$$

(see Theorem 5 of [15]). Now note that for β, γ fixed, $\alpha \to -\infty$ means $t \to 0$. Since according to Theorem 3.4 (alternatively, compare the proof of Theorem 5 of [15]) n_1 has a positive limit as $t \to 0$, we obtain the desired limiting behavior as $\alpha \to -\infty$.

Let us now prove iii). ii) and iii) of Theorem 3.5 imply that for $\gamma > 0, \beta \in \mathbf{R}$ the function $\alpha \mapsto \lambda_1(\alpha, \beta, \gamma)$ is strictly decreasing with limit β for $\alpha \to \infty$. By choice of β, this means that the function has exactly one root for each $\gamma > 0$, and in fact for $\gamma = 0$ as well, since we have

$$\lambda_1(\alpha,\beta,0) = \begin{cases} \beta + \sqrt{\beta^2-\alpha}, & \text{for} \quad \alpha < \beta^2, \\ \beta, & \text{for} \quad \alpha \geq \beta^2. \end{cases}$$

Hence the null line can be described by the graph of a function $f : \mathbf{R}_+ \to \mathbf{R}$ which takes the value 0 for $\gamma = 0$. Let us investigate further properties of f.

We begin with studying the behavior of f near $\alpha = \gamma = 0$. To do this, we compute the partial derivatives of λ_1 with respect to the variables α and γ. We obtain using moment equations discussed in Theorem 6 of [15]

$$\begin{aligned} \frac{\partial \lambda_1}{\partial \alpha} &= -\frac{\gamma}{2}(m_2 - m_1^2) = -\frac{1}{2}\frac{\beta^2-\alpha}{\gamma}[n_2 - n_1^2], \qquad (41) \\ \frac{\partial \lambda_1}{\partial \gamma} &= \frac{1}{6}m_1 - \frac{1}{3}(\beta^2-\alpha)[m_2 - m_1^2] \\ &= \frac{\sqrt{\beta^2-\alpha}}{6\gamma}n_1 - \frac{(\beta^2-\alpha)^2}{3\gamma^2}[n_2 - n_1^2], \end{aligned}$$

where

$$n_2 = \frac{\int_0^\infty \sqrt{v}^3 \exp(-\frac{1}{t}[\frac{v^3}{12} - v])\,dv}{\int_0^\infty \frac{1}{\sqrt{v}} \exp(-\frac{1}{t}[\frac{v^3}{12} - v])\,dv}.$$

We will now use (41) and the asymptotic expansion of n_1 of Theorem 3.4 and n_2 analogously as in the proof of Theorem 6 of [15] in t near $t = 0$ to study the behavior of $\frac{\partial \lambda_1}{\partial \gamma}$ and $\frac{\partial \lambda_1}{\partial \alpha}$ near and on the null line for $\gamma = 0$. In fact, we have

$$\begin{aligned} n_1 &= a_0 + a_1 t + a_2 t^2 + O(t^3), \\ n_2 &= b_0 + b_1 t + b_2 t^2 + O(t^3). \end{aligned}$$

Let us briefly describe how to compute the coefficients in these asymptotic expansions. For n_1, it is just a reminder of the statement of Theorem 3.4, for n_2 a straightforward extension of it. There exist a $\mathcal{C}^\infty$ function ϕ defined on a neighborhood of $x_0 = \sqrt{2}$ whose derivatives at x_0 can be computed via the recursion

$$\phi(x_0) = \sqrt{2}, \quad \phi'(x_0) = \frac{1}{2\sqrt{2}}, \quad \phi'' = -\frac{4(4+6\phi^2+\phi^4)}{3\phi^3(2+\phi^2)^3} \qquad (42)$$

and $\mathcal{C}^\infty$ functions $f, g, h : \mathbf{R}^+ \to \mathbf{R}$ satisfying

$$\begin{aligned} f^{(n)}(0) &= \frac{1}{2^n}(\phi^2\phi')^{(2n)}(x_0), \qquad (43) \\ g^{(n)}(0) &= \frac{1}{2^n}(\phi^4\phi')^{(2n)}(x_0), \\ h^{(n)}(0) &= \frac{1}{2^n}\phi^{(2n+1)}(x_0) \end{aligned}$$

such that

$$a_n = \frac{1}{n!} \left(\frac{d}{dt}\right)^n \frac{f(t)}{h(t)}\bigg|_{t=0}, \quad b_n = \frac{1}{n!} \left(\frac{d}{dt}\right)^n \frac{g(t)}{h(t)}\bigg|_{t=0}, \quad n \geq 0.$$

After some algebra one obtains

$$\begin{aligned} n_1 &= 2 - \frac{1}{2}\,t - \frac{5}{16}\,t^2 + O(t^2), \\ n_2 &= 4 - t - \frac{1}{2}\,t^2 + O(t^2), \\ n_1^2 &= 4 - 2\,t - t^2 + O(t^2). \end{aligned}$$

Substituting this into (41) gives

$$\frac{\partial \lambda_1}{\partial \alpha} = -\frac{1}{2}\,\frac{(\beta^2 - \alpha)}{\gamma}\,[n_2 - n_1^2] = -\frac{1}{2}\,\frac{1}{\sqrt{\beta^2 - \alpha}} \quad + O(t), \tag{44}$$

$$\frac{\partial \lambda_1}{\partial \gamma} = \frac{\sqrt{\beta^2 - \alpha}}{6\gamma}\,n_1 - \frac{(\beta^2 - \alpha)^2}{3\,\gamma^2}[n_2 - n_1^2] = -\frac{1}{4}\,\frac{1}{\beta^2 - \alpha} + O(t). \tag{45}$$

(44) and (45) immediately allow us to compute the slope of the null line at $\alpha = \gamma = 0$. We have

$$\lim_{\alpha,\gamma \to 0} \frac{\partial \gamma}{\partial \alpha} = - \lim_{\alpha,\gamma\to 0} \frac{\frac{\partial \lambda_1}{\partial \alpha}}{\frac{\partial \lambda_1}{\partial \gamma}}| = -2|\beta|.$$

Hence it is clear that for small positive γ the slope is negative, and consequently that f takes the local maximum 0 at $\gamma = 0$.

Using the scaling property of λ_1 and Theorem 6 of [15] we conclude that for each $\alpha > 0$ the function $\gamma \mapsto \lambda_1(\alpha, \beta, \gamma)$ has a unique root. In particular $f(\gamma)$ must be nonnegative for large γ, hence f is bounded below. Let us next find its global minimum. We know that $n_1 = n_1(t), t \geq 0$, fulfills the equations

$$\frac{d}{dt} n_1|_{t=0} = -\frac{1}{2}, \quad \lim_{t\to\infty} n_1(t) = \infty.$$

Hence n_1 must have a global minimum, which is taken for some $t_0 > 0$. Let $c = n_1(t_0)$. The *minimal line* will accordingly be given by the equation $\gamma = t_0\,(\beta^2 - \alpha)^{\frac{3}{2}}$ for $\beta^2 \geq \alpha$. This line is characterized by the condition $\frac{\partial \lambda_1}{\partial \gamma} = 0$. For (α, γ) on the null line, we must have

$$m_1 = \frac{1}{\beta^2 - \alpha}\frac{1}{t}\,n_1(t) = -\frac{2\beta}{\gamma}. \tag{46}$$

Therefore by choice of c,

$$0 < c \leq n_1 = -2\,\frac{\beta(\beta^2 - \alpha)}{(\beta^2 - \alpha)^{\frac{3}{2}}}.$$

This evidently implies

$$\alpha \geq \beta^2(1 - \frac{4}{c^2}). \tag{47}$$

Let us next show that this lower bound is in fact taken by the function f. Since the null line must be a smooth curve and is bounded below, it must cross the minimum line, say for the pair (α_0, γ_0). So we have the equation

$$\frac{1}{\beta^2 - \alpha_0} \frac{1}{t_0} c = -\frac{2\beta}{\gamma_0},$$

which implies

$$\alpha_0 = \beta^2 (1 - \frac{4}{c^2}). \tag{48}$$

Let us next investigate the behavior of the null line for $\alpha > 0$. Recall the Lyapunov exponent $\hat{\lambda}_1$ of the rescaled equation with $\hat{\alpha} = 1, \hat{\beta} = \frac{\beta}{\sqrt{\alpha}}, \hat{\gamma} = \frac{\gamma}{\alpha^{\frac{3}{2}}}$, and denote by $\hat{m}_1, \hat{m}_2, \ldots$ the respective quantities. Let $\hat{f} : \mathbf{R}_- \to \mathbf{R}_+$ be the function whose graph gives the null line of $\hat{\lambda}_1$ in the $(\hat{\beta}, \hat{\gamma})-$plane. Then (33) shows that for $\beta < 0$ fixed, the mapping

$$\alpha \mapsto \alpha^{\frac{3}{2}} \hat{f}(\frac{\beta}{\sqrt{\alpha}})$$

gives the portion of the graph of the null line for which $\alpha > 0$. To investigate its behavior as $\alpha \to \infty$, we therefore may study the asymptotics of $\hat{f}$ for $\hat{\beta}$ small. It will in fact be sufficient to find the slope of $\hat{f}$ at $\beta = 0$. To do this, we shall use the asymptotic expansion for $\hat{m}_1$ given in Theorem 3.3 for $\hat{\alpha} = 1$ and an analogous one for $\hat{m}_2$. We have

$$\hat{m}_1 = \frac{1}{2} \frac{1}{1 - \hat{\beta}^2} - \frac{15}{32} \frac{\hat{\gamma}^2}{(1 - \hat{\beta}^2)^4} + O(\hat{\gamma}^3), \quad \hat{m}_1^2 = \frac{1}{4} \frac{1}{(1 - \hat{\beta}^2)^2} - \frac{15}{32} \frac{\hat{\gamma}^2}{(1 - \hat{\beta}^2)^5} + O(\hat{\gamma}^3). \tag{49}$$

To get an expansion for $\hat{m}_2$, note first that

$$\begin{aligned} & \int_0^\infty \sqrt{v}^3 \exp(-\frac{v^3 \hat{\gamma}^2}{12} - v(1 - \hat{\beta}^2))\, dv \\ = & \frac{1}{(1 - \hat{\beta}^2)^{\frac{5}{2}}} [\Gamma(\frac{5}{2}) - \frac{\hat{\gamma}^2}{12(1 - \hat{\beta}^2)^3} \Gamma(3 + \frac{5}{2}) + O(\hat{\gamma}^3)], \end{aligned}$$

$$\begin{aligned} & \int_0^\infty \frac{1}{\sqrt{v}} \exp(-\frac{v^3 \hat{\gamma}^2}{12} - v(1 - \hat{\beta}^2))\, dv \\ = & \frac{1}{(1 - \hat{\beta}^2)^{\frac{1}{2}}} [\Gamma(\frac{1}{2}) - \frac{\hat{\gamma}^2}{12(1 - \hat{\beta}^2)^3} \Gamma(3 + \frac{1}{2}) + O(\hat{\gamma}^3)]. \end{aligned}$$

Therefore

$$\begin{aligned} \hat{m}_2 &= \frac{\int_0^\infty \sqrt{v}^3 \exp(-\frac{v^3 \hat{\gamma}^2}{12} - v(1 - \hat{\beta}^2))\, dv}{\int_0^\infty \frac{1}{\sqrt{v}} \exp(-\frac{v^3 \hat{\gamma}^2}{12} - v(1 - \hat{\beta}^2))\, dv} \qquad (50) \\ &= \frac{1}{(1 - \hat{\beta}^2)^2} [\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})} - \frac{\hat{\gamma}^2}{12(1 - \hat{\beta}^2)^3} \frac{\Gamma(3 + \frac{5}{2})\, \Gamma(\frac{1}{2}) - \Gamma(3 + \frac{1}{2})\, \Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})^2} + O(\hat{\gamma}^3)] \\ &= \frac{1}{(1 - \hat{\beta}^2)^2} [\frac{3}{4} - \frac{\hat{\gamma}^2}{(1 - \hat{\beta}^2)^3} \frac{75}{32}] + O(\hat{\gamma}^3). \end{aligned}$$

So we finally obtain

$$\frac{\partial \hat{\lambda}_1}{\partial \hat{\beta}} = 1 + \hat{\beta}\hat{\gamma}[\hat{m}_2 - \hat{m}_1^2] = 1 + \frac{\hat{\beta}\hat{\gamma}}{(1-\hat{\beta}^2)^2}[\frac{1}{2} - \frac{\hat{\gamma}^2}{(1-\hat{\beta}^2)^3}\frac{60}{32}] + O(\hat{\gamma}^4), \tag{51}$$

$$\frac{\partial \hat{\lambda}_1}{\partial \hat{\gamma}} = \frac{1}{1-\hat{\beta}^2}\frac{5}{6} - \frac{\hat{\gamma}^2}{(1-\hat{\beta}^2)^4}\frac{45}{32} + O(\hat{\gamma}^3). \tag{52}$$

The consequence of (51) and (52) we are most interested in is the following statement on the slopes of level lines in the $(\hat{\beta}, \hat{\gamma})-$plane

$$\frac{\partial \hat{\gamma}}{\partial \hat{\beta}} = -\frac{6}{5}(1-\hat{\beta}^2) + O(\hat{\gamma}). \tag{53}$$

¿From (53) we obtain

$$\frac{d}{d\hat{\beta}}\hat{f}|_{\hat{\beta}=0} = -\frac{6}{5}.$$

Hence we conclude

$$\lim_{\gamma\to\infty} f(\gamma) = \lim_{\alpha\to\infty} \alpha^{\frac{3}{2}}\hat{f}(\frac{\beta}{\sqrt{a}}) = \infty.$$

In fact, we even know that f becomes asymptotically linear as $\gamma \to \infty$ and the slope of the line is $\frac{5}{6}$. This completes the proof. □

Remark:
Numerical values for the minimum of n_1 are available. We have $t_0 = 1.69461$, and $c = 1.45677$. This gives a numerical estimate for the critical restoring force below which there is no more stabilization by noise. It is given by $\alpha_0 = -0.88486\ \beta^2$.

3.3 The rotation number

Using the notation of the preceding section, we see that Theorem 2.2 implies

$$\begin{aligned}\tilde{\rho} &= -\frac{1}{|\sigma|\sqrt{2\pi}}\frac{1}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(-\frac{1}{6\sigma^2}v^3 + \frac{4v}{\sigma^2}(\beta^2-\alpha))\,dv} \\ &= -\sqrt{\frac{2}{\pi}}\frac{1}{\gamma}\frac{1}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(-\frac{\gamma^2 v^3}{12} + v(\beta^2-\alpha))\,dv}\end{aligned}$$

hence

$$\rho = \sigma^2\,\tilde{\rho} = -2\sqrt{\frac{2}{\pi}}\frac{1}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(-\frac{\gamma^2 v^3}{12} + v(\beta^2-\alpha))\,dv}. \tag{54}$$

An expansion of ρ in terms of hypergeometric functions is given by the following Theorem, the proof of which uses the idea of the proof of Theorem 2 of [15].

Theorem 3.7 *Assume* $\gamma = \frac{\sigma^2}{2} > 0$ *and define* $\delta = \frac{4(\beta^2-\alpha)^3}{9\gamma^2}$. *Then for* $|\beta| \neq \sqrt{\alpha}$, *setting*

$$\begin{aligned} H(\alpha,\beta,\gamma) \;=\; & \Big[\ \gamma^{4/3} \cdot 2^{1/3}\, \Gamma(\tfrac{1}{6})\ {}_1F_2(\tfrac{1}{6}; \tfrac{1}{3}, \tfrac{2}{3}; \delta) \\ & + \gamma^{2/3}(\beta^2-\alpha) \cdot 2 \cdot 3^{1/3} \cdot \sqrt{\pi}\ {}_1F_2(\tfrac{1}{2}; \tfrac{2}{3}, \tfrac{4}{3}; \delta) \\ & + (\beta^2-\alpha)^2\ 6^{2/3}\Gamma(\tfrac{5}{6})\ {}_1F_2(\tfrac{5}{6}; \tfrac{4}{3}, \tfrac{5}{3}; \delta)\ \Big], \end{aligned}$$

we have

$$\rho = -2\sqrt{\frac{2}{\pi}}\, 3^{\frac{5}{6}}\, \gamma^{\frac{5}{3}}\, \frac{1}{H(\alpha,\beta,\gamma)}.$$

For $|\beta| = \sqrt{\alpha}$ *we have*

$$\rho = -6\sqrt{\frac{2}{\pi}}\, \frac{1}{12^{1/6}\,\Gamma(\frac{1}{6})}\, \gamma^{1/3}$$

We next obtain an asymptotic expansion of the rotation number in γ near $\gamma = \infty$.

Theorem 3.8 *Let* $\gamma > 0$. *Then we have*

$$\rho \;=\; -2\sqrt{\frac{2}{\pi}} \cdot \frac{3}{12^{1/6}} \sum_{n=0}^{\infty} \left((\beta^2-\alpha) \cdot \frac{12^{1/3}}{\gamma^{2/3}} \right)^n c_n,$$

where

$$\begin{aligned} c_0 &= \frac{1}{\Gamma(\frac{1}{6})}, \\ c_n &= \sum_{m=1}^{n} \frac{(-1)^m}{\Gamma(\frac{1}{6})^{m+1}} \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\ldots+l_m = n}} \prod_{i=1}^{m} \frac{\Gamma(\frac{l_i}{3}+\frac{1}{6})}{l_i!}, \quad n \geq 1. \end{aligned}$$

Proof:
We have to apply Lemma 1 of [15] with $b_0 = 1, b_n = 0, n \geq 1$, and $a_n = \frac{\Gamma(\frac{n}{3}+\frac{1}{6})}{n!}, n \geq 0$. □

Here is an asymptotic expansion for the rotation number in γ near $\gamma = 0$ for the case $|\beta| < \sqrt{\alpha}$.

Theorem 3.9 *Suppose* $|\beta| < \sqrt{\alpha}$. *Let*

$$\begin{aligned} c_0 &= 1, \\ c_n &= \sum_{m=1}^{n} (-1)^m \sum_{\substack{l_1,\ldots,l_m \geq 1 \\ l_1+\ldots+l_m = n}} \prod_{i=1}^{m} \frac{(6l_i)!}{(3l_i)!\,l_i!}, \quad n \geq 1. \end{aligned}$$

Then the formal (but not convergent) asymptotic expansion of ρ *in* $\gamma = \frac{\sigma^2}{2}$ *at* $\gamma = 0$ *is given by*

$$\rho \;=\; -\frac{8\sqrt{2}}{\pi}\sqrt{\alpha-\beta^2} \sum_{n=0}^{\infty} c_n \left(\frac{\gamma^2}{3 \cdot 2^8(\beta^2-\alpha)^3} \right)^n$$

Proof:
Apply Lemma 1 of [15] with $b_0 = 1, b_n = 0, n \geq 1$, and $a_n = \frac{(6n)!}{(3n)!\,n!}, n \geq 0$. □

We finally turn to the case $|\beta| > \sqrt{\alpha}$. In this case we obviously have $\rho = 0$ on the axis $\gamma = 0$.

Theorem 3.10 *Assume* $|\beta| > \sqrt{\alpha}$. *Then the formal (but not convergent) asymptotic expansion of the quantity* $\exp(\frac{4(\beta^2-\alpha)^{3/2}}{3\gamma}) \cdot \rho$ *in* $\gamma = \frac{\sigma^2}{2}$ *at* $\gamma = 0$ *is given by*

$$\exp(\frac{4(\beta^2-\alpha)^{3/2}}{3\,\gamma}) \cdot \rho \;=\; -2\sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} c_n (\beta^2-\alpha)^{-(3n-1)/2}\,\gamma^n, \tag{55}$$

where the c_n *can be computed as follows:*
There exist a $\mathcal{C}^\infty$ *function* ϕ *defined on a neighborhood of* $x_0 = \sqrt{2}$ *whose derivatives at* x_0 *are given via the recursion*

$$\phi(x_0) \;=\; \sqrt{2}, \quad \phi'(x_0) \;=\; \frac{1}{2\sqrt{2}}, \quad \phi'' \;=\; -\frac{4(4+6\phi^2+\phi^4)}{3\phi^3(2+\phi^2)^3} \tag{56}$$

and a $\mathcal{C}^\infty$ *function* $f : \mathbf{R}^+ \to \mathbf{R}$ *satisfying*

$$f^{(n)}(0) \;=\; \frac{1}{2^n}\phi^{(2n+1)}(x_0) \tag{57}$$

such that

$$c_n \;=\; \frac{1}{n!}\left(\frac{d}{dt}\right)^n \frac{1}{f(t)}\bigg|_{t=0}, \quad n \geq 0.$$

Proof:
Setting as before $t = \frac{\gamma}{(\beta^2-\alpha)^{3/2}}$ we may write

$$\begin{aligned}
\rho &= -2\sqrt{\frac{2}{\pi}}\frac{1}{\int_0^\infty \frac{1}{\sqrt{v}}\exp(-\frac{\gamma^2 v^3}{12} + v(\beta^2-\alpha))\,dv}. \\
&= -\sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\beta^2-\alpha}}{\sqrt{2\pi}} \cdot \exp(-\frac{4}{3\;t})\;\frac{1}{\frac{1}{\sqrt{2\pi t}}\int_0^\infty \exp(-\frac{1}{t}[\frac{x^6}{12} - x^2 + 4/3])\,dx} \\
&= -\frac{1}{\pi}\sqrt{\beta^2-\alpha} \cdot \exp(-\frac{4}{3\,t}) \sum_{n=0}^{\infty} c_n (\beta^2-\alpha)^{-(3n-1)/2}\,\gamma^n,
\end{aligned}$$

with c_n as described. The derivation of the recursion formulas for the coefficients was given in Theorem 5 of [15]. □

Plots

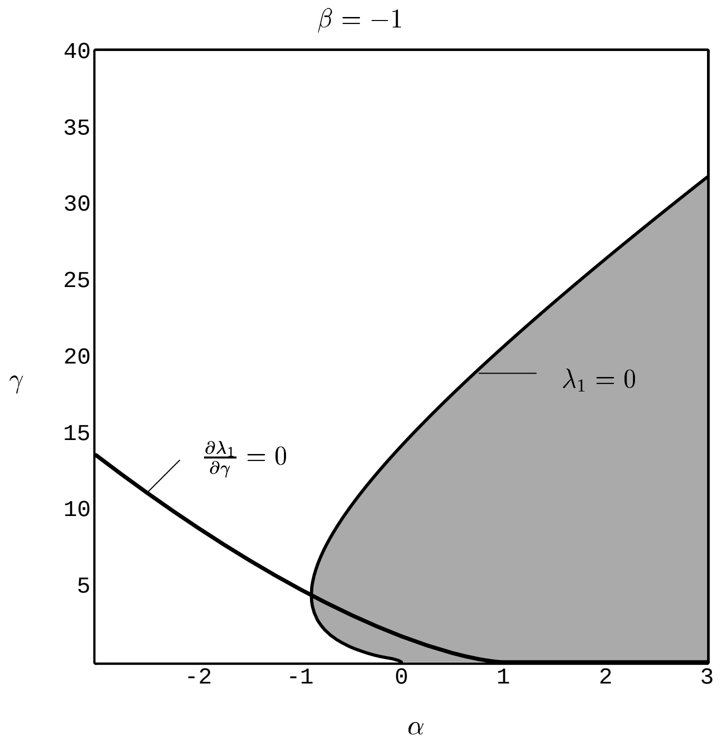

Figure 1: Roots of λ_1 as a function of (α, γ) for $\beta = -1$

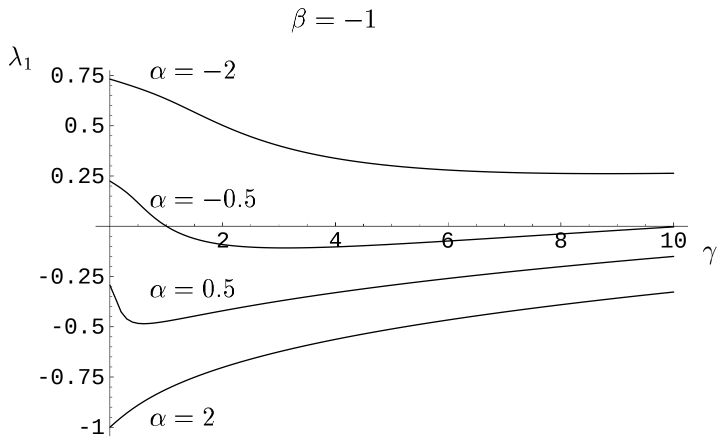

Figure 2: λ_1 as a function of γ for fixed (α, β)

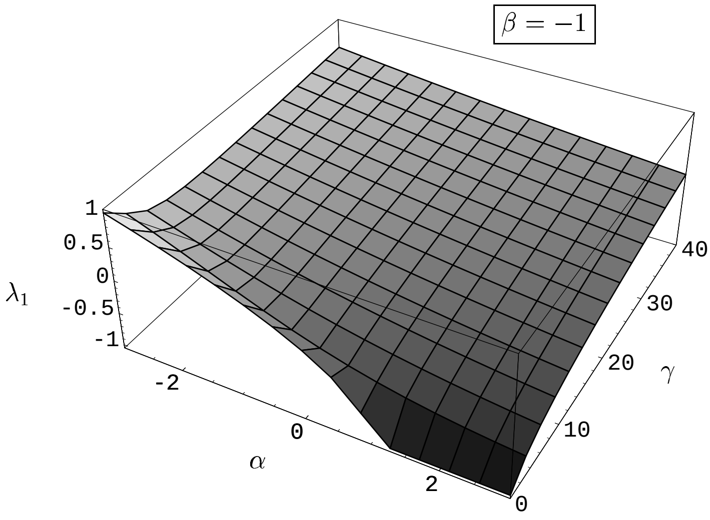

Figure 3: λ_1 as a function of α and γ for fixed β

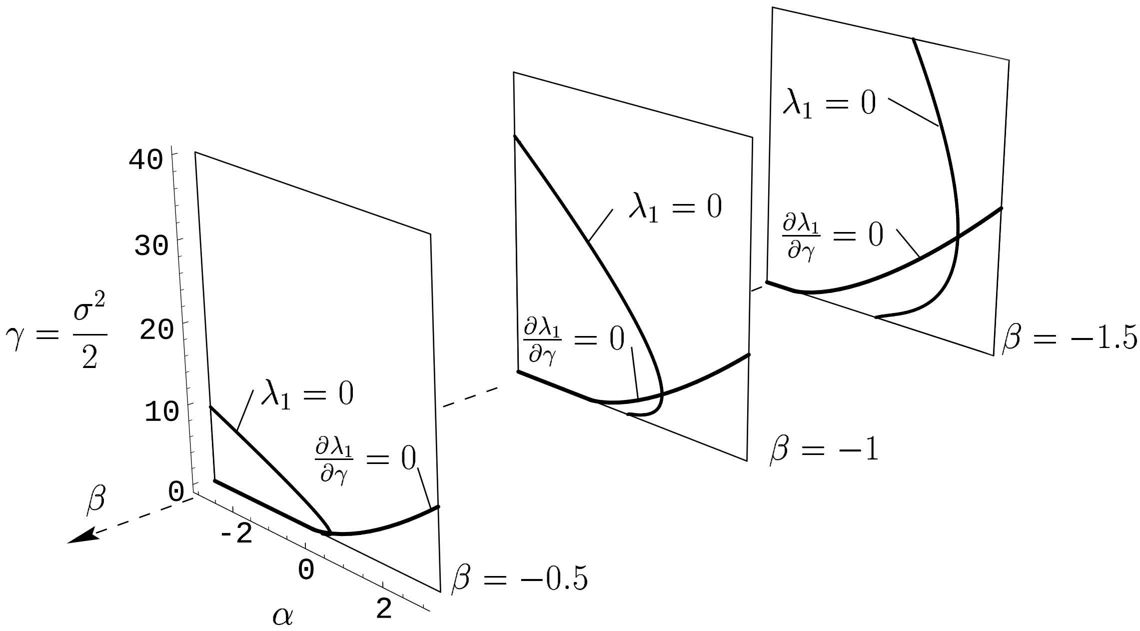

Figure 4: Dependency of the stability diagram on β

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