Simple SDE dynamical models interpreting climate data and their meta-stability

PETER IMKELLER
(joint work with Claudia Hein, Ilya Pavlyukevich)

Simple models of the earth's energy balance are able to interpret some qualitative aspects of the dynamics of paleo-climatic data. In the 1980s this led to the investigation of periodically forced dynamical systems of the reaction-diffusion type with small Gaussian noise, and a rough explanation of glacial cycles by Gaussian meta-stability. A spectral analysis of Greenland ice time series performed at the end of the 1990s representing average temperatures during the last ice age suggest an $\alpha-$stable noise component with an $\alpha \sim 1.75$. ([3], [4]). Based on this observation, papers in the physics literature attempted an interpretation featuring dynamical systems perturbed by small Lévy noise. The typical 1-dimensional system is given by the stochastic differential equation

$$X^\epsilon(t) = x - \int_0^t U'(X^\epsilon(s-))\,ds + \epsilon L(t), \quad t \geq 0, \epsilon > 0. \tag{1}$$

Here L is symmetric α-stable Lévy process, with $\alpha \in (0,2)$, U a potential function, at least C^2, typically with finitely many local minima where U has positive curvature, separated by saddles where U has negative curvature. We study the exit time and transition asymptotics in the small noise limit $\epsilon \to 0$ between the meta-stable local minima of the potential function for the solution trajectories of (1). Due to the heavy-tail nature of its α-stable component, the results for Lévy noise differ strongly from the well known case of purely Gaussian perturbations. For a comparison of Gaussian and α-stable noise asymptotics in a simple case, consider a potential function with exactly one well at 0, where $U(0) = 0$, assume that $U' < 0$ on $]-\infty, 0[$, and $U' > 0$ on $]0, \infty[$, consider the exit times from an interval $[-b, a]$ containing 0, and let $h = \max\{U(-b), U(a)\}$. Denote by σ resp. τ the exit time of the trajectories of X^ϵ resp. the analogue of (1) with a Wiener process W replacing L. Then the following type of asymptotic behavior results, for which the Gaussian part is classical.

Theorem 0.1. *For any $\delta > 0$, $x \in]-b, a[$ with respect to the probability measure $\mathbf{P_x}$ of X^ϵ perturbed by W starting at x*

$$\mathbf{P_x}(e^{(2h-\delta)/\epsilon^2} < \tau < e^{(2h+\delta)/\epsilon^2}) \to 1 \quad (\epsilon \to 0) \quad \textit{(Freidlin-Wentzell)},$$

$$\mathbf{E_x}\tau \approx \frac{\epsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}} e^{2h/\epsilon^2} \quad \textit{(Kramers, Williams, Bovier et al.)},$$

$$\mathbf{P_x}(\frac{\tau}{\mathbf{E_x}\tau} > u) \sim \exp(-u) \quad \textit{(Day, Bovier et al.)}.$$

Theorem 0.2. *([8]) For any $\delta > 0$, $x \in]-b, a[$ with respect to the probability measure $\mathbf{P_x}$ of X^ϵ perturbed by L starting at x*

$$\mathbf{P_x}(\frac{1}{\epsilon^{\alpha-\delta}} < \sigma < \frac{1}{\epsilon^{\alpha+\delta}}) \to 1 \quad (\epsilon \to 0),$$

$$\mathbf{E_x}\sigma \approx \frac{1}{\epsilon^\alpha}\Big(\int_{\mathbb{R}\setminus[-b,a]} \frac{dy}{|y|^{1+\alpha}}\Big)^{-1},$$

$$\mathbf{P_x}(\frac{\sigma}{\mathbf{E_x}\sigma} > u) \sim \exp(-u).$$

Ditlevsen's ([3], [4]) interpretation of paleo-climatic time series by simple dynamical systems with noise presents a typical statistical model selection problem. In the parametric version involved, one needs an efficient testing method for the parameter α corresponding to the best fitting α-stable noise component. We develop a statistical testing method based on the p-variation of the solution trajectories of SDE with Lévy noise which have been used before in model selection problems for high frequency financial time series (Corcuera et al. [2], Jacod [9], Barndorff-Nielsen, Shephard [1]). If $X^\epsilon = Y^\epsilon + L^\epsilon$ denotes the solution of (1), where Y^ϵ is the absolutely continuous part related to the potential gradient, and $L^\epsilon = \epsilon L$, and if for some stochastic process Z we let

$$V_t^{p,n}(Z) = \sum_{i=1}^{[nt]} |Z(\frac{i}{n}) - Z(\frac{i-1}{n})|^p, \quad V_t^p(Z) = \lim_{n\to\infty} V_t^{p,n}(Z), \quad t \geq 0,$$

we take $V^p(X^\epsilon)$ as a test statistic for α. In fact, it is well known that the stability index α of a stable process L can be identified with the least p for which $V^p(L) = 0$. Our elementary testing results for the real data from the Greenland ice core are based on the following limit theorems. Special cases of the first one have been treated in Greenwood [5] and Greenwood, Fristedt [6].

Theorem 0.3. *([7]) Let $(L_t)_{t\geq 0}$ be an α-stable Lévy process. If $p > \alpha/2$ then*

$$\big(V_p^n(L)_t - nt\, B_n(\alpha, p)\big)_{t\geq 0} \xrightarrow{\mathcal{D}} (L'_t)_{t\geq 0} \quad \text{as } n \to \infty, \tag{2}$$

where L' is an independent $\frac{\alpha}{p}$-stable Lévy process, and $\xrightarrow{\mathcal{D}}$ denotes convergence in the Skorokhod topology. The normalizing sequence $(B_n(\alpha, p))_{n\geq 1}$ is deterministic and given by

$$B_n(\alpha, p) = \begin{cases} n^{-p/\alpha}\mathbf{E}|L_1|^p, & p \in (\alpha/2, \alpha), \\ \mathbf{E}\sin\big(n^{-1}|L_1|^\alpha\big), & p = \alpha, \\ 0, & p > \alpha. \end{cases} \tag{3}$$

Intuitively, adding absolutely continuous processes to a Lévy process L should not alter its power variations. This intuition is confirmed by the following theorems.

Theorem 0.4. *([7]) Let $(L_t)_{t\geq 0}$ be an α-stable Lévy process, and $(Y_t)_{t\geq 0}$ be another stochastic process that satisfies*

$$V_p^n(Y) \to 0, \quad n \to \infty, \tag{4}$$

uniformly on compacts of $\mathbb{R}_+$ in probability for some $p \in (\alpha/2, 1) \cup (\alpha, \infty)$. Then

$$(V_p^n(L+Y)_t - nt\, B_n(\alpha, p))_{t\geq 0} \xrightarrow{\mathcal{D}} (L'_t)_{t\geq 0} \quad \text{as } n \to \infty, \tag{5}$$

with L' an independent $\frac{\alpha}{p}$-stable process, $(B_n(\alpha, p))_{n\geq 1}$ defined in (3).

Theorem 0.5. *([7]) Let $(L_t)_{t\geq 0}$ be an α-stable Lévy process, $\alpha \in (1,2)$ and let $(Y_t)_{t\geq 0}$ be another stochastic process. Let $p \in (1, \alpha]$ and $T > 0$. If Y is such that for every $\delta > 0$ there exists $K(\delta) > 0$ that satisfies*

$$\mathbf{P}(|Y_s(\omega) - Y_t(\omega)| \leq K(\delta)|s-t| \text{ for all } s,t \in [0,T]) \geq 1 - \delta, \tag{6}$$

the process Y does not contribute to the limit of $V_p^n(L+Y)$, i.e.

$$(V_p^n(L+Y)_t - nt\, B_n(\alpha, p))_{0\leq t\leq T} \xrightarrow{\mathcal{D}} (L'_t)_{0\leq t\leq T}, \quad \text{as } n \to \infty, \tag{7}$$

with L' an independent $\frac{\alpha}{p}$-stable process, $(B_n(\alpha, p))_{n\geq 1}$ defined in (3).

We apply these limit theorems to test the real data from the Greenland ice core for the best fitting stability index α. To this end, we use the Kolmogorov-Smirnov distance between the empirical law of $V^{2\alpha,n}(X^\epsilon)$ and the $\frac{1}{2}$-stable law, stipulating that this distance has to be minimal in case the noise in the data contains an α-stable component. We obtain $\alpha = 0.73$ as best fit, which is nicely confirmed in a comparison with simulated data, and surprisingly differs from the result obtained by Ditlevsen ([3], [4]) by a quantity very close to 1.

References

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