# Stochastic Resonance in Two-State Markov Chains

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In this paper we introduce a model which provides a new approach to the phenomenon of stochastic resonance. It is based on the study of the properties of the stationary distribution of the underlying stochastic process. We derive the formula for the spectral power aplification coefficient, study its asymptotic properties and dependence on parameters.

## Introduction

The notion of *Stochastic Resonance* appeared about twenty years ago in the works of Benzi et al. [1] and Nicolis [2] in the context of an attempt to explain the phenomenon of ice ages. The modern methods of acquiring and interpreting climate records indicate at least seven major climate changes in the last 700,000 years. These changes occured with the periodicity of about $100,000$ years and are characterized by a substantial variation of the average Earth's temperature of about $10K$.

The effect can be explained with the help of a simple energy balance model (for an extended review on the subject see [3]). The Earth is considered as a point in space, and its temporally and spatially averaged temperature $X(t)$ satysfies the equation

$$\dot{X}(t) = -U'(X(t)) - Q\sin\left(\frac{2\pi t}{T}\right), \tag{1}$$

where $U(X)$ is a double-well potential with minima at $278.6K$ and $288.6K$ and saddle point at $283.3K$ and wells of equal depth. The second term in (1) corresponds to a small variation of the solar constant of about $0.1\%$ with a period of $T = 100,000$ years due to the periodic change of the eccentricity of the Earth's orbit caused by Jupiter. The influence of this term reflects itself in small periodic changes of the depths of the potential wells. In this setting, the left well is deeper during the time intervals $(kT, (k+\frac{1}{2})T)$, whereas the right one is deeper during the intervals $((k+\frac{1}{2})T, (k+1)T)$, $k = 0, 1, 2, \ldots$

The trajectories of the deterministic equation (1) have two metastable states given by the minima of the wells. Due to the smallness of the solar constant $Q$ no transition between these states is possible. In order to obtain such transitions Benzi et al. [1] and Nicolis [2] suggested to add noise to the system which results in considering the stochastic differential equation

$$\dot{X}^{\epsilon,T}(t) = -U'(X^{\epsilon,T}(t)) - Q\sin\left(\frac{2\pi t}{T}\right) + \sqrt{\epsilon}\dot{W}_t, \tag{2}$$

$\epsilon > 0$, $\dot{W}$ a white noise.

Now one can observe the following effect. Fix all parameters of the system except $\epsilon$ and consider the typical behaviour of the solutions of (2) for different values of $\epsilon$. If the noise intensity is very small, the trajectory only occasionally can escape from the minimum of the well in which it is staying, and one can hardly detect any periodicity in this motion. If the intensity is very large, the trajectory jumps rapidly but randomly between the two wells and therefore also lacks periodicity properties. An interesting effect appears when the noise level takes a certain value $\epsilon_0$: the trajectory always tends to be near the minimum of the deepest well and consequently follows the deterministic periodic jump function which describes the location of the deepest well's minimum. It is very important to note that to produce this effect one needs all three of the following components to be present in the system (2): the double-well potential for bi-stability, the noise to pass the potential barriers, and a small periodic perturbation to change the wells' depths.

The following are natural questions arising in the context of these qualitative considerations: how can one measure *periodicity* of the trajectories and, consequently, how does the quality of *tuning* of the noisy output to the periodic input be improved by adjusting the noise intensity $\epsilon$?

The formulation of the latter question suggests to consider the system (2) as a random amplifier. The random system receives the harmonic signal of small amplitude $Q$ and usually large period $T$ as input. The stochastic process $X^{\epsilon,T}(t)$ is observed as the output. The input signal carries power $Q^2$ at frequency $1/T$. The random output has continuous spectrum and thus carries power at all frequencies. Benzi et al. [1] considered the power spectrum of the output for different values of $\epsilon$ and discovered a sharp peak at the input frequency for a certain optimal value of $\epsilon_0$. This means that the random process $X^{\epsilon_0,T}(t)$ has a big component of frequency $1/T$. The effect of amplification of the power carried by the harmonic considered as a response of the nonlinear system (2) to optimally chosen noise was called *stochastic resonance*.

In the past twenty years more than three hundred papers on this subject were published. An extensive description of the phenomenon from the physical point of view can be found in [4] and [5]. The notion *stochastic resonance* is now used in a much broader sense. It describes a wide class of effects with the common underlying property: the presence of noise induces a qualitatively new behaviour of the system and improves some of its characteristics.

Although stochastic resonance was observed and studied in many physical systems, only few mathematically rigorous results are known. The approach of M. Freidlin is briefly outlined in the next section of this paper. In sections 2, 3 and 4 we introduce discrete-time Markov chains with transition probabilities chosen in such a way, that on a large temporal scale the *attractor hopping* behaviour of the underlying diffusion process is imitated in the limit $\epsilon \to 0$. We investigate stochastic resonance for the Markov chains. The last section is devoted to generalizations and discussion.

# 1 Large deviations approach

In this section we briefly survey rigorous mathematical results obtained by M. Freidlin in [6] using the theory of large deviations for randomly perturbed dynamical systems, developped in Freidlin and Wentzell (see [7]). Though the results of [6] are valid in a quite general framework, we confine our attention to a simple example of a diffusion with weak noise.

Consider the SDE in $\mathbb{R}$

$$\dot{X}^{\epsilon,T}(t) = -U'(X^{\epsilon,T}(t), \frac{t}{T}) + \sqrt{\epsilon}\dot{W}(t), \tag{3}$$

where $\dot{W}$ is a white noise and $U'(x,t) = \frac{\partial}{\partial x}U(x,t)$, with a time dependent potential just periodically switching between two symmetric double well states, i.e.

$$U(x,t) = \sum_{k\geq 0} U(x)\mathbf{1}_{[k,k+\frac{1}{2})}(t) + U(-x)\mathbf{1}_{[k+\frac{1}{2},k+1)}(t),$$

where $U(x)$ has local minima in $x = \pm 1$ and a saddle point in $x = 0$, $\lim_{|x|\to\infty} U(x) = \infty$. We also fix the depths of the wells by two numbers $0 < v < V$, assuming that $U(-1) = -V/2$, $U(1) = -v/2$, and $U(0) = 0$. Note, that $X^{\epsilon,T}$ is a Markov process which is not time homogeneous. In the following Theorem time scales are determined in which some form of periodicity is observed.

**Theorem 1** *Suppose $T = T(\epsilon)$ is given such that*

$$\lim_{\epsilon\to 0} \epsilon \ln T(\epsilon) = \lambda > 0.$$

*a) If $\lambda < v$, then the Lebesgue measure of the set*

$$\{t \in [0,1] \,:\, |X^{\epsilon,T(\epsilon)}(T(\epsilon)t) - \mathrm{sgn}X_0| > \delta\}$$

*converges to 0 in $P_{X_0}$ probability as $\epsilon \to 0$, for any $\delta > 0$.*

*b) If $\lambda > v$, then the Lebesque measure of the set*

$$\{t \in [0,1] \,:\, |X^{\epsilon,T(\epsilon)}(T(\epsilon)t) - \phi(t)| > \delta\}$$

*converges to 0 in $P_{X_0}$ probability as $\epsilon \to 0$, for any $\delta > 0$, where*

$$\phi(t) = \sum_{k\geq 0} -\mathbf{1}_{[k,k+\frac{1}{2})}(t) + \mathbf{1}_{[k+\frac{1}{2},k+1)}(t)$$

*and $P_{X_0}$ denotes the law of the diffusion starting in $X_0$.* •

It is nessesary to explain why $\lambda = v$ is critical for the long time behaviour of the diffusion. At least intuitevely, the answer follows from the asymptotics of the mean exit time from a potential well for the time-homogeneous diffusion. If the diffusion starts in the potential well with the depth $v/2$, its mean time $\mathbf{E}(\tau(\epsilon))$ needed to leave the well satisfies

$$\epsilon \ln \mathbf{E}(\tau(\epsilon)) \to v, \quad \epsilon \to 0.$$

according to Freidlin and Wentzell [7]. This means, $X^{\epsilon,T(\epsilon)}$ can leave neither the deep well with the depth $V/2$ nor the shallow one with the depth $v/2$ in time $T(\epsilon)$ of order $e^{\lambda/\epsilon}$ if $\lambda < v$. Therefore, $X^{\epsilon,T(\epsilon)}$ stays in the $\delta$-neighbourhood of the minimum of the initial well. On the other hand, if $\lambda > v$, $X^{\epsilon,T(\epsilon)}$ has always enough time to reach the deepest well. In both cases, the Lebesgue measure of excursions leaving the $\delta$-tube of the deterministic periodic function $\phi$ is exponentially negligible on the time scale $T(\epsilon)$ as $\epsilon \to 0$.

The Theorem suggests the time scale which induces periodic and deterministic behaviour of the system (3), and the Lebesgue measure as a measure of quality. In fact, it only gives a lower bound for the scale. In the next section, in the framework of discrete Markov chains approximating the diffusion processes just considered, we investigate different measures of quality which provide unique optimal tuning.

# 2 Markov chains with time-periodic transition probabilities

For $m \in \mathbf{N}$, consider a Markov chain $X_m = (X_m(k))_{k\geq 0}$ on the state space $\mathcal{S} = \{-1, 1\}$. Let $P_m(k)$ be the matrix of one-step transition probabilities at time $k$. If we denote $\pi_m^-(k) = \mathbf{P}(X_m(k) = -1)$, $\pi_m^+(k) = \mathbf{P}(X_m(k) = 1)$, and write $P^*$ for the transposed matrix, we have

$$\begin{pmatrix} \pi_m^-(k+1) \\ \pi_m^+(k+1) \end{pmatrix} = P_m^*(k) \begin{pmatrix} \pi_m^-(k) \\ \pi_m^+(k) \end{pmatrix}.$$

In order to model the periodic switching of the double-well potential in our Markov chains, we define the transition matrix $P_m$ to be periodic in time with half-period $m$. More precisely,

$$P_m(k) = \begin{cases} P_1, & 0 \leq k (\mathrm{mod}\ 2m) \leq m-1, \\ P_2, & m \leq k (\mathrm{mod}\ 2m) \leq 2m-1, \end{cases}$$

with

$$
\begin{aligned}
P_1 &= \begin{pmatrix} 1-\phi & \phi \\ \psi & 1-\psi \end{pmatrix} = \begin{pmatrix} 1-px^V & px^V \\ qx^v & 1-qx^v \end{pmatrix}, \\
P_2 &= \begin{pmatrix} 1-\psi & \psi \\ \phi & 1-\phi \end{pmatrix} = \begin{pmatrix} 1-qx^v & qx^v \\ px^V & 1-px^V \end{pmatrix}.
\end{aligned} \tag{4}
$$

where $\phi = \phi(\epsilon, p, V) = pe^{-V/\epsilon}$, $\psi = \psi(\epsilon, q, v) = qe^{-v/\epsilon}$, $x = e^{-1/\epsilon}$, $0 \le p, q \le 1$, $0 < v < V < +\infty$, $0 < \epsilon < +\infty$. Sometimes, it will be convenient to consider $x \in [0, 1]$. In these cases the ends of the interval will correspond to the limits $\epsilon \to 0$ and $\epsilon \to \infty$.

In this setting, the numbers $V/2$ and $v/2$ clearly have to be associated with the depths of the potential wells, $\epsilon$ with the level of noise. According to the Freidlin-Wentzel theory, the exponential factors in the one-step transition probabilities just correspond to the inverses of the expected transition times between the respective wells for the diffusion considered in the preceding section. This is what should be expected for a Markov chain *in equilibrium*, modulo the phenomenological *prefactors* $p$ and $q$. They model the prefactors appearing in large deviation statements, and add assymetry to the picture.

It is well known that for a time-homogeneuos Markov chain on $S$ with transition matrix $P$ one can talk about *equilibrium*, given by the stationary distribution, to which the law of the chain converges exponentially fast. The stationary distribution can be found by solving the matrix equation $\pi = P^*\pi$ with norming condition $\pi^- + \pi^+ = 1$.

For non time homogeneous Markov chains with time periodic transition matrix, the situation is quite similar. Enlarging the state space $S$ to $S_m = \{-1, 1\} \times \{0, 1, \ldots, 2m-1\}$, we recover a time homogeneous chain by setting

$$
Y_m(k) = (X_m(k), k(\text{mod } 2m)), \quad k \ge 0,
$$

to which the previous remarks apply. For convenience of notation, we assume $S_m$ to be ordered in the following way:
$\mathcal{S}_m = [(-1,0), (1,0), (-1,1), (1,1), \ldots, (-1, 2m-1), (1, 2m-1)]$. Writing $\mathrm{A}_m$ for the matrix of one-step transition probabilities of $Y_m$, the stationary distribution $Q = (q(i,j))^*$ is obtained as a normalized solution of the matrix equation $(\mathrm{A}_m^* - E)Q = 0$, $E$ being the unit matrix. We shall be dealing with the following variant of stationary measure, which is not normalized in time.

**Definition 1** *Let* $\pi_m(k) = (\pi_m^-(k), \pi_m^+(k))^* = 2m(q(-1,k), q(1,k))^*$, $0 \le k \le 2m-1$. *We call the set* $\pi_m = (\pi_m(k))_{0 \le k \le 2m-1}$ *the stationary distribution of the Markov chain* $X_m$.

The matrix $\mathrm{A}_m$ of one-step transition probabilities of $Y_m$ is explicitly given by

$$
\mathrm{A}_m = \begin{pmatrix}
0 & P_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & P_1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & & & & & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & P_2 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & P_2 \\
P_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}.
$$

$\mathrm{A}_m$ has block structure. In this notation 0 means a $2\times 2$-matrix with all entries equal to zero, $P_1$, and $P_2$ are the 2-dimensional matrices defined in (4).

Applying some algebra we see that $(\mathrm{A}_m^* - E)Q = 0$ is equivalent to $A'_m\, Q = 0$, where

$$\mathrm{A}'_m = \begin{pmatrix} \widehat{P}-E & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ P_1^* & -E & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -E & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & P_2^* & -E & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & P_2^* & -E \end{pmatrix}$$

and $\widehat{P} = P_2^* P_2^* \cdots P_1^* = (P_2^*)^m (P_1^*)^m$. But $\mathrm{A}'_m$ is a block-wise lower diagonal matrix, and so $A'_m Q = 0$ can be solved in the usual way to give

**Theorem 2** *For every $m \geq 1$, the stationary distribution $\pi_m$ of $X_m$ with matrices of one-step probabilities defined in (4) is:*

$$\begin{cases} \pi_m^-(l) = \dfrac{\psi}{\phi+\psi} + \dfrac{\phi-\psi}{\phi+\psi}\,\dfrac{(1-\phi-\psi)^l}{1+(1-\phi-\psi)^m}, \\ \pi_m^+(l) = \dfrac{\phi}{\phi+\psi} - \dfrac{\phi-\psi}{\phi+\psi}\,\dfrac{(1-\phi-\psi)^l}{1+(1-\phi-\psi)^m}; \end{cases}$$
$$\begin{cases} \pi_m^-(l+m) = \pi_m^+(l), \\ \pi_m^+(l+m) = \pi_m^-(l), \qquad 0 \leq l \leq m-1. \end{cases} \tag{5}$$

*Proof.* $\pi_m(0)$ satisfies the matrix equation $((P_2^*)^m(P_1^*)^m - E)\pi_m(0) = 0$ with additional condition $\pi_m^-(0) + \pi_m^+(0) = 1$. To calculate $(P_2^*)^m(P_1^*)^m$, we use a formula for the $m$-th power of $2\times 2$-matrices, which results in

$$\begin{pmatrix} p_{-1,-1} & p_{-1,1} \\ p_{1,-1} & p_{1,1} \end{pmatrix}^m = \frac{1}{2-p_{-1,-1}-p_{1,1}} \begin{pmatrix} 1-p_{1,1} & 1-p_{-1,-1} \\ 1-p_{1,1} & 1-p_{-1,-1} \end{pmatrix}$$
$$+ \frac{(p_{-1,-1}+p_{1,1}-1)^m}{2-p_{-1,-1}-p_{1,1}} \begin{pmatrix} 1-p_{-1,-1} & -(1-p_{-1,-1}) \\ -(1-p_{1,1}) & 1-p_{1,1} \end{pmatrix}$$

Using some more elementary algebra we find

$$\begin{aligned} (P_2^*)^m(P_1^*)^m &= ((P_1)^m(P_2)^m)^* = \begin{pmatrix} 1-\psi & \psi \\ \phi & 1-\phi \end{pmatrix}^m \begin{pmatrix} 1-\phi & \phi \\ \psi & 1-\psi \end{pmatrix}^m = \\ &= \frac{1}{\phi+\psi} \begin{pmatrix} \phi & \phi \\ \psi & \psi \end{pmatrix} + (1-\phi-\psi)^m \frac{\phi-\psi}{\phi+\psi} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \\ &+ \frac{(1-\phi-\psi)^{2m}}{\phi+\psi} \begin{pmatrix} \phi & -\psi \\ -\phi & \psi \end{pmatrix}, \end{aligned}$$

from which a straightforward calculation yields

$$\begin{cases} \pi_m^-(0) = \dfrac{\phi+\psi(1-\phi-\psi)^m}{(\phi+\psi)(1+(1-\phi-\psi)^m)}, \\ \pi_m^+(0) = \dfrac{\psi+\phi(1-\phi-\psi)^m}{(\phi+\psi)(1+(1-\phi-\psi)^m)}. \end{cases}$$

To compute the remaining entries, we use $\pi_m(l) = (P_1^*)^l \pi_m(0)$ for $0 \le l \le m-1$, and $\pi_m(l) = (P_2^*)^l (P_1^*)^m \pi_m(0)$ for $m \le l \le 2m-1$ to obtain (5). Note also the symmetry $\pi_m^-(l+m) = \pi_m^+(l)$ and $\pi_m^+(l+m) = \pi_m^-(l)$, $0 \le l \le m-1$. •

# 3 Spectral power amplification

The chain $X_m$ can be interpreted as amplifier of a signal. Our stochastic system may be seen to receive a deterministic periodic input signal which switches the double depths of the potential wells in (4), i.e.

$$I_m(l) = \begin{cases} V, & 0 \le l (\text{mod } 2m) \le m-1, \\ v, & m \le l (\text{mod } 2m) \le 2m-1. \end{cases}$$

The output is a random process $X_m(k)$.

The input signal $I_m$ admits a spectral representation

$$I_m(k) = \frac{1}{2m} \sum_{a=0}^{2m-1} c_m(a) e^{-\frac{2\pi i k}{2m} a},$$

where $c_m(a) = (1/2m) \sum_{l=0}^{2m-1} I_2(l) e^{\frac{2\pi i a}{2m} l}$ is the Fourier coefficient of frequency $a/2m$. The quantity $|c_{2m}(a)|^2$ measures the power carried by this Fourier component. We are only interested in the component of the input frequency $1/2m$. Its power is given by

$$|c_m(1)|^2 = \frac{(V-v)^2}{4m^2} \csc^2 \left(\frac{\pi}{2m}\right). \tag{6}$$

In the stationary regime, i.e. if the law of $X_m$ is given by the measure $\pi_m$, the power carried by the output at frequency $a/2m$ is a random variable

$$\xi_m(a) = \frac{1}{2m} \sum_{l=0}^{2m-1} X_m(l) e^{\frac{2\pi i a}{2m} l}.$$

We define the *spectral power amplification* as the relative expected power carried by the component of the output with frequency $\frac{1}{2m}$.

**Definition 2** *The* spectral power amplification (SPA) coefficient *of the Markov chain $X_m$ with half period $m \ge 1$ is given by*

$$\eta_m = \frac{|\mathbf{E}_{\pi_m}(\xi_m(1))|^2}{|c_m(1)|^2}.$$

*Here $\mathbf{E}_{\pi_m}$ denotes expectation w.r.t. the stationary distribution $\pi_m$.*

The explicit description of the invariant measure now readily yields the following formula for the spectral power amplification.

**Theorem 3** *Let $m \geq 1$. The spectral power amplification coefficient of the Markov chain $X_m$ with one-step transition probabilities (4) equals*

$$\eta_m = \left(\frac{(\phi-\psi)\sin\left(\frac{\pi}{2m}\right)}{V-v}\right)^2 \frac{(2-\phi-\psi)^2 + 2\cot^2\left(\frac{\pi}{2m}\right)(\phi+\psi)^2}{\left((2-\phi-\psi)^2 - 4\cos^2\left(\frac{\pi}{2m}\right)(1-\phi-\psi)\right)^2}.$$

*Proof:* Using (5) one immediately gets

$$\begin{aligned}\mathbf{E}_{\pi_m}\xi_m(1) &= \frac{1}{2m}\sum_{k=0}^{2m-1}\mathbf{E}_{\pi_m}X_m(k)e^{\frac{2\pi i}{2m}k} = \frac{1-e^{\pi i}}{2m}\sum_{k=0}^{m-1}(\pi_m^+(k)-\pi_m^-(k))e^{\frac{2\pi i}{2m}k} \\ &= \frac{2}{m}\frac{\phi-\psi}{\phi+\psi}\left(\frac{1}{1-e^{\frac{\pi i}{m}}} - \frac{1}{1-(1-\phi-\psi)^m e^{\frac{\pi i}{m}}}\right).\end{aligned}$$

Some algebra and an appeal to (6) finish the proof. •

Recall now that the one-step probabilities $P_1$ and $P_2$ depend on the parameters $0 \leq p, q \leq 1$ and, what is especially important, on $0 < \epsilon < \infty$ which is interpreted as noise level. Our next goal is to *tune* the parameter $\epsilon$ to a value which maximizes the amplification coefficient $\eta_m = \eta_m(\epsilon)$ as a function of $\epsilon$.

# 4 Extrema and zeros of $\eta_m(\epsilon)$.

In this section we study some features of the function $\eta_m(\epsilon)$ and its dependence on $m \in \mathbf{N}, 0 < v < V < \infty$ and the prefactors $0 \leq p \leq 1$, $0 \leq q \leq 1$.

After substituting $e^{-1/\epsilon} = x$ and writing $\eta_m(\epsilon) = \eta_m(x)$, this function takes the form

$$\begin{aligned}\eta_m(x) &= \left(\frac{\sin\left(\frac{\pi}{2m}\right)}{V-v}\right)^2 \frac{(px^V - qx^v)^2}{(2-px^V-qx^v)^2 - 4\cos^2\left(\frac{\pi}{2m}\right)(1-px^V-qx^v)^2)^2} \\ &\quad \times \left((2-px^V-qx^v)^2 + 2\cot^2\left(\frac{\pi}{2m}\right)(px^V+qx^v)^2\right). \qquad (7)\end{aligned}$$

In what follows, we assume $x \in [0,1]$. The boundaries $x = 0$ and $x = 1$ correspond to the limiting cases $\epsilon = 0$ and $\epsilon = \infty$. Denote $a_m = \csc\left(\frac{\pi}{2m}\right)^2 \geq 1$, $m \geq 1$.

Our main result on optimal tuning is contained in the following theorem.

**Theorem 4** *a) We have $\eta_m(x) \geq 0$, $\eta_m(0) = 0$.*

*b) Let $0 < \beta = v/V < 1$ and $m \geq 1$ be fixed. There exists a continuous function*

$$p_-(q) = p_-(q;\beta,m) = \frac{b(q;m,\beta) - \sqrt{b(q;m,\beta)^2 - 4a(q;m,\beta)(2-q)q}}{2a(q;m,\beta)},$$

*where $a(q;m,\beta) = 1 - a_m q(1-\beta)$, $b(q;m,\beta) = 2 - 3(1-\beta)q + a_m(1-\beta)q^2$, with following properties:*

*i)* $p_-(q) \geq 0,\ q \in [0,1]$ *and* $p_-(q) = 0 \Leftrightarrow q = 0$*;*
*ii)* $p_-(q) \leq q,\ q \in [0,1]$ *and* $p_-(q) = q \Leftrightarrow q = 0$ *or* $q = 1,\ m = 1$*;*
*iii)* $\left.\dfrac{dp_-(q; m, \beta)}{dq}\right|_{q=0} = \beta.$

*Moreover for* $m \geq 2$

*1) If* $(p,q) \in U_0 = \{(p,q) : 0 < q \leq 1, 0 \leq p < p_-(q)\}$, $\eta_m(x)$ *is strictly increasing on* $[0,1]$.

*2) If* $(p,q) \in U_1 = \{(p,q) : 0 < q \leq 1, p_-(q) < p \leq q\}$, $\eta_m(x)$ *has a unique local maximum on* $[0,1]$.

*3) If* $(p,q) \in U_2 = \{(p,q) : 0 < q \leq 1, q \leq p \leq 1\}$, $\eta_m(x)$ *has a unique local maximum on* $[0,1]$ *and a unique root on* $(0,1]$. *(See* Fig. *)*

*c)For any* $\delta > 0$ *there exists* $M_0 = M_0(p,q,\beta,\delta)$ *such that for* $m > M_0$ *the coordinate of the local maximum* $\widehat{x}_m \in [x_m(1-\delta), x_m]$, *where*

$$x_m = \left(\frac{\pi^2}{2m^2pq}\frac{v}{V-v}\right)^{\frac{1}{V+v}}$$

**Proof:**

Differentiate the explicit formula (7) with respect to $x$ to determine the critical points and sets $U_0, U_1, U_2$. The calculation of the resonance point in $U_1, U_2$ requires to find two points in some neighborhood such that the derivative is strictly monotone on the interval between them, and has different signs at the extremities. •

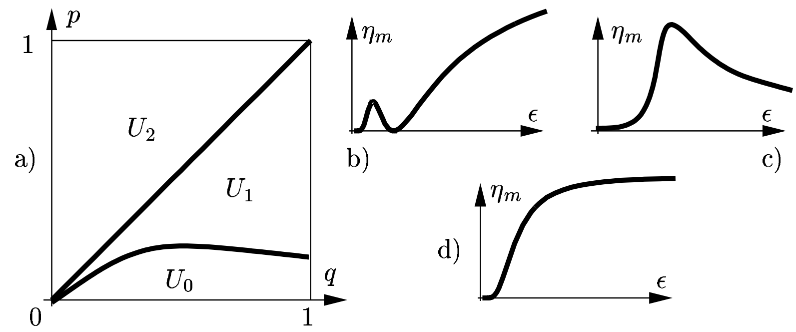


Fig. 1: a) Typical form of the domains $U_0$, $U_1$ and $U_2$. Typical form of $\eta_m(\epsilon)$ when $(p,q)$ belongs $U_2$ (b), $U_1$ (c) and $U_0$ (d).

**Remarks:**

1. The optimal tuning rule can be rewritten in the form

$$m(\epsilon) \cong \frac{1}{\pi}\sqrt{2pq}\sqrt{\frac{V-v}{v}}\exp\left(\frac{V+v}{2\epsilon}\right).$$

The maximal value of amplification is found as

$$\lim_{\epsilon \to 0} \eta_{[m(\epsilon)]}(\epsilon) = \frac{4}{\pi^2 (V-v)^2}.$$

2. We also see that the spectral power amplification as a measure of quality of stochastic resonance allows to distinguish a unique time scale, find its exponential rate ($\lambda = (V+v)/2$) together with the pre-exponential factor.

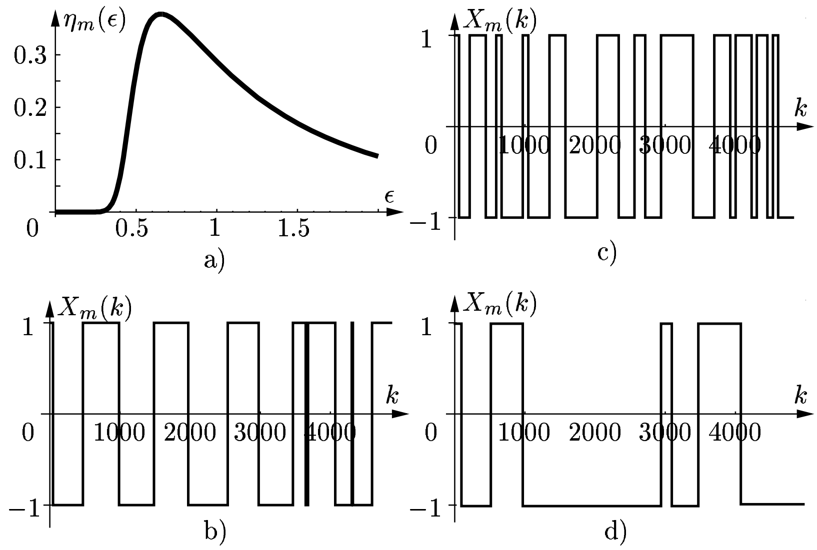


Fig. 2: a) $\eta_m(\epsilon)$ for $p = q = 0.5$, $m = 500$, $v = 2$, $V = 4$. Numerical simulations of $X_m(k)$ for b) $\epsilon = 0.65$, c) $\epsilon = 0.9$ and d) $\epsilon = 0.4$.

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