DOUBLE POINTS OF THE BROWNIAN SHEET IN $\mathbf{R}^d$ AND THE GEOMETRY OF THE PARAMETER SPACE

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Abstract. For the Brownian sheet W with values in $\mathbf{R^d}$ the set of double points $(s,t) \in A \times B$, i.e. points for which $W_s = W_t$, is investigated, in terms of the corresponding self intersection local time. Its existence is seen to depend sensitively upon the geometric constellation of the compact sets A, B in $[0,1]^2$. We suppose that A and B intersect at exactly one point p, and can be separated locally by an axial parallel line. We further assume that their boundaries in a vicinity of p are given by power type functions. We compute the critical dimension below which self intersection local time exists and describe it in terms of the powers of the boundary curves of A and B.

Key words and phrases: Brownian sheet; self-intersection local time; multiple stochastic integrals; canonical Dirichlet structure.

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Introduction

To investigate the local behaviour of stochastic processes or random fields $X_t, t \in T$, with values in $\mathbf{R}^d$, local time has proved to be an appropriate instrument. For $x \in \mathbf{R}^d$ it is given by the differentiated occupation measure

$$L(x) = \lim_{\epsilon \to 0} \frac{1}{\lambda(K_\epsilon(x))} \int_T 1_{K_\epsilon(x)}(X_s)\, ds,$$

where $K_\epsilon(x)$ denotes the ball of radius ϵ centered at x. For instance, if $L(x)$ exists, the level set of x, i. e. the set of time points $t \in T$ for which $X_t = x$, has Lebesgue measure 0.

More generally, also the size of the random set of double points, i. e. pairs $(s,t) \in T^2$ of points for which $X_s = X_t$ may be investigated by a local time type functional, the *self intersection local time*

$$\alpha = \lim_{\epsilon \to 0} \frac{1}{\lambda(K_\epsilon(0))} \int_T \int_T 1_{K_\epsilon(x)}(X_t - X_s)\, ds\, dt.$$

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If α exists, at least in a reasonable distributional sense, this means that the Lebesgue measure of the set of points (s,t) for which X_s and X_t are within a ball of radius ϵ of each other, remains of controllable size in proportion to the (of course d-dependent) Lebesgue measure of this ball. In a way this says that the set of double points remains small.

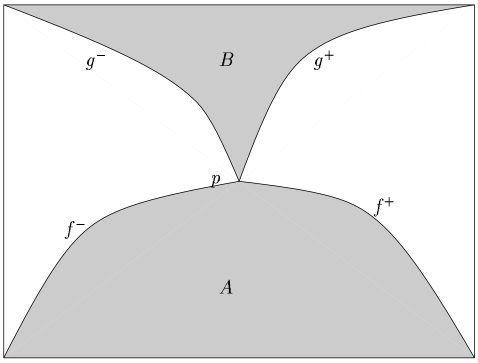
In particular, if X consists of d independent copies of one-dimensional processes, increasing d will mean to increase the proportional size of the set of double points. Consequently, there will be a *critical dimension* d_0 below which α exists, and above which the proportional size of the set of double points is beyond bounds.

Of course, this critical dimension will also sensitively depend on the degree of autocorrelation of the process X. It will be smaller, the closer the cumulative correlation between X_s and X_t as s and t run through T. The stochastic field we shall consider in this paper is the Brownian sheet W in $\mathbf{R}^d$. W has independent increments over disjoint rectangles of $[0,1]^2$, and is well known to have an essentially more complicated local structure than the Brownian motion in $\mathbf{R}^d$ (see for example Dalang, Walsh [4], [5], or Dalang, Mountford [3]).

Now suppose that instead of the whole parameter space, s and t are allowed to vary in A and B respectively, where A and B are compact sets in $[0,1]^2$. Then the critical dimension for the existence of self intersection local time will be a sensitive function of the geometric constellation of A and B. Increasing *closeness* of A and B will be expressed in higher cumulative correlation of the process running in the two sets, and consequently in lower critical dimension.

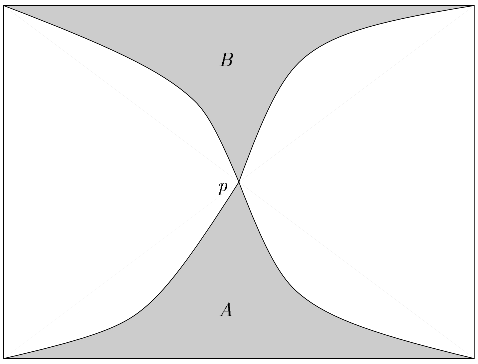
How this relationship between closeness and critical dimension may be expressed in geometrical terms more quantitatively, emerged in Imkeller, Weisz [8]. In this paper we considered axial parallel rectangles for A and B, and exhibited the following critical dimensions d. If $A \cap B = \emptyset$, then $d = \infty$, if $A \cap B$ is one point, then $d = 8$, if it is a line segment, $d = 6$, and if it contains a rectangle, $d = 4$.

The present paper is devoted to clarify further this crucial dependence of critical dimensions on the geometry of A and B. In particular, we are interested in an interpolation in geometrical terms between the cases in which A and B are disjoint, intersect in a point and in a line segment. We therefore suppose that $A \cap B = \{p\}$, and that they can be separated by an axial parallel line through p. We take this line to be horizontal, thereby taking into account the crucial dependence of the covariance structure of W on the axes, which has been encountered in many properties of the Brownian sheet. But we emphasize at this point that for rectangles intersecting in one point and with boundaries taking an angle of 45 degrees with the axes, our results will provide the critical dimension 8 as well. As in the Maltese cross condition for the sharp Markov property of W in space dimension 1 (see Dalang, Walsh [5]), it turns out that the critical dimension sensitively depends on the way in which the boundary curves tend away from the common intersection point. To measure this more quantitatively, we suppose throughout the paper that in a vicinity of p the boundary curves are of power type, as in the following sketch:

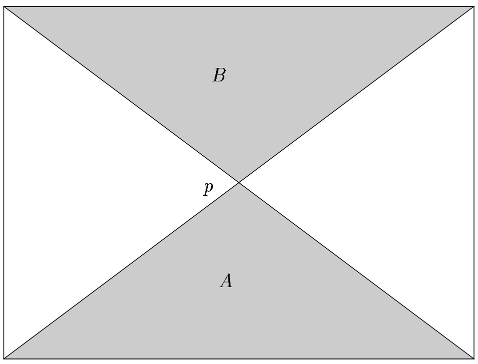


The exponents of the power functions g^+, g^- are denoted by μ^+, μ^-, those of f^+, f^- by ν^+, ν^-. Our main results, given in Theorem 8, may be summarized in the following way.

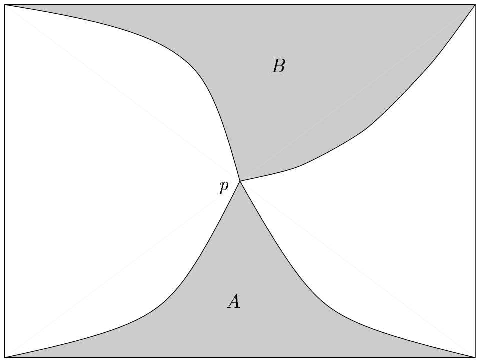
If all four exponents are less than one, i. e. the curves tend away fast from the point and the dividing line, the self intersection local time exists for $d < 4+2(\frac{1}{\mu^+ \vee \mu^-} + \frac{1}{\nu^+ \vee \nu^-})$:



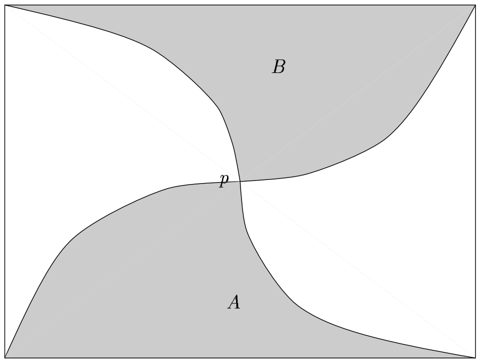
Note that depending on the size of the exponents, this critical dimension interpolates the dimensions ∞ (for disjoint A and B) and 8 (for rectangles intersecting in one point):



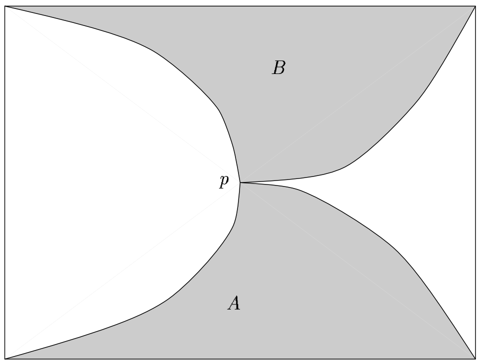
If 3 exponents are less than one, whereas the remaining one (say $\mu^+ or \mu^-$) exceeds 1, or if $\mu^+, \mu^- \geq 1, \nu^+, \nu^- < 1$ or $\mu^+, \mu^- < 1, \nu^+, \nu^- \geq 1$, the critical dimension is given by $6 + \frac{2}{\nu^+ \vee \nu^-}, 6 + \frac{2}{\mu^+ \vee \mu^-}$ respectively:



If $\mu^+ \wedge \nu^- \geq 1, \mu^- \vee \nu^+ < 1$ or $\mu^+ \vee \nu^- < 1, \mu^- \wedge \nu^+ \geq 1$, the critical dimension is 8:



If at least $\mu^+ \wedge \nu^+ \geq 1$ or $\nu^+ \wedge \nu^- \geq 1$, the critical dimension is given by $6 + \frac{2}{\mu^+ \wedge \mu^- \wedge \nu^+ \wedge \nu^-}$:



Note that this value interpolates the critical dimensions 8 (for rectangles with a common point) and 6 (for rectangles with a common line segment).

Our method of investigation is - as in Imkeller, Weisz [8] - a series decomposition of functionals approximating the self intersection local time by multiple Wiener-Itô integrals in the framework of the spectral decomposition of the infinite dimensional Ornstein-Uhlenbeck operator. This leads us to establish finiteness (for existence) or infiniteness (for divergence) of *characteristic integrals* of different types. The estimation of these integrals is in most cases rather straightforward, but complicated. So as a rule we only give the complete set of arguments in the most involved cases.

Already at this place we add a remark concerning the convergence (divergence) results natural for our setting. Smoothness being defined by the infinite dimensional Ornstein-Uhlenbeck operator, statements on convergence (divergence) appear in the sense of appropriate Sobolev norms. We could improve for example divergence results in this sense to divergence results in the *a.s.* sense in the following way. After renormalizing the approximate self intersection local times appropriately, we could show in many cases convergence of the renormalized quantities in statements of the type of laws of large numbers to constants. Using independence by cutting the parameter space into sequences of disjoint sets and Borel-Cantelli arguments these convergence results could slightly be improved to statements about *a.s.* convergence, and consequently *a.s.* divergence of the non renormalized quantities. We however refrain from making these steps precise, because a reasonable treatment of renormalization questions would inflate this already long paper beyond bounds.

The paper is organized as follows. In section 1 we recall the details of the spectral decomposition of self intersection local times. Section 2 is entirely devoted to the estimations of the various characteristic integrals, for functionals related to pairs of opposite regions bounded by power type curves. The particular results for different pairs of regions are presented in Theorems 2 - 7, and are finally combined to yield the main result (Theorem 8).

Preliminaries and notations

In this paper we consider the canonical Brownian sheet indexed by $[0,1]^2$ with values in $\mathbf{R}^d$ on the canonical Wiener space $(\Omega, \mathcal{F}, P)$. P is the probability measure under which W_t possesses the probability density

$$p^d_{t_1 t_2}(x) = \frac{1}{\sqrt{2\pi t_1 t_2}^d} \exp(-\frac{|x|^2}{2t_1 t_2}), \qquad x \in \mathbf{R}^d,\ t = (t_1, t_2) \in [0,1]^2.$$

The ordering of the parameter space is supposed to be coordinatewise linear ordering on $\mathbf{R}_+$. Intervals with respect to this partial ordering are defined in the usual way, and $s < t$ means $s_i < t_i$, $i = 1, 2$.

For $d = 1$ it is well known that $L^2(\Omega, \mathcal{F}, P)$ possesses an orthogonal decomposition by the eigenspaces of the Ornstein-Uhlenbeck operator on Wiener space, which are generated by the multiple Wiener-Ito integrals I_n, defined on $L^2(([0,1]^2)^n)$, $n \geq 0$ (see for example Bouleau, Hirsch [2], pp. 78-80). The multiple integrals possess the orthogonality property

$$E(I_n(f) I_m(g)) = \begin{cases} 0, & \text{if } n \neq m, \\ n! \int_{([0,1]^2)^n} fg \, d\lambda, & \text{if } n = m, \end{cases}$$

where λ denotes Lebesgue measure without reference to the dimension of the space on which it is defined. If H_n is the n^{th} Hermite polynomial defined by

$$H_n(x) = \frac{(-1)^n}{\sqrt{n!}} \exp(\frac{x^2}{2})(\frac{d}{dx})^n[\exp(-\frac{x^2}{2})],$$

$x \in \mathbf{R}$, $n \geq 0$, and if

$$W(h) = \int_{[0,1]^2} h \, dW$$

denotes the Gaussian stochastic integral of a function $h \in L^2([0,1]^2)$, the relation

$$H_n(W(h)) = \frac{1}{\sqrt{n!}} I_n(h^{\otimes_n})$$

holds true whenever $\|h\| = 1$. Here $h^{\otimes_n}$ denotes the n-fold tensor product of h with itself, while $\|\cdot\|$ is the norm in $L^2([0,1]^2)$. We write $W(D) = W(1_D)$ for $D \in \mathcal{B}(\mathbf{R}^2_+)$, so $W_t = W(R_t)$ for $R_t = [0,t]$. For $\rho \in \mathbf{R}$ we may define the *Sobolev space* of order ρ on Wiener space by introducing the norm

$$\|F\|_{2,\rho} = (\sum_{n=0}^{\infty} (1+n)^{\rho} \|I_n(f_n)\|_2^2)^{1/2}$$

on the space

$$\{F = \sum_{n=0}^{m} I_n(f_n) : f_n \in L^2(([0,1]^2)^n), \qquad 0 \le n \le m,\ m \in \mathbf{N}\}$$

which is dense in $L^2(\Omega, \mathcal{F}, P)$ and completing with respect to $\|\cdot\|_{2,\rho}$. We denote this space by $\mathbf{D}_{2,\rho}$. In case $\rho = 1$ we just recover the domain of the gradient operator of the canonical Dirichlet form on Wiener space, for $\rho < 0$ we obtain a space of distributions over Wiener space (see Watanabe [26], Bouleau, Hirsch [2], Nualart [14]).

To denote multiple Wiener-Ito integrals with respect to the independent components W^i of W in $\mathbf{R}^d$, we use the symbol I_n^i, $1 \le i \le d$, $n \ge 0$. Corresponding Sobolev spaces are defined for functionals of the d-dimensional Brownian sheet (see Watanabe [26]).

1. Two series representations and the characteristic integrals

In this section we consider approximations of self-intersection local times and give two different chaos decompositions for them.They will serve later to get upper and lower limits for stating convergence or non-convergence in Sobolev spaces of the canonical Dirichlet structure, i.e. the existence or non-existence of self-intersection local time, where intersection times are allowed to vary in two compact sets A, B of parameter space. Taking therefore Sobolev norms of the approximations will make appear "characteristic integrals" of degree k for $k \in \mathbf{N}_0$. These integrals are in general rather complicated to evaluate. For this reason we shall give a few decompositions which will then be seen in the following sections to make the estimates more transparent and tractable.

We now fix two compact sets A and B off the boundary of $\mathbf{R}^2_+$. We shall consider a canonical approximation $\alpha_\epsilon(x,\cdot)$, $\epsilon \to 0$, of self-intersection local time of the Brownian sheet in $\mathbf{R}^d$ corresponding to these sets.

For $\epsilon > 0$, $x \in \mathbf{R}^d$ let

$$\alpha_\epsilon(x,\cdot) = \int_B \int_A p_\epsilon^d(W_t - W_s - x)\, ds\, dt$$

where

$$p_\epsilon^d(x) = \frac{1}{\sqrt{2\pi\epsilon}^d} \exp(-\frac{|x|^2}{2\epsilon}), \quad x \in \mathbf{R}^d, \quad \epsilon > 0,$$

is the density function of the d-dimensional Wiener process at time ϵ.

We proved in [8] that for $x \in \mathbf{R}^d$ and $\epsilon > 0$ we have

$$\alpha_\epsilon(x,\cdot) = \sum_{n_i=0}^{\infty} \int_B \int_A \prod_{i=1}^{d} \frac{1}{\sqrt{n_i!}} I_{n_i}^i([\frac{1_{R_t} - 1_{R_s}}{\sqrt{\epsilon + \lambda(R_t \triangle R_s)}}]^{\otimes n_i}) \\ H_{n_i}(\frac{x_i}{\sqrt{\epsilon + \lambda(R_t \triangle R_s)}}) p^d_{\epsilon+\lambda(R_t\triangle R_s)}(x)\, ds\, dt,$$

where $Q \triangle R = (Q \setminus R) \cup (R \setminus Q)$ denotes the symmetric difference of the sets Q, R.

Choose $\eta > 0$ such that $\underline{\eta} = (\eta, \eta) < x$ for each $x \in A, B$. To simplify the computations we introduce the sets

$$C_t := R_t \setminus [\underline{\eta}, t], \qquad D_t := [\underline{\eta}, t] \qquad (t \in [\underline{\eta}, 1]).$$

Thus $R_t = C_t \cup D_t$. Then we have

$$\alpha_\epsilon(x,\cdot) = \sum_{n_i=0}^{\infty} \int_B \int_A \prod_{i=1}^{d} \frac{1}{\sqrt{n_i!}} I_{n_i}^i([\frac{1_{C_t} - 1_{C_s}}{\sqrt{\epsilon + \lambda(C_t \triangle C_s)}}]^{\otimes n_i}) \\ H_{n_i}(\frac{\xi(s,t,x)_i}{\sqrt{\epsilon + \lambda(C_t \triangle C_s)}}) p^d_{\epsilon+\lambda(C_t\triangle C_s)}(x)\, ds\, dt$$

for $x \in \mathbf{R}^d$ and $\epsilon > 0$ (see Imkeller, Weisz [8]), where

$$\xi(s,t,x) = x - (W(D_t) - W(D_s)).$$

Let us now fix $\rho \in \mathbf{R}$ and compute the norm of $\alpha_\epsilon(x,\cdot)$ in the two representations with respect to the quadratic Sobolev space of order ρ. For $x \in \mathbf{R}^d$, $\epsilon > 0$ we have

$$\|\alpha_\epsilon(x,\cdot)\|_{2,\rho}^2 \tag{1}$$

$$= \sum_{k=0}^{\infty} (1+k)^\rho \int_B \int_A \int_B \int_A \frac{[\int_{[0,1]^2} (1_{R_t} - 1_{R_s})(1_{R_v} - 1_{R_u})\, d\lambda]^k}{[(\epsilon + \lambda(R_t \triangle R_s))(\epsilon + \lambda(R_v \triangle R_u))]^{\frac{k}{2}}} \\ \sum_{n_1+\cdots+n_d=k} \prod_{i=1}^{d} H_{n_i}(\frac{x_i}{\sqrt{\epsilon + \lambda(R_t \triangle R_s)}}) H_{n_i}(\frac{x_i}{\sqrt{\epsilon + \lambda(R_v \triangle R_u)}}) \\ p^d_{\epsilon+\lambda(R_t\triangle R_s)}(x) p^d_{\epsilon+\lambda(R_v\triangle R_u)}(x)\, ds\, dt\, du\, dv$$

and

$$\|\alpha_\epsilon(x,\cdot)\|_{2,\rho}^2 \\ \leq c \sum_{k=0}^{\infty} (1+k)^\rho k^{d/2-1} \\ \int_B \int_A \int_B \int_A \frac{\lambda((C_t \triangle C_s) \cap (C_v \triangle C_u))^k}{[(\epsilon + \lambda(C_t \triangle C_s))(\epsilon + \lambda(C_v \triangle C_u))]^{\frac{k+d}{2}}}\, ds\, dt\, du\, dv$$

(see Imkeller, Weisz [8]). In the following section, the convergence of $\alpha_\epsilon(x,\cdot)$ as $\epsilon \to 0$ will be investigated. We have seen in [8] that $\alpha_\epsilon(x,\cdot)$ converges to the self-intersection local time for all $x \in \mathbf{R}^d$ (in particular for 0),

$$\alpha(x,\cdot) = \sum_{n_i=0}^{\infty} \int_B \int_A \prod_{i=1}^{d} \frac{1}{\sqrt{n_i!}} I_{n_i}^i\Big(\Big[\frac{1_{R_t} - 1_{R_s}}{\sqrt{\lambda(R_t \triangle R_s)}}\Big]^{\otimes n_i}\Big) \\ H_{n_i}\Big(\frac{x_i}{\sqrt{\lambda(R_t \triangle R_s)}}\Big) p^d_{\lambda(R_t\triangle R_s)}(x)\, ds\, dt \tag{2}$$

in $\mathbf{D}_{2,\rho}$ if

$$I(k) \le c_k \quad \text{and} \quad \sum_{k=0}^{\infty} c_k (1+k)^\rho k^{d/2-1} < \infty \tag{3}$$

where

$$I(k) = \int_B \int_A \int_B \int_A \frac{\lambda((C_t \triangle C_s) \cap (C_v \triangle C_u))^k}{[(\lambda(C_t\triangle C_s))(\lambda(C_v \triangle C_u))]^{\frac{k+d}{2}}}\, ds\, dt\, du\, dv,$$

$k \in \mathbf{N}_0$, denote the characteristic integrals.

Let us first discuss *characteristic integrals* needed to state convergence. Observe that

$$\lambda(C_t \triangle C_s) = \eta(|t_1 - s_1| + |t_2 - s_2|)$$

and

$$\lambda((C_t\triangle C_s) \cap (C_v \triangle C_u)) = \eta[\lambda([s_1 \wedge t_1, s_1 \vee t_1] \cap [u_1 \wedge v_1, u_1 \vee v_1]) \\ + \lambda([s_2 \wedge t_2, s_2 \vee t_2] \cap [u_2 \wedge v_2, u_2 \vee v_2])].$$

Denote

$$S_i = [s_i \wedge t_i, s_i \vee t_i], \quad T_i = [u_i \wedge v_i, u_i \vee v_i], \qquad i = 1,2.$$

We will investigate special sets A and B for which $S_2 \cap T_2 \neq \emptyset$. Hence up to an irrelevant factor $\eta^{(k-d)/2}$, $I(k)$ is equal to the sum of the following two integrals

$$I(k,1) := \int_B \cdots \int_A 1_{\{S_1\cap T_1 = \emptyset\}} \frac{(t_2 \wedge v_2 - s_2 \vee u_2)^k}{[(|t_1 - s_1| + t_2 - s_2)(|v_1 - u_1| + v_2 - u_2)]^{\frac{k+d}{2}}}\, ds \dots dv,$$

$$I(k,2) := \int_B \cdots \int_A 1_{\{S_1\cap T_1 \neq \emptyset\}} \\ \frac{[(s_1 \vee t_1) \wedge (u_1 \vee v_1) - (s_1 \wedge t_1) \vee (u_1 \wedge v_1) + t_2 \wedge v_2 - s_2 \vee u_2]^k}{[(|t_1 - s_1| + t_2 - s_2)(|v_1 - u_1| + v_2 - u_2)]^{\frac{k+d}{2}}}\, ds \dots dv.$$

Since these integrals will be used for upper estimates, we sometimes may replace them by good upper estimates, such as the following ones. We verified in [8] that

$$[\lambda(S_1 \cap T_1) + \lambda(S_2 \cap T_2)][\lambda(S_1 \cup T_1) + \lambda(S_2 \cup T_2)] \le 2[\lambda(S_1) + \lambda(S_2)][\lambda(T_1) + \lambda(T_2)].$$

Using this we can see that $I(k,i) \le J(k,i)$ $(i = 1,2)$, where

$$J(k,1) := \int_B \cdots \int_A 1_{\{S_1 \cap T_1 = \emptyset\}} \frac{(t_2 \wedge v_2 - s_2 \vee u_2)^{\frac{k-d}{2}}}{(|v_1 - u_1| + |t_1 - s_1| + t_2 \vee v_2 - s_2 \wedge u_2)^{\frac{k+d}{2}}}\, ds \dots dv,$$

and

$$J(k,2) := \int_B \cdots \int_A 1_{\{S_1 \cap T_1 \neq \emptyset\}}$$
$$\frac{[(s_1 \vee t_1) \wedge (u_1 \vee v_1) - (s_1 \wedge t_1) \vee (u_1 \wedge v_1) + t_2 \wedge v_2 - s_2 \vee u_2]^{\frac{k-d}{2}}}{[(s_1 \vee t_1) \vee (u_1 \vee v_1) - (s_1 \wedge t_1) \wedge (u_1 \wedge v_1) + t_2 \vee v_2 - s_2 \wedge u_2]^{\frac{k+d}{2}}}\, ds \dots dv.$$

The case $k = 0$ is much simpler, since

$$I(0) = (\int_B \int_A 1_{\{s_1 < t_1, s_2 < t_2\}} \frac{1}{(t_1 - s_1 + t_2 - s_2)^{\frac{d}{2}}}\, ds\, dt)^2.$$

Finally, let us state by means of which characteristic integrals we will obtain divergence of $\alpha_\epsilon(x,\cdot)$, $\epsilon \to 0$, for all $x \in \mathbf{R}^d$. To get divergence it is enough to show that, at least for one $k \in \mathbf{N}_0$, the characteristic integral

$$K(k) = \int_B \int_A \int_B \int_A \frac{[\int_{[0,1]^2} (1_{R_t} - 1_{R_s})(1_{R_v} - 1_{R_u})\, d\lambda]^k}{[(\lambda(R_t \triangle R_s))(\lambda(R_v \triangle R_u))]^{\frac{k+d}{2}}}\, ds\, dt\, du\, dv$$

appearing in (1), is infinite.

In [8] we investigated the case when A and B are rectangles and proved that the critical dimension below which self-intersection local time exists, is given by $d = 4$ if $A = B$. If $A \cap B$ consists of an axial parallel line, it is 6, if it consists of a point, 8, and if it is empty, infinity, i.e., in this case the self-intersection local time exists for all dimensions (at least as a well behaved distribution).

In the next section we characterize the critical dimension of the self-intersection local time for more general compact sets A and B. In fact, we shall mainly be concerned with the case in which A and B touch at one point, and the boundaries containing this common point are given by graphs of power functions. We shall see that the behaviour of the corresponding self-intersection local time sensitively depends on the geometry of these boundaries.

2. Self-intersections of the Brownian sheet in different compact sets

Now suppose that the sets A and B are compact and $A, B \subset \mathbf{R}^2_+ \setminus \partial \mathbf{R}^2_+$. If $A \cap B = \emptyset$ then the critical dimension of the self-intersection local time is infinity, as for rectangles.

Theorem 1. *Suppose that $A \cap B = \emptyset$ and d is arbitrary. Then*

$$\alpha(x,\cdot) = \lim_{\epsilon \to 0} \alpha_\epsilon(x,\cdot) \qquad (x \in \mathbf{R}^d) \tag{4}$$

exists in $\mathbf{D}_{2,\rho}$ for any $\rho < 4 - d/2$ and is given by (2). Moreover, $\alpha(x,\cdot)$ is a function iff $d < 8$.

Proof. We can find pairwise disjoint, closed rectangles R_i, $i = 1,\ldots,n$ and S_j, $j = 1,\ldots,m$, the vertices of which are parallel to the axis, such that

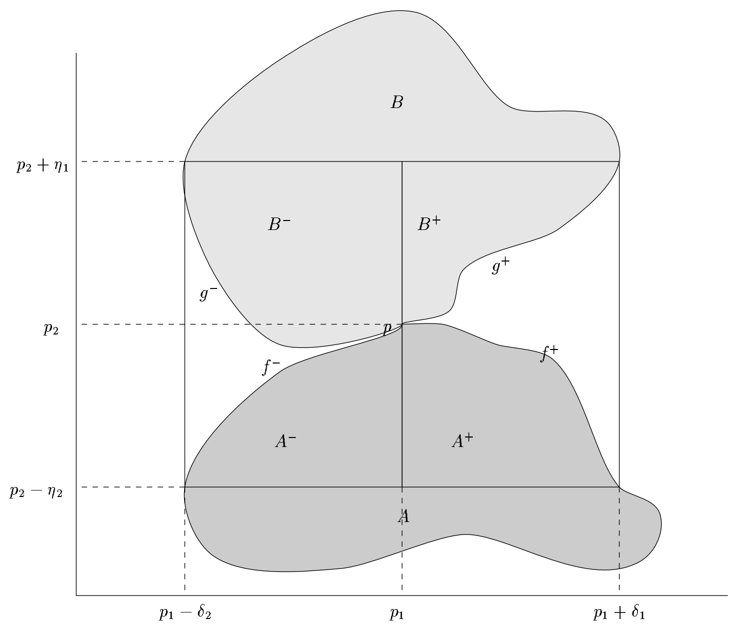
$$A \subset \bigcup_{i=1}^{n} R_i, \qquad B \subset \bigcup_{j=1}^{m} S_j.$$

Then we apply Theorem 9 of Imkeller, Weisz [8] for each pair (R_i, S_j) of rectangles and obtain that the critical dimension of the self-intersection local time is infinite. This proves the theorem. ■

In what follows we suppose that A and B have one common point, $\{p\} = A \cap B$. We shall further suppose that the boundaries of A and B containing the common point p are given by graphs of power functions. We shall essentially be concerned throughout the section with exhibiting the extremly fine and sensitive dependence of the behaviour of the self-intersection local time on the structure of these boundaries. Of course, one should expect that the closer the boundaries are, the rougher the behaviour of α. Our calculations allow us to state this more quantitatively. Let us now state more precisely the geometrical setting of our investigations.

Suppose that there is a rectangle $K := I \times J$, $I = [p_1 - \delta_2, p_1 + \delta_1]$, $J = [p_2 - \eta_2, p_2 + \eta_1]$, such that $\partial A \cap K$ and $\partial B \cap K$ are graphs of functions, more exactly,

$$\partial A \cap K = \{(x, f(x)) : p_1 - \delta_2 \le x \le p_1 + \delta_1\},$$
$$\partial B \cap K = \{(x, g(x)) : p_1 - \delta_2 \le x \le p_1 + \delta_1\}.$$



Here $0 < \delta_1, \delta_2, \eta_1, \eta_2 < 1$. Let $f \le g$. We can suppose that $g(p_1+\delta_1) = g(p_1-\delta_2) = p_2+\eta_1$ and $f(p_1+\delta_1) = f(p_1-\delta_2) = p_2-\eta_2$. Denote

$$f^+ := f 1_{[p_1, p_1+\delta_1]}, \qquad f^- := f 1_{[p_1-\delta_2, p_1]},$$
$$g^+ := g 1_{[p_1, p_1+\delta_1]}, \qquad g^- := g 1_{[p_1-\delta_2, p_1]},$$

and

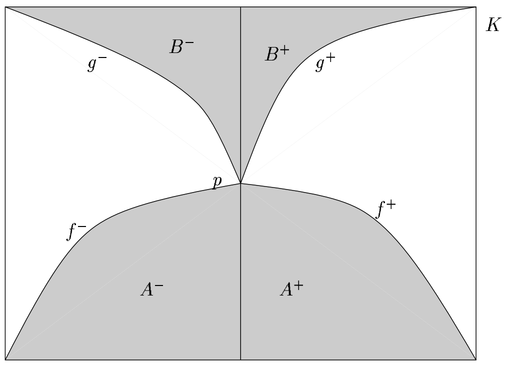
$$A^+ := A \cap K \cap \{(x,y) : x \ge p_1\}, \qquad A^- := A \cap K \cap \{(x,y) : x \le p_1\},$$
$$B^+ := B \cap K \cap \{(x,y) : x \ge p_1\}, \qquad B^- := B \cap K \cap \{(x,y) : x \le p_1\}.$$

To get the self-intersection local time over A and B, it is obviously enough to consider the self-intersection local time over the sets $A^\pm$ and $B^\pm$. This leads us to the investigation of characteristic integrals $I_{\pm,\pm}(k,i)$ and $J_{\pm,\pm}(k,i)$ instead of $I(k,i)$ and $J(k,i)$ $(i=1,2)$, if A and B are replaced by $A^\pm$ and $B^\pm$. Similar notations are introduced for $I(k)$ and $K(k)$, $k \in \mathbf{N}_0$.

As announced above, we shall assume that the boundaries of A and B are given by power-type functions. More precisely, assume that

$$g^+(x) = p_2 + b^+(x-p_1)^{\mu^+}, \qquad f^+(x) = p_2 - a^+(x-p_1)^{\nu^+},$$
$$g^-(x) = p_2 + b^-(p_1-x)^{\mu^-}, \qquad f^-(x) = p_2 - a^-(p_1-x)^{\nu^-},$$
$$g = g^+ 1_{[p_1,p_1+\delta_1]} + g^- 1_{[p_1-\delta_2,p_1]}, \qquad f = f^+ 1_{[p_1,p_1+\delta_1]} + f^- 1_{[p_1-\delta_2,p_1]},$$

for some $0 < \mu^\pm, \nu^\pm \le \infty$, $0 < a^\pm, b^\pm < \infty$.



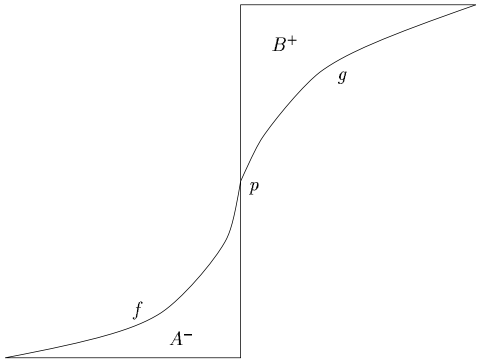
Note that the case $a^+ = 0$ is covered by $\mu^+ = \infty$ etc. We shall use also the notation

$$B^+ = B^+_{\mu^+} \qquad \text{and} \qquad A^+ = A^+_{\nu^+}. \tag{5}$$

We can suppose that $a^\pm = b^\pm = 1$ (see the computations below). **The symmetry of the integrals $I(k)$ and $J(k)$ allows us to make the following general assumption which is valid throughout the paper: $\mu^+ \le \nu^\pm$ and $\mu^- \le \nu^\pm$.** If $A^\pm$ and $B^\pm$ are both rectangles, then the problem is solved in Imkeller and Weisz [8]. So we suppose that $\mu^\pm < \infty$.

The constant C appearing in estimations below may vary from line to line and does not depend on k.

2.1. The case $A^+ = B^- = \emptyset$ and $\max(\mu^+, \nu^-) \le 1$



We first consider the case in which A and B are contained in two opposite quadrants and in which the boundary curves "tend away" fast from the separating line, i.e. both exponents of the power-type functions are smaller than one. In this case, it turns out that the critical dimension depends on both exponents. It is given by $4 + \frac{2}{\mu^+} + \frac{2}{\nu^-}$.

In the following calculations we shall always verify the finiteness of an integral of the form $\int_0^1 x^\alpha \, dx$ by comparing the exponent α with -1. The exponents α appearing will always be more or less complicated functions $\alpha(\mu^\pm, \nu^\pm)$ of $\mu^\pm$ and $\nu^\pm$ and $\alpha(\mu^\pm, \nu^\pm) > -1$ will be a consequence of the hypothesis $d < d_0$, where d_0 is given in the corresponding propositions.

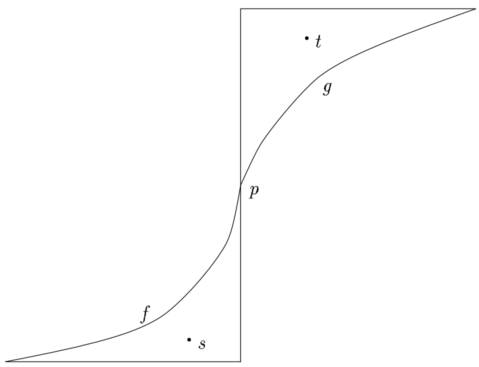
In estimates of integrals of the form $\int_0^1 (x+y)^\alpha \, dy$ the exponent α may be -1. In order to avoid having to work with log-terms in consequence of this, we estimate tacitly by $\int_0^1 (x+y)^{\alpha-\epsilon} \, dy$ (for upper bounds) resp. by $\int_0^1 (x+y)^{\alpha+\epsilon} \, dy$ (for lower bounds). Obviously, this does not affect the generality of the results.

Proposition 1. *Suppose that* $\mu^+ \le \nu^- \le 1$ *and*

$$d < 4 + \frac{2}{\mu^+} + \frac{2}{\nu^-}. \tag{6}$$

Then there exists a constant C *such that* $k^2 J_{-,+}(k) \le C$ *for all* $k \in \mathbf{N}_0$.

Proof.



For $I_{-,+}(0)$ we have (see sketch)

$$\sqrt{I_{-,+}(0)} = \int_{B^+}\int_{A^-} 1_{\{s_1<t_1,s_2<t_2\}} \frac{1}{(t_1-s_1+t_2-s_2)^{\frac{d}{2}}}\,ds\,dt$$
$$\approx \int\int 1_{\{s_1<p_1<t_1\}} \frac{1}{(t_1-s_1+g(t_1)-f(s_1))^{\frac{d}{2}-2}}\,ds_1\,dt_1,$$

where $\approx$ means asymptotic equivalence. It is easy to see that the transformation

$$x_1 = p_1 - s_1, \qquad x_2 = t_1 - p_1 \tag{7}$$

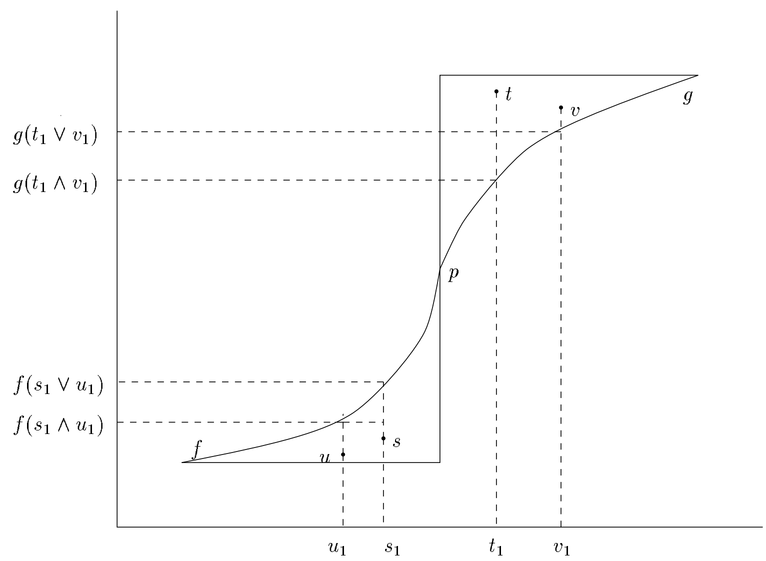
implies

$$\begin{aligned}\sqrt{I_{-,+}(0)} &\approx \int_0^1\int_0^1 \frac{1}{(x_1+x_2+x_2^{\mu^+}+x_1^{\nu^-})^{\frac{d}{2}-2}}\,dx_1\,dx_2\\ &\approx \int_0^1\int_0^1 \frac{1}{(x_2^{\mu^+}+x_1^{\nu^-})^{\frac{d}{2}-2}}\,dx_1\,dx_2\\ &= \int_0^1\int_0^1 1_{\{x_2^{\mu^+}<x_1^{\nu^-}\}}\frac{1}{(x_2^{\mu^+}+x_1^{\nu^-})^{\frac{d}{2}-2}}\,dx_1\,dx_2\\ &\quad + \int_0^1\int_0^1 1_{\{x_2^{\mu^+}>x_1^{\nu^-}\}}\frac{1}{(x_2^{\mu^+}+x_1^{\nu^-})^{\frac{d}{2}-2}}\,dx_1\,dx_2.\end{aligned} \tag{8}$$

On the set $x_2^{\mu^+} < x_1^{\nu^-}$ we use the substitution $x_2^{\mu^+} = x_1^{\nu^-}z$ to derive that the first term is equivalent to

$$\int_0^1\int_0^1 (x_1^{\nu^-}z + x_1^{\nu^-})^{2-d/2} x_1^{\nu^-/\mu^+} z^{1/\mu^+-1}\,dx_1\,dz < \infty \tag{9}$$

if and only if (6) is true. The second term can be handled similarly.



Let us investigate the integrals $J_{-,+}(k)$. Observe that $J_{-,+}(k,1) = 0$. The part of the integral $J_{-,+}(k,2)$ over the set $\{g(t_1 \vee v_1) < t_2 \wedge v_2\} \cap \{s_2 \vee u_2 < f(s_1 \wedge u_1)\}$ (see sketch) is denoted by $J_{-,+}(k,2,1)$. More precisely,

$$J_{-,+}(k,2,1) := \int_{B^+} \cdots \int_{A^-} 1_{\{g(t_1 \vee v_1) < t_2 \wedge v_2, s_2 \vee u_2 < f(s_1 \wedge u_1)\}} \frac{[t_1 \wedge v_1 - s_1 \vee u_1 + t_2 \wedge v_2 - s_2 \vee u_2]^{\frac{k-d}{2}}}{[t_1 \vee v_1 - s_1 \wedge u_1 + t_2 \vee v_2 - s_2 \wedge u_2]^{\frac{k+d}{2}}} \, ds \ldots dv. \tag{10}$$

Similarly, we define the integrals $J_{-,+}(k,2,2)$, $J_{-,+}(k,2,3)$ and $J_{-,+}(k,2,4)$ using the sets

$$\begin{aligned} &\{g(t_1 \vee v_1) < t_2 \wedge v_2\} \cap \{f(s_1 \wedge u_1) < s_2 \vee u_2 < f(s_1 \vee u_1)\}, \\ &\{g(t_1 \wedge v_1) < t_2 \wedge v_2 < g(t_1 \vee v_1)\} \cap \{s_2 \vee u_2 < f(s_1 \wedge u_1)\}, \\ &\{g(t_1 \wedge v_1) < t_2 \wedge v_2 < g(t_1 \vee v_1)\} \cap \{f(s_1 \wedge u_1) < s_2 \vee u_2 < f(s_1 \vee u_1)\}, \end{aligned} \tag{11}$$

respectively. Then, obviously,

$$J_{-,+}(k,2) = J_{-,+}(k,2,1) + J_{-,+}(k,2,2) + J_{-,+}(k,2,3) + J_{-,+}(k,2,4).$$

Integrating in $t_2 \vee v_2$ and $s_2 \wedge u_2$ we have

$$J_{-,+}(k,2,1) \le \frac{C}{k^2} \int_{B^+} \cdots \int_{A^-} 1_{\{g(t_1 \vee v_1) < t_2 \wedge v_2, s_2 \vee u_2 < f(s_1 \wedge u_1)\}} \frac{[t_1 \wedge v_1 - s_1 \vee u_1 + t_2 \wedge v_2 - s_2 \vee u_2]^{\frac{k-d}{2}}}{[t_1 \vee v_1 - s_1 \wedge u_1 + t_2 \wedge v_2 - s_2 \vee u_2]^{\frac{k+d}{2}-2}} \, ds \ldots dv.$$

By the definition of f and g and the transformation

$$\begin{aligned} x_1 &= p_1 - s_1 \vee u_1, & x_2 &= p_1 - s_1 \wedge u_1, & x_3 &= t_1 \wedge v_1 - p_1, \\ x_4 &= t_1 \vee v_1 - p_1, & y_1 &= p_2 - s_2 \vee u_2, & y_2 &= t_2 \wedge v_2 - p_2, \end{aligned} \tag{12}$$

$$J_{-,+}(k,2,1) \le \frac{C}{k^2} \int_0^1 \cdots \int_0^1 1_{\{x_1 < x_2 < y_1^{1/\nu^-}, x_3 < x_4 < y_2^{1/\mu^+}\}} \frac{(x_3 + x_1 + y_1 + y_2)^{\frac{k-d}{2}}}{(x_4 + x_2 + y_1 + y_2)^{\frac{k+d}{2}-2}} \, dx_1 \ldots dy_2.$$

Then the coordinate change

$$x_2 = y_1^{1/\nu^-} t_1, \qquad x_1 = y_1^{1/\nu^-} t_1 t_2, \qquad x_4 = y_2^{1/\mu^+} u_1, \qquad x_3 = y_2^{1/\mu^+} u_1 u_2$$

gives

$$J_{-,+}(k,2,1) \le \frac{C}{k^2} \int_0^1 \cdots \int_0^1 \frac{(y_2^{1/\mu^+} u_1 u_2 + y_1^{1/\nu^-} t_1 t_2 + y_1 + y_2)^{\frac{k-d}{2}}}{(y_2^{1/\mu^+} u_1 + y_1^{1/\nu^-} t_1 + y_1 + y_2)^{\frac{k+d}{2}-2}} y_1^{2/\nu^-} y_2^{2/\mu^+} t_1 u_1 \, dt_1 \ldots dy_2.$$

On the set $\{y_1 < y_2\}$ let $y_1 = y_2 z$. Then the contribution of this set to the integral is estimated by

$$\int_0^1 \cdots \int_0^1 \frac{(y_2^{1/\mu^+} u_1 u_2 + y_2^{1/\nu^-} z^{1/\nu^-} t_1 t_2 + y_2 z + y_2)^{\frac{k-d}{2}}}{(y_2^{1/\mu^+} u_1 + y_2^{1/\nu^-} z^{1/\nu^-} t_1 + y_2 z + y_2)^{\frac{k+d}{2}-2}} \\ y_2^{2/\nu^-} z^{2/\nu^-} y_2^{2/\mu^+ +1}\, dt_1 \dots dy_2 \le C$$

with a constant C independent of k, provided that (6) holds. This is seen by comparing numerator and denominator, extracting y_2 and integrating in y_2. On the set $\{y_2 < y_1\}$ we can argue similarly.

The integral $J_{-,+}(k, 2, 3)$ is more complicated than $J_{-,+}(k, 2, 2)$. So we concentrate on the former. We integrate in $t_2 \vee v_2$ and $s_2 \wedge u_2$, use the transform (12) and

$$x_2 = y_1^{1/\nu^-} t_1, \qquad x_1 = y_1^{1/\nu^-} t_1 t_2, \qquad y_2^{1/\mu^+} = x_4 u_1, \qquad x_3 = x_4 u_1 u_2$$

to get

$$\begin{aligned} J_{-,+}(k, 2, 3) \le & \frac{C}{k^2} \int_{B^+} \cdots \int_{A^-} 1_{\{g(t_1 \wedge v_1) < t_2 \wedge v_2 < g(t_1 \vee v_1), s_2 \vee u_2 < f(s_1 \wedge u_1)\}} \\ & \frac{[t_1 \wedge v_1 - s_1 \vee u_1 + t_2 \wedge v_2 - s_2 \vee u_2]^{\frac{k-d}{2}}}{[t_1 \vee v_1 - s_1 \wedge u_1 + g(t_1 \vee v_1) - s_2 \vee u_2]^{\frac{k+d}{2}-2}}\, ds \dots dv \\ \le & \frac{C}{k^2} \int_0^1 \cdots \int_0^1 1_{\{x_1 < x_2 < y_1^{1/\nu^-}, x_3 < y_2^{1/\mu^+} < x_4\}} \\ & \frac{(x_3 + x_1 + y_1 + y_2)^{\frac{k-d}{2}}}{(x_4 + x_2 + y_1 + x_4^{\mu^+})^{\frac{k+d}{2}-2}}\, dx_1 \dots dy_2 \\ = & \frac{C}{k^2} \int_0^1 \cdots \int_0^1 \frac{(x_4 u_1 u_2 + y_1^{1/\nu^-} t_1 t_2 + y_1 + x_4^{\mu^+} u_1^{\mu^+})^{\frac{k-d}{2}}}{(x_4 + y_1^{1/\nu^-} t_1 + y_1 + x_4^{\mu^+})^{\frac{k+d}{2}-2}} \\ & y_1^{2/\nu^-} x_4^{\mu^+ +1} t_1 u_1^{\mu^+}\, dt_1 \dots dy_1. \end{aligned}$$

On the set $\{y_1^{1/\mu^+} < x_4\}$ the transformation $y_1^{1/\mu^+} = x_4 z$ yields the estimate

$$\begin{aligned} & \frac{C}{k^2} \int_0^1 \cdots \int_0^1 \frac{(x_4 u_1 u_2 + x_4^{\mu^+/\nu^-} z^{\mu^+/\nu^-} t_1 t_2 + x_4^{\mu^+} z^{\mu^+} + x_4^{\mu^+} u_1^{\mu^+})^{\frac{k-d}{2}}}{(x_4 + x_4^{\mu^+/\nu^-} z^{\mu^+/\nu^-} t_1 + x_4^{\mu^+} z^{\mu^+} + x_4^{\mu^+})^{\frac{k+d}{2}-2}} \\ & \qquad x_4^{2\mu^+/\nu^-} z^{2\mu^+/\nu^-} t_1 x_4^{\mu^+ +1} u_1^{\mu^+} x_4^{\mu^+} z^{\mu^+ -1}\, dt_1 \dots dz \\ \le & \frac{C}{k^2} \int_0^1 \cdots \int_0^1 \frac{(x_4^{1-\mu^+} u_1 u_2 + x_4^{\mu^+/\nu^- - \mu^+} z^{\mu^+/\nu^-} t_1 t_2 + z^{\mu^+} + u_1^{\mu^+})^{\frac{k-d}{2}}}{(x_4^{1-\mu^+} + x_4^{\mu^+/\nu^- - \mu^+} z^{\mu^+/\nu^-} t_1 + z^{\mu^+} + 1)^{\frac{k+d}{2}-2}} \\ & \qquad x_4^{\mu^+(4-d) + 2\mu^+/\nu^- + 1} z^{2\mu^+/\nu^- + \mu^+ - 1} u_1^{\mu^+}\, dt_1 \dots dz. \end{aligned}$$

The integral of the x_4-factor after the fraction is finite by hypothesis. If $k \geq d$ then we can see immediately that the other integral is also finite and has a bound independent of k. For $k < d$ we have to investigate the integral

$$\begin{aligned}
&\int_0^1 \int_0^1 (z^{\mu^+} + u_1^{\mu^+})^{\frac{k-d}{2}} z^{2\mu^+/\nu^- + \mu^+ - 1} u_1^{\mu^+} \, du_1 \, dz \\
&= \int_0^1 \int_0^1 1_{\{z < u_1\}} (z^{\mu^+} + u_1^{\mu^+})^{\frac{k-d}{2}} z^{2\mu^+/\nu^- + \mu^+ - 1} u_1^{\mu^+} \, du_1 \, dz \\
&\quad + \int_0^1 \int_0^1 1_{\{z > u_1\}} (z^{\mu^+} + u_1^{\mu^+})^{\frac{k-d}{2}} z^{2\mu^+/\nu^- + \mu^+ - 1} u_1^{\mu^+} \, du_1 \, dz.
\end{aligned}$$

On the set $\{z < u_1\}$ let $z = u_1 v$. So for the first term we get the estimate

$$\int_0^1 \int_0^1 (u_1^{\mu^+} v^{\mu^+} + u_1^{\mu^+})^{\frac{k-d}{2}} u_1^{2\mu^+/\nu^- + 2\mu^+} \, du_1 \, dv$$

and this is finite if $d < k + 4 + 2/\mu^+ + 2/\nu^-$. The second term is treated analogously.

To finish the treatment of $J_{-,+}(k, 2, 3)$, we still have to consider the contribution of the set $\{x_4 < y_1^{1/\mu^+}\}$, on which we use the transform $x_4 = y_1^{1/\mu^+} z$. However, this is much simpler than the contribution above and is left to the reader.

For $J_{-,+}(k, 2, 4)$ after integrating in $t_2 \vee v_2$ and $s_2 \wedge u_2$ as above, we use (12) and

$$y_1^{1/\nu^-} = x_2 t_1, \qquad x_1 = x_2 t_1 t_2, \qquad y_2^{1/\mu^+} = x_4 u_1, \qquad x_3 = x_4 u_1 u_2.$$

We conclude that

$$\begin{aligned}
J_{-,+}(k, 2, 4) &\leq \frac{C}{k^2} \int_0^1 \cdots \int_0^1 1_{\{x_1 < y_1^{1/\nu^-} < x_2, x_3 < y_2^{1/\mu^+} < x_4\}} \\
&\qquad \frac{(x_3 + x_1 + y_1 + y_2)^{\frac{k-d}{2}}}{(x_4 + x_2 + x_4^{\mu^+} + x_2^{\nu^-})^{\frac{k+d}{2} - 2}} \, dx_1 \dots dy_2 \\
&= \frac{C}{k^2} \int_0^1 \cdots \int_0^1 \frac{(x_4 u_1 u_2 + x_2 t_1 t_2 + x_2^{\nu^-} t_1^{\nu^-} + x_4^{\mu^+} u_1^{\mu^+})^{\frac{k-d}{2}}}{(x_4 + x_2 + x_4^{\mu^+} + x_2^{\nu^-})^{\frac{k+d}{2} - 2}} \\
&\qquad x_2^{\nu^- + 1} t_1^{\nu^-} x_4^{\mu^+ + 1} u_1^{\mu^+} \, dt_1 \dots dx_4.
\end{aligned}$$

On the set $\{x_2^{\nu^-} < x_4^{\mu^+}\}$, let $x_2^{\nu^-} = x_4^{\mu^+} z$. Then the integral has the estimate

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 \frac{(x_4 u_1 u_2 + x_4^{\mu^+/\nu^-} z^{1/\nu^-} t_1 t_2 + x_4^{\mu^+} z t_1^{\nu^-} + x_4^{\mu^+} u_1^{\mu^+})^{\frac{k-d}{2}}}{(x_4 + x_4^{\mu^+/\nu^-} z^{1/\nu^-} + x_4^{\mu^+} + x_4^{\mu^+} z)^{\frac{k+d}{2} - 2}} \\
&\qquad x_4^{\mu^+(\nu^- + 1)/\nu^-} z^{(\nu^- + 1)/\nu^-} t_1^{\nu^-} x_4^{\mu^+ + 1} u_1^{\mu^+} x_4^{\mu^+/\nu^-} z^{1/\nu^- - 1} \, dt_1 \dots dz \\
&\leq \int_0^1 \cdots \int_0^1 \frac{(x_4^{1-\mu^+} u_1 u_2 + x_4^{\mu^+/\nu^- - \mu^+} z^{1/\nu^-} t_1 t_2 + z t_1^{\nu^-} + u_1^{\mu^+})^{\frac{k-d}{2}}}{(x_4^{1-\mu^+} + x_4^{\mu^+/\nu^- - \mu^+} z^{1/\nu^-} + 1 + z)^{\frac{k+d}{2} - 2}} \\
&\qquad x_4^{\mu^+(4-d) + 2\mu^+/\nu^- + 1} z^{2/\nu^-} t_1^{\nu^-} u_1^{\mu^+} \, dt_1 \dots dz.
\end{aligned}$$

The integral of the x_4-factor alone is finite if (6) holds. If $k \geq d$ then the integral of the fraction is finite, too, as is seen by estimating the numerator by the denominator, with a bound independent of k. The case $k < d$ is not quite simple. We may estimate by

$$\int_0^1 \cdots \int_0^1 (z t_1^{\nu^-} + u_1^{\mu^+})^{\frac{k-d}{2}} z^{2/\nu^-} t_1^{\nu^-} u_1^{\mu^+} \, dt_1 \ldots dz.$$

On $\{t_1^{\nu^-} < u_1^{\mu^+}\}$ we use $t_1^{\nu^-} = u_1^{\mu^+} v$ to obtain the estimate

$$\int_0^1 \cdots \int_0^1 (z u_1^{\mu^+} v + u_1^{\mu^+})^{\frac{k-d}{2}} u_1^{2\mu^+} u_1^{\mu^+/\nu^-} \, du_1 \ldots dz < \infty$$

provided that $d < k + 4 + 2/\mu^+ + 2/\nu^-$.

On $\{u_1^{\mu^+} < t_1^{\nu^-}\}$, putting $u_1^{\mu^+} = t_1^{\nu^-} v$ yields the estimate

$$\int_0^1 \cdots \int_0^1 (z+v)^{\frac{k-d}{2}} t_1^{\nu^-(k-d)/2 + 2\nu^- + \nu^-/\mu^+} z^{2/\nu^-} v^{1/\mu^+} \, dt_1 \ldots dz.$$

The integral in t_1 is finite. For the integral in the remaining variables we need another transformation. On $\{z < v\}$ we get from $z = vx$ the estimate

$$\int_0^1 \int_0^1 (vx + v)^{\frac{k-d}{2}} v^{2/\nu^- + 1/\mu^+ + 1} \, dv \, dx < \infty.$$

On the set $\{v < z\}$ the argument is analogous.

Finally, the contribution of the integral $J_{-,+}(k, 2, 4)$ over the set $\{x_4^{\mu^+} < x_2^{\nu^-}\}$ can be estimated in a completely analogous way. This finishes the proof of Proposition 1. ∎

Proposition 2. *Suppose that* $\mu^+ \leq \nu^- \leq 1$. *If* $d \geq 4 + \frac{2}{\mu^+} + \frac{2}{\nu^-}$ *then* $K_{-,+}(0) = \infty$.

Proof. Since

$$t_1 t_2 - s_1 s_2 = (t_1 - s_1) t_2 + s_1 (t_2 - s_2) \begin{cases} \leq t_1 - s_1 + t_2 - s_2, \\ \geq \eta [t_1 - s_1 + t_2 - s_2], \end{cases} \tag{13}$$

with η as chosen in Section 1, we can conclude that $K_{-,+}(0)$ is equivalent to $I_{-,+}(0)$. Hence the assertion follows from (8) and (9). ∎

The next theorem is a consequence of Propositions 1 and 2.

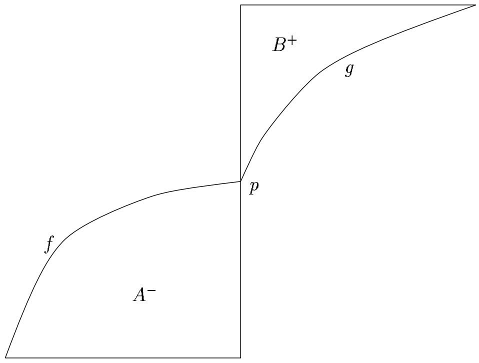
Theorem 2. *Suppose that* $\mu^+ \leq \nu^- \leq 1$. *The self-intersection local time in (4) exists in the Sobolev space* $\mathbf{D}_{2,\rho}$ *for any* $\rho < 2 - d/2$ *if and only if* $d < 4 + \frac{2}{\mu^+} + \frac{2}{\nu^-}$.

Proof. The order of smoothness follows from (3) with $c_k = ck^{-2}$ and from the fact that

$$\sum_{k=1}^{\infty} (1+k)^\rho k^{d/2-1} k^{-2} < \infty$$

if $\rho < 2 - d/2$. ■

2.2. The case $A^+ = B^- = \emptyset$ and $\min(\mu^+, \nu^-) \le 1 \le \max(\mu^+, \nu^-)$



We consider a similar case to the preceding one, just with one of the two boundaries given by a power function with exponent bigger than 1. Here the result depends only on the smaller exponent. We show that the critical dimension is given by $6 + \frac{2}{\mu^+}$.

Proposition 3. *Suppose that $\mu^+ \le 1 \le \nu^-$ and*

$$d < 6 + \frac{2}{\mu^+}. \tag{14}$$

Then there exists a constant C such that $k^3 J_{-,+}(k) \le C$ for all $k \in \mathbf{N}_0$.

Proof. Similarly to (8) we have

$$\sqrt{I_{-,+}(0)} \approx \int_0^1 \int_0^1 \frac{1}{(x_2^{\mu^+} + x_1)^{\frac{d}{2}-2}} \, dx_1 \, dx_2. \tag{15}$$

Applying (9) for $\nu^- = 1$ we get $I_{-,+}(0) < \infty$ if and only if (14) holds.

For $k > 0$ we have again $J_{-,+}(k,1) = 0$. To estimate $J_{-,+}(k,2)$, we replace A^- by the rectangle A^-_∞, where A^-_∞ is defined in (5). Obviously,

$$J_{-,+}(k,2) \le J_{-,+}(k,2,1) + J_{-,+}(k,2,2)$$

where (by integration in $t_2 \vee v_2$, $s_2 \wedge u_2$ and $s_1 \wedge u_1$)

$$\begin{aligned} J_{-,+}(k,2,1) &:= \int_{B^+} \cdots \int_{A^-_\infty} 1_{\{g(t_1 \vee v_1) < t_2 \wedge v_2\}} \\ &\qquad \frac{[t_1 \wedge v_1 - s_1 \vee u_1 + t_2 \wedge v_2 - s_2 \vee u_2]^{\frac{k-d}{2}}}{[t_1 \vee v_1 - s_1 \wedge u_1 + t_2 \vee v_2 - s_2 \wedge u_2]^{\frac{k+d}{2}}} \, ds \dots dv \\ &\le \frac{C}{k^3} \int \cdots \int 1_{\{s_1 \vee u_1 < p_1 < t_1 \wedge v_1 < t_1 \vee v_1 < g^{-1}(t_2 \wedge v_2)\}} \\ &\qquad \frac{[t_1 \wedge v_1 - s_1 \vee u_1 + t_2 \wedge v_2 - s_2 \vee u_2]^{\frac{k-d}{2}}}{[t_1 \vee v_1 - s_1 \vee u_1 + t_2 \wedge v_2 - s_2 \vee u_2]^{\frac{k+d}{2}-3}} \, ds \dots dv \end{aligned}$$

and

$$J_{-,+}(k,2,2) := \int_{B^+} \cdots \int_{A_\infty^-} 1_{\{g(t_1\wedge v_1)<t_2\wedge v_2<g(t_1\vee v_1)\}}$$
$$\frac{[t_1 \wedge v_1 - s_1 \vee u_1 + t_2 \wedge v_2 - s_2 \vee u_2]^{\frac{k-d}{2}}}{[t_1 \vee v_1 - s_1 \wedge u_1 + t_2 \vee v_2 - s_2 \wedge u_2]^{\frac{k+d}{2}}} \, ds \dots dv$$
$$\le \frac{C}{k^3} \int \cdots \int 1_{\{s_1\vee u_1<p_1<t_1\wedge v_1<g^{-1}(t_2\wedge v_2)<t_1\vee v_1\}}$$
$$\frac{[t_1 \wedge v_1 - s_1 \vee u_1 + t_2 \wedge v_2 - s_2 \vee u_2]^{\frac{k-d}{2}}}{[t_1 \vee v_1 - s_1 \vee u_1 + g(t_1 \vee v_1) - s_2 \vee u_2]^{\frac{k+d}{2}-3}} \, ds \dots dv.$$

By the transformations

$$x_1 = p_1 - s_1 \vee u_1, \qquad x_2 = t_1 \wedge v_1 - p_1,$$
$$x_3 = t_1 \vee v_1 - p_1, \qquad y_1 = p_2 - s_2 \vee u_2, \qquad y_2 = t_2 \wedge v_2 - p_2, \tag{16}$$

and

$$x_3 = y_2^{1/\mu^+} u_1, \qquad x_2 = y_2^{1/\mu^+} u_1 u_2$$

we obtain

$$J_{-,+}(k,2,1) \le \frac{C}{k^3} \int_0^1 \cdots \int_0^1 1_{\{x_2<x_3<y_2^{1/\mu^+}\}} \frac{(x_2+x_1+y_1+y_2)^{\frac{k-d}{2}}}{(x_3+x_1+y_1+y_2)^{\frac{k+d}{2}-3}} \, dx_1 \dots dy_2$$
$$\le \frac{C}{k^3} \int_0^1 \cdots \int_0^1 \frac{(y_2^{1/\mu^+} u_1 u_2 + x_1 + y_1 + y_2)^{\frac{k-d}{2}}}{(y_2^{1/\mu^+} u_1 + x_1 + y_1 + y_2)^{\frac{k+d}{2}-3}} y_2^{2/\mu^+} u_1 \, du_1 \dots dy_2.$$

We argue only for the contribution of the set $\{x_1 < y_1 < y_2\}$. The arguments for the other sets are similar. Let $y_1 = y_2 t_1$ and $x_1 = y_2 t_1 t_2$. Then the estimate is

$$\int_0^1 \cdots \int_0^1 \frac{(y_2^{1/\mu^+} u_1 u_2 + y_2 t_1 t_2 + y_2 t_1 + y_2)^{\frac{k-d}{2}}}{(y_2^{1/\mu^+} u_1 + y_2 t_1 t_2 + y_2 t_1 + y_2)^{\frac{k+d}{2}-3}} y_2^{2/\mu^+ + 2} \, du_1 \dots dy_2 \le C$$

with a constant independent of k, as is seen by extracting y_2 from numerator and denominator.

For the second integral (16) and $y_2^{1/\mu^+} = x_3 u_1$, $x_2 = x_3 u_1 u_2$ imply

$$J_{-,+}(k,2,2) \le \frac{C}{k^3} \int_0^1 \cdots \int_0^1 1_{\{x_2<y_2^{1/\mu^+}<x_3\}} \frac{(x_2+x_1+y_1+y_2)^{\frac{k-d}{2}}}{(x_3+x_1+x_3^{\mu^+}+y_1)^{\frac{k+d}{2}-3}} \, dx_1 \dots dy_2$$
$$\le \frac{C}{k^3} \int_0^1 \cdots \int_0^1 \frac{(x_3 u_1 u_2 + x_1 + y_1 + x_3^{\mu^+} u_1^{\mu^+})^{\frac{k-d}{2}}}{(x_3 + x_1 + x_3^{\mu^+} + y_1)^{\frac{k+d}{2}-3}} x_3^{\mu^+ + 1} u_1^{\mu^+} \, du_1 \dots dy_1.$$

Here we compute only the contribution of the set $\{x_1 < y_1 < x_3^{\mu^+}\}$, the other sets being treated similarly. With the help of $y_1 = x_3^{\mu^+} t_1$ and $x_1 = x_3^{\mu^+} t_1 t_2$ we estimate the contribution by

$$\frac{C}{k^3} \int_0^1 \cdots \int_0^1 \frac{(x_3^{1-\mu^+} u_1 u_2 + t_1 t_2 + t_1 + u_1^{\mu^+})^{\frac{k-d}{2}}}{(x_3^{1-\mu^+} + t_1 t_2 + 1 + t_1)^{\frac{k+d}{2}-3}} x_3^{\mu^+(6-d)+1} u_1^{\mu^+} t_1 \, du_1 \ldots dx_3.$$

The integral of the x_3-factor alone is obviously finite. For the rest, we may estimate in the following way by an expression not containing x_3. If $k \geq d$, we may estimate the numerator of the fraction by the denominator, hence lose dependence on k and see the integral is bounded by a finite constant independent of k. For $k < d$ we integrate in t_2 and t_1 to get the estimate

$$\begin{aligned}
&\frac{C}{k^3} \int_0^1 \cdots \int_0^1 (t_1 t_2 + t_1 + u_1^{\mu^+})^{\frac{k-d}{2}} u_1^{\mu^+} t_1 \, du_1 \ldots dt_2 \\
&\leq \frac{C}{k^3} \int_0^1 \int_0^1 (t_1 + u_1^{\mu^+})^{\frac{k-d}{2}+1} u_1^{\mu^+} \, du_1 \ldots dt_1 \\
&\leq \frac{C}{k^3} \int_0^1 u_1^{\mu^+(\frac{k-d}{2}+2)} u_1^{\mu^+} \, du_1 < \infty,
\end{aligned}$$

if $d < k + 6 + 2/\mu^+$. The proof of Proposition 3 is complete. ■

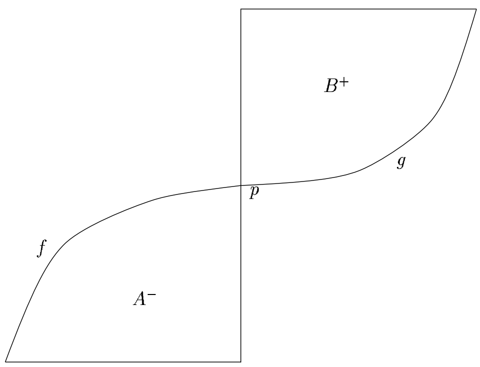
The next result follows from the estimates in (8) and (15).

Proposition 4. *Suppose that* $\mu^+ \leq 1 \leq \nu^-$. *If* $d \geq 6 + \frac{2}{\mu^+}$ *then* $K_{-,+}(0) = \infty$.

From Propositions 3 and 4 we obtain

Theorem 3. *Suppose that* $\mu^+ \leq 1 \leq \nu^-$. *The self-intersection local time in (4) exists in the Sobolev space* $\mathbf{D}_{2,\rho}$ *for any* $\rho < 3 - d/2$ *if and only if* $d < 6 + \frac{2}{\mu^+}$.

2.3. The case $A^+ = B^- = \emptyset$ and $1 \leq \min(\mu^+, \nu^-)$



In this section we consider the case in which both boundaries approach the horizontal through p rather fast, i.e. where the exponents are both bigger than 1. Here the result is the same as for rectangles.

Proposition 5. *Suppose that $1 \le \mu^+ \le \nu^-$ and $d < 8$. Then there exists a constant C such that $k^4 J_{-,+}(k) \le C$ for all $k \in \mathbf{N}_0$.*

Proof. If we replace A^- by A^-_∞ and B^+ by B^+_∞ then the proposition follows from Theorem 6 in Imkeller and Weisz [8]. ■

Proposition 6. *Suppose that $1 \le \mu^+ \le \nu^-$. If $d \ge 8$ then $K_{-,+}(0) = \infty$.*

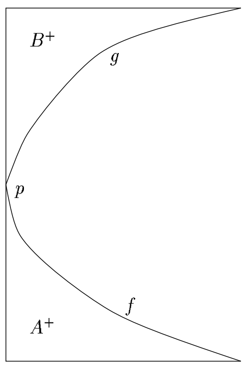
Proof. Similarly to (8) we have

$$\sqrt{K_{-,+}(0)} \approx \int_0^1 \int_0^1 \frac{1}{(x_2 + x_1)^{\frac{d}{2}-2}} \, dx_1 \, dx_2.$$

Applying (9) for $\mu^+ = \nu^- = 1$ we get that $K_{-,+}(0) = \infty$ if and only if $d \ge 8$. ■

Theorem 4. *Suppose that $1 \le \mu^+ \le \nu^-$. The self-intersection local time exists in some Sobolev space if and only if $d < 8$. In this case (4) holds in $\mathbf{D}_{2,\rho}$ for any $\rho < 4 - d/2$. In particular, $\alpha(x, \cdot)$ is a function.*

2.4. The case $A^- = B^- = \emptyset$ and $\max(\mu^+, \nu^+) \le 1$



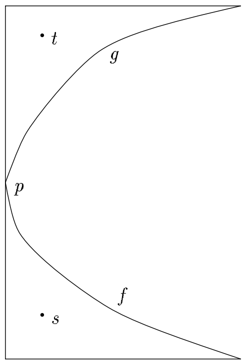
We study the case of two power-type functions opposite to each other for which both exponents are less than 1. We prove that in this case the critical dimension is given by $4 + \frac{2}{\mu^+} + \frac{2}{\nu^+}$.

Proposition 7. *Suppose that $\mu^+ \le \nu^+ \le 1$ and*

$$d < 4 + \frac{2}{\mu^+} + \frac{2}{\nu^+}. \tag{17}$$

Then there exists a constant C such that $k^2 J_{+,+}(k) \le C$ for all $k \in \mathbf{N}_0$.

Proof.



For $I_{+,+}(0)$ we have (see sketch)

$$
\begin{aligned}
\sqrt{I_{+,+}(0)} &= \int_{B^+}\int_{A^+} \frac{1}{(|t_1-s_1|+t_2-s_2)^{\frac{d}{2}}}\,ds\,dt \\
&\approx \int_{B^+}\int_{A^+} \frac{1}{(|t_1-s_1|+g(t_1)-f(s_1))^{\frac{d}{2}-2}}\,ds_1\,dt_1 \\
&= \int_{B^+}\int_{A^+} 1_{\{s_1<t_1\}}\frac{1}{(t_1-s_1+g(t_1)-f(s_1))^{\frac{d}{2}-2}}\,ds_1\,dt_1 \\
&\quad + \int_{B^+}\int_{A^+} 1_{\{s_1>t_1\}}\frac{1}{(s_1-t_1+g(t_1)-f(s_1))^{\frac{d}{2}-2}}\,ds_1\,dt_1 \\
&:= I_{+,+}(0,1)+I_{+,+}(0,2).
\end{aligned} \tag{18}
$$

It is easy to see that the transformations

$$
x_1 = s_1 \wedge t_1 - p_1, \qquad x_2 = s_1 \vee t_1 - p_1 \tag{19}
$$

and $x_1 = x_2 z$ imply

$$
\begin{aligned}
I_{+,+}(0,1) &= \int_0^1\int_0^1 1_{\{x_1<x_2\}}\frac{1}{(x_2-x_1+x_2^{\mu^+}+x_1^{\nu^+})^{\frac{d}{2}-2}}\,dx_1\,dx_2 \\
&= \int_0^1\int_0^1 \frac{x_2}{(x_2(1-z)+x_2^{\mu^+}+x_2^{\nu^+}z^{\nu^+})^{\frac{d}{2}-2}}\,dx_2\,dz \\
&\approx \int_0^1\int_0^1 (x_2^{\mu^+}+x_2^{\nu^+}z^{\nu^+})^{2-d/2}x_2\,dx_2\,dz < \infty
\end{aligned} \tag{20}
$$

if $d < 4+\frac{4}{\mu^+} (\geq 4+\frac{2}{\mu^+}+\frac{2}{\nu^+})$.

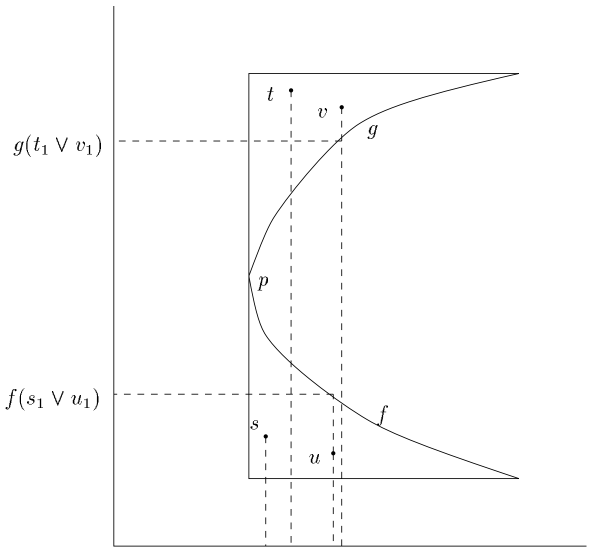
For $I_{+,+}(0,2)$ we obtain

$$
\begin{aligned}
I_{+,+}(0,2) &= \int_0^1\int_0^1 1_{\{x_1<x_2\}}\frac{1}{(x_2-x_1+x_1^{\mu^+}+x_2^{\nu^+})^{\frac{d}{2}-2}}\,dx_1\,dx_2 \\
&= \int_0^1\int_0^1 \frac{x_2}{(x_2(1-z)+x_2^{\mu^+}z^{\mu^+}+x_2^{\nu^+})^{\frac{d}{2}-2}}\,dx_2\,dz \\
&\approx \int_0^1\int_0^1 (z^{\mu^+}+x_2^{\nu^+-\mu^+})^{2-d/2}x_2^{\mu^+(2-d/2)+1}\,dx_2\,dz.
\end{aligned} \tag{21}
$$

On the set $\{z^{\mu^+} < x_2^{\nu^+-\mu^+}\}$ the transformation $z^{\mu^+} = x_2^{\nu^+-\mu^+} t$ yields the estimate

$$\int_0^1 \int_0^1 (x_2^{\nu^+-\mu^+} t + x_2^{\nu^+-\mu^+})^{2-d/2} x_2^{\mu^+(2-d/2)+1} x_2^{\frac{\nu^+-\mu^+}{\mu^+}} t^{\frac{1}{\mu^+}-1}\, dx_2\, dt < \infty \qquad (22)$$

if and only if $d < 4 + \frac{2}{\mu^+} + \frac{2}{\nu^+}$. The contribution of the set $\{z^{\mu^+} > x_2^{\nu^+-\mu^+}\}$ can be treated analogously.



Now suppose $k > 0$. We define $J_{+,+}(k,i,j)$, $i = 1,2$, $j = 1,2,3,4$, in the same way as in Section 1.2 $J_{-,+}(k,2,j)$ (cf. (10) and (11)). Then we have again

$$J_{+,+}(k,i) = J_{+,+}(k,i,1) + J_{+,+}(k,i,2) + J_{+,+}(k,i,3) + J_{+,+}(k,i,4) \qquad (i = 1,2),$$

where $J_{-,+}(k,2,j)$, $j = 1,2,3,4$, are defined by using the sets (see sketch)

$$\{g(t_1 \vee v_1) < t_2 \wedge v_2\} \cap \{s_2 \vee u_2 < f(s_1 \vee u_1)\}$$
$$\{g(t_1 \vee v_1) < t_2 \wedge v_2\} \cap \{f(s_1 \vee u_1) < s_2 \vee u_2 < f(s_1 \wedge u_1)\},$$
$$\{g(t_1 \wedge v_1) < t_2 \wedge v_2 < g(t_1 \vee v_1)\} \cap \{s_2 \vee u_2 < f(s_1 \vee u_1)\},$$
$$\{g(t_1 \wedge v_1) < t_2 \wedge v_2 < g(t_1 \vee v_1)\} \cap \{f(s_1 \vee u_1) < s_2 \vee u_2 < f(s_1 \wedge u_1)\}.$$

Integrating in $t_2 \vee v_2$ and $s_2 \wedge u_2$ we can see that

$$\begin{aligned} J_{+,+}(k,1,1) :=& \int_{B^+} \cdots \int_{A^+} 1_{\{S_1 \cap T_1 = \emptyset\}} 1_{\{g(t_1 \vee v_1) < t_2 \wedge v_2, s_2 \vee u_2 < f(s_1 \vee u_1)\}} \\ & \frac{(t_2 \wedge v_2 - s_2 \vee u_2)^{\frac{k-d}{2}}}{(|v_1 - u_1| + |t_1 - s_1| + t_2 \vee v_2 - s_2 \wedge u_2)^{\frac{k+d}{2}}}\, ds \ldots dv \\ \leq& \frac{C}{k^2} \int_{B^+} \cdots \int_{A^+} 1_{\{g(v_1) < t_2 \wedge v_2, s_2 \vee u_2 < f(u_1)\}} 1_{\{s_1 \vee t_1 < u_1 \wedge v_1\}} \\ & \frac{(t_2 \wedge v_2 - s_2 \vee u_2)^{\frac{k-d}{2}}}{(|v_1 - u_1| + |t_1 - s_1| + t_2 \wedge v_2 - s_2 \vee u_2)^{\frac{k+d}{2}-2}}\, ds \ldots dv, \end{aligned}$$

the latter by symmetry. It is easy to see that the transformation

$$x_1 = s_1 \wedge t_1 - p_1, \qquad x_2 = s_1 \vee t_1 - p_1, \qquad x_3 = u_1 \wedge v_1 - p_1,$$
$$x_4 = u_1 \vee v_1 - p_1, \qquad y_1 = p_2 - s_2 \vee u_2, \qquad y_2 = t_2 \wedge v_2 - p_2$$

implies

$$J_{+,+}(k,1,1) \le \frac{C}{k^2}\int_0^1 \cdots \int_0^1 (1_{\{x_4<y_2^{1/\mu^+},x_3<y_1^{1/\nu^+}\}} + 1_{\{x_4<y_1^{1/\nu^+},x_3<y_2^{1/\mu^+}\}})$$
$$1_{\{x_1<x_2<x_3<x_4\}} \frac{(y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_3+x_2-x_1+y_1+y_2)^{\frac{k+d}{2}-2}}\, dx_1 \dots dy_2.$$

Similarly,

$$J_{+,+}(k,1,2) \le \frac{C}{k^2}\int_{B^+} \cdots \int_{A^+} 1_{\{S_1\cap T_1=\emptyset\}} 1_{\{g(t_1\vee v_1)<t_2\wedge v_2, f(s_1\vee u_1)<s_2\vee u_2<f(s_1\wedge u_1)\}}$$
$$\frac{(t_2\wedge v_2 - s_2\vee u_2)^{\frac{k-d}{2}}}{(|v_1-u_1|+|t_1-s_1|+t_2\wedge v_2 - f(s_1\vee u_1))^{\frac{k+d}{2}-2}}\, ds\dots dv$$
$$\le \frac{C}{k^2}\int_0^1 \cdots \int_0^1 1_{\{x_1<x_2<x_3<x_4, x_4<y_2^{1/\mu^+}, x_1<y_1^{1/\nu^+}<x_3\}}$$
$$\frac{(y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_3+x_2-x_1+y_2+x_3^{\nu^+})^{\frac{k+d}{2}-2}}\, dx_1\dots dy_2$$
$$+\frac{C}{k^2}\int_0^1 \cdots \int_0^1 1_{\{x_1<x_2<x_3<x_4, x_3<y_2^{1/\mu^+}, x_1<y_1^{1/\nu^+}<x_4\}}$$
$$\frac{(y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_3+x_2-x_1+y_2+x_4^{\nu^+})^{\frac{k+d}{2}-2}}\, dx_1\dots dy_2,$$

$$J_{+,+}(k,1,3) \le \frac{C}{k^2}\int_{B^+} \cdots \int_{A^+} 1_{\{S_1\cap T_1=\emptyset\}} 1_{\{g(t_1\wedge v_1)<t_2\wedge v_2<g(t_1\vee v_1), s_2\vee u_2<f(s_1\vee u_1)\}}$$
$$\frac{(t_2\wedge v_2 - s_2\vee u_2)^{\frac{k-d}{2}}}{(|v_1-u_1|+|t_1-s_1|+g(t_1\vee v_1) - s_2\vee u_2)^{\frac{k+d}{2}-2}}\, ds\dots dv$$
$$\le \frac{C}{k^2}\int_0^1 \cdots \int_0^1 1_{\{x_1<x_2<x_3<x_4, x_1<y_2^{1/\mu^+}<x_3, x_4<y_1^{1/\nu^+}\}}$$
$$\frac{(y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_3+x_2-x_1+x_3^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\, dx_1\dots dy_2$$
$$+\frac{C}{k^2}\int_0^1 \cdots \int_0^1 1_{\{x_1<x_2<x_3<x_4, x_1<y_2^{1/\mu^+}<x_4, x_3<y_1^{1/\nu^+}\}}$$
$$\frac{(y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_3+x_2-x_1+x_4^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\, dx_1\dots dy_2$$

and

$$
\begin{aligned}
&J_{+,+}(k,1,4)\\
&\le \frac{C}{k^2}\int_{B^+}\cdots\int_{A^+} 1_{\{S_1\cap T_1=\emptyset\}} 1_{\{g(t_1\wedge v_1)<t_2\wedge v_2<g(t_1\vee v_1),f(s_1\vee u_1)<s_2\vee u_2<f(s_1\wedge u_1)\}}\\
&\qquad \frac{(t_2\wedge v_2-s_2\vee u_2)^{\frac{k-d}{2}}}{(|v_1-u_1|+|t_1-s_1|+g(t_1\vee v_1)-f(s_1\vee u_1))^{\frac{k+d}{2}-2}}\,ds\ldots dv\\
&\le \frac{C}{k^2}\int_0^1\cdots\int_0^1 (1_{\{x_1<y_2^{1/\mu^+}<x_4,x_2<y_1^{1/\nu^+}<x_3\}}+1_{\{x_2<y_2^{1/\mu^+}<x_4,x_1<y_1^{1/\nu^+}<x_3\}})\\
&\qquad 1_{\{x_1<x_2<x_3<x_4\}}\frac{(y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_3+x_2-x_1+x_4^{\mu^+}+x_3^{\nu^+})^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2\\
&\quad+\frac{C}{k^2}\int_0^1\cdots\int_0^1 (1_{\{x_1<y_2^{1/\mu^+}<x_3,x_2<y_1^{1/\nu^+}<x_4\}}+1_{\{x_2<y_2^{1/\mu^+}<x_3,x_1<y_1^{1/\nu^+}<x_4\}})\\
&\qquad 1_{\{x_1<x_2<x_3<x_4\}}\frac{(y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_3+x_2-x_1+x_3^{\mu^+}+x_4^{\nu^+})^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2.
\end{aligned}
$$

For the integrals $J_{+,+}(k,2,j)$ we use the transformation

$$
\begin{aligned}
x_1&=(s_1\wedge t_1)\wedge(u_1\wedge v_1)-p_1, & x_2&=(s_1\wedge t_1)\vee(u_1\wedge v_1)-p_1,\\
x_3&=(s_1\vee t_1)\wedge(u_1\vee v_1)-p_1, & x_4&=(s_1\vee t_1)\vee(u_1\vee v_1)-p_1,\\
y_1&=p_2-s_2\vee u_2, & y_2&=t_2\wedge v_2-p_2
\end{aligned}
\tag{23}
$$

to get

$$
\begin{aligned}
&J_{+,+}(k,2,1)\le \frac{C}{k^2}\int_{B^+}\cdots\int_{A^+} 1_{\{S_1\cap T_1\neq\emptyset\}}1_{\{g(t_1\vee v_1)<t_2\wedge v_2,s_2\vee u_2<f(s_1\vee u_1)\}}\\
&\frac{((s_1\vee t_1)\wedge(u_1\vee v_1)-(s_1\wedge t_1)\vee(u_1\wedge v_1)+t_2\wedge v_2-s_2\vee u_2)^{\frac{k-d}{2}}}{((s_1\vee t_1)\vee(u_1\vee v_1)-(s_1\wedge t_1)\wedge(u_1\wedge v_1)+t_2\wedge v_2-s_2\vee u_2)^{\frac{k+d}{2}-2}}\,ds\ldots dv\\
&\qquad\le \frac{C}{k^2}\int_0^1\cdots\int_0^1 (1_{\{x_2<y_2^{1/\mu^+},x_4<y_1^{1/\nu^+}\}}+1_{\{x_4<y_2^{1/\mu^+},x_2<y_1^{1/\nu^+}\}})\\
&\qquad\qquad 1_{\{x_1<x_2<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_1+y_1+y_2)^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2,
\end{aligned}
\tag{24}
$$

$$J_{+,+}(k,2,2) \le \frac{C}{k^2}\int_{B^+}\cdots\int_{A^+} 1_{\{S_1\cap T_1\neq\emptyset\}} 1_{\{g(t_1\vee v_1)<t_2\wedge v_2, f(s_1\vee u_1)<s_2\vee u_2<f(s_1\wedge u_1)\}}$$

$$\frac{((s_1\vee t_1)\wedge(u_1\vee v_1)-(s_1\wedge t_1)\vee(u_1\wedge v_1)+t_2\wedge v_2-s_2\vee u_2)^{\frac{k-d}{2}}}{((s_1\vee t_1)\vee(u_1\vee v_1)-(s_1\wedge t_1)\wedge(u_1\wedge v_1)+t_2\wedge v_2-f(s_1\vee u_1))^{\frac{k+d}{2}-2}}\,ds\ldots dv$$

$$\le \frac{C}{k^2}\int_0^1\cdots\int_0^1 1_{\{x_4<y_2^{1/\mu^+}, x_1<y_1^{1/\nu^+}<x_3\}}$$

$$1_{\{x_1<x_2<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_1+y_2+x_3^{\nu^+})^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2$$

$$+\frac{C}{k^2}\int_0^1\cdots\int_0^1 1_{\{x_4<y_2^{1/\mu^+}, x_1<y_1^{1/\nu^+}<x_2\}}$$

$$1_{\{x_1<x_2<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_1+y_2+x_2^{\nu^+})^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2$$

$$+\frac{C}{k^2}\int_0^1\cdots\int_0^1 (1_{\{x_3<y_2^{1/\mu^+}, x_1<y_1^{1/\nu^+}<x_4\}}+1_{\{x_2<y_2^{1/\mu^+}, x_3<y_1^{1/\nu^+}<x_4\}})$$

$$1_{\{x_1<x_2<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_1+y_2+x_4^{\nu^+})^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2,$$

$$J_{+,+}(k,2,3) \le \frac{C}{k^2}\int_{B^+}\cdots\int_{A^+} 1_{\{S_1\cap T_1\neq\emptyset\}} 1_{\{g(t_1\wedge v_1)<t_2\wedge v_2<g(t_1\vee v_1), s_2\vee u_2<f(s_1\vee u_1)\}}$$

$$\frac{((s_1\vee t_1)\wedge(u_1\vee v_1)-(s_1\wedge t_1)\vee(u_1\wedge v_1)+t_2\wedge v_2-s_2\vee u_2)^{\frac{k-d}{2}}}{((s_1\vee t_1)\vee(u_1\vee v_1)-(s_1\wedge t_1)\wedge(u_1\wedge v_1)+g(t_1\vee v_1)-s_2\vee u_2)^{\frac{k+d}{2}-2}}\,ds\ldots dv$$

$$\le \frac{C}{k^2}\int_0^1\cdots\int_0^1 1_{\{x_1<y_2^{1/\mu^+}<x_3, x_4<y_1^{1/\nu^+}\}}$$

$$1_{\{x_1<x_2<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_1+x_3^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2$$

$$+\frac{C}{k^2}\int_0^1\cdots\int_0^1 1_{\{x_1<y_2^{1/\mu^+}<x_2, x_4<y_1^{1/\nu^+}\}}$$

$$1_{\{x_1<x_2<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_1+x_2^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2$$

$$+\frac{C}{k^2}\int_0^1\cdots\int_0^1 (1_{\{x_1<y_2^{1/\mu^+}<x_4, x_3<y_1^{1/\nu^+}\}}+1_{\{x_3<y_2^{1/\mu^+}<x_4, x_2<y_1^{1/\nu^+}\}})$$

$$1_{\{x_1<x_2<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_1+x_4^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2 \tag{25}$$

and

$$
\begin{aligned}
&J_{+,+}(k,2,4)\\
&\le \frac{C}{k^2}\int_{B^+}\cdots\int_{A^+} 1_{\{S_1\cap T_1\neq\emptyset\}} 1_{\{g(t_1\wedge v_1)<t_2\wedge v_2<g(t_1\vee v_1),f(s_1\vee u_1)<s_2\vee u_2<f(s_1\wedge u_1)\}}\\
&\quad \frac{((s_1\vee t_1)\wedge(u_1\vee v_1)-(s_1\wedge t_1)\vee(u_1\wedge v_1)+t_2\wedge v_2-s_2\vee u_2)^{\frac{k-d}{2}}\,ds\ldots dv}{((s_1\vee t_1)\vee(u_1\vee v_1)-(s_1\wedge t_1)\wedge(u_1\wedge v_1)+g(t_1\vee v_1)-f(s_1\vee u_1))^{\frac{k+d}{2}-2}}\\
&\le \frac{C}{k^2}\int_0^1\cdots\int_0^1 \left(1_{\{x_1<y_2^{1/\mu^+}<x_4,x_2<y_1^{1/\nu^+}<x_3\}}+1_{\{x_2<y_2^{1/\mu^+}<x_4,x_1<y_1^{1/\nu^+}<x_3\}}\right)\\
&\qquad 1_{\{x_1<x_2<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}\,dx_1\ldots dy_2}{(x_4-x_1+x_4^{\mu^+}+x_3^{\nu^+})^{\frac{k+d}{2}-2}}\\
&\quad +\frac{C}{k^2}\int_0^1\cdots\int_0^1 1_{\{x_3<y_2^{1/\mu^+}<x_4,x_1<y_1^{1/\nu^+}<x_2\}}1_{\{x_1<x_2<x_3<x_4\}}\\
&\qquad \frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}\,dx_1\ldots dy_2}{(x_4-x_1+x_4^{\mu^+}+x_2^{\nu^+})^{\frac{k+d}{2}-2}}\\
&\quad +\frac{C}{k^2}\int_0^1\cdots\int_0^1 \left(1_{\{x_1<y_2^{1/\mu^+}<x_3,x_2<y_1^{1/\nu^+}<x_4\}}+1_{\{x_2<y_2^{1/\mu^+}<x_3,x_1<y_1^{1/\nu^+}<x_4\}}\right)\\
&\qquad 1_{\{x_1<x_2<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}\,dx_1\ldots dy_2}{(x_4-x_1+x_3^{\mu^+}+x_4^{\nu^+})^{\frac{k+d}{2}-2}}\\
&\quad +\frac{C}{k^2}\int_0^1\cdots\int_0^1 1_{\{x_1<y_2^{1/\mu^+}<x_2,x_3<y_1^{1/\nu^+}<x_4\}}1_{\{x_1<x_2<x_3<x_4\}}\\
&\qquad \frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}\,dx_1\ldots dy_2}{(x_4-x_1+x_2^{\mu^+}+x_4^{\nu^+})^{\frac{k+d}{2}-2}}.
\end{aligned}
$$

Here we compute only one of the integrals, because the computations are quite long and we use always the same idea. The third terms of the integrals $J_{+,+}(k,1,4)$ and $J_{+,+}(k,2,4)$ are the most complicated ones. Therefore we consider one type of these integrals, say, $J_{+,+}(k,2,4)$ over the set $\{x_1<y_2^{1/\mu^+}<x_2<y_1^{1/\nu^+}<x_3<x_4\}$. All other integrals can be handled with the same method. If $k\le d$ then

$$
\begin{aligned}
&\int_0^1\cdots\int_0^1 1_{\{x_1<y_2^{1/\mu^+}<x_2<y_1^{1/\nu^+}<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}\,dx_1\ldots dy_2}{(x_4-x_1+x_3^{\mu^+}+x_4^{\nu^+})^{\frac{k+d}{2}-2}}\\
&\le \int_0^1\cdots\int_0^1 1_{\{x_1<y_2^{1/\mu^+}<x_2<y_1^{1/\nu^+}<x_3<x_4\}}\frac{(y_1+y_2)^{\frac{k-d}{2}}\,dx_1\ldots dy_2}{(x_3^{\mu^+}+x_4^{\nu^+})^{\frac{k+d}{2}-2}}\\
&\le \int_0^1\cdots\int_0^1 1_{\{y_2^{1/\mu^+}<y_1^{1/\nu^+}<x_3<x_4\}}\frac{(y_1+y_2)^{\frac{k-d}{2}}}{(x_3^{\mu^+}+x_4^{\nu^+})^{\frac{k+d}{2}-2}}y_2^{1/\mu^+}y_1^{1/\nu^+}\,dx_3\ldots dy_2.
\end{aligned}
\tag{26}
$$

By the transformation

$$
x_3=x_4t_1,\qquad y_1^{1/\nu^+}=x_4t_1t_2,\qquad y_2^{1/\mu^+}=x_4t_1t_2t_3
$$

which has the Jacobian $x_4^{\mu^++\nu^++1} t_1^{\mu^++\nu^+} t_2^{\mu^++\nu^+-1} t_3^{\mu^+-1}$, we get the estimate

$$\int_0^1 \cdots \int_0^1 \frac{(x_4^{\nu^+-\mu^+} t_1^{\nu^+-\mu^+} t_2^{\nu^+-\mu^+} + t_3^{\mu^+})^{\frac{k-d}{2}}}{(x_4^{\nu^+-\mu^+} + t_1^{\mu^+})^{\frac{k+d}{2}-2}} \\ x_4^{\mu^+(3-d)+\nu^++3} t_1^{\mu^+\frac{k-d}{2}+\mu^++\nu^++2} t_2^{\mu^+\frac{k-d}{2}+\mu^++\nu^++1} t_3^{\mu^+}\, dx_4 \dots dt_3. \tag{27}$$

On the set $\{x_4^{\nu^+-\mu^+} < t_1^{\mu^+}\}$ we use the transformation $x_4^{\nu^+-\mu^+} = t_1^{\mu^+} y$. So we get the estimate

$$\int_0^1 \cdots \int_0^1 (t_1^{\nu^+} t_2^{\nu^+-\mu^+} y + t_3^{\mu^+})^{\frac{k-d}{2}} t_1^{\frac{(\mu^+)^2(3-d)+\mu^+\nu^++4\mu^+}{\nu^+-\mu^+}+3\mu^++\nu^++2-\mu^+d} \\ t_2^{\mu^+\frac{k-d}{2}+\mu^++\nu^++1} t_3^{\mu^+} y^{\frac{\mu^+(3-d)+\nu^++4}{\nu^+-\mu^+}-1}\, dt_1 \dots dy. \tag{28}$$

Using the transformation $y = t_3^{\mu^+} z$ on the set $\{y < t_3^{\mu^+}\}$ we have the estimate

$$\int_0^1 \cdots \int_0^1 (t_1^{\nu^+} t_2^{\nu^+-\mu^+} z + 1)^{\frac{k-d}{2}} t_1^{\frac{(\mu^+)^2(3-d)+\mu^+\nu^++4\mu^+}{\nu^+-\mu^+}+3\mu^++\nu^++2-\mu^+d} \\ t_2^{\mu^+\frac{k-d}{2}+\mu^++\nu^++1} t_3^{\frac{(\mu^+)^2(3-d)+\mu^+\nu^++4\mu^+}{\nu^+-\mu^+}+\mu^+\frac{k-d}{2}+\mu^+} z^{\frac{\mu^+(3-d)+\nu^++4}{\nu^+-\mu^+}-1}\, dt_1 \dots dz.$$

In consequence of $\mu^+ \le \nu^+$ one can now see that this integral is finite if $d < 4 + \frac{2}{\mu^+} + \frac{2}{\nu^+}$.

On the set $\{y > t_3^{\mu^+}\}$ let $t_3^{\mu^+} = yz$. Then this contribution can be estimated by

$$\int_0^1 \cdots \int_0^1 (t_1^{\nu^+} t_2^{\nu^+-\mu^+} + z)^{\frac{k-d}{2}} t_1^{\frac{(\mu^+)^2(3-d)+\mu^+\nu^++4\mu^+}{\nu^+-\mu^+}+3\mu^++\nu^++2-\mu^+d} \\ t_2^{\mu^+\frac{k-d}{2}+\mu^++\nu^++1} y^{\frac{\mu^+(3-d)+\nu^++4}{\nu^+-\mu^+}+\frac{k-d}{2}+\frac{1}{\mu^+}} z^{\frac{1}{\mu^+}}\, dt_1 \dots dz.$$

Since the integral in y is finite by hypothesis, it remains to consider the integral

$$\int_0^1 \cdots \int_0^1 (t_1^{\nu^+} t_2^{\nu^+-\mu^+} + z)^{\frac{k-d}{2}} t_1^{\frac{(\mu^+)^2(3-d)+\mu^+\nu^++4\mu^+}{\nu^+-\mu^+}+3\mu^++\nu^++2-\mu^+d} \\ t_2^{\mu^+\frac{k-d}{2}+\mu^++\nu^++1} z^{\frac{1}{\mu^+}}\, dt_1 \dots dz.$$

On the set $\{t_2^{\nu^+-\mu^+} < z\}$ we use the transformation $t_2^{\nu^+-\mu^+} = zv$ to estimate the last integral by

$$\int_0^1 \cdots \int_0^1 (t_1^{\nu^+} v + 1)^{\frac{k-d}{2}} t_1^{\frac{(\mu^+)^2(3-d)+\mu^+\nu^++4\mu^+}{\nu^+-\mu^+}+3\mu^++\nu^++2-\mu^+d} \\ z^{\frac{k-d}{2}+\frac{\mu^+\frac{k-d}{2}+\mu^++\nu^++1}{\nu^+-\mu^+}+\frac{1}{\mu^+}+\frac{1}{\nu^+-\mu^+}} v^{\frac{\mu^+\frac{k-d}{2}+\mu^++\nu^++2}{\nu^+-\mu^+}}\, dt_1 \dots dv$$

which is finite by hypothesis.

With $z = t_2^{\nu^+-\mu^+} v$ the contribution of the set $\{t_2^{\nu^+-\mu^+} > z\}$ can be estimated by

$$\int_0^1 \cdots \int_0^1 (t_1^{\nu^+} + v)^{\frac{k-d}{2}} t_1^{\frac{(\mu^+)^2(3-d)+\mu^+\nu^++4\mu^+}{\nu^+-\mu^+}+3\mu^++\nu^++2-\mu^+ d}$$
$$t_2^{\nu^+\frac{k-d}{2}+2\nu^++\frac{\nu^+}{\mu^+}} v^{\frac{1}{\mu^+}}\, dt_1 \dots dv.$$

The integral in t_2 is finite. The remaining integral can be estimated on the set $\{t_1^{\nu^+} < v\}$ by

$$\int_0^1 \cdots \int_0^1 (x+1)^{\frac{k-d}{2}} v^{\frac{k-d}{2}+\frac{(\mu^+)^2(3-d)+\mu^+\nu^++4\mu^+}{\nu^+(\nu^+-\mu^+)}+\frac{3\mu^++\nu^++2-\mu^+ d}{\nu^+}+\frac{1}{\mu^+}+\frac{1}{\nu^+}}$$
$$x^{\frac{(\mu^+)^2(3-d)+\mu^+\nu^++4\mu^+}{\nu^+(\nu^+-\mu^+)}+\frac{3\mu^++\nu^++2-\mu^+ d}{\nu^+}+\frac{1}{\nu^+}-1}\, dv\, dx.$$

where we used that $t_1^{\nu^+} = vx$. This integral is finite by hypothesis, as is seen by comparing the exponents with -1.

For the contribution of the set $\{t_1^{\nu^+} > v\}$ we use $v = t_1^{\nu^+} x$ to get the estimate

$$\int_0^1 \cdots \int_0^1 (1+x)^{\frac{k-d}{2}} t_1^{\nu^+\frac{k-d}{2}+\frac{(\mu^+)^2(3-d)+\mu^+\nu^++4\mu^+}{\nu^+-\mu^+}+3\mu^++2\nu^++2-\mu^+ d+\frac{\nu^+}{\mu^+}} x^{\frac{1}{\mu^+}}\, dt_1\, dx < \infty.$$

On the set $\{x_4^{\nu^+-\mu^+} > t_1^{\mu^+}\}$ we estimate the integral in (27) by

$$\int_0^1 \cdots \int_0^1 (x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} y^{\frac{\nu^+-\mu^+}{\mu^+}} t_2^{\nu^+-\mu^+} + t_3^{\mu^+})^{\frac{k-d}{2}} x_4^{\nu^+(3-d)+\frac{\nu^+(\nu^++3)}{\mu^+}}$$
$$t_2^{\mu^+\frac{k-d}{2}+\mu^++\nu^++1} t_3^{\mu^+} y^{\frac{\mu^+\frac{k-d}{2}+\mu^++\nu^++3}{\mu^+}-1}\, dx_4 \dots dy,$$

where we put $t_1^{\mu^+} = x_4^{\nu^+-\mu^+} y$.

On the set $\{y^{\frac{\nu^+-\mu^+}{\mu^+}} < t_3^{\mu^+}\}$ let $y^{\frac{\nu^+-\mu^+}{\mu^+}} = t_3^{\mu^+} z$. Then we can estimate by

$$\int_0^1 \cdots \int_0^1 (x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} t_2^{\nu^+-\mu^+} z + 1)^{\frac{k-d}{2}} x_4^{\nu^+(3-d)+\frac{\nu^+(\nu^++3)}{\mu^+}} t_2^{\mu^+\frac{k-d}{2}+\mu^++\nu^++1}$$
$$t_3^{\mu^+\frac{k-d}{2}+\mu^++\frac{(\mu^+)^2\frac{k-d}{2}+(\mu^+)^2+\mu^+\nu^++3\mu^+}{\nu^+-\mu^+}} z^{\frac{\mu^+\frac{k-d}{2}+\mu^++\nu^++3}{\nu^+-\mu^+}-1}\, dx_4 \dots dz < \infty.$$

Using the coordinate change $t_3^{\mu^+} = y^{\frac{\nu^+-\mu^+}{\mu^+}} z$ on the set $\{y^{\frac{\nu^+-\mu^+}{\mu^+}} > t_3^{\mu^+}\}$ we obtain the estimate

$$\int_0^1 \cdots \int_0^1 (x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} t_2^{\nu^+-\mu^+} + z)^{\frac{k-d}{2}} x_4^{\nu^+(3-d)+\frac{\nu^+(\nu^++3)}{\mu^+}}$$
$$t_2^{\mu^+\frac{k-d}{2}+\mu^++\nu^++1} y^{\frac{\nu^+}{\mu^+}\frac{k-d}{2}+\frac{2\nu^++3}{\mu^+}+\frac{\nu^+-\mu^+}{(\mu^+)^2}-1} z^{\frac{1}{\mu^+}}\, dx_4 \dots dz.$$

Since the integral in y is finite we have to investigate the integral

$$\int_0^1 \cdots \int_0^1 (x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} t_2^{\nu^+-\mu^+} + z)^{\frac{k-d}{2}} x_4^{\nu^+(3-d)+\frac{\nu^+(\nu^++3)}{\mu^+}} t_2^{\mu^+\frac{k-d}{2}+\mu^++\nu^++1} z^{\frac{1}{\mu^+}}\, dx_4 \ldots dz.$$

On the set $\{t_2^{\nu^+-\mu^+} < z\}$, setting $t_2^{\nu^+-\mu^+} = zv$ we estimate by

$$\int_0^1 \cdots \int_0^1 (x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} v + 1)^{\frac{k-d}{2}} x_4^{\nu^+(3-d)+\frac{\nu^+(\nu^++3)}{\mu^+}} z^{\frac{k-d}{2}+\frac{\mu^+\frac{k-d}{2}+\mu^++\nu^++2}{\nu^+-\mu^+}+\frac{1}{\mu^+}} v^{\frac{\mu^+\frac{k-d}{2}+\mu^++\nu^++2}{\nu^+-\mu^+}-1}\, dx_4\, dz\, dv < \infty.$$

On the set $\{t_2^{\nu^+-\mu^+} > z\}$ $(z = t_2^{\nu^+-\mu^+} v)$ we may estimate by

$$\int_0^1 \cdots \int_0^1 (x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} + v)^{\frac{k-d}{2}} x_4^{\nu^+(3-d)+\frac{\nu^+(\nu^++3)}{\mu^+}} t_2^{\nu^+\frac{k-d}{2}+2\nu^++\frac{\nu^+}{\mu^+}} v^{\frac{1}{\mu^+}}\, dx_4\, dt_2\, dv.$$

The integral in t_2 is finite. The remaining integral on the set $\{x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} < v\}$ can be estimated by $(x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} = vu)$

$$\int_0^1 \cdots \int_0^1 (u+1)^{\frac{k-d}{2}} v^{\frac{k-d}{2}+\frac{\mu^+\nu^+(3-d)+\nu^+(\nu^++3)+\mu^+}{\nu^+(\nu^+-\mu^+)}+\frac{1}{\mu^+}} u^{\frac{\mu^+\nu^+(3-d)+\nu^+(\nu^++3)+\mu^+}{\nu^+(\nu^+-\mu^+)}-1}\, dv\, du < \infty.$$

Finally, on the set $\{x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} > v\}$ let $v = x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} u$. The contribution can be estimated by

$$\int_0^1 \cdots \int_0^1 (1+u)^{\frac{k-d}{2}} x_4^{\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}\frac{k-d}{2}+\nu^+(3-d)+\frac{\nu^+(\nu^++3)}{\mu^+}+\frac{\nu^+(\nu^+-\mu^+)}{(\mu^+)^2}+\frac{\nu^+(\nu^+-\mu^+)}{\mu^+}} u^{\frac{1}{\mu^+}}\, dx_4\, du < \infty.$$

If $k \geq d$ then we estimate the first line of (26) by

$$\int_0^1 \cdots \int_0^1 1_{\{x_1<y_2^{1/\mu^+}<x_2<y_1^{1/\nu^+}<x_3<x_4\}} (x_4 - x_3 + x_3^{\mu^+} + x_4^{\nu^+})^{2-d}\, dx_1 \ldots dy_2$$
$$\leq \int_0^1 \cdots \int_0^1 1_{\{y_2^{1/\mu^+}<y_1^{1/\nu^+}<x_3<x_4\}} (x_3^{\mu^+} + x_4^{\nu^+})^{2-d} y_2^{1/\mu^+} y_1^{1/\nu^+}\, dx_3 \ldots dy_2.$$

Using the methods above (after (26)) we can show in the same way that this integral is finite, too. ∎

To get a lower bound for the critical dimension, we now investigate the integral $K(0)$.

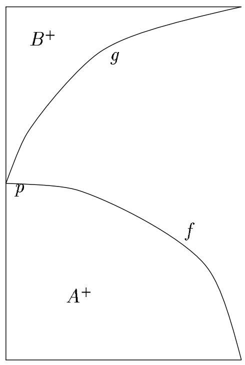
Proposition 8. *Suppose that* $\mu^+ \le \nu^+ \le 1$. *If* $d \ge 4 + \frac{2}{\mu^+} + \frac{2}{\nu^+}$ *then* $K_{+,+}(0) = \infty$.

Proof. The proof follows from (13), (21) and (22). ■

Theorem 5. *Suppose that* $\mu^+ \le \nu^+ \le 1$. *The self-intersection local time exists in some Sobolev space if and only if* $d < 4 + \frac{2}{\mu^+} + \frac{2}{\nu^+}$. *In this case (4) holds in* $\mathbf{D}_{2,\rho}$ *for any* $\rho < 2 - d/2$.

The proof follows from Propositions 7 and 8.

2.5. The case $A^- = B^- = \emptyset$ and $\min(\mu^+, \nu^-) \le 1 \le \max(\mu^+, \nu^-)$



We consider the case where the boundaries consist of one convex and one concave power-type curves opposite to each other. We prove that in this case the critical dimension is given by $6 + \frac{2}{\mu^+}$, if μ^+ is the smaller of the two powers.

Proposition 9. *Suppose that* $\mu^+ \le 1 \le \nu^+$ *and*

$$d < 6 + \frac{2}{\mu^+}. \tag{29}$$

Then there exists a constant C *such that* $k^3 I_{+,+}(k) \le C$ *for all* $k \in \mathbf{N}_0$.

Proof. First we consider $I_{+,+}(0)$. Similarly to (20) $I_{+,+}(0,1)$ is finite if $d < 4 + \frac{4}{\mu^+} (\ge 6 + \frac{2}{\mu^+})$. From the second integral in (21) we derive that

$$I_{+,+}(0,2) \le \int_0^1 \int_0^1 (x_2^{1-\mu^+}(1-z) + z^{\mu^+})^{2-d/2} x_2^{\mu^+(2-d/2)+1} \, dx_2 \, dz.$$

On the set $\{z > 1/2\}$ this integral is finite for $d < 4 + \frac{4}{\mu^+}$. On the set $\{z < 1/2\}$ we can continue as in (21) (with $\nu^+ = 1$) to see that the integral is finite if (29) holds.

If $k > 0$ then we replace A^+ by A^+_∞. We compute now the integrals $I_{+,+}(k,1)$ and $J_{+,+}(k,2)$. It is enough to estimate the integrals $I_{+,+}(k,1,j)$ and $J_{+,+}(k,1,j)$, $j = 1, 2$, where these integrals are defined like $J_{-,+}(k,2,j)$ in the proof of Proposition

3. Again by integration in $t_2 \vee v_2$, $s_2 \wedge u_2$, s_1 and t_1 we have, with $i = 0$ if $t_2 < v_2$, $s_2 > u_2$, $i = 1$ if $t_2 > v_2$, $s_2 > u_2$ or $t_2 < v_2$, $s_2 < u_2$, $i = 2$ if $t_2 > v_2$, $s_2 < u_2$,

$$\begin{aligned} I_{+,+}(k,1,1) &:= \int_{B^+} \cdots \int_{A_\infty} 1_{\{S_1 \cap T_1 = \emptyset, g(t_1 \vee v_1) < t_2 \wedge v_2\}} \\ &\qquad \frac{(t_2 \wedge v_2 - s_2 \vee u_2)^k}{[(|t_1 - s_1| + t_2 - s_2)(|v_1 - u_1| + v_2 - u_2)]^{\frac{k+d}{2}}} \, ds \ldots dv \\ &\le \frac{C}{k^2} \int_{B^+} \cdots \int_{A_\infty} 1_{\{S_1 \cap T_1 = \emptyset, g(t_1 \vee v_1) < t_2 \wedge v_2\}} \\ &\qquad \frac{(t_2 \wedge v_2 - s_2 \vee u_2)^k \, ds \ldots dv}{(|t_1 - s_1| + t_2 \wedge v_2 - s_2 \vee u_2)^{\frac{k+d}{2} - i} (|v_1 - u_1| + t_2 \wedge v_2 - s_2 \vee u_2)^{\frac{k+d}{2} - (2-i)}} \\ &\le \frac{C}{k^4} \int_{B^+} \cdots \int_{A_\infty} 1_{\{S_1 \cap T_1 = \emptyset, g(t_1 \vee v_1) < t_2 \wedge v_2\}} (t_2 \wedge v_2 - s_2 \vee u_2)^{4-d} \, ds \ldots dv. \end{aligned}$$

Using the transformation

$$\begin{aligned} x_1 &= t_1 \wedge v_1 - p_1, \qquad & x_2 &= t_1 \vee v_1 - p_1, \\ y_1 &= p_2 - s_2 \vee u_2, \qquad & y_2 &= t_2 \wedge v_2 - p_2, \end{aligned}$$

we conclude

$$\begin{aligned} I_{+,+}(k,1,1) &\le \frac{C}{k^4} \int_0^1 \cdots \int_0^1 1_{\{x_1 < x_2 < y_2^{1/\mu^+}\}} (y_1 + y_2)^{4-d} \, dx_1 \ldots dy_2 \\ &\le \frac{C}{k^4} \int_0^1 \cdots \int_0^1 1_{\{x_1 < x_2 < y_2^{1/\mu^+}\}} y_2^{5-d} \, dx_1 \ldots dy_2 \\ &\le \frac{C}{k^4} \int_0^1 x_2^{\mu^+ (6-d) + 1} \, dx_1, \end{aligned} \tag{30}$$

which is finite by hypothesis. Similarly,

$$\begin{aligned} I_{+,+}(k,1,2) &:= \int_{B^+} \cdots \int_{A_\infty} 1_{\{S_1 \cap T_1 = \emptyset, g(t_1 \wedge v_1) < t_2 \wedge v_2 < g(t_1 \vee v_1)\}} \\ &\qquad \frac{(t_2 \wedge v_2 - s_2 \vee u_2)^k}{[(|t_1 - s_1| + t_2 - s_2)(|v_1 - u_1| + v_2 - u_2)]^{\frac{k+d}{2}}} \, ds \ldots dv \\ &= \frac{C}{k^4} \int_{B^+} \cdots \int_{A_\infty} 1_{\{S_1 \cap T_1 = \emptyset, g(t_1 \wedge v_1) < t_2 \wedge v_2 < g(t_1 \vee v_1)\}} \\ &\qquad \frac{(t_2 \wedge v_2 - s_2 \vee u_2)^{\frac{k-d}{2} + i} \, ds \ldots dv}{(g(t_1 \vee v_1) - s_2 \vee u_2)^{\frac{k+d}{2} - (4-i)}} \\ &\le \frac{C}{k^4} \int_0^1 \cdots \int_0^1 1_{\{x_1 < y_2^{1/\mu^+} < x_2\}} \frac{(y_1 + y_2)^{\frac{k-d}{2} + i}}{(x_2^{\mu^+} + y_1)^{\frac{k+d}{2} - (4-i)}} \, dx_1 \ldots dy_2, \end{aligned} \tag{31}$$

where i is equal to 1 or 2. If $\frac{k-d}{2}+i+1<0$ then

$$I_{+,+}(k,1,2) \le \frac{C}{k^5}\int_0^1 \cdots \int_0^1 1_{\{x_1<x_2\}} \frac{(x_1^{\mu^+}+y_1)^{\frac{k-d}{2}+i+1}}{(x_2^{\mu^+}+y_1)^{\frac{k+d}{2}-(4-i)}}\,dx_1\,dx_2\,dy_1. \tag{32}$$

Substituting $x_1 = x_2 t$ we conclude

$$I_{+,+}(k,1,2) \le \frac{C}{k^5}\int_0^1 \cdots \int_0^1 \frac{(x_2^{\mu^+}t^{\mu^+}+y_1)^{\frac{k-d}{2}+i+1}x_2}{(x_2^{\mu^+}+y_1)^{\frac{k+d}{2}-(4-i)}}\,dt\,dx_2\,dy_1.$$

On the set $\{x_2^{\mu^+}<y_1\}$ put $x_2 = y_1^{1/\mu^+}z^{1/\mu^+}$ and estimate the integral by

$$\begin{aligned} I_{+,+}(k,1,2) &\le \frac{C}{k^5}\int_0^1 \cdots \int_0^1 \frac{(y_1 z t^{\mu^+}+y_1)^{\frac{k-d}{2}+i+1}y_1^{1/\mu^+}z^{1/\mu^+}}{(y_1 z+y_1)^{\frac{k+d}{2}-(4-i)}} y_1^{1/\mu^+}z^{1/\mu^+-1}\,dt\,dz\,dy_1 \\ &\le \frac{C}{k^5}\int_0^1 y_1^{5-d+2/\mu^+}\,dy < \infty. \end{aligned}$$

On the set $\{y_1 < x_2^{\mu^+}\}$ we use $y_1 = x_2^{\mu^+}z$ to get the estimation

$$I_{+,+}(k,1,2) \le \frac{C}{k^5}\int_0^1 \cdots \int_0^1 \frac{(t^{\mu^+}+z)^{\frac{k-d}{2}+i+1}}{(1+z)^{\frac{k+d}{2}-(4-i)}} x_2^{\mu^+(6-d)+1}\,dt\,dx_2\,dz.$$

Since the x_2-integral is finite, it remains to estimate

$$\int_0^1\int_0^1 (t^{\mu^+}+z)^{\frac{k-d}{2}+i+1}\,dt\,dz \le \int_0^1 t^{\mu^+(\frac{k-d}{2}+i+2)}\,dt < \infty$$

whenever $d < k+2i+4+2/\mu^+$.

For $\frac{k-d}{2}+i+1=0$ the estimation is similar. If $\frac{k-d}{2}+i+1>0$ then (31) implies

$$I_{+,+}(k,1,2) \le \frac{C}{k^5}\int_0^1 \cdots \int_0^1 1_{\{x_1<x_2\}}(x_2^{\mu^+}+y_1)^{5-d}\,dx_1\,dx_2\,dy_1 < \infty.$$

For $J_{+,+}(k,2)$ we integrate twice and use the transformation (23) (cf. (24)). Then in the integral $J_{+,+}(k,2,1)$ we get the indicator function of the set $\{x_2 < y_2^{1/\mu^+}\}$, $\{x_3 < y_2^{1/\mu^+}\}$ or $\{x_4 < y_2^{1/\mu^+}\}$. We estimate these by $1_{\{x_2<y_2^{1/\mu^+}\}}$. If in the integral $J_{+,+}(k,2,2)$ the preceding sets appear, we estimate them again by $\{x_2 < y_2^{1/\mu^+}\}$ and thus the integral $J_{+,+}(k,2,2)$ by $J_{+,+}(k,2,1)$. It remains to investigate $J_{+,+}(k,2,2)$ on the set $\{x_1 < y_2^{1/\mu^+} < x_2\}$. This means that it is enough to consider the following two integrals (cf. (24) and (25))

$$\frac{C}{k^2}\int_0^1 \cdots \int_0^1 1_{\{x_1<x_2<x_3<x_4,\,x_2<y_2^{1/\mu^+}\}} \frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_1+y_1+y_2)^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2, \tag{33}$$

$$\frac{C}{k^2}\int_0^1\cdots\int_0^1 1_{\{x_1<y_2^{1/\mu^+}<x_2<x_3<x_4\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}}{(x_4-x_1+x_2^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2. \tag{34}$$

In the first integral we integrate in x_4 and use that $x_3-x_2<x_3-x_1$. Then (33) can be estimated by

$$\begin{aligned}
&\frac{C}{k^3}\int_0^1\cdots\int_0^1 1_{\{x_1<x_2<x_3,x_2<y_2^{1/\mu^+}\}}\frac{(x_3-x_2+y_1+y_2)^{\frac{k-d}{2}}}{(x_3-x_1+y_1+y_2)^{\frac{k+d}{2}-3}}\,dx_1\ldots dy_2,\\
&\quad\le\frac{C}{k^3}\int_0^1\cdots\int_0^1 1_{\{x_1<x_2<x_3,x_2<y_2^{1/\mu^+}\}}(x_3-x_2+y_1+y_2)^{3-d}\,dx_1\ldots dy_2,\\
&\quad\le\frac{C}{k^3}\int_0^1\cdots\int_0^1 1_{\{x_1<x_2<y_2^{1/\mu^+}\}}(y_1+y_2)^{4-d}\,dx_1\ldots dy_2<\infty,
\end{aligned}$$

as above in (30).

If $\frac{k-d}{2}+1<0$ then we integrate in (34) with respect to x_3 to get the estimate

$$\frac{C}{k^3}\int_0^1\cdots\int_0^1 1_{\{x_1<y_2^{1/\mu^+}<x_2<x_4\}}\frac{(y_1+y_2)^{\frac{k-d}{2}+1}}{(x_4-x_1+x_2^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2. \tag{35}$$

If $\frac{k-d}{2}+2<0$ then this can be estimated by

$$\begin{aligned}
&\frac{C}{k^4}\int_0^1\cdots\int_0^1 1_{\{x_1<x_2<x_4\}}\frac{(y_1+x_1^{\mu^+})^{\frac{k-d}{2}+2}}{(x_4-x_1+x_2^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\,dx_1\ldots dx_4\\
&\quad\le\frac{C}{k^5}\int_0^1\cdots\int_0^1 1_{\{x_1<x_2\}}\frac{(x_1^{\mu^+}+y_1)^{\frac{k-d}{2}+2}}{(x_2^{\mu^+}+y_1)^{\frac{k+d}{2}-3}}\,dx_1\ldots dy_1<\infty
\end{aligned}$$

because of (32). If $\frac{k-d}{2}+2>0$ then we estimate (35) by

$$\begin{aligned}
&\frac{C}{k^4}\int_0^1\cdots\int_0^1 1_{\{x_1<x_2<x_4\}}\frac{(y_1+x_2^{\mu^+})^{\frac{k-d}{2}+2}}{(x_4-x_1+x_2^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\,dx_1\ldots dx_4\\
&\quad\le\frac{C}{k^5}\int_0^1\cdots\int_0^1 1_{\{x_1<x_2\}}(x_2^{\mu^+}+y_1)^{5-d}\,dx_1\ldots dy_1<\infty.
\end{aligned}$$

If $\frac{k-d}{2}+1>0$ then we estimate (34) by

$$\begin{aligned}
&\frac{C}{k^3}\int_0^1\cdots\int_0^1 1_{\{x_1<y_2^{1/\mu^+}<x_2<x_4\}}\frac{(x_4-x_2+y_1+y_2)^{\frac{k-d}{2}+1}}{(x_4-x_1+x_2^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_2\\
&\quad\le\frac{C}{k^4}\int_0^1\cdots\int_0^1 1_{\{x_1<x_2<x_4\}}\frac{(x_4-x_2+y_1+x_2^{\mu^+})^{\frac{k-d}{2}+2}}{(x_4-x_1+x_2^{\mu^+}+y_1)^{\frac{k+d}{2}-2}}\,dx_1\ldots dy_1\\
&\quad\le\frac{C}{k^4}\int_0^1\cdots\int_0^1 1_{\{x_1<x_2<x_4\}}(x_4-x_2+x_2^{\mu^+}+y_1)^{4-d}\,dx_1\ldots dy_1\\
&\quad\le\frac{C}{k^4}\int_0^1\cdots\int_0^1 1_{\{x_1<x_2\}}(x_2^{\mu^+}+y_1)^{5-d}\,dx_1\,dx_2\,dy_1<\infty.
\end{aligned}$$

The proof of Proposition 9 is complete. ■

Now we give lower estimates, i.e. we consider the integrals $K(k)$.

Proposition 10. *Suppose that $\mu^+ \leq 1 \leq \nu^+$. If $d \geq 6 + \frac{2}{\mu^+}$ then $K_{+,+}(0) = \infty$.*

Proof. Using (13) and (21) we can see that

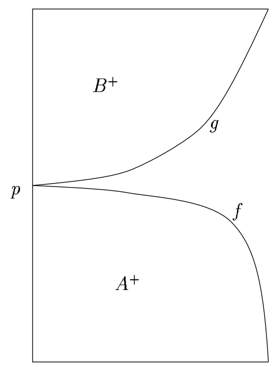
$$K_{+,+}(0) \geq \int_0^1 \int_0^1 \frac{x_2}{(x_2^{\mu^+} z^{\mu^+} + x_2)^{\frac{d}{2}-2}} \, dx_2 \, dz.$$

We can finish the proof as in Proposition 7. ■

Theorem 6. *Suppose that $\mu^+ \leq 1 \leq \nu^+$. The self-intersection local time exists in some Sobolev space if and only if $d < 6 + \frac{2}{\mu^+}$. In this case (4) holds in $\mathbf{D}_{2,\rho}$ for any $\rho < 3 - d/2$. Moreover, $\alpha(x, \cdot)$ is a function.*

The proof follows from Propositions 9 and 10.

2.6. The case $A^- = B^- = \emptyset$ and $1 \leq \min(\mu^+, \nu^-)$



We consider the case where the boundaries consist of two convex power-type curves opposite to each other. We prove that in this case the critical dimension is again given by $6 + \frac{2}{\mu^+}$, if μ^+ is the smaller exponent. Remember that if both A^+ and B^+ are rectangles, then the self-intersection local time exists if and only if $d < 6$ (see Theorem 3 of Imkeller and Weisz [8]).

Proposition 11. *Suppose that $1 \leq \mu^+ \leq \nu^+$ and $d < 6 + \frac{2}{\mu^+}$. Then there exists a constant C such that $k^4 I_{+,+}(k) \leq C$ for all $k \in \mathbf{N}_0$.*

Proof. We replace A^+ again by A^+_∞. Similarly to (18) we have

$$\begin{aligned} I_{+,+}(0,1) &\leq \int\int 1_{\{s_1 < t_1\}} \frac{1}{(t_1 - s_1 + g(t_1) - p_2)^{\frac{d}{2}-2}} \, ds_1 \, dt_1 \\ &\leq \int (g(t_1) - p_2)^{3-\frac{d}{2}} \, dt_1 < \infty. \end{aligned}$$

$I_{+,+}(0,2)$ can be handled similarly.

Notice that in case $k > 0$ the proof of Proposition 9 works also for $1 \le \mu^+ \le \nu^+$. The problem is only, that in the estimate of (33) we have k^{-3} instead of k^{-4}. Let us consider this integral again. Of course, we can suppose that $k \ge d$. Integrating in x_3 and x_1 we estimate (33) by

$$\begin{aligned}
&\frac{C}{k^3}\int_0^1 \cdots \int_0^1 1_{\{x_1<x_2<x_4, x_2<y_2^{1/\mu^+}\}} \frac{(x_4-x_2+y_1+y_2)^{\frac{k-d}{2}+1}}{(x_4-x_1+y_1+y_2)^{\frac{k+d}{2}-2}}\, dx_1 \dots dy_2 \\
&\le \frac{C}{k^4}\int_0^1 \cdots \int_0^1 1_{\{x_2<x_4, x_2<y_2^{1/\mu^+}\}} (x_4-x_2+y_1+y_2)^{4-d}\, dx_1 \dots dy_2 \\
&\le \frac{C}{k^4}\int_0^1 \cdots \int_0^1 1_{\{x_2<y_2^{1/\mu^+}\}} (y_1+y_2)^{5-d}\, dx_2\, dy_1\, dy_2 \\
&\le \frac{C}{k^4}\int_0^1 \cdots \int_0^1 1_{\{x_2<y_2^{1/\mu^+}\}} y_2^{6-d}\, dx_2\, dy_2.
\end{aligned}$$

This integral is finite if $d < 7 + \frac{1}{\mu^+} (\ge 6 + \frac{2}{\mu^+})$ which finishes the proof of Proposition 11. ■

Remark. In case $\mu^+ \ge 1$ the integral $J_{+,+}(k,1,2)$ can be infinite even if (29) holds. More exactly, one can show that $J_{+,+}(k,1,2) = \infty$ for $d = 7$, $k = 1$ and $(1+\sqrt{5})/2 \le \mu^+ < 2$, which contradicts (29). On the other hand, $J_{+,+}(k,4,2)$ is finite for $k > 1$ and $d < 6 + 2/\mu^+$.

Proposition 12. *Suppose that* $1 \le \mu^+ \le \nu^+$. *If* $d \ge 6 + \frac{2}{\mu^+}$ *then* $K_{+,+}(0) = \infty$.

Proof. In the lower estimates we replace $A^+_{\nu^+}$ by $A^+_{\mu^+}$, i.e. we replace f by $f_0 := 2p_2 - g$ on the interval $[p_1, p_1+\delta_1]$. By (13) and by integration in s_2 and t_2 we get

$$\begin{aligned}
\sqrt{K_{+,+}(0)} &\ge \int\int 1_{\{s_1<t_1\}} \frac{1}{(t_1-s_1+g(t_1)-f_0(s_1))^{\frac{d}{2}-2}}\, ds_1\, dt_1 \\
&= \int_0^1\int_0^1 1_{\{x_1<x_2\}} \frac{1}{(x_2-x_1+x_2^{\mu^+}+x_1^{\mu^+})^{\frac{d}{2}-2}}\, dx_1\, dx_2 \\
&\ge \int_0^1\int_0^1 1_{\{x_1<x_2\}} \frac{1}{(x_2-x_1+2x_2^{\mu^+})^{\frac{d}{2}-2}}\, dx_1\, dx_2
\end{aligned}$$

where $x_1 = s_1 - p_1$ and $x_2 = t_1 - p_1$. Let us integrate in x_1 to obtain

$$\sqrt{K_{+,+}(0)} \ge \int_0^1 \frac{1}{(2x_2^{\mu^+})^{\frac{d}{2}-3}} - \frac{1}{(x_2+2x_2^{\mu^+})^{\frac{d}{2}-3}}\, dx_2. \tag{36}$$

The first integral is infinite if $d \ge 6 + \frac{2}{\mu^+}$ and the second is finite if $d < 8$. This means that for $\mu^+ \ge 2$, $K_{+,+}(0) = \infty$ for $d = 7$. By monotonity, $K_{+,+}(0) = \infty$ for all $d \ge 7$. If $1 \le \mu^+ < 2$ then it is enough to show that $K_{+,+}(0) = \infty$ for $d = 8$. For $d = 8$ (36) implies that

$$\sqrt{K_{+,+}(0)} \ge \int_0^1 \frac{x_2}{(2x_2^{\mu^+})(x_2+2x_2^{\mu^+})}\, dx_2 = \infty$$

whenever $\mu^+ \geq 1$. This completes the proof of Proposition 12. ■

Proposition 11 and 12 imply

Theorem 7. *Suppose that $1 \leq \mu^+ \leq \nu^+$. The self-intersection local time exists in some Sobolev space if and only if $d < 6 + \frac{2}{\mu^+}$. In this case (4) holds in $\mathbf{D}_{2,\rho}$ for any $\rho < 4 - d/2$. Moreover, $\alpha(x, \cdot)$ is a function.*

Theorem 8. *Let $A, B \subset \mathbf{R}_+^2 \setminus \partial \mathbf{R}_+^2$ be compact such that $A \cap B = \{p\}$. Suppose that there is a rectangle $K := I \times J$, $I = [p_1 - \delta_2, p_1 + \delta_1]$, $J = [p_2 - \eta_2, p_2 + \eta_1]$ such that $\partial A \cap K$ is given by graphs of functions*

$$g^+(x) = p_2 + b^+(x - p_1)^{\mu^+}, \qquad p_1 \leq x \leq p_1 + \delta_1,$$
$$g^-(x) = p_2 + b^-(p_1 - x)^{\mu^-}, \qquad p_1 - \delta_2 \leq x \leq p_1,$$

$\partial B \cap K$ by graphs of functions

$$f^+(x) = p_2 - a^+(x - p_1)^{\nu^+}, \qquad p_1 \leq x \leq p_1 + \delta_1,$$
$$f^-(x) = p_2 - a^-(p_1 - x)^{\nu^-}, \qquad p_1 - \delta_2 \leq x \leq p_1.$$

Then self-intersection local time (as a function or distribution) exists below the following critical dimensions:

a) if $\mu^+, \mu^-, \nu^+, \nu^- \leq 1$, for $d < 4 + 2(\frac{1}{\mu^+ \vee \mu^-} + \frac{1}{\nu^+ \vee \nu^-})$,
b) if $\nu^+ \vee \nu^- > 1$ and $\mu^+ \vee \mu^- \leq 1$, for $d < 6 + \frac{2}{\mu^+ \vee \mu^-}$, analogously if $\nu^+ \vee \nu^- \leq 1$ and $\mu^+ \vee \mu^- > 1$,
c) if $\mu^+ \wedge \nu^- > 1$ and $\mu^- \vee \nu^+ \leq 1$ or $\mu^+ \vee \nu^- \leq 1$ and $\mu^- \wedge \nu^+ > 1$, for $d < 8$,
d) if $\nu^+ \wedge \nu^+ > 1$ and $\mu^- \vee \nu^- \leq 1$, for $d < 6 + \frac{2}{\mu^+ \wedge \nu^+}$, analogously if $\mu^- \wedge \nu^- > 1$ and $\mu^+ \vee \nu^+ \leq 1$,
e) if $\mu^+ \wedge \nu^+ > 1$ or $\nu^+ \wedge \nu^- > 1$, for $d < 6 + \frac{2}{\mu^+ \wedge \nu^+ \wedge \mu^- \wedge \nu^-})$.

Proof. This is a combination of Theorems 2–7 and versions of Theorems 2–4 for $A^- = B^+ = \emptyset$ instead of $A^+ = B^- = \emptyset$, versions of Theorems 5–7 for $A^+ = B^+ = \emptyset$ instead of $A^- = B^- = \emptyset$, the compactness of A, B and the fact that for disjoint rectangles the critical dimension is infinite (see Imkeller and Weisz [8]). ■

References

[1] Bass, R., Koshnevisan, D.: Intersection local times and Tanaka formulas. Ann. Inst. Henri Poincare, Probab. Stat. 29 (1993), 419-451.

[2] Bouleau, N., Hirsch, F.: Dirichlet forms and analysis on Wiener space. W. de Gruyter, Berlin (1991).

[3] Dalang, R.C., Mountford, T.: Points of increase of the Brownian sheet. Probab. Th. Rel. Fields. 108 (1997), 1-27.

[4] Dalang, R.C., Walsh, J.B.: Geography of the level sets of the Brownian sheet. Probab. Th. Rel. Fields. 96 (1993), 153-176.

[5] Dalang, R.C., Walsh, J.B.: The sharp Markov property of the Brownian sheet and related processes. Acta Math. 168 (1992), 153-218.

[6] Dynkin, E.: Self-intersection gauge for random walks and for Brownian motion. Ann. Probab. 16 (1988), 1-57.

[7] Imkeller, P., Perez-Abreu, V., Vives J.: Chaos expansions of double intersection local time of Brownian motion in $\mathbf{R}^d$ and renormalization. Stochastic Processes Appl. 56 (1995), 1-34.

[8] Imkeller, P., Weisz, F.: Critical dimensions for the existence of self-intersection local times of the Brownian sheet in $\mathbf{R}^d$. Progress in Probability, Vol. 36. Birkhäuser: Basel, Boston 1995, 151-168.

[9] Imkeller, P., Weisz, F.: The asymptotic behaviour of local times and occupation integrals of the N-parameter Wiener process in $\mathbf{R}^d$. Probab. Th. Rel. Fields 98 (1994), 47-75.

[10] Imkeller, P., Weisz, F.: Critical dimensions for the existence of self-intersection local times of the N-parameter Brownian motion in $\mathbf{R}^d$. J. Theor. Probab. (submitted)

[11] Kuo, H.H.: Donsker's delta function as a generalized Brownian functional and its application. Lecture Notes in Control and Information Sciences, vol. 49 (1983), 167-178. Springer, Berlin.

[12] Le Gall, J.F.: Sur le temps local d'intersection du mouvement brownien plan et la méthode de renormalisation de Varadhan. Sém. de Prob. XIX, 1983/84. LNM 1123 (1985), 314-331. Springer, Berlin.

[13] Mountford, T.: Estimates of the Hausdorf dimension of the boundary of positive Brownian sheet components. Sem. de Probabilites XXVII, Lect. Notes in Math. vol. 1557, pp. 233-255. Springer, Berlin (1993).

[14] Nualart, D.: Malliavin calculus. Springer, Berlin (1995).

[15] Nualart, D., Vives, J.: Chaos expansion and local times. Publicacions Matematiques 36 (2) (1992), 827-836.

[16] Rosen, J.: A local time approach to the self-intersections of Brownian paths in space. Comm. Math. Phys. 88 (1983), 327-338.

[17] Rosen, J.: Stochastic integrals and intersections of Brownian sheets. Preprint, Univ. of Massachusetts.

[18] Rosen, J.: Self-intersections of random fields. Ann. Probab. 12 (1984), 108-119.

[19] Rosen, J.: Tanaka's formula and renormalization for intersections of planar Brownian motion. Ann. Probab. 14 (1986), 1245-1251.

[20] Rosen, J., Yor, M.: Tanaka formulae and renormalization for triple intersections of Brownian motion in the plane. Ann. Probab. 19 (1991), 142-159.

[21] Shieh, N.R.: White noise analysis and Tanaka formula for intersections of planar Brownian motion. Nagoya Math. J. 122 (1991), 1-17.

[22] Szymanzik, K.: Euclidean quantum field theory. In "Local Quantum Theory" (R. Jost, ed.). Academic, New-York (1969).

[23] Varadhan, S.R.S.: Appendix to "Euclidean quantum field theory" by K. Szymanzik. In "Local Quantum Theory" (R. Jost, ed.). Academic, New York (1969).

[24] Walsh, J.B.: An introduction to stochastic partial differential equations. Ecole d'ete de probabilites de Saint-Flour XIV - 1984, Lect. Notes Math. 1180, 265-437 (1986).

[25] Watanabe, H.: The local time of self-intersections of Brownian motions as generalized Brownian functionals. Letters in Math. Physics 23 (1991), 1-9.

[26] Watanabe, S.: Lectures on stochastic differential equations and Malliavin calculus. Springer, Berlin (1984).

[27] Wolpert, R.: Wiener path intersection and local time. J. Func. Anal. 30 (1978), 329-340.

[28] Yor, M.: Sur la représentation comme intégrales stochastiques des temps d'occupation du mouvement Brownian dans $\mathbf{R}^d$. Sém. de Prob. XX. 1984/85. LNM 1204, 543-552. Springer, Berlin (1986).