# Simple models for trading climate and weather risk

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In these lectures we shall present some elementary models by which market external risk can be traded, or at least *redistributed* among market participants. These models will have to be replaced by more complex ones, and will in particular include large traders such as re-insurance companies issuing climate or catastrophe bonds for instance. Some illustrations are provided in the accompanying file ENSO.pdf.

## 1 A partial equilibrium model for trading market external risk

The material is from Hu, Imkeller, Müller [21]. In this model we create a market of small agents who share a common interest in trading some risk which affects their usual business. Let us first explain the main risk source we are interested in, climate risk, in one typical example. See ENSO.pdf, slides 1-4. Our particular focus is agents who have a negatively correlated exposure to this risk source. Examples will be given below. The agents are all allowed to invest into a capital market.

We shall present the model in a very simple situation, with the lowest dimensions possible. We shall discuss generalizations at the end of the presentation.

## 1.1 The stock market

Let a stochastic basis be given: $(\Omega, \mathcal{F}, P)$ a probability space, $W = (W^1, W^2)$ a 2-dimensional Brownian motion on this space, indexed by $t \in [0, T]$, where $T > 0$ is a time horizon. Assume that $(\mathcal{F}_t)_{t\in[0,T]}$ is the completed filtration of $W$ on this space.

The price process for the (single) risky asset on the stock market is given by

$$dX_t = X_t \, (b_t^X dt + \sigma_t^X dW_t^1), \quad t \in [0, T].$$

We assume that $b^X, \sigma^X$ are bounded $(\mathcal{F}_t)$-adapted processes, and that

$$\sigma_t^X \geq \epsilon > 0$$

for all $0 \leq t \leq T$.

## 1.2 External risk

Here is the first main task of this problem: we have to model the risk factors. In our simple setting we shall use equally simple low-dimensional models for the risk process. We shall usually think of a temperature process $K = (K_t)_{t\in[0,T]}$ which is also assumed to be adapted with respect to $(\mathcal{F}_t)_{t\in[0,T]}$.

**Example 1:** *Ornstein-Uhlenbeck process*

$$dK_t = c(\kappa - K_t)dt + \sigma_t^K dW_t^2,$$

with an $(\mathcal{F}_t)-$adapted $\sigma^K \neq 0$. The drift of this process is mean reverting around some *temperature state* $\kappa$. This model is not suitable to represent temperature realistically in a bi-stable situation as for example the above explained El Niño sea surface temperature anomaly.

**Example 2:** *periodically perturbed Ornstein-Uhlenbeck process*

Here temperature is described by

$$dK_t = [c_1(\kappa - K_t) + c_2 \cos(\frac{2\pi}{t_0} t)]dt + \sigma_t^K dW_t^2,$$

again with an $(\mathcal{F}_t)-$adapted $\sigma^K \neq 0$. This model could describe a temperature process which receives a periodic external signal with period $t_0$ such as for example daily or seasonally varying solar radiation. As such it has been used for modeling temperature in simple mathematical models for *HDD (heating degree days) swaps*, an instrument to hedge the risk of energy producers due to unusually high temperatures during heating seasons.

**Example 3:** *periodically forced bi-stable temperature*
We now come to models which are more realistic for describing the basic features of a bi-stable phenomenon such as El Niño. There are two-dimensional models for the SST of the South Pacific, attempting to explain the event by a non-linear interaction between *temperature* and the *thermocline depth*, the depth of the surface layer of the ocean separated from the deep water by a boundary of a significant temperature gradient. This model is explained for example in Wang, Barcilon, Fang [1].

Another possibility to describe the El Niño phenomenon, a linear model (!), which in addition is the most widely used in prediction models for El Niño, is investigated by Penland [35]. In this linear regression model the sea surface temperature is described by a linear sde of the type

$$dK_t = B\, K_t dt + dN_t, \quad t \geq 0,$$

with a Gaussian noise $N$, and a 15-dimensional process $K$ one of whose components is identified with sea surface temperature, and a $15 \times 15-$matrix $B$ with a non-trivial rotational part.

A third interesting attempt uses a delay effect caused by the reflection of Kelvin waves travelling through the Pacific emitted at the South American coast, and reflected at the Asian coast. The delay time $T_0$ is the travelling time of the waves. The system can be described by a one-dimensional stochastic delay equation of the form (see Battisti [6])

$$dK_t = aK_t - bK_t^3 + c\, K_{t-T_0} dt + dN_t,$$

again with a Gaussian noise $N$ and some constants $a, b, c \in \mathbf{R}$.

For our purposes we prefer a phenomenological model which retains the crucial feature of climate transitions which usually are very sharp and spontaneous. We study a kind of one-dimensional projection of the two-dimensional model by Wang, Barcilon, Fang [1]. It is given by a *potential diffusion* with a double-well potential. Let $U(t,k), t \geq 0, k \in \mathbf{R}$, be a time varying potential, whose time variation is periodic of periodicity, say, $t_0$. It possesses two wells at the positions $k = \overline{k}, \underline{k}$ which do not vary with time. The depths of the wells vary with time, and perform periodic motions with the same period, and a fixed phase for example. These diffusions are given by equations of the form

$$dK_t = -\frac{\partial}{\partial x} U(t, K_t) dt + \sigma^K dW_t^2,$$

with a constant $\sigma^K \neq 0$. See ENSO.pdf, slide 5. Their solution curves are, for suitably chosen $\sigma^K$, *stochastically periodic*, and show typically sharp transitions between vicinities of the two *meta-stable states* $\underline{k}, \overline{k}$ (see ENSO.pdf, slide 6). Of course, we may imagine situations in which the temperature process may depend on $W^1$ as well.

## 1.3 Agents, composition of the market

Irrespective of liquidity issues - these markets are illiquid - we assume that there is a finite set of agents $I$ interested in trading the risk. The set of agents usually should tend to be "diversified" with respect to risk exposure, i.e. contain traders who profit while others suffer from the risk, and vice versa. This aspect of the market will only be seen in the numerical simulations of its performance, and is not important in the mathematical development that follows.

To illustrate this idea briefly (we shall give more details in the simulation part), let us return to the El Niño sea surface temperature anomaly, viewed from the perspective of parts of the economy of Peru for instance. Recall that $\underline{k}$ corresponds to the average sea surface temperature under usual conditions, while $\overline{k}$ corresponds to the average temperature during an El Niño event. For a *fisher*, the income of his usual business may be maximal if temperatures are near $\underline{k}$, while the opposite is the case, if the temperature is near $\overline{k}$, due to bad fishing conditions during the event. From the perspective of a *rice farmer*, the situation looks quite different. Due to the higher evaporation during El Niño events, precipitation rates in his farm land will be higher then, improving the conditions for rice farming, while under usual conditions precipitation is much less favorable. So we may suppose that the farmer's income is optimal for $\overline{k}$, while it is minimal for $\underline{k}$. Farmer and fisher thus exhibit negatively correlated exposure to El Niño risk.

The internet abounds of examples of pairs of agents with negatively correlated El Niño risk exposure. For example, precipitation rates in Indonesia and in (South) America are known to be negatively correlated in El Niño years. So rice farmers in Indonesia and in America possess negatively correlated El Niño exposure. See for example Naylor, Falcon, Rochberg, Wada [34].

Each one of the agents $a \in I$ has two natural sources of income. Firstly, he receives *income from his usual business*. We assume it to be a function of both climate process $K$ and stock price process $X$:

$$H^a = g^a(X, K).$$

Here $g^a$ is a measurable function defined on the product of the set of continuous functions on $[0, T]$ with itself. We might take a simpler dependence

$$H^a = f^a(X_T, K),$$

where $f^a$ is a function defined on a space in which the first component is just $\mathbf{R}$.

The second source of income is based on *investments on the financial market*. Suppose that $(\pi_t^{X,a})_{t\in[0,T]}$ denotes an adapted investment strategy on $X$. Then the wealth process due to investment using this strategy is given by

$$dV^X(\pi^{X,a})_t = \pi_t^{X,a} \frac{dX_t}{X_t} = \pi_t^{X,a} [b_t^X dt + \sigma_t^X dW_t^1],$$

$t \in [0, T]$. We assume that

$$\int_0^T (\pi_t^{X,a} \sigma_t^X)^2 dt < \infty \quad P - a.s.$$

for any trading strategy $\pi^X$. So far, we face a typical incomplete market situation, due to the non-tradable risk inherent in $W^2$. To make this external risk source tradable, we introduce another security on the market which serves as a third source of income for interested traders.

## 1.4 Market completion

The price process of the *additional security* needed for risk trading is given by

$$dY_t = Y_t[b_t^Y dt + \sigma_t^Y dW_t^2], \quad t \in [0,T].$$

The coefficients are supposed to be variable, and depend on the demand of the market participants for this risk trading security. Each agent $a \in I$ can invest on the asset by choosing an investment process $(\pi_t^{Y,a})_{t\in[0,T]}$ which satisfies

$$\int_0^T (\pi_t^{Y,a}\sigma_t^Y)^2 dt < \infty \quad P-a.s.$$

The wealth process for investment into this asset is again given by an sde of the form

$$dV^Y(\pi^{Y,a})_t = \pi_t^{Y,a}\frac{dY_t}{Y_t} = \pi_t^{Y,a}[b_t^Y dt + \sigma_t^Y dW_t^2],$$

$t \in [0,T]$.
Summarizing, the income of each agent is composed of the three sources

- $V_T^X(\pi^{X,a})_T$: the terminal wealth from investment into the capital market,
- $H^a$: the (terminal) income from his usual business,
- $V_T^Y(\pi^{Y,a})_T$: the terminal wealth from investment into the risk trading asset.

The *price of external risk* $\theta^Y = \frac{b^Y}{\sigma^Y}$ emerges from the formula

$$dY_t = \sigma_t^Y Y_t[\theta_t^Y dt + dW_t^2].$$

It has to be determined by the preferences of the agents, and the balance of supply and demand following from this. This will be done in the sequel. We denote by $\theta^X = \frac{b^X}{\sigma^X}$ the corresponding *price of market risk* which is given by the formula

$$dX_t = \sigma_t^X X_t[\theta_t^X dt + dW_t^1].$$

Let $\theta = (\theta^X, \theta^Y)$. The prices of risk determine risk neutral measures in the usual way by the Girsanov densities

$$Z_T^\theta = \frac{dQ^\theta}{dP} = \exp(-\int_0^T \theta_t^X dW_t^1 - \int_0^T \theta_t^Y dW_t^2 - \frac{1}{2}\int_0^T [(\theta_t^X)^2 + (\theta_t^Y)^2]dt).$$

Under $Q^\theta$, both $X$ and $Y$ are (local) martingales. Recall that $\theta^X$ is bounded. To have a meaningful calculus for the risk neutral measures, we assume that

$$\textbf{(H)} \quad \theta^Y \in \mathcal{K} = \{\theta | \theta \text{ is adapted square integrable, } \int_0^\cdot \theta_s dW_s^2 \text{ is a } P-BMO \text{ martingale}\}$$

and search to determine the equilibrium price of external risk on this set.

Let us briefly recall some facts about BMO martingales.

**Definition 1.1** *Let $M = (M_t)_{t\in[0,T]}$ be a square integrable martingale for $(\mathcal{F}_t)_{t\in[0,T]}$ and $P$. $M$ is called $P - BMO$ martingale if*

$$||M||_{BMO}^2 = \sup_{t\in[0,T]} E(|M_T - M_t|^2|\mathcal{F}_t) = \sup_{t\in[0,T]} E(\langle M\rangle_T - \langle M\rangle_t|\mathcal{F}_t) < \infty.$$

The significance of BMO martingales for our purposes stems from the following Theorem, proved in Kazamaki [27], Thm. 2.3.

**Theorem 1.1** *If $M \in BMO$, then $\exp(M - \frac{1}{2}\langle M\rangle)$ is a uniformly integrable martingale.*

According to Theorem 1.1, the fact that $\theta \in \mathcal{K}$ makes sure that our market prices of risk always lead to equivalent risk-neutral measures $Q^\theta$.

In our setting of three sources of income, each individual agent is supposed to maximize his expected utility from terminal wealth according to his individual preferences.

## 1.5 Individual utility maximization

The preferences of agent $a \in I$ will be described by an exponential utility function

$$U_a(x) = -\exp(-\alpha_a\, x), \quad x \in \mathbf{R},$$

with individual risk aversion factor $\alpha_a > 0$. His total income at terminal time $T$ obtained from his three sources, provided that he trades using the strategy $\pi^a = (\pi^{X,a}, \pi^{Y,a})$ is given by

$$V_T^X(\pi^{X,a}) + V_T^Y(\pi^{Y,a}) + H^a =: V_T(\pi^a) + H^a.$$

Therefore our optimization problem reads

$$\text{find} \quad \max_{\pi^a \in \mathcal{A}} E(U_a(V_T(\pi^a) + H^a)), \tag{1}$$

with *admissibility set*

$$\mathcal{A} = \{\pi | \pi = (\pi^X, \pi^Y), \int_0^T [(\pi_t^X \sigma_t^X)^2 + (\pi_t^Y \sigma_t^Y)^2] dt < \infty \ P - a.s.\}.$$

Since for any $a \in I$ the function $U_a$ satisfies the Inada conditions, this utility maximization problem can be solved using well known techniques. They are provided in the classical paper by Karatzas, Lehotzky, Shreve [25]. From these results we obtain

**Theorem 1.2** *Let $H^a$ be a bounded $\mathcal{F}_t$-measurable random variable, $v_0^a \geq 0$ an initial capital. Let $\lambda_a \in \mathbf{R}$ be such that*

$$[U_a']^{-1}(\lambda_a) = v_0^a + E^\theta(H^a).$$

*Define*

$$M^{\theta,a} = [U_a']^{-1}(\lambda_a Z_T^\theta) = -\frac{1}{\alpha_a}\log(\frac{1}{\alpha_a}\lambda_a Z_T^\theta).$$

*Then $M^{\theta,a}$ solves (1) for agent $a \in I$.*

**Proof:**
We sketch the duality arguments leading to the representation of the solution variable.

Just note that due to $\theta \in \mathcal{K}$ we have for an admissible strategy $\pi$

$$E^\theta(V_T(\pi) + H^a)) = v_0^a + E^\theta(H^a).$$

Therefore we may write

$$\begin{aligned}
&\max_{\pi^a \in \mathcal{A}} E(U_a(V_T(\pi^a) + H^a)) \\
&\quad = \max_{\pi^a \in \mathcal{A}}[E(U_a(V_T(\pi^a) + H^a)) - Z_T^\theta(V_T(\pi^a) + H^a)] + v_0^a + E^\theta(H^a).
\end{aligned}$$

Since $U_a$ satisfies the Inada conditions, the Legendre transform of $U_a$, however, has the form

$$\max_{x \in \mathbf{R}}[U_a(x) - z\,x] = [U_a']^{-1}(z).$$

Hence we obtain by definition of $\lambda_a$

$$\max_{\pi^a \in \mathcal{A}} E(U_a(V_T(\pi^a)+H^a)) = E([U_a']^{-1}(Z_T^\theta))+v_0^a+E^\theta(H^a) = E([U_a']^{-1}(\lambda_a\, Z_T^\theta)).$$

□

Let us next explain what the optimal trading strategy of agent $a \in I$ into the individual assets is. For convenience, we introduce the following notation for the Brownian motions shifted by the prices of risk which become Brownian motions under the risk neutral measures $Q^\theta$. Let

$$W^{\theta^X} = W^1 + \int_0^\cdot \theta_s^X ds, \quad W^{\theta^Y} = W^2 + \int_0^\cdot \theta_s^Y ds.$$

Denote the optimal trading strategies by $\hat{\pi}^{X,a}$ and $\hat{\pi}^{Y,a}$.The preceding theorem obviously gives for each $a \in I$

$$\begin{aligned} -\frac{1}{\alpha_a}\log(\frac{1}{\alpha_a}\lambda_a Z_T^\theta) &= M^{\theta,a} \\ &= v_o^a + \int_0^T \hat{\pi}_t^{X,a}\,\sigma_t^X\,dW_t^{\theta^X} + \int_0^T \hat{\pi}_t^{Y,a}\,\sigma_t^Y\,dW_t^{\theta^Y} + H^a. \end{aligned}$$

These strategies exist due to martingale representation theorems. By summing over all $a \in I$, we obtain

$$\sum_{a\in I}[M^{\theta,a} - H^a] = \sum_{a\in I} v_0^a + \int_0^T [\sum_{a\in I}\hat{\pi}_t^{X,a}]\,\sigma_t^X\,dW_t^{\theta^X} + \int_0^T [\sum_{a\in I}\hat{\pi}_t^{Y,a}]\,\sigma_t^Y\,dW_t^{\theta^Y}. \quad (2)$$

Here is the idea of partial equilibrium.

## 1.6 Partial equilibrium

**Definition 1.2** *Let $v_0^a, H^a, a \in I$, and $X$ be given. A market price of risk process $\hat{\theta}^Y$ is called* partial equilibrium, *if the individual optimal trading strategies satisfy*

*a) $\hat{\pi}^{X,a}, \hat{\pi}^{Y,a}$ solve the individual utility optimization problem (1),*

*b) there is* market clearing *for the security of risk trading $Y$, i.e.*

$$\sum_{a\in I}\hat{\pi}^{Y,a} = 0$$

*$P \otimes \lambda$-a.e.*

In partial equilibrium, according to (2) we have to get

$$\sum_{a\in I}[M^{\hat{\theta},a} - H^a] = \sum_{a\in I} v_0^a + \int_0^T [\sum_{a\in I}\hat{\pi}_t^X]\,\sigma_t^X\,dW_t^{\hat{\theta}^X},$$

where

$$\hat{\pi}^X = \sum_{a\in I}\hat{\pi}^{X,a}.$$

In other words, by lumping together some constants, introducing $c_1$ for this purpose, and writing $Z^{\hat{\theta}}$ explicitly we obtain

$$\sum_{A\in I}\frac{1}{\alpha_a}[-\int_0^T \hat{\theta}_t^X dW_t^1 - \int_0^T \hat{\theta}_t^Y dW_t^2 - \frac{1}{2}\int_0^T [(\hat{\theta}_t^X)^2 + (\hat{\theta}_t^Y)^2]dt]$$
$$= c_1 + \sum_{a\in I} H^a + \int_0^T \hat{\pi}_t^X \sigma_t^X dW_t^1 + \int_0^T \hat{\pi}_t^X \sigma_t^X \hat{\theta}_t^X dt.$$

Now denote

$$\overline{H} = \sum_{a\in I} H^a, \quad \overline{\alpha} = [\sum_{a\in I}\frac{1}{\alpha_a}]^{-1}.$$

Recall that $\hat{\theta}^X = \theta^X$, since this process is given from the beginning. Write further

$$H = c_1 + \overline{\alpha}\overline{H} - \frac{1}{2}\overline{\alpha}\int_0^T (\hat{\theta}_t^X)^2 dt.$$

Then we can rephrase the preceding equation by

$$\begin{aligned} H &= -\int_0^T [\hat{\theta}_t^X + \overline{\alpha}\hat{\pi}_t^X \sigma_t^X] dW_t^1 - \int_0^T \hat{\theta}_t^Y dW_t^2 \\ &\quad -\int_0^T \hat{\theta}_t^X (\hat{\theta}_t^X + \overline{\alpha}\,\hat{\pi}_t^X \sigma_t^X)\,dt - \frac{1}{2}\int_0^T [(\hat{\theta}_t^Y)^2] dt. \end{aligned} \tag{3}$$

Now let $z^X = \hat{\theta^X} + \overline{\alpha}\,\hat{\pi}^X \sigma^X, \quad z^Y = \hat{\theta}^Y$. Then we may derive a dynamic version of (3) in the form

$$h_t = H - \int_t^T z_s^X dW_s^1 - \int_t^T z_s^Y dW_s^2 - \frac{1}{2}\int_t^T (z_s^Y)^2 ds - \int_t^T \hat{\theta}_s^X z_s^X ds. \tag{4}$$

This is a BSDE with quadratic non-linearity in the control process.

We briefly recall the notion of BSDE, not in the most general case.

**Definition 1.3** *Let $f : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a continuous function, $H$ an $\mathcal{F}_T-$measurable random variable. Then a pair of $(\mathcal{F}_t)-$adapted processes $(h,z)$ satisfying*

$$h_t = H - \int_t^T z_s dW_s - \int_t^T f(\cdot, s, h_s, z_s) ds, \tag{5}$$

*is called solution of the BSDE (5).*

If the *generator* $f$ vanishes, and $H$ is square integrable, the BSDE reduces to the martingale representation theorem

$$h_t = \int_0^t z_s dW_s, \quad h_T = H.$$

There exists a still developing theory for the solutions of BSDE of the type we encounter.

**Theorem 1.3** *Let $H$ be bounded. Then the BSDE (4) possesses a unique solution. The process $h$ is bounded, and*

$$E(\int_0^T [(z_t^X)^2 + (z_t^Y)^2] dt) < \infty.$$

*The choice $\hat{\theta}^Y = z^Y$ provides the equilibrium price of external risk.*

**Proof:**
For $H$ bounded, Kobylanski [28] proves existence and uniqueness of the BSDE.

The only thing to verify additionally is the BMO property of the martingale $M = \int_0^\cdot z_t^Y dW_t^2$.

Without loss of generality, modulo translation by a constant we may suppose $H$ nonnegative. Now by a comparison argument ( see Theorem 2.6 in [28]) $h \geq 0$. For every $t \in [0, T]$ Itô's formula yields

$$E\left[ H^2 - h_t^2 - \int_t^T (2h_s \hat{\theta}_s^X z_s^X + (z_s^X)^2) ds \middle| \mathcal{F}_t \right]$$
$$= E\left[ \int_t^T (h_s + 1)(z_s^Y)^2 ds \middle| \mathcal{F}_t \right] \geq E\left[ \int_t^T (z_s^Y)^2 ds \middle| \mathcal{F}_t \right].$$

To find also an upper bound for the left hand side in the inequality above we note

$$-2h_s \hat{\theta}_s^X z_s^X - (z_s^X)^2 = (\hat{\theta}_s^X)^2 h_s^2 - (\hat{\theta}_s^X h_s + z_s^X)^2.$$

Let $S_1$ denote an upper bound for $H^2$ and $S_2$ an upper bound for $(\hat{\theta}_s^X)^2 h_s^2$ over all times $s$. Then we get for every $t \in [0,T]$

$$\begin{aligned} S_1 + TS_2 &\geq E\left[ \int_t^T (z_s^Y)^2 ds \middle| \mathcal{F}_t \right] \\ &= E\left[ \langle M \rangle_T - \langle M \rangle_t | \mathcal{F}_t \right]. \end{aligned}$$

Therefore $M$ is a $P$–BMO martingale. □

**Remarks:** 1. Given recent results by Briand, Hu [8], it is possible to extend the existence and uniqueness result to random variables $H$ for which some $\gamma > 1$ exists such that $E(\exp(\gamma|H|)) < \infty$. It remains to establish the BMO property in this case, which appears very plausible.
2. We can also prove a converse of Theorem 1.3, stating that any partial equilibrium will be given by the solution of a BSDE of the type (4).
3. The model exhibited can be generalized to an arbitrary finite number of Brownian motions replacing $W^2$. See [21].

# 2 Numerical simulation of market performance in simple scenarios

The aim of this numerical model is to gain insight into the dynamics and performance of the market of risk redistribution created above. Apart from the existence and uniqueness results for equilibrium market price of risk, the theory provides no evident clues about further properties. For this reason, we fix simple scenarios, and use appropriate numerical schemes to simulate this price and individual optimal strategies of trading with the asset of risk trading by model agents. In the attempt briefly presented here, we start with translating the BSDE for the description of the equilibrium market price of risk into a nonlinear BSDE. This is done by using the generelized Feynman-Kac approach.

## 2.1 Generalized Feynman-Kac approach

Let us briefly reconsider the basic link between BSDE and PDE (with viscosity solutions in general). Let $m, d \in \mathbf{N}$. Let $W = (W^1, \cdots, W^m)$ an

$m-$dimensional Brownian motion, and let $b : \mathbf{R}^d \to \mathbf{R}^d$ and $\sigma : \mathbf{R}^d \to \mathbf{R}^{d\times m}$ be Lipschitz continuous and bounded. For $0 \le t \le T, x \in \mathbf{R}^d$ let $X^{t,x}$ be the solution of the stochastic differential equation

$$\begin{aligned} dX_s^{t,x} &= b(X_s^{t,x})dt + \sigma(X_s^{t,x})dW_s, \quad s \in [t,T] \\ X_t^{t,x} &= x. \end{aligned} \tag{6}$$

Let $\mathcal{O}$ be an open set in $]0,T[\times\mathbf{R}^d$, and denote by $\tau$ the first exit by $(X_s^{t,x})_{t\le s\le T}$ from $\mathcal{O}$. Let, moreover, $L$ be the infinitesimal generator associated with the SDE, i.e. for $\phi$ with continuous second partial derivatives we have

$$L\phi(x) = \sum_{i=1}^d b_i(x)\frac{\partial\phi}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^d \sigma\sigma^*_{ij}(x)\frac{\partial^2\phi}{\partial x_i \partial x_j}(x).$$

The classical Feynman-Kac formula in the backward direction states a stochastic representation for linear PDE related to the infinitesimal generator of our SDE. Our interest is in establishing the PDE-BSDE link in its basic framework. So we refrain from striving for the most general hypotheses.

**Theorem 2.1** *Let $h : \mathbf{R}^d \to \mathbf{R}$ be Lipschitz continuous and of (sub-)linear growth. Let $g : \mathbf{R}^d \to \mathbf{R}$ be continuous with at most polynomial growth. Suppose $v$ is the classical solution of the boundary value problem*

$$\begin{cases} \frac{\partial v}{\partial t} + Lv + hv = 0, \\ v = g \quad on \quad \partial\mathcal{O}. \end{cases}$$

*Then $v$ possesses the representation*

$$v(t,x) = E[g(X_\tau^{t,x})\exp(-\int_t^\tau h(X_s^{t,x})ds)], \quad (t,x) \in [0,T]\times\mathbf{R}^d.$$

**Proof:**
The proof is based on an application of Itô's formula to the processes $v(s, X_s^{t,x})$, and taking expectations thereafter.□

Theorem 2.1 has an extension to PDE in which a non-linear term replaces $hv$ on the side of $\frac{\partial v}{\partial t} + Lv$. This extension is most easily seen by interpreting

$$Y_s^{t,x} = v(s, X_s^{t,x}), \quad s \in [t, T],$$

in the spirit of BSDE. To see this, one just has to apply Itô's formula, with the result

$$\begin{aligned} dY_s^{t,x} &= [\frac{\partial v}{\partial t} + Lv](s, X_s^{t,x})\, ds + \sigma^*\, Dv(s, X_s^{t,x})\, dW_s, \\ Y_\tau^{t,x} &= g(X_\tau^{t,x}). \end{aligned}$$

The first line of this formula represents a BSDE with generator $hv(\cdot, X^{t,x})$ and control process $Z = \sigma^* Dv(\cdot, X^{t,x})$, since the former is given by $\frac{\partial v}{\partial t} + Lv$ due to the PDE. If the PDE reads

$$\begin{cases} \frac{\partial v}{\partial t} + Lv + F(v, \sigma^* Dv) = 0, \\ v = g \quad \text{on} \quad \partial\mathcal{O}, \end{cases}$$

all we have to do in the just derived BSDE is to replace $hv$ with $F(v, \sigma^* Dv)$. This gives the extended Feynman-Kac formula in the following Theorem.

**Theorem 2.2** *Let $F : \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ be $C^\infty$ and of (sub-)linear growth. Let $g : \mathbf{R}^d \to \mathbf{R}$ be continuous and bounded. Suppose $v$ is the classical solution of the boundary value problem*

$$\begin{cases} \frac{\partial u}{\partial t} + Lu + F(., u, \sigma^* Du) = 0, \\ u = g \quad on \quad \partial\mathcal{O}. \end{cases}$$

*Then*

$$\begin{aligned} Y_s^{t,x} &= u(s, X_s^{t,x}), \quad s \in [t, T], \\ Z_s^{t,x} &= \sigma^* Du(s, X_s^{t,x}), s \in [t, T], \end{aligned}$$

*solve the BSDE*

$$\begin{aligned} dY_s^{t,x} &= -F(x_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds + Z_s^{t,x}\, dW_s, \quad t \le s \le T, \\ Y_\tau^{t,x} &= g(X_\tau^{t,x}). \end{aligned}$$

**Proof:**
See Ma, Yong [30]. □

If the data are smooth enough, we can of course also go in the opposite direction in Theorem 2.2, and associate with a BSDE a non-linear PDE which in general has viscosity solutions. The correspondence is of course one-to-one, if the equations have unique solutions. See Fleming, Soner [17].

We use the access via Theorem 2.2 to derive non-linear PDE for our model established in the previous section. We take the PDE describing

- the individual optimal investments on the risk trading asset,
- the equilibrium price of risk.

We shall not go into further details here. For these equations we use simple numerical schemes, just finite difference methods, to do numerical simulations. This is possible in very low dimensions. More details will be given below. These simulations provide insight into the dynamics and performance of the market.

## 2.2 Numerical examples

For the **stock market price**, we take an ordinary geometric Brownian motion.

For the **temperature process**, we use examples explained in (1.2). The most interesting temperature process is undoubtedly a bi-stable one. See ENSO.pdf, slide 7.

We focus on **three model agents**: (Peruvian) fishers, farmers, and banks wishing to diversify their portfolios. Fisher and farmer are exposed to the El Niño risk, while the bank is not. Recall that the temperature process of the South Pacific in our simple model has two meta-stable states, given by $\underline{k}$, where the fisher has his maximal income, and $\overline{k}$, where the farmer profits most. The income functions of fisher and farmer are therefore modeled by taking

$$
\begin{aligned}
\phi^f(k) &= \exp(-(k-\underline{k})^2),\\
\phi^r(k) &= \exp(-(k-\overline{k})^2),
\end{aligned}
$$

and set

$$
\begin{aligned}
H^f &= \int_0^T \phi^f(K_s)ds,\\
H^r &= \int_0^T \phi^r(K_s)ds.
\end{aligned}
$$

For the bank, we set $H^b = 0$. See ENSO.pdf, slide 8.

Numerical experiments are conducted for the following three particular models, in which risk aversion of the agents is taken to be 1:

**Model A** $K$ is an Ornstein-Uhlenbeck process with mean reverting drift around 0; time horizon $T = 2$

**Model B** $K$ is a bi-stable periodic potential diffusion with $\overline{k} = 2.5, \underline{k} = -2.5$; the period is 1, $T = 2$

**Model C** $K$ is a bi-stable periodic potential diffusion with $\overline{k} = 2.5, \underline{k} = -2.5$; the period is 1, $T = \frac{3}{2}$

We show simulation results for the most interesting models B and C, and exhibit for both cases

- the optimal trading strategies in $Y$ for the model agents, as functions of $k$
- the *appreciation rate* for trading in the insurance asset $Y$, indicated by $E(Y_t)$ when starting from $k$

See ENSO.pdf, slides 9,10.

We only show the farmer's **optimal strategy**. One first notices, by taking into account the estimates for the expectation of $Y_t$, that the appreciation of risk trading is very low compared to model A.

The general feature of all diagrams is this: the agent has to invest into the risk trading asset, as long as his usual business gives sufficient yields. As soon as the temperature is in domains where his business is not doing well, he can make up for the losses by selling shares. There are slight differences, if trading is done at different times during one period. So for example the critical temperature at which the agent has to start investing, is shifted considerably to the right if investment is done after the first half period than after the whole period. This is explained by the fact that during the first half period the farmer is doing worse.

The qualitative difference between models B and C is not very big. We just observe that the farmer invests a little more in C than in B. This is explained by the fact that the farmer is worse off on average during the trading interval, since the bi-stable process takes more time near $\underline{k}$ than near $\overline{k}$ before the trading period has elapsed after $\frac{3}{2}$ periods.

As for **appreciation of trading** with $Y$, we notice that if temperature starts near 0 in models B and C, appreciation is maximal. This corresponds with the fact, that near 0 both agents are far from optimal business conditions, while this gets better for at least one of them if the temperature approaches -2.5 or 2.5. Also here models B and C behave quite similarly. We just observe that maximal appreciation is lower for model C, accounting for the fact that one of the agents is doing much better than the other in this model.

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