class 1 (October 28, 2008)

The space $C^1([a,b])$ is complete with respect to the norm

$$\|x\|_{C^1([a,b])} := \max_{t\in[a,b]} |x(t)| + \max_{t\in[a,b]} |x'(t)|$$

<u>Proof</u> Let $(x_k) \subset C^1([a,b])$ be a Cauchy sequence, i.e. $\forall\, \varepsilon > 0\ \exists\ k_0 = k_0(\varepsilon)$:

$$\|x_k - x_l\|_{C^1([a,b])} \le \varepsilon \quad \forall\, k,l \ge k_0 .$$

$C([a,b])$ complete $\Rightarrow$ $\exists\, x, y \in C([a,b])$:

$$\max_{[a,b]} |x - x_k| \to 0, \quad \max_{[a,b]} |x_k' - y| \to 0 ;$$

$$(*) \qquad x_k(t) = x_k(a) + \int_a^t x_k'(s)\,ds , \quad t \in [a,b] ,$$

$$\left| \int_a^t y\,ds - \int_a^t x_k'\,ds \right| \le (b-a) \max_{[a,b]} |y - x_k'| \xrightarrow[\text{as } k\to\infty]{} 0$$

passage to the limit $k \to \infty$ in $(*)$ $\Rightarrow$ claim. ∎

Exercise 1.3 / 2 : solution

Let $(x^{(n)})$ be a Cauchy sequence in s w.r. to the metric d ; let $k \in \mathbb{N}$ be fixed, let $\varepsilon > 0$ be arbitr.; define

$$\eta := \frac{1}{2^k} \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$$

$\Rightarrow \exists\ n_0 = n_0(\eta)$:

$$\sum_{j=1}^{\infty} \frac{|x_j^{(m)} - x_j^{(n)}|}{2^j\left(1 + |x_j^{(m)} - x_j^{(n)}|\right)} = d\left(x^{(m)}, x^{(n)}\right) \leq \eta \quad \forall\ m, n \geq n_0$$

$$\Rightarrow \frac{|x_k^{(m)} - x_k^{(n)}|}{2^k\left(1 + |x_k^{(m)} - x_k^{(n)}|\right)} \leq \eta$$

$$\Rightarrow |x_k^{(m)} - x_k^{(n)}| \leq 2^{k+1}\eta \leq \varepsilon,$$

i.e. $(x_k^{(n)})$ is Cauchy in $\mathbb{K}$ $\Rightarrow$ $x_k \in \mathbb{K}$ s.t. $x_k^{(n)} \longrightarrow x_k$ in $\mathbb{K}$. Clearly, $\forall\ N \in \mathbb{N}$:

$$\sum_{j=1}^{N} \frac{|x_j^{(m)} - x_j^{(n)}|}{2^j\left(1 + |x_j^{(m)} - x_j^{(n)}|\right)} \leq \eta \quad \forall\ m, n \geq n_0$$

$m \to \infty$, then $N \to \infty$ $\Rightarrow$ $d(x, x^{(n)}) \leq \eta \leq \dfrac{\varepsilon}{2^{k+1}} < \varepsilon$

∎

Exercise 1.3/4 : solution

Let $(x^{(n)}) \subset M$ be any sequence.

$k=1$: $|x_1^{(n)}| \leq 1 \quad \forall n \in \mathbb{N}$

$\Rightarrow \exists$ subsequence $(x_1^{(n_1)})$: $x_1^{(n_1)} \longrightarrow \xi_1$, $|\xi_1| \leq 1$;

$k=2$: clearly, $|x_2^{(n_1)}| \leq 1 \quad \forall n_1 \in \mathbb{N}$

$\Rightarrow \exists \ (x_2^{(n_2)}) \subset (x_1^{(n_1)})$: $x_2^{(n_2)} \longrightarrow \xi_2$, $|\xi_2| \leq 1$,

clearly, $x_1^{(n_2)} \longrightarrow \xi_1$.

Consider the diagonal sequence

$$\left(x^{(n_n)}\right) = \left(x_1^{(n_n)}, x_2^{(n_n)}, \ldots\right)$$

$$\Rightarrow \quad x_k^{(n_n)} \longrightarrow \xi_k, \quad |\xi_k| \leq 1 \quad \forall k \in \mathbb{N}.$$

By 1.3/3 $\Rightarrow$ $x^{(n_n)} \longrightarrow \xi$ w.r. to $d(\cdot,\cdot)$,

and $\xi \in M$. ∎