class 2 (November 04, 2008)

Exercise 2.2/2) : solution

equivalence of $\|\cdot\|_1$ and $|||\cdot|||$:

a) clearly, $\|x\|_1 \leq |||x|||$;

b) $x(t) = x(0) + \int_0^t x'(s)\,ds$

$\Rightarrow |x(t)| \leq |x(0)| + \max_{[0,1]} |x'| = \|x\|_1$

$\Rightarrow |||x||| \leq 2\|x\|_1$.

---

equivalence of $\|\cdot\|_2$ and $|||\cdot|||$:

a) $\int_0^1 x(t)\,dt = x(0) + \int_0^1 \left(\int_0^t x'(s)\,ds \right) dt$

$\Rightarrow \left| \int_0^1 x(t)\,dt \right| \leq |x(0)| + \int_0^1 \left(\int_0^t |x'(s)|\,ds \right) dt$

$\leq \max_{[0,1]} |x| + \max_{[0,1]} |x'|$

$\Rightarrow \|x\|_2 \leq |||x|||$,

b) $$|x(0)| = \left| \int_0^1 x\,dt - \int_0^1 \left(\int_0^t x'\,ds \right) dt \right|$$

$$\leq \left| \int_0^1 x\,dt \right| + \max_{[0,1]} |x'|$$

$$\Rightarrow |x(t)| \leq |x(0)| + \int_0^1 |x'|\,ds$$

$$\leq \left| \int_0^1 x\,d\tau \right| + 2 \max_{[0,1]} |x'|$$

$$\Rightarrow \max_{[0,1]} |x| + \max_{[0,1]} |x'|$$

$$\leq \left| \int_0^1 x\,d\tau \right| + 3 \max_{[0,1]} |x'| \leq 4\, \|x\|_2 .$$

$\|\cdot\|_3$ and $|||\cdot|||$ are not equivalent:

a) clearly, $\|x\|_3 \leq \sqrt{2}\; |||x|||$;

b) assume there exists $C_0 = \text{const} < +\infty$ s. t.

$$|||x||| \leq C_0 \|x\|_3 \quad \forall\, x \in C^1([0,1]) .$$

Consider $x_k(t) := t^k$, $t \in [0,1]$, $k \in \mathbb{N}$;

$$\Rightarrow \quad |||x_k||| = 1 + k;$$

$$\int_0^1 x_k^2\,dt = \int_0^1 t^{2k}\,dt = \frac{1}{2k+1}$$

$$\int_0^1 (x_k')^2\,dt = k^2 \int_0^1 t^{2k-2}\,dt = \frac{k^2}{2k-1}$$

$$\Rightarrow \quad 1+k \leq C_0 \left(\frac{1}{2k+1} + \frac{k^2}{2k-1} \right)^{1/2}$$

$$\Rightarrow \quad 1 < 1 + \frac{1}{k} \leq C_0 \left(\frac{1}{k^2(2k+1)} + \frac{1}{2k-1} \right)^{1/2} \xrightarrow[k \to \infty]{} 0$$

contradiction.

Problem $\boxed{2}$, part 2:

Let $(u_n) \subset C([a,b])$ be a bounded sequence, i.e.

$$\|u_n\|_{C([a,b])} \le c_1 = \text{const} \quad \forall n \in \mathbb{N}.$$

K is uniformly continuous on $[a,b] \times [a,b]$:

$$\forall \varepsilon > 0 \;\; \exists \delta = \delta(\varepsilon) > 0 : \left|K(t,s) - K(\bar{t},\bar{s})\right| \le \varepsilon$$

$$\forall (t,s), (\bar{t},\bar{s}) \in [a,b] \times [a,b] \text{ s.t. } |t-\bar{t}|, |s-\bar{s}| \le \delta$$

$$\Rightarrow \quad \left|(Tu_n)(t) - (Tu_n)(\bar{t})\right|$$

$$\le \int_a^b \left|K(t,s) - K(\bar{t},s)\right| |u_n(s)| \, ds$$

$$\le (b-a) c_1 \cdot \varepsilon \quad \forall |t-\bar{t}| \le \delta, \; \forall n \in \mathbb{N};$$

clearly, $\left|(Tu_n)(t)\right| \le (b-a) \max |K| c_1 \quad \forall t \in [a,b], \; \forall n \in \mathbb{N}$

Arzela-Ascoli: $\exists \; (u_{n_k})$:

$$Tu_{n_k} \to w \text{ in } C([a,b]).$$