Class 3 (November 11, 2008) on web: Nov. 17

1. Let $f \in C([-\pi,\pi])$, $f(-\pi) = f(\pi)$. Assume there holds the formal expansion

$$(*) \quad f(t) \sim \alpha_0 + \sum_{k=1}^{\infty} \left( \alpha_k \cos kt + \beta_k \sin kt \right), \quad t \in [-\pi,\pi]$$

If the series $(*)$ converges uniformly to $f$ on $[-\pi,\pi]$ we obtain

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,dt,$$

$$\alpha_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt\,dt,$$

$$\beta_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt\,dt; \quad k = 1,2,\ldots \, .$$

2. The system of functions

$$(+) \quad \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \ldots, \frac{1}{\sqrt{\pi}} \cos kt, \frac{1}{\sqrt{\pi}} \sin kt \ldots \right\}$$

is an ONS with respect to $L^2(-\pi,\pi)$.

Fourier coefficients of $f \in L^2(-\pi, \pi)$ with respect to the ONS (+):

$$a_0 = \int_{-\pi}^{\pi} f(t) \cdot \frac{1}{\sqrt{2\pi}}\, dt = \sqrt{2\pi}\, \alpha_0 ,$$

$$a_k = \int_{-\pi}^{\pi} f(t) \cdot \frac{\cos kt}{\sqrt{\pi}}\, dt = \sqrt{\pi}\, \alpha_k ,$$

$$b_k = \int_{-\pi}^{\pi} f(t) \cdot \frac{\sin kt}{\sqrt{\pi}}\, dt = \sqrt{\pi}\, \beta_k ;$$

thus:

$$f(t) \sim a_0 \cdot \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \left( a_k \cdot \frac{\cos kt}{\sqrt{\pi}} + b_k \cdot \frac{\sin kt}{\sqrt{\pi}} \right) ;$$

$$a_0^2 + \sum_{k=1}^{\infty} \left( a_k^2 + b_k^2 \right) \leq \int_{-\pi}^{\pi} |f(t)|^2 dt \quad \text{(Bessel)} .$$

3. The ONS is complete in $L^2(-\pi,\pi)$ [$\Rightarrow$ Parseval]

Sketch of proof: (see also pp. 3.7, 3.8)

3.1 Theorem (Weierstraß) Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and $2\pi$ periodic. Let $\varepsilon>0$. Then there exists

$$s_n(t) = s_{n,\varepsilon}(t) = \alpha_0 + \sum_{k=1}^{n}\left(\alpha_k \cos kt + \beta_k \sin kt\right)$$

s.t.

$$|f(t)-s_n(t)| \leq \varepsilon \quad \forall t\in\mathbb{R}.$$

3.2 Let $f\in C([-\pi,\pi])$. Set $M := \max\limits_{t\in[-\pi,\pi]}|f(t)|$.
Let $\varepsilon>0$, and $0<h<\dfrac{\varepsilon}{16M^2}$. Let $g=g_h\in C([-\pi,\pi])$ be defined by

$$g(t) := \begin{cases} f(t) & \text{if } t\in[-\pi,\pi-h], \\ f(-\pi) & \text{if } t=\pi, \\ \text{linear} & \text{if } t\in[\pi-h,\pi]. \end{cases}$$

Clearly, $\max\limits_{[-\pi,\pi]}|g| = M$. We obtain

$$\int_{-\pi}^{\pi} (f-g)^2 dt = \int_{\pi-h}^{\pi} (f-g)^2 dt \leq 4M^2 h \leq \frac{\varepsilon}{4}.$$

Apply 3.1 with $\frac{\varepsilon}{4}$ in place of $\varepsilon$ to obtain

$$\int_{-\pi}^{\pi} (f - s_{n,\varepsilon})^2 dt \leq \varepsilon.$$

3.3 The space of continuous functions with compact support in $]a,b[$ is dense in $L^2(a,b)$;

$$\operatorname{supp}(f) := \overline{\{t \in ]a,b[;\ f(t) \neq 0\}}.$$

This result is a special case of a more general consideration. Let $1 \leq p < +\infty$:

1) separability of $L^p$;

2) approximation of $L^p$ functions by continuous functions.

Ad 1)

Theorem 1 For every $u \in L^p(\mathbb{R}^n)$ and every $\varepsilon > 0$ there exist

$$\begin{cases} \lambda_i \in \mathbb{Q}, \\ W_i \subset \mathbb{R}^n \text{ half-open cube with rational corners} \\ (i = 1, \dots, m), \end{cases}$$

such that

$$\| u - \sum_{i=1}^m \lambda_i \chi_{W_i} \|_{L^p(\mathbb{R}^n)} \le \varepsilon .$$

Proof

- approximate $u$ by $u \chi_{B_r(0)} =: u_r$
- approximate $u_r$ by simple functions $\sum_{i=1}^m \alpha_i \chi_{A_i}$
- approximate $\alpha_i$ by rational numbers, approximate $A_i$ by open sets $U_i \supset A_i$
- represent $U_i = \bigcup_{l=1}^\infty W_i^{(l)}$, $W_i^{(l)}$ half-open cube.

<u>Corollary</u> Let $E \subseteq \mathbb{R}^n$ be measurable, $1 \leq p < +\infty$. Then $L^p(E)$ is separable.

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Ad 2)

<u>Theorem 2</u> Let $1 \leq p < +\infty$. Then for every $u \in L^p(\mathbb{R}^n)$ and every $\varepsilon > 0$ there exists $\varphi_\varepsilon \in C_c(\mathbb{R}^n)$ such that

$$\| u - \varphi_\varepsilon \|_{L^p(\mathbb{R}^n)} \leq \varepsilon.$$

<u>Proof</u> Approximate $\chi_{W_i}$ by continuous functions.

<u>Corollary</u> Let $E \subset \mathbb{R}^n$ be open, let $1 \leq p < +\infty$. Then for every $u \in L^p(E)$ and $\varepsilon > 0$ there exists $\psi_\varepsilon \in C_c(E)$ such that

$$\| u - \psi_\varepsilon \|_{L^p(E)} \leq \varepsilon.$$

<u>Proof</u>

- $\tilde{u}$ = extension of $u$ by zero onto $\mathbb{R}^n \setminus E$; let $\varphi_\varepsilon$ be as in Theorem 2 with $\tilde{u}$ in place of $u$
- $\psi_\varepsilon = \varphi_\varepsilon \zeta$, where $\zeta \in C_c(E)$ cut-off function with respect to $K \subset E$, $K$ compact.

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# Funktionalanalysis - WS 2008/2009
# Übungsblatt 6

(Abgabe der mit* versehenen Aufgaben zur Vorlesung am 25. November 2008)

**Aufgabe 1***

Sei $\Omega \subset \mathbb{R}^n$ offen und beschränkt. Zeigen Sie:

(a) Für $1 \leq p < q \leq \infty$ ist $L^q(\Omega) \subset L^p(\Omega)$.

(b) Ist $f \in L^\infty(\Omega)$, so gilt: $\lim\limits_{p\to\infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)}$.

(c) Die Inklusion $L^\infty(0,1) \subset \bigcap\limits_{1\leq p<\infty} L^p(0,1)$ ist echt.

(d) Für $1 \leq p < \infty$ gilt weder $L^p(\mathbb{R}) \subset L^\infty(\mathbb{R})$ noch $L^\infty(\mathbb{R}) \subset L^p(\mathbb{R})$.

**Aufgabe 2**

Für $n \in \mathbb{N}$ seien $f_n, g_n : [0,1] \to \mathbb{R}$ definiert durch

$$f_n(x) = x^n - x^{n+1}, \quad g_n(x) = n(x^n - x^{n+1}).$$

Untersuchen Sie die Folgen $(f_n)_{n\in\mathbb{N}}$ und $(g_n)_{n\in\mathbb{N}}$ auf Beschränktheit und Konvergenz in den Räumen $L^1([0,1]), L^2([0,1])$ und $L^\infty([0,1])$.

**Aufgabe 3***

Betrachten Sie den Hilbertraum $H = L^2((-\pi,\pi);\mathbb{C})$.

(a) Zeigen Sie, dass $\left\{\frac{1}{\sqrt{2\pi}}e^{ikx}; k \in \mathbb{Z}\right\}$ ein Orthonormalsystem in $H$ ist.

(b) Für $n \in \mathbb{N}$ betrachte man den $2n+1$-dimensionalen Teilraum

$$U_n := \mathrm{span}\{e^{ikx}; k = -n, -n+1, \dots, n-1, n\}.$$

Sei $f \in H$. Zeigen Sie für $f_n \in U_n$ ist äquivalent.

$$\|f - f_n\|_H = \min_{g\in U_n} \|f - g\|_H \iff f_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right)(x-y)\right)}{\sin\left(\frac{1}{2}(x-y)\right)} f(y)dy, \quad x \in [-\pi,\pi].$$

**Hinweis**: Zeigen Sie: $\sin\left(\frac{x}{2}\right)\sum\limits_{k=-n}^{n} e^{ikx} = \sin\left(\left(n+\frac{1}{2}\right)x\right), \ x \in \mathbb{R}, \ n \in \mathbb{N}.$

**Aufgabe 4**

Sei $f \in C^1(a,b)$ und $a < c < d < b$. Zeigen Sie, dass

$$\lim_{\varepsilon\to 0} \|f_\varepsilon - f\|_{C^1([c,d])} = 0.$$

More on the trigonometric system (+):

[1] Alt, H.W., Lineare Funktionalanalysis.
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Springer-Verlag, New York 2002;
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[3] Walter, W., Analysis II. Springer-Verlag,
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