Class 4 (November 18, 2008) on web: Nov. 20

Exercise 4.1 : solution

Let $(u_k) \subset C^{(\alpha)}([a,b])$ be a bounded sequence, i.e. $\|u_k\|_{C^{(\alpha)}([a,b])} \leq m_0 = \text{const} \ \forall k \in \mathbb{N}$

$$\Rightarrow \quad |u_k(t)| \leq m_0 \quad \forall k \in \mathbb{N}, \ \forall t \in [a,b];$$

$$(*) \qquad |u_k(s) - u_k(t)| \leq m_0 |s-t|^\alpha \ \forall k \in \mathbb{N}, \forall s,t \in [a,b]$$

By Arzela/Ascoli: $\exists$ subsequence of (u_k) (not relabelled) s.t.

$$u_k \to u \quad \text{in } C([a,b]) \text{ as } k \to \infty.$$

Passage to the limit $k \to \infty$ in $(*)$ gives

$$(**) \qquad |u(s) - u(t)| \leq m_0 |s-t|^\alpha \ \forall s,t \in [a,b],$$

i.e. $u \in C^{(\alpha)}([a,b])$.

Fix $0 < \beta < \alpha$. We prove:

$$\left\{\begin{array}{l} \forall \varepsilon > 0 \ \exists \ k_0 = k_0(\varepsilon) \text{ s.t.} \\ \sup\limits_{\substack{s,t \in [a,b], \\ s \neq t}} \dfrac{\left|(u_k - u)(s) - (u_k - u)(t)\right|}{|s-t|^{\beta}} \leq \varepsilon \ \forall \ k \geq k_0 . \end{array}\right.$$

Indeed, define $\lambda := \frac{\beta}{\alpha}$. Then $\exists \ k_0 = k_0(\varepsilon)$ s.t.

$$(+) \qquad \| u_k - u \|_{C([a,b])} \leq \left(\frac{\varepsilon}{2 m_0^{\lambda}} \right)^{1/(1-\lambda)} \ \forall \ k \geq k_0 .$$

By $(*)$ and $(**)$, and $(+)$

$$\left| (u_k - u)(s) - (u_k - u)(t) \right| \leq$$

$$\leq \left| \left(u_k(s) - u_k(t) \right) - \left(u(s) - u(t) \right) \right|^{\lambda} \cdot$$

$$\cdot \left(|u_k(s) - u(s)| + |u_k(t) - u(t)| \right)^{1-\lambda}$$

$$\leq \left(2 m_0 |s-t|^{\alpha} \right)^{\lambda} \cdot \left(2 \| u_k - u \|_{C([a,b])} \right)^{1-\lambda}$$

$$\leq \varepsilon |s-t|^{\beta} \quad \forall \ k \geq k_0 , \ \forall \ s,t \in [a,b] .$$

Hence

$$u_k \to u \ \text{ in } C^{(\beta)}([a,b]) \ \text{ as } k \to \infty .$$