Class 5 (December 02, 2008) on web: Dec. 08

1. Integralrepresentation of the partial sum

$$s_{n,f}(t) = \frac{a_0}{2} + \sum_{m=1}^{n} \left(a_m \cos mt + b_m \sin mt \right), \quad t \in \mathbb{R}$$

Observing that

$$2 \cos\alpha \sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$$

we obtain

$$2 \sin\frac{t}{2} \left(\frac{1}{2} + \cos t + \cos 2t + \dots + \cos nt \right) =$$

$$= \sin\frac{t}{2} + 2\cos t \sin\frac{t}{2} + 2\cos 2t \sin\frac{t}{2} + \dots + 2\cos nt \sin\frac{t}{2}$$

$$= \sin\frac{t}{2} + \left(\sin\frac{3t}{2} - \sin\frac{t}{2} \right) + \left(\sin\frac{5t}{2} - \sin\frac{3t}{2} \right)$$

$$+ \dots + \left[\sin\left(\left(n+\frac{1}{2}\right)t\right) - \sin\left(\left(n-\frac{1}{2}\right)t\right) \right]$$

$$= \sin\left(\left(n+\frac{1}{2}\right)t\right).$$

Thus

$$(1) \quad \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{2\sin\frac{t}{2}}, \quad t \neq 0.$$

Integrating (1) over $[-\pi, \pi]$ gives

$$(2) \quad \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{2\sin\frac{t}{2}}\, dt = 1.$$

Clearly,

$$(3) \quad \frac{1}{\pi} \int_{0}^{\pi} \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{2\sin\frac{t}{2}}\, dt = \frac{1}{\pi} \int_{-\pi}^{0} \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{2\sin\frac{t}{2}}\, dt = \frac{1}{2}.$$

Let $f \in C(\mathbb{R})$ be 2π-periodic. For every $t \in \mathbb{R}$ we obtain

$$s_{n,f}(t) = \frac{a_0}{2} + \sum_{m=1}^{n} \left(a_m \cos mt + b_m \sin mt\right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)\, ds +$$

$$+ \frac{1}{\pi} \sum_{m=1}^{n} \left(\int_{-\pi}^{\pi} f(s) \cos ms\, ds \cos mt + \int_{-\pi}^{\pi} f(s) \sin ms\, ds \sin mt \right)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \left[\frac{1}{2} + \sum_{m=1}^{n} \left(\cos ms \cos mt + \sin ms \sin mt \right) \right] ds$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \left[\frac{1}{2} + \sum_{m=1}^{n} \cos m(s-t) \right] ds$$

by (1)

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \frac{\sin\left(\left(n+\frac{1}{2}\right)(s-t)\right)}{2\sin\frac{s-t}{2}} ds$$

$$= \frac{1}{\pi} \int_{-\pi-t}^{\pi-t} f(t+\tau) \frac{\sin\left(\left(n+\frac{1}{2}\right)\tau\right)}{2\sin\frac{\tau}{2}} d\tau$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+\tau) \frac{\sin\left(\left(n+\frac{1}{2}\right)\tau\right)}{2\sin\frac{\tau}{2}} d\tau \quad [\text{for } f \text{ is } 2\pi\text{-periodic}].$$

The function

$$D_n(s) := \frac{\sin\left(\left(n+\frac{1}{2}\right)s\right)}{2\sin\frac{s}{2}} \quad \text{if } s \in [-\pi, \pi],\ s \neq 0,$$

$$D_n(0) := n + \frac{1}{2}$$

is called the Dirichlet kernel.

2. Estimation of $\int_{-\pi}^{\pi} |D_n(s)|\,ds$ from below

$$\int_{-\pi}^{\pi} \frac{|\sin((n+\frac{1}{2})s)|}{|\sin\frac{s}{2}|}\,ds \geq 2\int_{-\pi}^{\pi} \frac{|\sin((n+\frac{1}{2})s)|}{|s|}\,ds$$

$\left(\text{for } |\sin\tau| \leq |\tau|\right)$ $\left[\text{now: } \sigma = \left(n+\frac{1}{2}\right)s \ldots\right]$

$$= 2\int_{-\pi(n+\frac{1}{2})}^{\pi(n+\frac{1}{2})} \frac{|\sin\sigma|}{|\sigma|}\,d\sigma \geq$$

$$\geq 2\int_{0}^{n\pi} \frac{|\sin\sigma|}{|\sigma|}\,d\sigma =$$

$$= 2\sum_{k=1}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin\sigma|}{\sigma}\,d\sigma \quad \left[\text{observe: } \frac{1}{k\pi} \leq \frac{1}{\sigma} \leq \frac{1}{(k-1)\pi} \ldots\right]$$

$$\geq \frac{2}{\pi}\sum_{k=1}^{n} \frac{1}{k} \underbrace{\int_{(k-1)\pi}^{k\pi} |\sin\sigma|\,d\sigma}_{=2}$$

$$= \frac{4}{\pi}\sum_{k=1}^{n}\frac{1}{k}.$$

Thus $\lim_{n\to\infty} \int_{-\pi}^{\pi} |D_n(s)|\,ds = +\infty.$