Humboldt-Universität zu Berlin
Institut für Mathematik/Angewandte Mathematik

EXERCISES

**BMS Basic Course "Functional Analysis"/Höhere Analysis I**

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(last series)

**12.1** For $x = (x_1, x_2, \ldots) \in l^2$ ($\mathbb{K} = \mathbb{C}$), define

$$Sx := \Big(x_2, \frac{x_3}{2}, \frac{x_4}{3}, \cdots\Big), \quad Tx = \Big(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdot\Big).$$

Clearly, $S, T \in \mathcal{L}(l^2, l^2)$. Determine:

1) $\sigma(S), \sigma_p(S)$; **(2 P)**

2) $\sigma(T), \sigma_p(T)$. **(2 P)**

[*Hint* First show: $S$ and $T$ are compact operators.]

**12.2** Let $a \in L^{p'}(0,1)$ ($\mathbb{K} = \mathbb{R}$; $p' = \dfrac{p}{p-1}$, $1 < p < +\infty$) be fixed. For $u \in L^p(0,1)$, define

$$(Tu)(t) := \int_0^t a(s)u(s)ds, \quad t \in [0,1].$$

Prove: $T \in \mathcal{K}(L^p(0,1),\ C([0,1])$. **(2 P)**

**12.3** DEFINITION *Let* $\mathbb{K} = \mathbb{R}$. *Define*

$$H^1(0,1) := \Big\{u \in L^2(0,1);\ \exists\, w \in L^2(0,1) \text{ s. t.} \int_0^1 u\varphi' dt = -\int_0^1 w\varphi dt\ \ \forall\, \varphi \in C_c^\infty(]0,1[)\Big\}.$$

$H^1(0,1)$ is called **Sobolev space**. Clearly,

- $w$ is uniquely determined by $u$. *Notation*: $u' := w =$ weak derivative of $u$;
- $C^1([0,1])$ can be identified with a subspace of $H^1(0,1)$;

- the mapping $Tu := u'$, $u \in H^1(0,1)$ is unbounded in $L^2(0,1)$, i. e. $\exists\, u_n \in H^1(0,1)$ s. t. $\|u_n\|_{L^2} \le 1$ and $\|Tu_n\|_{L^2} \to +\infty$ as $n \to \infty$.

Prove:

1) $H^1(0,1)$ is a Hilbert space w. r. to the scalar product

$$(u,v)_{H^1} := \int_0^1 (uv + u'v')dt.$$

**(2 P)**

[*Remark.* It suffices to prove the completeness of $H^1(0,1)$.]

2) If $(u_n) \subset H^1(0,1)$ s. t.

$$u_n \rightharpoonup u, \;\; Tu_n \rightharpoonup z \;\text{ in }\; L^2(0,1),$$

then: $u \in H^1(0,1), \; Tu = z$. **(1 P)**

The following is known. *For every* $u \in H^1(0,1) \;\exists\; \tilde{u} \in C([0,1])$, $\gamma = \gamma_u = \text{const}$ *s. t.*

$$\tilde{u} \in u, \quad \tilde{u}(t) = \gamma + \int_0^t u' ds \quad \forall\, t \in [0,1].$$

Use this to prove: 3) For $(u_n) \subset H^1(0,1)$ s. t. $u_n \rightharpoonup u$ in $H^1(0,1)$, there exists a subsequence $(u_{n_k})$ s. t. $(\tilde{u}_{n_k}) \subset C([0,1])$, $\tilde{u}_{n_k} \to \tilde{u}$ in $C([0,1])$ and $\tilde{u} \in u$. **(5 P)**

Hand in solutions: **January 29, 2009; first lecture**.