Humboldt-Universität zu Berlin
Institut für Mathematik/Angewandte Mathematik

EXERCISES

BMS Basic Course "Functional Analysis"/Höhere Analysis I

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3.1 For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, define

$$(Tx)_k := \sum_{l=1}^{n} a_{kl} x_l \quad (a_{kl} \in \mathbb{R} \text{ fixed}).$$

Consider

$X_1 := (\mathbb{R}^n, \|\cdot\|_1)$, where $\|x\|_1 := \sum_{k=1}^{n} |x_k|$;

$X_\infty := (\mathbb{R}^n, \|\cdot\|_\infty)$, where $\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}$.

1) Prove $T \in \mathcal{L}(X_1, X_\infty)$, calculate $\|T\|_{\mathcal{L}(X_1, X_\infty)}$; **(2 P)**

2) Prove $T \in \mathcal{L}(X_\infty, X_1)$, calculate $\|T\|_{\mathcal{L}(X_\infty, X_1)}$. **(2 P)**

3.2 Let X be a vector space. Let $f : X \to \mathbb{K}$ be a linear mapping, $f \not\equiv 0$. Prove: for every $x \in X$ there exists a unique representation

$$x = x_0 + x_1,$$

where

$x_0 \in X_0 := \ker(f), \quad x_1 \in X_1, \quad \dim X_1 = 1.$ **(2 P)**

The exercises 3.3 - 3.6 refer to a Hilbert space H with scalar product $(\cdot, \cdot)$ and norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$.

3.3 Prove:

1) $|(x, y)| = \|x\| \, \|y\| > 0 \Longrightarrow \exists\, \lambda \neq 0$ s. t. $x = \lambda y$; **(2 P)**

2) if $x \neq y$, $y \neq 0$ and $\|x+y\| = \|x\| + \|y\|$, then $\exists\, \lambda \neq 0$ s. t. $x = \lambda y$; (**1 P**)

3) the following statements are equivalent:
1° $x_k \to x$; 2° $\|x_k\| \to \|x\|$ and $(x_k, y) \to (x, y)$ $\forall\ y \in H$. (**2 P**)

3.4 Let G be a subspace of H, let $x \in H$. Prove the equivalence of the following two statements:
1° $x \in G^\perp$; 2° $\|x\| \leq \|x - y\|$ $\forall\, y \in G$. (**2 P**)

3.5 Let K be a non-empty, closed, convex subset of H.
1) Let $P_K : H \to K$ denote the convex projection onto K, i. e. for every $x \in H$ there holds

$$(*) \qquad \|x - P_K x\| = \min_{w \in K} \|x - w\|.$$

Prove: Let $x \in H$. Then $(*)$ holds if and only if

$$(+) \qquad \mathrm{Re}(x - P_K x, w - P_K x) \leq 0 \quad \forall\, w \in K.$$

[*Hint* $(*) \implies (+)$: for $t \in]0,1[$, $w \in K$, define $z = (1-t)P_K x + tw$; then $\|x - P_K x\| \leq \|x - z\|$.] (**2 P**)

The implication $(+) \implies (*)$ is obvious.

2) Prove: $\|P_K x_1 - P_K x_2\| \leq \|x_1 - x_2\|$ $\forall\, x_1, x_2 \in H$. (**3 P**)

[*Hint* Make use of $(+)$ with $x = x_1$ and $x = x_2$; choose w appropriately].

3.6 Differentiability of the norm $\|\cdot\|$

Let $1 \leq p < +\infty$. For $x, y \in H$, $x \neq 0$, and $t \in \mathbb{R}$, define $\phi_p(t) := \|x + ty\|^p$.
Calculate the derivative $\phi_p'(0)$

[*Hints* Proceed as follows:

1) calculate $\phi_2'(0)$ (**1 P**)

2) calculate $\phi_1'(0)$ (**2 P**)

3) calculate $\phi_p'(0)$ $(1 < p < +\infty)$ (**1 P**)

Remark $\phi_p'(0) = \lim_{t \to 0} \frac{1}{t}(\|x + ty\|^p - \|x\|^p)$ is called the *directional derivative* of $\|\cdot\|^p$ at $x \in H$ with respect to y.

Hand in solutions: November 6, 2008; first lecture.