Humboldt-Universität zu Berlin
Institut für Mathematik/Angewandte Mathematik

EXERCISES

BMS Basic Course "Functional Analysis"/Höhere Analysis I

Fall 2008/09

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Series 7, December 04, 2008

7.1 Let H be a real Hilbert space, let $f \in H^*$. Define

$$F(v) := \frac{1}{2}\|v\|^2 - \langle f, v \rangle, \quad v \in H.$$

Clearly, $\inf\limits_{v \in H} F(v) =: d \geq -\frac{1}{2}\|f\|^2_{H^*}$.

7.1.1 Let (u_m) be any sequence in H s. t. $\lim\limits_{m\to\infty} F(u_m) = d$. Prove: $u_m \to u_0$ and $F(u_0) = d$.

[*Hint* cf. projection on a closed convex subset of H] **(2 P)**

7.1.2 Let $u_0 \in H$. Prove the equivalence of the following two statements:

1° $F(v) \geq F(u_0) \quad \forall\ v \in H$;

2° $\langle f, v \rangle = (u_0, v) \quad \forall\ v \in H$ **(2 P)**

(in other words, the minimizer of F over H gives the element according to the Riesz representation theorem for the continuous linear functional $f \in H^*$).

7.2 Let $(X, \|\cdot\|)$ be a normed space. Let $f : [0, +\infty[\to X$ be a continuous function s. t.

- $f(0) = 0$,
- $\forall\ \varepsilon > 0, \quad \forall\ t > 0\ \exists\, k_0 = k_0(\varepsilon, t) \in \mathbb{N} : \|f(kt)\| \leq \varepsilon \quad \forall\ k \in \mathbb{N},\ k \geq k_0$.

Prove: $\forall\ \varepsilon > 0\ \ \exists\, t_0 = t_0(\varepsilon) \in]0, +\infty[: \quad \|f(t)\| \le \varepsilon\ \ \forall\, t \ge t_0$.

[*Hints* verify: 1) for $\varepsilon > 0$ and $k \in \mathbb{N}$, the set

$$A_{\varepsilon,k} := \{t \in [0, +\infty[; \quad \|f(nt)\| \le \varepsilon \quad \forall\ n \ge k\}$$

(5 P)

is closed in $[0, +\infty[$;

2) $[0, +\infty[= \bigcup\limits_{k=1}^{\infty} A_{\varepsilon,k}$].

7.3 Let X and Y be normed spaces. Prove: If $\mathcal{L}(X, Y)$ is complete, then Y is complete. **(2 P)**

7.4 Let X be a real Banach space. Let (x_k^*) be a sequence of elements of X^* s. t.

$$\forall\ x \in X: \quad \sup_{k\in\mathbb{N}} \Big|\langle x_k^*, x\rangle\Big| < +\infty.$$

Prove: the set

$$X_0 := \{x \in X; \lim_{k\to\infty} \langle x_k^*, x\rangle \textit{ exists in } \mathbb{R}\}$$

is closed. **(2 P)**

7.5 DEFINITION *Let X and Y be normed spaces, let $D \subset X$ be a subspace. A linear mapping $T : D \to Y$ is called* **closed**, *if for every sequence $(x_k) \subset D$ s. t. $x_k \to x_0$ and $Tx_k \to y_0$ as $k \to \infty$ there holds*

$$x_0 \in D, \quad y_0 = Tx_0.$$

Prove the equivalence of the following two statements:

1° $T : D \to Y$ is closed;

2° $G(T) := \{\{x, y\} \in X \times Y; x \in D, y = Tx\}$ is closed in $X \times Y$. **(2 P)**

Hand in solution: December 11, 2008; first lecture.