WORKING PAPER

CONDITIONS FOR OPTIMALITY AND STRONG STABILITY IN NONLINEAR PROGRAMS WITHOUT ASSUMING TWICE DIFFERENTIABILITY OF DATA

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November 1989
WP-89-089

IIASA
International Institute
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Foreword

The paper provides second order sufficient conditions for optimality and for strong stability of local minimizers of optimization problems for which twice differentiability fails but the data are $C^{1,1}$ functions.

The results were obtained in the frame of the IIASA Contracted Study "The Development of Parametric Optimization and its Applications."

Alexander B. Kurzhanski
Chairman
System and Decision Sciences Program

CONDITIONS FOR OPTIMALITY AND STRONG STABILITY IN NONLINEAR PROGRAMS WITHOUT ASSUMING TWICE DIFFERENTIABILITY OF DATA

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December 1988

Abstract. The present paper is concerned with optimization problems in which the data are differentiable functions having a continuous or locally Lipschitzian gradient mapping. Its main purose is to develop second-order sufficient conditions for a stationary solution to a programm with $C^{1,1}$ data to be a strict local minimizer or to be a local minimizer which is even strongly stable with respect to certain perturbations of the data. It turns out that some concept of a set-valued directional derivative of a Lipschitzian mapping is a suitable tool to extend well-known results in the case of programs with twice differentiable data to more general situations. The local minimizers being under consideration have to satisfy the Mangasarian-Fromovitz CQ. An application to iterated local minimization is sketched.

Key words. Second-order sufficient conditions, programs with $C^{1,1}$-data, Lipschitzian mappings, directional derivatives, strongly stable stationary solution, local minimizer, iterated local minimization

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1. Introduction

Optimality conditions and sensitivity analysis of optimal solutions play an important role in theory and applications of nonlinear optimization problems. Motivations for the study of sensitivity and stability of optimization problems come from the development of numerical methods, from the convergence analysis of solution procedures, from semi-infinite programming and from the analysis of inexact models. The aim of the present paper is to give second-order sufficient conditions for optimality and for strong stability of local minimizers (under data perturbations), where the optimization problems being under consideration include functions for which twice differentiability fails. Our main tool used in the following is a set-valued directional derivative of Lipschitz continuous mappings, which was introduced by Kummer [19]. The second-order conditions concern optimization problems in which the data are differentiable functions having a locally Lipschitzian gradient mapping (so-called $C^{1,1}$-functions).

Given a metric space T, an open subset Q of R^n and functions $f_i\colon Q \times T \longrightarrow R$ $(i=0,1,\ldots,m)$, we consider the following family of optimization problems,

$$P(t): \qquad \min_x \; \{ f_o(x,t) \;/\; x \in M(t) \} \;, \quad t \in T \,,$$

where the multifunction $M\colon T \rightrightarrows R^n$ is defined by

$$M(t) := \{ x \in R^n \;/\; f_i(x,t) = 0,\ i \in I_1;\ \ f_j(x,t) \le 0,\ j \in I_2 \},$$
$$t \in T \,, \quad I_1 := \{1,\ldots,p\} \,, \quad I_2 := \{p+1,\ldots,m\} \,.$$

Throughout the paper we shall suppose that for each $i \in \{0,1,\ldots,m\}$ and for each $t \in T$,

$$\begin{aligned} &f_i(\cdot,t) \text{ is Fréchet differentiable on } Q, \text{ and} \\ &f_i \text{ and } D_x f_i(\cdot,\cdot) \text{ are continuous on } Q \times T \,, \end{aligned} \tag{1.1}$$

where $D_x f_i(x,t)$ denotes the gradient of $f_i(\cdot,t)$ at x for fixed t. Put for $(x,u,t) \in Q \times R^m \times T$,

$$l(x,u,t) := f_o(x,t) + \sum\nolimits_{i=1}^{m} u_i\, f_i(x,t) \,.$$

Given $t \in T$, each point $x \in Q$ satisfying with some $u \in R^m$ the

Karush-Kuhn-Tucker system

$$\begin{aligned} &D_x\, l(x,u,t) = 0 \quad , \quad f_i(x,t) = 0\, ,\ (i=1,\ldots,p)\ , \\ &f_j(x,t) \leqq 0\ ,\quad u_j \geq 0\ ,\quad u_j f_j(x,t) = 0\, ,\ (j=p+1,\ldots,m), \end{aligned} \tag{1.2}$$

is said to be a <u>stationary solution</u> of $P(t)$, in symbols: $x \in S(t)$. For each (x,t), the set of all vectors u with the property that (x,u,t) satisfies (1.2) will be denoted by $LM(x,t)$. A point $x \in M(t)$ is said to be a <u>local minimizer</u> of $P(t)$ if there is some neighborhood V of x such that $f_o(x,t) \leqq f_o(z,t)$ for all $z \in M(t) \cap V$ holds. A stationary solution x (or a local minimizer x) of $P(t)$ is called <u>isolated</u> if there is some neighborhood of x which does not contain any other stationary solution (or local minimizer) of $P(t)$. An isolated local minimizer of $P(t)$ is also <u>strict</u>, i.e., $f_o(x,t) < f_o(z,t)$ for all $z \in M(t) \cap V$, $z \neq x$.

In this paper, the notion of a strongly stable stationary solution plays a central role. Let $B(y,r)$ and $\overset{o}{B}(y,r)$ denote the closed and the open r-neighborhood of y, respectively, where we use the same notation no matter whether $y \in R^n$ or $y \in T$. Adapting Kojima's definition [15] to the parametric problem $\{P(t), t \in T\}$, we shall say that a stationary solution x^o of the problem $P(t^o)$ for fixed $t=t^o$ is <u>strongly stable</u> (w.r. to $\{P(t), t \in T\}$) if for some real number $r > 0$ and each $r' \in (0,r]$, there exists a real number $a=a(r')$ such that whenever $t \in B(t^o,a)$, $B(x^o,r')$ contains a stationary solution of the problem $P(t)$ which is unique in $B(x^o,r)$. A local minimizer which is also a strongly stable stationary solution will briefly be called a strongly stable local minimizer.

The concept of strong stability has been essentially used in homotopy methods, multi-level methods and statements on local convergence in nonlinear optimization, cf., for example, Guddat, Wacker and Zulehner [8], Jongen, Möbert and Tammer [11], Kojima [15], Lehmann [20]. It has been introduced and developed by Kojima [15] for optimization problems with twice differentiable data. We note that, in this case, strong stability is closely related to the concept of strong regularity in Robinson's sense [22], provided that the corresponding stationary solution satisfies the Linear Independence Constraint Qualification, we refer to [11].

In the case of non-C^2 or non-differentiable data there are several approaches to sensitivity studies in nonlinear programming via nonsmooth analysis. These concepts are often based on implicit function theorems for nonsmooth functions. Robinson [25] gives an implicit-function theorem for B-differentiable functions. Based on these ideas, Newton type methods for nonsmooth functions are developed, cf. Robinson [26] and Pang [21]. An implicit-function theorem for Lipschitzian mappings under the basic assumption that Clarke's [6] generalized Jacobian matrix is nonsingular is presented in Jongen, Klatte and Tammer [10]. It has applications in the sensitivity analysis of programs with C^2-data. Generalized Newton methods for various classes of nonsmooth functions are also given by Kojima and Shindo [16] and Kummer [18]. Second-order sufficient conditions for optimality and strong stability in $C^{1,1}$-optimization problems, by using Clarke's concept of a generalized Jacobian matrix, can be found in Klatte and Tammer [14] and Klatte [13], second-order necessary optimality conditions are presented in Hiriart-Urruty, Strodiot and Nguyen [9]. More general results concerning the sensitivity of local minimzers and stationary solutions in the non-C^2 case, but without aiming at the s t r o n g stability, are published, e.g., in Robinson [23], Alt [1], Auslender [2], Klatte [12] and Kummer [17].

The paper is organized as follows. In Section 2, we shall derive simple consequences of the strong stability of stationary solutions and local minimizers, using only first-order information. For motivation and application of strong stability we in particular give a theorem on iterated local minimization, extending a result of Jongen, Möbert and Tammer [11]. In Section 3, we present the main results of the paper: second-order sufficient conditions for a stationary solution to a program with $C^{1,1}$ data to be isolated or to be even a strongly stable local minimizer. Using Kummer's concept [19] of a set-valued directional derivative, we extend second-order conditions well-known for programs with twice differentiable data.

We have chosen a unified approach to both optimality and stability results. Finally, Section 4 discusses some particular cases of the (rather abstract) conditions given in Section 3.

Now we introduce some further notation. In what follows each $x \in R^k$ is considered to be a column vector, $x^T y$ is the scalar product of $x, y \in R^k$. If X and Y are subsets of R^k, then conv X (bd X , cl X) denote the convex hull (the boundary resp. the closure) of X, and, with $\beta \in R$, we write $\beta X + Y$ to denote the set $\{\beta x + y \;/\; x \in X,\ y \in Y\}$. For $x \in R^k$ and $X \subset R^k$ we often use the symbol $x + X$ instead of $\{x\} + X$. B_n and $\overset{o}{B}_n$ are the closed and the open unit ball of R^n. The linear space of (m,n)-matrices is identified with $R^{m \times n}$.

We use the symbols $C^1(Y)$, $C^1(Y,R^s)$, $C^2(Y)$ and $C^2(Y,R^s)$ to denote the classes of functions $f\colon Y \subset R^n \longrightarrow R$ or $F=(F_1,\dots,F_s)$ with $F_i\colon Y \subset R^n \longrightarrow R$ $(i=1,\dots,s)$, respectively, which are once or twice continuously differentiable on Y. By $Df(x)$, $DF(x)$ and $D^2 f(x)$ we symbolize the corresponding first and second derivatives, where $DF(x)$ is considered to be an (s,m)-matrix with the row vectors $DF_i(x)^T$ $(i=1,\dots,s)$. If f is a function of two variables x and y, we also take the notation $f(\cdot,\cdot)$, and we denote by $f(\cdot,y)$ the function $x \longmapsto f(x,y)$ for fixed y.

A multifunction $F\colon T \Longrightarrow R^n$ is said to be closed at t^o if $\limsup_{t \longrightarrow t^o} F(t) \subset F(t^o)$, or equivalently, if for any two sequences $\{t^k\} \subset T$ and $\{x^k\} \subset R^n$, $t^k \longrightarrow t^o$, $x^k \longrightarrow x^o$ and $x^k \in F(t^k)$ $(\forall k)$ imply that $x^o \in F(t^o)$. F is said to be locally bounded at t^o if for some neighborhood U of t^o, the union of all sets $F(t)$, $t \in U$, is a bounded set. A closed and locally bounded at t^o multifunction is also upper semicontinuous (u.s.c.) in Berge's sense, i.e., for each open set $Q \supset F(t^o)$ there is some neighborhood U of t^o such that $F(t) \subset Q$ holds for each $t \in U$. We shall say that F is closed (locally bounded, u.s.c.) on $T_o \subset T$ if F has this property at each element t of T_o. For a discussion of semicontinuity of multifunctions we refer, e.g., to the book [3], Section 2.2 .

2. Strong stability of stationary solutions under the Mangasarian-Fromovitz Constraint Qualification

Throughout this section we consider the parametric program $\{P(t),\ t\in T\}$ introduced above, and we suppose that the general assumption (1.1) is satisfied. We note that the analysis of perturbations via a parametric program also allows to treat special classes of perturbations, such as the classes $\mathcal{F}$ (C^2-perturbations of all data) and $\mathcal{F}^*$ (perturbation of the objective function by a quadratic function and right-hand side perturbations of the constraints) which appear in Kojima's paper [15]. This means that our studies of this section and of the following ones can be applied to many questions arising in programs with C^2 data, which are considered in [15],[22 , 23],[7],[11].

In Section 2, we first recall some basic sensitivity results for stationary solutions and local minimizers. Then we show that the property of strong stability of stationary solutions persists under small perturbations. Finally we give an interesting motivation and application of strong stability: the extension of a result of Jongen, Möbert and Tammer [11] on local iterated minimization, which is crucial for decomposition methods in nonconvex optimization. As a common regularity assumtion in these investigations, we require that the Mangasarian-Fromovitz Constraint Qualification holds at the points of interest.

Given for fixed $t=t^o$ the nonlinear program $P(t^o)$ introduced in §1, we shall say that $x^o\in M(t^o)$ satisfies the <u>Mangasarian-Fromovitz CQ</u> (w.r. to $M(t^o)$) if

(a) $D_xf_1(x^o,t^o),\dots,D_xf_p(x^o,t^o)$ are linearly independent, and

(b) there is some $h\neq 0$ satisfying $h^TD_xf_i(x^o,t^o)=0$, $i=1,\dots,p$, and $h^TD_xf_j(x^o,t^o)<0$ for all $j\in\{p+1,\dots,m\}$ with $f_j(x^o,t^o)=0$.

It is well-known that if x^o is a local minimizer of $P(t^o)$ which satisfies the Mangasarian-Fromovitz CQ, then $x^o\in S(t^o)$.

However, this CQ is also an important stability condition: Robinson [23, Th. 2.3] has shown the following basic properties of feasible points and stationary solutions of $P(t^o)$ under perturbations.

Proposition 2.1: Consider the parametric program $\{P(t), t \in T\}$, suppose (1.1), let $t^o \in T$ and $x^o \in M(t^o)$. Suppose that x^o satisfies the Mangasarian-Fromovitz CQ w.r. to $M(t^o)$.
Then there exist neighborhoods U_1 of t^o and V_1 of x^o such that for each $t \in U_1$ and for each $x \in M(t) \cap V_1$, x satisfies the Mangasarian-Fromovitz CQ w.r. to $M(t)$. Moreover, if $x^o \in S(t^o)$ then there are neighborhoods U_2 of t^o and V_2 of x^o such that the multifunctions

$$t \in U_2 \rightrightarrows S(t) \cap V_2 \quad \text{and} \quad (x,t) \in V_2 \times U_2 \rightrightarrows LM(x,t)$$

are closed and locally bounded (and hence u.s.c.) on U_2 and $V_2 \times U_2$, respectively.

Further, we recall a result on the stability of strict local minimizers under perturbations. It is, in fact, an adaptation of Berge's classical continuity theorems (cf., e.g., [3], §4.2) concerning global minimizing sets to the situation of local minimization. The formulation of the following proposition is a particular case of Th. 4.3 in Robinson [24] and of Th. 1 in [12]. For $X \subset R^n$ and $t \in T$, denote the set of all global minimizing points of $f_o(\cdot,t)$ subject to the feasible set $M(t) \cap X$ by $\operatorname{argmin}_x\{f_o(x,t) \ / \ x \in M(t) \cap X\}$.

Proposition 2.2: Consider the parametric program $\{P(t), t \in T\}$, assume (1.1), let $t^o \in T$, and let x^o be a strict local minimizer of $P(t^o)$ which satisfies the Mangasarian-Fromovitz CQ w.r. to $M(t^o)$. Then for some $\bar{r} > 0$ and for each $r \in (0, \bar{r}]$ there is some $a = a(r) > 0$ such that for each $t \in B(t^o, a)$, $X(t) := \operatorname{argmin}_x\{f_o(x,t) / x \in M(t) \cap B(x^o, r)\}$ is nonempty, and each element of $X(t)$ is a local minimizer of $P(t)$.

Note: By the first part of Proposition 2.1 and by the fact that under Mangasarian-Fromovitz CQ, a local minimizer is also a stationary solution, we have $X(t) \subset S(t)$ for $t \in B(t^o, a)$ if $\bar{r}$ is small.

Lemma 2.3: Consider $\{P(t), t\in T\}$, assume (1.1), let $t^o \in T$ and $x^o \in S(t^o)$. Suppose that x^o satisfies the Mangasarian-Fromovitz CQ w.r. to $M(t^o)$. Then x^o is strongly stable w.r. to $\{P(t), t\in T\}$ if and only if there are real numbers $r_o > 0$ and $a_o > 0$ and a mapping $x(\cdot)$: $B(t^o, a_o) \longrightarrow B(x^o, r_o)$ which is continuous on $B(t^o, a_o)$ and which fulfils

$$x(t^o) = x^o \quad \text{and} \quad S(t) \cap B(x^o, r_o) = \{x(t)\} \quad (\forall t\in B(t^o, a_o)). \qquad (2.1)$$

Proof: The "if"-direction of the proof is trivial. Now let U_2 and V_2 be as in Proposition 2.1, and let r_o be small enough such that $B(x^o, r_o) \subset V_2$. If x^o is strongly stable w.r. to $\{P(t), t\in T\}$, then there exists some $a(r_o)$ and some mapping $x(\cdot)$ with $x(t^o) = x^o$ and

$$B(x^o, r_o) \cap S(t) = \{x(t)\} \qquad (\forall t \in B(t^o, a(r_o))).$$

Choose $a_o < a(r_o)$ such that $B(t^o, a_o) \subset U_2$. Hence, by Proposition 2.1, $x(\cdot)$ is continuous on $B(t^o, a_o)$, and so the "only if"-direction of the lemma is shown. //

The very simple fact stated in Lemma 2.3 (i.e., continuity of $x(\cdot)$ at t^o implies continuity of $x(\cdot)$ in some neighborhood of t^o) turns out to be useful in many situations, such as in the proof of the following two theorems. The next theorem says that the strong stability property persists under small perturbations, provided that the Mangasarian-Fromovitz CQ holds. This fact has been already observed in the case of programs with twice differentiable data, cf. Robinson [22, Th. 2.4] and Kojima [15, Corollary 7.8]. However, our arguments use only first-order information.

Theorem 2.4: Consider $\{P(t), t\in T\}$, assume (1.1), let $t^o \in T$ and $x^o \in S(t^o)$. Suppose that x^o is strongly stable w.r. to $\{P(t), t\in T\}$ and satisfies the Mangasarian-Fromovitz CQ. Then there exist real numbers $r_1 > 0$ and $a_1 > 0$ and a continuous mapping $x(\cdot)$ from T to R^n with $x(t^o) = x^o$ such that for each $t' \in B(t^o, r_1)$, $x(t')$ is a stationary solution of $P(t')$ which is strongly stable w.r. to $\{P(t), t\in T\}$ too.

<u>Proof</u>: By Lemma 2.3, there are numbers $r_o > 0$, $a_o > 0$ and a continuous mapping $x(\cdot)$ from $B(t^o, a_o)$ to $B(x^o, r_o)$ satisfying (2.1). Choose a_o in such a way that for $t \in B(t^o, a_o)$, $x(t)$ satisfies the Mangasarian-Fromovitz CQ w.r. to $M(t)$; this can be done because of Proposition 2.1. Let $r_1 := \frac{1}{4} r_o$. By the continuity of $x(\cdot)$ there is some $0 < a_1 \leq a_o$ such that

$$x(t) \in S(t) \cap B(x^o, r_1) \quad \text{for all } t \in B(t^o, 2a_1).$$

Let $t' \in B(t^o, a_1)$ and $x' := x(t')$, hence $x' \in B(x^o, r_1)$. Then for each $t \in B(t', a_1)$, one also has $x(t) \in S(t) \cap B(x^o, r_1)$, and therefore $x(t) \in S(t) \cap B(x', 2r_1)$. On the other hand, since $B(x', 2r_1) \subset B(x^o, r_o)$ holds,

$$S(t) \cap B(x', 2r_1) = \{x(t)\} \quad (\forall t \in B(t', a_1))$$

follows. Using the "if"-part of Lemma 2.3 with x' instead of x^o and with $2r_1$ and a_1 instead of r_o and a_o, we obtain the desired result. //

In order to motivate the study of s t r o n g stability and, moreover, to show the applicability of the results which will be presented in the following sections, now we give a theorem on a general principle of iterated local minimization. It extends Th. 3.1 in [11]. We note that Theorem 2.5 does not remain true, when strong stability of x^o fails. An example illustrating this fact can be found in [11], §1 ; there the data are polynomial functions in two variables.

Given the functions $f_o, f_1, \ldots, f_m$ as above, we consider the optimization problem

$$(P): \qquad \min_{(x,t)} \left\{ f_o(x,t) \;\middle|\; \begin{array}{l} f_i(x,t) = 0\,,\ i=1,\ldots,p \\ f_j(x,t) \leq 0,\ j=p+1,\ldots,m \\ t \in T \end{array} \right\}$$

which is intended to be solved by a two-phases method, and where we look for local minimizers of (P). Further, let $P(t^o)$ and $\{P(t), t \in T\}$ be given as in Section 1, and suppose that the general assumption (1.1) is satisfied.

We emphasize that the following theorem holds without additional assumptions on T.

<u>Theorem 2.5</u>: Let t^o T, and let x^o be a local minimizer of $P(t^o)$. Suppose that x^o is a stationary solution of $P(t^o)$ being strongly stable w.r. to $\{P(t), t \in T\}$ and satisfying the Mangasarian-Fromovitz CQ. Further, let U be a neighborhood of t^o, and let $\bar{x}(\cdot): U \longrightarrow R^n$ be a vector function which is continuous at t^o and which fulfils $\bar{x}(t) \in S(t)$ for $t \in U$ and $\bar{x}(t^o) = x^o$.
Then (x^o, t^o) is a local minimizer of (P) if t^o is a local minimizer of the problem $(\tilde{P})$: $f_o(\bar{x}(t), t) \longrightarrow \min$ s.t. $t \in T$.

<u>Proof</u>: By the assumptions on x^o and by Lemma 2.3 there are real numbers $a_o > 0$ and $r_o > 0$ and a continuous mapping $x(\cdot)$ from $B(t^o, a_o)$ to $B(x^o, r_o)$ such that

$$x(t^o)=x^o \quad \text{and} \quad S(t) \cap B(x^o, r_o) = \{x(t)\} \ (\forall t \in B(t^o, a_o)). \qquad (2.2)$$

We may assume that U is a subset of $B(t^o, a_o)$, without loss of generality let $U = B(t^o, a_o)$. Hence, $\bar{x}(\cdot)$ and $x(\cdot)$ coincide on $B(t^o, a_o)$. Taking Proposition 2.1 and the continuity of $x(\cdot)$ into account, we may further assume that a_o and r_o are small enough to ensure that both the property (2.2) holds and for each $t \in B(t^o, a_o)$ and for each $x \in M(t) \cap B(x^o, r_o)$, the Mangasarian-Fromovitz CQ is satisfied at x w.r. to M(t).
In particular, it follows that x^o is a strict local minimizer of $P(t^o)$. Moreover, the continuity of $x(\cdot)$, Proposition 2.2 and the note following Proposition 2.2 provide that there exists some $a = a(r_o) \leq a_o$ such that for all $t \in B(t^o, a)$,

$$\emptyset \neq X(t) := \operatorname{argmin}_x \{f_o(x,t) \,/\, x \in M(t) \cap B(x^o, r_o)\} \subset S(t).$$

Thus, we obtain from (2.2)

$$X(t) = \{x(t)\} \quad \text{for all } t \in B(t^o, a),$$

and hence,

$$f_o(x(t), t) < f_o(x, t) \text{ for all } t \in B(t^o, a) \text{ and } x \in M(t) \cap B(x^o, r_o).$$

Since t^o is a local minimizer of $(\tilde{P})$, there is some neighborhood U_o of t_o, $U_o \subset B(t^o, a)$, such that

$$f_o(x(t^o), t^o) \leq f_o(x(t), t)$$

for all $t \in U_o$, and so we have for all $t \in U_o$ and for all x with

$x \in M(t) \cap B(x^o, r_o)$, i.e., for all feasible points (x,t) of (P) which belong to the neighborhood $U_o \subset B(x^o, r_o)$ of (x^o, t^o),

$$f_o(x^o, t^o) = f_o(x(t^o), t^o) \leq f_o(x(t), t) < f_o(x, t). \qquad (2.3)$$

This completes the proof. //

By (2.3), we have that, under the assumptions of Theorem 2.5, (x^o, t^o) is even a strict local minimizer of (P). A careful inspection of the proof shows that the differentiability assumptions on $f_i(\cdot, t)$ could be omitted, if we would require that for each t near t^o, $x(t)$ is a local minimizer of $P(t)$ being isolated in some neighborhood of x^o (independent of t). In order to remain within the framework of this paper, we have preferred the formulation used above.

3. Second-order sufficient conditions for optimality and strong stability

The main purpose of this section is to give a second-order sufficient condition for strong stability of local minimizers to nonlinear optimization problems, avoiding the assumption of twice differentiability of the problem data. Before presenting this result, we shall study the related question of second-order sufficient optimality conditions. Using a concept of a set-valued directional derivative for Lipschitzian mappings (cf. [19]) and assuming generalized second-order conditions, we extend existence and stability results which are known from the case of nonlinear programs with twice differentiable data, cf., e.g., Fiacco and McCormick [7], Robinson [22, 23], Kojima [15]) to $C^{1,1}$-optimization problems. Concerning $C^{1,1}$-programs our approach allows to modify and to generalize the results in [13] and [14]. Similar to Section 2, we again use the Mangasarian-Fromovitz CQ as first-order regularity condition if necessary.

Given an open set $Y \subset R^q$, $C^{1,1}(Y)$ will denote the class of all functions $f: Y \longrightarrow R$ which are differentiable on Y and whose gradient mapping $Df(\cdot)$ is locally Lipschitzian on Y.

Throughout this section we consider the parametric program $\{P(t), t \in T\}$ introduced in Section 1, and we suppose that (1.1) holds and that the following assumption is additionally satisfied:

$$Q \text{ is convex and } f_i(\cdot,t) \in C^{1,1}(Q) \quad (\forall i \in \{0,1,\dots,m\}\ \forall t \in T). \quad (3.1)$$

The convexity of the open set Q is required in view of the use of some second-order Taylor expansion. It is easy to verify that, under (3.1), for all $t \in T$ the Lagrange function $l(\cdot,\cdot,t)$ belongs to $C^{1,1}(Q \times R^m)$. In order to analyze the stability of the Karush-Kuhn-Tucker system of P(t) under (1.1) and (3.1), we need some concept of generalized derivative of vector functions. In this context, Clarke's concept [6] of a generalized Jacobian matrix was used in [9], [13] and [14]: Given some open set $Y \subset R^q$ and a mapping $F: Y \longrightarrow R^d$ which is locally Lipschitzian on Y (i.e., for each $x \in Y$ there is some neighborhood V_x of x and some modulus $L(x) > 0$ such that for all x', x'' in V_x, $\| F(x')-F(x'')\| \leq L(x) \|x'-x''\|$), the set of (d,q)-matrices

$$J_{Cl}F(x^o) := \text{conv}\left\{M: \exists x^k \longrightarrow x^o \text{ with } x^k \in E_F\ (\forall k),\ DF(x^k) \longrightarrow M\right\}$$

is called the <u>generalized Jacobian matrix</u> of F at $x^o \in Y$ (in Clarke's sense), where $E_F \subset Y$ denotes the set of all points x for which the usual Jacobian DF(x) exists. The idea and the justification of this concept is given by Rademacher's theorem which ensures that a locally Lipschitzian mapping is almost everywhere differentiable on its domain. We note that $J_{Cl}F(x^o)$ is a nonempty compact convex subset of $R^{d \times q}$, the multifunction $J_{Cl}F(\cdot)$ is closed and locally bounded at x^o, and if F is continuously differentiable at x^o then $J_{Cl}F(x^o) = \{DF(x^o)\}$, cf. Clarke [6, §2.6].

Recently, in [19], the following notion of a set-valued (generalized) directional derivative of a continuous function $F: R^q \longrightarrow R^d$ was introduced. The set

$$F(x^o;h) := \left\{ z: \begin{array}{l} \exists x^k \longrightarrow x^o \quad \exists \lambda_k \longrightarrow +0 \quad \text{with} \\ \lambda_k^{-1}(F(x^k + \lambda_k h) - F(x^k)) \longrightarrow z \end{array} \right\}$$

is called the <u>directional derivative of F at x^o in direction h</u>. For simplicity, we use the notation

$$v^T \Delta F(x;h) := \{ v^T z \ / \ z \in \Delta F(x;h) \}$$

if $(x,h,v) \in R^q \times R^q \times R^d$, and we also write $v^T \Delta F(x;h) \geq c$ (with $c \in R$) to symbolize that $v^T z \geq c$ for all $z \in \Delta F(x;h)$ holds.

In the following we summarize several properties of this directional derivative, the proofs can be found in [19]. Let $C^{o,1}(Y,R^d)$ denote the set of all functions $F\colon Y \times R^q \longrightarrow R^d$ which are locally Lipschitzian on Y. Given $F,G \in C^{o,1}(Y,R^d)$, $Y \subset R^q$ open, $x \in Y$, $h \in R^q$, the following properties hold:

(P 1) $\Delta F(x; \beta h) = \beta \Delta F(x;h)$ for $\beta \geq 0$,
$\Delta (F+G)(x;h) \subset \Delta F(x;h) + \Delta G(x;h)$;

(P 2) $\Delta F(x;h)$ is nonempty and compact,
$\Delta F(\cdot;\cdot)$ is closed and locally bounded at (x,h);

(P 3) if $\tilde{G} \in C^{o,1}(Y,R^d)$, $\tilde{F}(x,u) := u^T \tilde{G}(x)$ $(\forall (x,u) \in Y \times R^d)$,
if $(\bar{x},\bar{u}) \in Y \times R^d$, $(h,0) \in R^q \times R^d$, then $F \in C^{o,1}(Y \times R^d, R)$
and $\Delta(F(\cdot,\bar{u}))(\bar{x};h) = \Delta(F(\cdot,\cdot))((\bar{x},\bar{u});(h,0))$;

(P 4) $\Delta F(x;-h) = -\Delta F(x;h)$;

(P 5) $\Delta F(x;h) \subset (J_{Cl}F(x))\,h := \{ Mh \ / \ M \in J_{Cl}F(x) \}$;

(P 6) if $F \in C^1(Y,R^d)$, then $F(x;h) = \{ DF(x)\,h \}$;

(P 7) if F has a (local) Lipschitz modulus $L(x)$ to some neighborhood V of x, then $\Delta F(x;h') \subset \Delta F(x;h'') + L(x)\,\|h'-h''\|\,B_d$ holds for all $h',h'' \in R^d$.

Based on a mean-value theorem for $C^{o,1}$-mappings, a second-order Taylor expansion for $C^{1,1}$-functions holds, namely

<u>Lemma 3.1 ([19, Proposition 5.1])</u>: Let Y be any open subset of R^q, let $f \in C^{1,1}(Y)$ and let $\mathrm{conv}\{x,x+h\} \subset Y$. Then there is some $\theta \in (0,1)$ such that

$$f(x+h) \in f(x) + Df(x)\,h + \tfrac{1}{2}\, h^T \Delta Df(x+\theta h\,;\,h).$$

Now we pass over to the presentation of second-order conditions. Considering the parametric optimization problem $\{P(t), t \in T\}$, we put for $(x,u,t) \in Q \times R^m \times T$,

$$\begin{aligned}
I_2(x,t) &:= \left\{ j \in \{p+1,\dots,m\} / \ f_j(x,t) = 0 \right\}, \\
I(x,t) &:= \left\{ 1,\dots,p \right\} \cup I_2(x,t), \\
I_2^+(u) &:= \left\{ j \in \{p+1,\dots,m\} / \ u_j > 0 \right\}, \\
I^+(u) &:= \left\{ 1,\dots,p \right\} \cup I_2^+(u), \\
W^+(x,u,t) &:= \left\{ h \in R^n / \ h^T D_x f_i(x,t) = 0, \ i \in I^+(u) \right\}, \\
W(x,u,t) &:= \left\{ h \in W^+(x,u,t) / \ h^T D_x f_j(x,t) \leq 0, \ j \in I_2(x,t) \setminus I_2^+(u) \right\}.
\end{aligned}$$

Now we formulate two types of second-order sufficient conditions for optimality or strong stability, respectively. The first condition is an immediate extension of the usual second-order sufficient optimality condition for C^2 data, cf., e.g., Fiacco and McCormick [7], Robinson [23].
Let $l(\cdot,u^o,t^o)$ denote the function $x \in Q \longmapsto l(x,u^o,t^o)$ for fixed $(u^o,t^o) \in R^m \times T$.

<u>Condition 3.2</u>: Given $P(t^o)$ for $t^o \in T$, $x^o \in S(t^o)$ and $u^o \in LM(x^o,t^o)$, we shall say that (x^o,u^o) satisfies Condition 3.2 with modulus $c > 0$ if for each vector h with $h \in W(x^o,u^o,t^o)$, one has

$$h^T \ \Delta(D_x l(\cdot,u^o,t^o))(x^o;h) \geq c \ \| h \|^2 .$$

The condition introduced next is a uniform strong second-order regularity condition which is, in the case of C^2 data, related to the corresponding conditions of Robinson [22] and Kojima [15 , Condition 7.3].

<u>Condition 3.3</u>: Given $\{P(t), t \in T\}$, $t^o \in T$ and $x^o \in S(t^o)$, we shall say that Condition 3.3 holds on $\{x^o\} \times LM(x^o,t^o)$ with modulus $c > 0$ if there exist a neighborhood U of t^o, a neighborhood V of x^o and open sets $N \supset LM(x^o,t^o)$ and $W \supset W^+(x^o,u^o,t^o) \cap \mathrm{bd} B_n$ such that one has

$$h^T \Delta(D_x l(\cdot,u,t))(x;h) \geq c \quad \text{for all } (x,u,t,h) \in V \times N \times U \times W.$$

Obviously, if Condition 3.3 holds on $\{x^o\} \times LM(x^o,t^o)$, then for each $u^o \in LM(x^o,t^o)$, (x^o,u^o) satisfies Condition 3.2.

The following technical lemma allows a unified approach to derive the second-order existence and stability results of this section. The proof is modeled after an idea used by Robinson [23 , Theorems 2.2 and 2.4] in the case of C^2 data.

<u>Lemma 3.4</u>: Consider the parametric program $\{P(t), t \in T\}$, assume (1.1) and (3.1). Given $t^o \in T$, $x^o \in S(t^o)$ and $u^o \in LM(x^o,t^o)$, let $\{t^k\} \subset T$, $\{x^k\}$, $\{y^k\} \subset Q$ and $\{u^k\} \subset R^m$ be any sequences such that

$$x^k \in S(t^k)\ ,\ u^k \in LM(x^k,t^k) \quad \text{and} \quad y^k \in M(t^k) \quad \text{for all } k$$

hold, and such that

$$(x^k,u^k,t^k) \longrightarrow (x^o,u^o,t^o) \quad \text{and} \quad y^k \longrightarrow x^o$$

are fulfilled. Moreover, suppose that for some positive real number c and for all k the following holds:

$$f_o(y^k,t^k) - f_o(x^k,t^k) < \frac{c}{4} \| y^k - x^k \|^2.$$

Then the sequence $\{h^k\}$ with $h^k := \| y^k - x^k \|^{-1}(y^k - x^k)$ has an accumulation point $h \in W^+(x^o,u^o,t^o)$, and for all k, there are real numbers $\theta_k > 0$ and vectors $z^k \in R^n$ such that $\theta_k \longrightarrow +0$ and

$$z^k \in \Delta\,(D_x l(\cdot,u^k,t^k))(x^k + \theta_k h^k; h^k) \quad \text{and} \quad {h^k}^T z^k < \frac{c}{2}\ . \tag{3.2}$$

Further, if $t^k \equiv t^o$ and $x^k \equiv x^o$, then $\{h^k\}$ even has an accumulation point in $W(x^o,u^o,t^o)$.

<u>Proof</u>: First we show that $\{h^k\}$ has an accumulation point h belonging to $W^+(x^o,u^o,t^o)$. Since $\{h^k\} \subset \text{bd } B_n$, we may assume, without loss of generality, that $\{h^k\}$ converges to some $h \in \text{bd } B_n$. By the continuity of the functions $f_1,\dots,f_m$, the assumption $(x^k,u^k,t^k) \longrightarrow (x^o,u^o,t^o)$ implies that

$$I^+(u^o) \subset I^+(u^k) \subset I(x^k,t^k) \subset I(x^o,t^o) \quad \text{for k large.} \tag{3.3}$$

For $j \in I^+(u^o)$ and for sufficiently large k, we thus obtain

$$f_j(y^k,t^k) = (y^k - x^k)^T D_x f_j(x^k,t^k) + o(\| y^k - x^k \|). \tag{3.4}$$

Since $h^k \longrightarrow h$ and $y^k \in M(t^k)$ $(\forall k)$, the continuity of $D_x f_i(\cdot,\cdot)$, $i=1,\dots,m$, then yields that

$$h^TD_xf_i(x^o,t^o) = 0,\ i=1,\dots,p\ ;\quad h^TD_xf_j(x^o,t^o) \leq 0,\ j \in I_2^+(u^o) \tag{3.5}$$

As $(x^o,u^o) \in S(t^o) \times LM(x^o,t^o)$, thus we have, with $J:=I^+(u^o)$,

$$\begin{aligned} & h^TD_xf_o(x^o,t^o) \\ \geq\ & h^TD_xf_o(x^o,t^o) + \sum_{j \in J} h^TD_xf_j(x^o,t^o) \\ =\ & h^TD_xl(x^o,u^o,t^o) \\ =\ & 0\ . \end{aligned}$$

Further, by hypothesis, we know that for all k,

$$\frac{c}{4}\|y^k-x^k\|^2 > f_o(y^k,t^k)-f_o(x^k,t^k) = (y^k-x^k)^TD_xf_o(x^k,t^k) + o(\|y^k-x^k\|)$$

which implies

$$h^TD_xf_o(x^o,t^o) \leq 0,$$

where $h^k \to h$, $y^k - x^k \to 0$ and the continuity of $D_xf_o(\cdot,\cdot)$ were taken into account. Hence,

$$\sum_{j \in J} u_j^o\ h^TD_xf_j(x^o,t^o) = 0,$$

and so, by (3.5) and in view of $u_j^o > 0$ for $j \in I_2^+(u^o)$,

$$h^TD_xf_j(x^o,t^o) = 0\ ,\quad j \in J\ .$$

Thus, we have shown $h \in W^+(x^o,u^o,t^o)$ with $h \in \mathrm{bd}\ B_n$.

At this place, we note that in the case $(x^k,t^k) \equiv (x^o,t^o)$ one has for all $j \in I_2(x^o,t^o) \setminus I_2^+(u^o)$,

$$0 \geq f_j(y^k,t^o) = (y^k - x^o)^T D_x f_j(x^o,t^o) + o(\|y^k - x^o\|) \qquad (\forall k),$$

which implies, by arguments similar to those used above,

$$h^T D_x f_j(x^o,t^o) \leq 0\ ,\quad j \in I_2(x^o,t^o) \setminus I_2^+(u^o).$$

This means that in our special case $h \in W(x^o,u^o,t^o)$ holds.

Now we show (3.2). By hypothesis, $\mathrm{conv}\ \{x^k,y^k\} \subset Q \quad (\forall k)$. Let k be fixed. For simplicity, we put $l_k := l(\cdot,u^k,t^k)$, and we denote by $H(x;\hat{h})$ the set $\Delta(Dl_k)(x;\hat{h})$ of directional derivatives of Dl_k at x in direction $\hat{h}$. Assumption (3.1) then allows a second-order Taylor expansion of l_k at x^k according to Lemma 3.1. By hypothesis and taking $y^k \in M(t^k)$, $x^k \in S(t^k)$ and

$u^k \in LM(x^k,t^k)$ into account, Lemma 3.1 hence implies the existence of some $\tilde{\theta}_k \in (0,1)$ and of some $\tilde{z}^k \in H(x^k+\tilde{\theta}_k(y^k-x^k);y^k-x^k)$ such that

$$\begin{aligned}\frac{c}{4}\| y^k - x^k \|^2 &> f_o(y^k,t^k) - f_o(x^k,t^k) \\ &\geq l_k(y^k) - l_k(x^k) \\ &= (y^k - x^k)^T D\, l_k(x^k) + \frac{1}{2}(y^k - x^k)^T \tilde{z}^k \\ &= \frac{1}{2}(y^k - x^k)^T \tilde{z}^k .\end{aligned}$$

Setting $\theta_k := \tilde{\theta}_k \| y^k - x^k \|$, we obtain, by property (P 1) of directional derivatives,

$$H(x^k+\tilde{\theta}_k(y^k-x^k) ; y^k-x^k) = \| y^k-x^k \| \; H(x^k+\theta_k h^k ; h^k),$$

and so, with $z^k := \| y^k - x^k \|^{-1} \tilde{z}^k$, the relations

$$z^k \in H(x^k + \theta_k h^k ; h^k) \qquad \text{and} \qquad {h^k}^T z^k < \frac{c}{2}$$

follow. Obviously, $(y^k - x^k) \longrightarrow 0$ implies that $\theta_k \longrightarrow +0$, hence (3.2) is shown. //

In the following theorem, Condition 3.2 turns out to be a second-order sufficient optimality condition for $C^{1,1}$-optimization problem. This theorem modifies a result in [14] and generalizes known results in the C^2 case which is discussed in Section 4 below.

<u>Theorem 3.5</u>: Consider for fixed $t^o \in T$ the nonlinear program $P(t^o)$ introduced in Section 1. Suppose that the functions $f_i(\cdot,t^o): Q \longrightarrow R$ $(i=0,1,\ldots,m)$ belong to the class $C^{1,1}(Q)$, where Q is some open convex subset of R^n.
If $(x^o,u^o) \in Q \times R^m$ satisfies both the Karush-Kuhn-Tucker conditions (1.2) with $t=t^o$ and Condition 3.2 with some modulus $c > 0$, then there exists a real number r 0 such that

$$f_o(x,t^o) - f_o(x^o,t^o) \geq \frac{c}{4}\| x - x^o \|^2 \quad (\forall x \in M(t^o) \cap B(x^o,r)) \qquad (3.6)$$

holds, i.e., x^o is a strict local minimizer with order 2 of $P(t^o)$.

Proof: If (3.6) is not true, then we have the situation of Lemma 3.4 in the case $(x^k, u^k, t^k) \equiv (x^o, u^o, t^o)$ with some sequence $\{y^k\}$ satisfying $y^k \in M(t^o)$ for all k and $y^k \longrightarrow x^o$. Hence, the sequence $\{h^k\}$ with $h^k := \| y^k - x^o \|^{-1}(y^k - x^o)$ has an accumulation point $h \in W(x^o, u^o, t^o) \cap \text{bd } B_n$, and there exist sequences $\{\theta_k\} \subset R$ and $\{z^k\} \subset R^n$ such that $\theta_k \longrightarrow +0$ and such that for all k

$$z^k \in \Delta(D_x l(\cdot, u^o, t^o))(x^o + \theta_k h^k; h^k) \quad \text{and} \quad h^{k^T} z^k < \frac{c}{2} .$$

By property (P 2) of directional derivatives, $\{z^k\}$ has an accumulation point z in $\Delta(D_x l(\cdot, u^o, t^o))(x^o; h)$, and hence

$$h^T z \leq \frac{c}{2} < c$$

holds, and the theorem now follows by contraposition. //

However, Theorem 3.5 does not give an answer to the question whether the strict local minimizer x^o is also an isolated one. In general, the assumptions of Theorem 3.5 are not sufficient to ensure that there is some neighborhood of x^o in which no other local minimizer of $P(t^o)$ exists: Robinson's counter-example [23 , p.206] presented in the case of programs with C^2-data also applies to our problem. As in the C^2 case one has to add a constraint qualification and to require that Condition 3.2 is satisfied on $\{x^o\} \times LM(x^o, t^o)$.

Corollary 3.6: Assume the hypotheses of Theorem 3.5, and further suppose that x^o satisfies the Mangasarian-Fromovitz CQ. If for each $u^o \in LM(x^o, t^o)$, (x^o, u^o) satisfies Condition 3.2 with some modulus $c(x^o, u^o) > 0$, then x^o is an isolated stationary solution of $P(t^o)$.

Note: Since the Mangasarian-Fromovitz CQ is satisfied at x^o, by Proposition 2.1, then x^o is also an isolated local minimizer of $P(t^o)$.

Proof: By contraposition. Suppose there is some sequence $\{v^k\} \subset S(t^o)$ with $v^k \neq x^o$ for all k and $v^k \longrightarrow x^o$. Since x^o is a strict local minimizer of $P(t^o)$ because of Theorem 3.5, then there is some index k' such that

$$f_o(v^k,t^o) > f_o(x^o,t^o) \quad \text{for all } k \geq k'.$$

For each k, let u^k be a Lagrange multiplier vector of $P(t^o)$ associated with v^k. Since the mapping $x \Longmapsto LM(x,t^o)$ is closed and locally bounded at x^o (Proposition 2.1), then by passing to a subsequence if necessary we have

$$u^k \longrightarrow u^o \in LM(x^o,t^o).$$

Now we can apply Lemma 3.4 (put there $c=c(x^o,u^o)$, $t^k \equiv t^o$, $x^k := v^k$, $y^k \equiv x^o$ for all $k \geq k'$), and we obtain that the sequence $\{h^k\}$ with $h^k := \| x^o - v^k \|^{-1}(x^o - v^k)$ has an accumulation point $h \in W^+(x^o,u^o,t^o) \cap \text{bd } B_n$, and there are sequences $\{\theta_k\} \subset R$ and $\{z^k\} \subset R^n$ such that $\theta_k \longrightarrow +0$ and such that for k sufficiently large

$$z^k \in \Delta(D_x l(\cdot,u^k,t^o))(v^k + \theta_k h^k; h^k) \quad \text{and} \quad h^{k^T} z^k < \frac{c}{2}$$

hold. Hence, the properties (P 2) and (P 3) of directional derivatives ensure the existence of some

$$z \in \Delta(D_x l(\cdot,u^o,t^o))(x^o;h) \quad \text{with} \quad h^T z \leq \frac{c}{2} \ .$$

By property (P 4),

$$-z \in \Delta(D_x l(\cdot,u^o,t^o))(x^o;-h) \quad \text{with} \quad (-h)^T(-z) \leq \frac{c}{2} \tag{3.7}$$

holds. Obviously, we have $-h \in W^+(x^o,u^o,t^o)$. Moreover, taking

$$0 \geq f_j(v^k,t^o) = (v^k - x^o)^T D_x f_j(x^o,t^o) + o(\| v^k - x^o \|)$$

(for all k and all $j \in I(x^o,t^o)$) into account and passing to the limit, we obtain that

$$(-h)^T D_x f_j(x^o,t^o) \leq 0 \quad \text{for all } j \in I(x^o,t^o)$$

is fulfilled. Hence,

$$-h \in W(x^o,u^o,t^o) \cap \text{bd } B_n.$$

Putting this and (3.7) together, we find a contradiction to Condition 3.2 and thereby complete the proof. //

We note that Corollary 3.6 is a modification and extension of Theorem 2 in [14].

Now we prove the main result of the paper: the strong stability of local minimizers of $C^{1,1}$-programs under the Mangasarian-Fromovitz CQ and under Condition 3.3. However, Condition 3.3 looks rather strong and hardly practicable, but we had to by-pass the difficulty that the "partial directional Hessian" $\Delta(D_x l(\cdot,u,t))(x;h)$ is not in general u.s.c. w.r. to all variables (x,u,t,h). The discussion in Section 4 will provide several specializations and simplifications which make more plausible and better usuable this second-order condition.

Theorem 3.7: Consider the parametric program $\{P(t), t \in T\}$, and suppose (1.1) and (3.1). Given $t^o \in T$, let x^o be a stationary solution of $P(t^o)$. Suppose that x^o satisfies the Mangasarian-Fromovitz CQ w.r. to $M(t^o)$ and that Condition 3.3 holds on $\{x^o\} \times LM(x^o,t^o)$ with some modulus $c_o > 0$. Then

(1) x^o is strongly stable w.r. to $\{P(t), t \in T\}$,

and there exist real numbers $r > 0$ and $a > 0$ and a mapping $x(\cdot)$ from T to R^n such that for each $t \in B(t^o,a)$, $S(t) \cap B(x^o,r) = \{x(t)\}$ and

(2) $f_o(x,t) - f_o(x(t),t) \geq \frac{c_o}{2} \| x - x(t) \|^2$
for all $x \in M(t) \cap B(x(t),r)$,

(3) $x(t)$ is a strongly stable local minimizer of $P(t)$.

Proof: By Theorem 3.5, x^o is a strict local minimizer of $P(t^o)$. Consequently, the assumptions of Proposition 2.2 and of the note following Proposition 2.2 are satisfied. This entails that for some $r' > 0$ and each $s \in (0,r']$ there exists some $a(s) > 0$ such that for $t \in B(t^o,a(s))$, $S(t) \cap B(x^o,s)$ is nonempty. Later on, this fact will be indicated by (+).

To show (1) and (2) it is sufficient to prove that for some $r > 0$ with $r \leq r'$ and some $a > 0$ with $a \leq a(r')$, the inequality (3.8) holds:

$$\begin{aligned} & f_o(x,t) - f_o(z,t) \geq (\tfrac{1}{2} c_o) \; \| x - z \|^2 \\ & \text{for all} \quad t \in B(t^o,a) \\ & \text{and all} \quad z \in S(t) \cap B(x^o,r) \\ & \text{and all} \quad x \in M(t) \cap B(x^o,2r). \end{aligned} \tag{3.8}$$

Assume, for the moment, (3.8) is shown. Then for each $t \in B(t^o,a)$ and any two points $x^1(t)$, $x^2(t) \in S(t) \cap B(x^o,r)$ with $x^1(t) \neq x^2(t)$, we have

$$f_o(x^1(t),t) - f_o(x^2(t),t) \geq (\tfrac{1}{2} \; c_o) \; \| x^1(t) - x^2(t) \|^2$$

and

$$f_o(x^2(t),t) - f_o(x^1(t),t) \geq (\tfrac{1}{2} \; c_o) \; \| x^1(t) - x^2(t) \|^2,$$

which is impossible. Thus, for each $t \in B(t^o,a)$, there is some point $x(t)$ such that

$$S(t) \cap B(x^o,r) = \{x(t)\} \; .$$

Property (+) derived before yields that $x(\cdot)$ is continuous at x^o, hence (1) is shown. Since $x \in M(t) \cap B(x(t),r)$ for $t \in B(t^o,a)$ belongs to $M(t) \cap B(x^o,2r)$, assertion (2) is a special case of (3.8).

Now we complete the proof of (1) and (2) by demonstrating (3.8). If (3.8) is not true, then there exist sequences $\{t^k\} \subset T$, $\{x^k\}$ and $\{y^k\}$ such that $x^k \in S(t^k)$ and $y^k \in M(t^k)$ for all k and both $\{x^k\}$ and $\{y^k\}$ converge to x^o, and such that for all k

$$f_o(y^k,t^k) - f_o(x^k,t^k) < (\tfrac{1}{2} c_o) \; \| y^k - x^k \|^2 \; .$$

For each k, let $u^k \in LM(x^k,t^k)$. Due to Proposition 2.1, the Mangasarian-Fromovitz CQ implies that $LM(\cdot,\cdot)$ is closed and locally bounded at (x^o,t^o). By using this fact and by passing to a subsequence if necessary, we have that $\{u^k\}$ converges to some $u^o \in LM(x^o,t^o)$. Put $c := 2\, c_o$, then Lemma 3.4 applies to our situation. Using the same notation as in the statement of Lemma 3.4, we have that for sufficiently large k,

$$x^k + \theta_k h^k \in V \, , \; u^k \in N \, , \; t^k \in U \quad \text{and} \quad h^k \in W$$

and property (3.2) hold, where V,N,U and W are taken from Condition 3.3. However, this provides us with a contradiction to Condition 3.3. Hence (3.8) and so (1) and (2) are shown.

Finally, we note that (3) is an immediate consequence of (1) and (2), one has to apply Theorem 2.4. This completes the proof. //

4. A discussion of second-order sufficient conditions

In this section we discuss how to replace the uniform strong second-order condition formulated in Condition 3.3 by requirements which contain only information taken from the initial problem $P(t^o)$. Further, we recall a special class of $C^{1,1}$-optimization problems for which the verification of the Conditions 3.2 and 3.3 reduces to checking whether finitely many matrices are positive definite.

Throughout this section we consider the parametric problem $\{P(t), t \in T\}$ introduced in Section 1, and we suppose that (1.1) and (3.1) are satisfied. Now we study a series of special cases.

4.1. We recall that the complicated form of Condition 3.3 is due to the fact that the multifunction which assigns to each (x,u,t,h) the set $\Delta(D_x l(\cdot,u,t))(x;h)$ is not u.s.c., in general. We can meet this difficulty even in the case that the mapping $Dl(\cdot,\cdot,\cdot)$ is Lipschitz continuous with respect to the triple (x,u,t) of variables (and $T \subset R^k$), cf. an example in [19]. However, we succeed in by-passing this difficulty and in formulating a second-order condition in terms of the initial problem, if, for example, an imbedding of this "bad" multifunction into a suitable u.s.c. multifunction is possible:

Let $t^o \in T$, $x^o \in S(t^o)$ and suppose that for some bounded open set $N \supset LM(x^o,t^o)$, some open set W containing

$$\bigcup_{u^o \in LM(x^o,t^o)} \left(W^+(x^o,u^o,t^o) \cap \text{bd } B_n\right)$$

and some multifunction

$$H: Q \times N \times T \times W \Longrightarrow R^n$$

the following hold:

H is closed and locally bounded on $\{x^o\} \times LM(x^o,t^o) \times \{t^o\} \times \text{bd } B_n$

$$(4.1)$$

and

$$\Delta(D_x l(\cdot,u,t))(x;h) \subset H(x,u,t,h) \quad (\forall (x,u,t,h) \in Q \times N \times T \times W). \tag{4.2}$$

<u>Condition 3.3'</u>: For each $u^o \in LM(x^o,t^o)$, for each $h \in W^+(x^o,u^o,t^o) \cap \text{bd } B_n$ and for each $z \in H(x^o,u^o,t^o,h)$, one has $h^T z > 0$.

<u>Proposition 4.1</u>: Assume (4.1) and (4.2). Then Condition 3.3' and Condition 3.3 are equivalent.

<u>Proof</u>: It suffices to show that Condition 3.3' implies Condition 3.3. Indeed, the general assumptions (1.1) and the boundedness of the set N ensure that $LM(x^o,t^o)$ is a compact set. By (3.3), the multifunction $W^+(x^o,\cdot,t^o)$ is closed on $LM(x^o,t^o)$, hence

$$W_o := \bigcup_{u \in LM(x^o,t^o)} (W^+(x^o,u,t^o) \cap \text{bd } B_n)$$

is a compact set. By (4.1), H is closed and locally bounded on $\{x^o\} \times LM(x^o,t^o) \times \{t^o\} \times W_o$, thus

$$H_o := \bigcup_{u \in LM(x^o,t^o)} \bigcup_{h \in W_o} H(x^o,u,t^o,h)$$

is a compact set too. Consequently, there exist open sets $W_1 \supset W_o$ and $H_1 \supset H_o$ and some $c > 0$ such that

$$h^T z \geq c \quad \text{for all } h \in W_1 \text{ and for all } z \in H_1. \tag{4.3}$$

Since (4.1) includes that H is u.s.c. on $\{x^o\} \times LM(x^o,t^o) \times \{t^o\} \times W_o$, there are neighborhoods V of x^o and U of t^o and open sets $N_1 \supset LM(x^o,t^o)$ and $W_2 \supset W_o$ such that

$$H(x,u,t,h) \subset H_1 \quad (\forall (x,u,t,h) \in V \times N_1 \times U \times W_2).$$

Hence, (4.3) and (4.2) imply that

$$h^T \Delta(D_x l(\cdot,u,t))(x;h) \geq c$$

holds for each $(x,u,t,h) \in (V \quad Q) \times (N_1 \cap N) \times U \times (W_2 \cap W)$, i.e., Condition 3.3 is satisfied on $\{x^o\} \times LM(x^o,t^o)$ with modulus c. //

4.2. Now we consider the case of twice differentiable data. The given parametric program satisfies, as assumed above, the requirements (1.1). Additionally, we suppose that for each $i \in \{0,1,\dots,m\}$,

$$f_i(\cdot,t) \text{ is twice differentiable on } Q \quad (\forall t \in T), \tag{4.4}$$

$$D_x^2 f_i(\cdot,\cdot) \text{ is continuous on } Q \times T. \tag{4.5}$$

By property (P 6) of directional derivatives, then we have for $(x,u,t,h) \in Y \times R^m \times T \times R^n$,

$$h^T \Delta(D_x l(\cdot,u,t))(x;h) = \{h^T D_x^2 l(x,u,t)\, h\},$$

which immediately implies that Condition 3.2 reduces to the well-known second-order sufficient optimality condition in the standard book of Fiacco and McCormick [7].

Moreover, (4.1) and (4.2) are automatically fulfilled with $H(x,u,t,h) := \{h^T D_x^2 l(x,u,t)\, h\}$ and with any bounded open set $N \supset LM(x^o,u^o)$ (provided that $LM(x^o,t^o)$ is bounded, which is equivalent to the assumption that the Mangasarian-Fromovitz CQ holds at x^o) and $W=R^n$. Thus, Condition 3.3 passes to a special version of Condition 3.3' which is also known, cf. Robinson [22 , §4] and Kojima [15 , Condition 7.3] .

4.3. The previous remarks immediately allow to specify Condition 3.3 in the case that a $C^{1,1}$-optimization problem is perturbed by C^2-functions. For the given parametric program, consider the case that for each $(x,t) \in Q \times T$ and for each $i \in \{0,1,\dots,m\}$, f_i has the representation

$$f_i(x,t) = \bar{g}_i(x) + g_i(x,t), \tag{4.6}$$

where g_i satisfies the assumptions (1.1), (4.4) and (4.5), and $\bar{g}_i : Q \longrightarrow R$ belongs to the class $C^{1,1}(Q)$. Then we have, obviously,

$$h^T\Delta(D_x l(\cdot,u,t))(x;h)$$
$$= h^T\Delta(D_x l_1(\cdot,u))(x;h) + \{h^T D_x^2 l_2(x,u,t)\,h\}, \tag{4.7}$$

where for $(x,u,t)\in Q\times R^m\times T$,

$$l_1(x,u) := \bar g_o(x) + \sum_{i=1}^m u_i\,\bar g_i(x)\ ,$$

$$l_2(x,u,t) := g_o(x,t) + \sum_{i=1}^m u_i\,g_i(x,t).$$

In virtue of the properties (P 2) and (P 3) of generalized directional derivatives, the multifunction which assigns to (x,u,h) the set $\Delta(D_x l_1(\cdot,u))(x;h)$ is closed and locally bounded on $Q\times R^m\times R^n$, and hence, by (4.7) and by the discussion in §4.2, the multifunction $H(x,u,t,h) := \Delta(D_x l(\cdot,u,t))(x;h)$ satisfies (4.1) and (4.2), and we can again replace Condition 3.3 by Condition 3.3'.

We note that literature on decomposition methods pays a special attention to optimization problems in which the objective function is separable w.r. to two groups of variables (cf., for example Bank, Mandel and Tammer [4] or Beer [5]), i.e., in (4.6) one has $f_o(x,t) = \bar g_o(x) + g_o(t)$. Assuming that $\bar g_i(x)\equiv 0$ $(i=1,\dots,m)$, we obtain a particular form of Condition 3.3' with

$$H(x^o,u^o,t^o,h) := \Delta(D\bar g_o)(x^o;h) + \sum_{i=1}^m u_i^{\,o}\,D_x^2 g_i(x^o,t^o)\,h\,.$$

4.4. The discussion in the previous special cases suggests to look for general conditions which guarantee directly the closedness of the multifunction $\Delta(D_x l)(\cdot;\cdot)$. To do this, we suppose again (1.1) and (3.1) for the given parametric program, and we additionally suppose that for some $t^o\in T$ and some $x^o\in S(t^o)$, there are a constant $\beta>0$ and neighborhoods U_o of t^o and V_o of x^o such that for $i\in\{0,1,\dots,m\}$,

$$\| D_x f_i(x',t) - D_x f_i(x'',t)\| \le \beta\,\| x'-x''\|\ (\forall x',x''\in V_o\ \forall t\in U_o) \tag{4.8}$$

and

$$\limsup_{\substack{t\to t^o\\ x\to x^o}} \Delta(D_x f_i(\cdot,t) - D_x f_i(\cdot,t^o))(x;h) = \{0\}\quad (\text{for all } h\in \mathrm{bd}\, B_n). \tag{4.9}$$

We note that in the case of C^2 data (4.9) corresponds to (4.5). In the following proposition we handle special problems for which the continuity and differentiability requirements on the data (1.1), the $C^{1,1}$ property (3.1) and both (4.8) and (4.9) are satisfied.

Proposition 4.2: Consider $\{P(t), t \in T\}$, let $t^o \in T$, $x^o \in S(t^o)$ and suppose that (1.1), (3.1), (4.8) and (4.9) hold. Further, suppose that the Mangasarian-Fromovitz CQ is satisfied at x^o w.r. to $M(t^o)$.
Then Condition 3.3' and Condition 3.3 are equivalent.

Proof: By Proposition 4.1, it suffices to show that (4.1) and (4.2) are fulfilled. Put

$$H(x,u,t,h) \quad := \quad \Delta\,(D_x l(\cdot,u,t))(x;h)$$

for $(x,u,t,h) \in Q \times R^m \times T \times R^n$, which implies that, by property (P 1) of generalized directional derivatives, the following inclusions hold:

$$\begin{aligned} &H(x,u,t,h) \\ &\subset H(x,u,t^o,h) + \Delta(D_x l(\cdot,u,t) - D_x l(\cdot,u,t^o))(x;h) \\ &\subset H(x,u,t^o,h) + \sum_{i=o}^{m} u_i \;\Delta(D_x f_i(\cdot,t) - D_x f_i(\cdot,t^o))(x;h) \end{aligned} \qquad (4.10),$$

where $u_o := 1$. Let U_o and V_o be as in (4.8).

By the properties (P 2) and (P 3) of generalized directional derivatives, the multifunction $H(\cdot,\cdot,\cdot,t^o)$ is closed and locally bounded on $\{x^o\} \times LM(x^o,t^o) \times \text{bd } B_n$. As $LM(x^o,t^o)$ is bounded (because of the Mangasarian-Fromovitz CQ which is satisfied at x^o), hence there exist an open neighborhood $V_1 \subset V_o$ of x^o, open bounded sets $N_1 \supset LM(x^o,t^o)$ and $W_1 \supset \text{bd } B_n$, and a bounded set $X \subset R^n$ such that

$$H(x,u,t^o,h) \subset X \quad \text{for all } (x,u,h) \in V_1 \times N_1 \times W_1 \;. \qquad (4.11)$$

Now let $i \in \{0,1,\ldots,m\}$, $t \in U_o$, $x \in V_1$ and $h \in W_1$ be fixed. For simplicity of notation, we put

$$F_{i,t}(x) \;:=\; D_x f_i(x,t) - D_x f_i(x,t^o). \qquad (4.12)$$

By definition of $\Delta F_{i,t}(x;h)$, we then have

$$z \in \Delta F_{i,t}(x;h) \text{ if and only if } z = \lim_{\substack{x^k \to x \\ \theta_k \to +0}} \theta_k^{-1} z(x^k+\theta_k h)$$

with $z(x^k+\theta_k h) := F_{i,t}(x^k+\theta_k h) - F_{i,t}(x^k)$. Hence, (4.8) and and (4.12) then imply that

$$\| z(x^k + \theta_k h) \| \le \theta_k \, \text{ß} \, \| h \|$$

and therefore (with $d(W_1) := \sup \{ \| h \| \;/\; h \in W_1 \}$),

$$\| z \| \le \text{ß} \cdot d(W_1) \qquad (\forall z \in \Delta F_{i,t}(x;h)). \qquad (4.13)$$

Property (P 7) and assumption (4.8) yield that for any $h^o \in W_1$ the inclusion

$$\Delta F_{i,t}(x;h) \subset \Delta F_{i,t}(x;h^o) + \text{ß} \, \| h-h^o \| \, B_n \qquad (4.14)$$

holds. From (4.10), (4.11) and (4.13) then we obtain that for all $(x,u,t,h) \in V_1 \times N_1 \times U_o \times W_1$, one has the boundedness:

$$H(x,u,t,h) \subset X \; + \beta \, d(W_1)(1 + m \, d(N_1)) \; B_n .$$

To show that for any $u^o \in LM(x^o,t^o)$ and any $h^o \in \text{bd } B_n$, H is also closed at (x^o,u^o,t^o,h^o), we shall use the closedness of $H(\cdot,\cdot,\cdot,t^o)$ and apply (4.10), (4.9) and (4.14). These facts imply the inclusions

$$\begin{aligned}
& \limsup_{(x,u,t,h) \to (x^o,u^o,t^o,h^o)} H(x,u,t,h) \\
\subset \; & \limsup_{(x,u,h) \to (x^o,u^o,h^o)} H(x,u,t^o,h) \\
& + \limsup_{(x,t) \to (x^o,t^o)} \Delta F_{o,t}(x;h^o) \\
& + \limsup_{(x,u,t) \to (x^o,u^o,t^o)} \sum_{i=1}^{m} u_i \Delta F_{i,t}(x;h^o) \\
= \; & H(x^o,u^o,t^o,h^o).
\end{aligned}$$

This completes the proof. //

4.5. Now we recall a broad class of $C^{1,1}$-functions g, for which a simple representation of Clarke's generalized Jacobian of Dg is possible, and which is of particular interest in several applications of $C^{1,1}$-optimization, cf. the discussions in [13], Remark 4 and [14], §4 .

Given an open set $Q \subset R^n$ and functions $g_i \in C^2(Q)$, $i=1,\dots,s$, let g be a continuous selection from $\{g_1,\dots,g_s\}$ satisfying the following properties:

(a) For each $x \in Q$ there is some $i(x) \in \{1,\dots,s\}$ such that $g(x) = g_{i(x)}(x)$,

(b) g is continuous on Q,

(c) for each pair $i,j \in \{1,\dots,s\}$ and each $x \in Q_i \cap Q_j$ one has $Dg_i(x) = Dg_j(x)$, where $Q_i := \{x \in Q \ / \ g(x) = g_i(x)\}$.

<u>Proposition 4.3 ([14 , Th. 4])</u>: The function g belongs to the class $C^{1,1}(Q)$, and for each $x \in Q$, there exists an index set $J(x) \subset \{i \in \{1,\dots,s\} \ / \ g(x) = g_i(x)\}$ such that

$$J_{Cl}\, Dg(x) = \operatorname{conv} \{ D^2 g_i(x) \ / \ i \in J(x)\}.$$

In what follows, g will be called a $C^{1,1}$-selection of $\{g_1,\dots,g_s\}$. Returning to the parametric problem $\{P(t), t \in T\}$, chosing $t^o \in T$, $x^o \in S(t^o)$ and assuming that the Mangasarian-Fromovitz CQ holds at x^o w.r. to $M(t^o)$, we now consider the following special case:

(1) For each $i \in \{0,1,\dots,m\}$, f_i is a continuous selection from $\{g_1,\dots,g_s\}$, where $g_j: Q \times T \longrightarrow R$ $(j=1,\dots,s)$ are continuous functions which are twice continuously differentiable with respect to x,

(2) for each $i \in \{0,1,\dots,m\}$ and each $t \in T$, $f_i(\cdot,t)$ is a $C^{1,1}$-selection of $\{g_1(\cdot,t),\dots,g_s(\cdot,t)\}$,

(3) $D_x g_j(\cdot,\cdot)$ and $D_x^2 g_j(\cdot,\cdot)$ are continuous on $Q \times T$ $(j=1,\dots,s)$,

(4) with $I(f_i,x,t) := \{j \in \{1,\dots,s\} \ / \ f_i(x,t) = g_j(x,t)\}$ and $H_i(x,t,h) := \operatorname{conv}\{D_x^2 g_j(x,t)\, h \ / \ j \in I(f_i,x,t)\}$, $i=0,1,\dots,m$, set

$$H(x,u,t,h) := H_o(x,t,h) + \sum_{i=1}^{m} u_i\, H_i(x,t,h).$$

As a direct consequence of the assumptions (1) ... (4) we obtain that H is closed and locally bounded on $\{x^o\} \times LM(x^o,t^o) \times \{t^o\} \times \mathrm{bd}\, B_n$. Hence, (4.1) holds. Property (4.2) follows from Proposition 4.3, property (P 5) of directional derivatives and assumption (4). So, Condition 3.3 may be replaced by Condition 3.3'.

Now consider the case p=0, i.e., there are no equality constraints. In order to verify in Condition 3.2 or Condition 3.3' that for some (x^0,u^0,t^0), $h^Tz>0$ holds for all h belonging to some set W and for all $z\in H(x^0,u^0,t^0,h)$, the following condition would suffice:
For some $i\in\{0\}\cup I^+(u^0)$ and some $j\in I(f_i,x^0,t^0)$ and for each $h\in W$, one has $h^TD_x^2\,g_j(x^0,t^0)h>0$ and $h^TD_x^2\,g_k(x^0,t^0)h\geq 0$ if $k\in\{1,\ldots,s\}\setminus j$.
This reduces the expense to the verfification of positive (semi-)definiteness of finitely many matrices.

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