Musterlösung Blatt 4

①

A1 Betrachte DGL $x''(t) = g(x(t))$.

(a) Äquivalentes DGLs 1. Ordnung:
Nach dem Ansatz $y(t) := x'(t)$ folgt für y: $y'(t) = x''(t) = g(x(t))$.
Also

$$\begin{cases} x'(t) = y(t) \\ y'(t) = g(x(t)) \end{cases}$$

bzw. mit $v(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$: $v'(t) = X(v(t))$, wo

$$X(x_1, x_2) = \begin{pmatrix} x_2 \\ g(x_1) \end{pmatrix}.$$

(b) (x_0, y_0) Gleichgewichtszustand $\Leftrightarrow$ $X(x_0, y_0) = 0 \Leftrightarrow \begin{cases} y_0 = 0 \\ g(x_0) = 0 \end{cases}$. Da $g(0) = 0$ und $g(x) \neq 0$ für $x \neq 0$ (wegen $x \cdot g(x) < 0$ für $x \neq 0$), ist dies der Fall genau für $(x_0, y_0) = (0,0)$.

(c) Mit $L(x,y) = \frac{1}{2} y^2 - \int_0^x g(s)\,ds$ gilt

$$L(X) = \frac{\partial L}{\partial x} \cdot y + \frac{\partial L}{\partial y} \cdot g(x)$$

$$\underset{\text{Hauptsatz}}{=} -g(x) \cdot y + y \cdot g(x) = 0.$$

Da $x g(x) < 0$ für $x \in \mathbb{R} \setminus \{0\}$, gilt $g(x) > 0$ für $x < 0$, $g(x) < 0$ für $x > 0$ und

Es folgt: $\int_0^x \underbrace{g(s)}_{\geq 0,\ > 0 \text{ für } s > 0} ds < 0$ für $x > 0$

und $\int_0^x g(s)\,ds = -\underbrace{\int_x^0 g(s)\,ds}_{\leq 0} < 0$ für $x < 0$

$\Rightarrow L(x,y) > L(0,y) = \frac{1}{2} y^2$ für $x \neq 0$.

Da weiterhin $\frac{1}{2} y^2 > 0$ für $y \neq 0$, folgt: $(0,0)$ ist striktes globales Minimum von L.

Also L Ljapunov-Fkt $\Rightarrow$ $(0,0)$ ~~[illegible]~~ stabil.

A2

(a)

$$[fX,Y]_i = \sum_{j=1}^n (fX)_j \frac{\partial Y_i}{\partial x_j} - \sum_{j=1}^n Y_j \frac{\partial (fX)_i}{\partial x_j}$$

$$= f \sum_{j=1}^n (X)_j \frac{\partial Y_i}{\partial x_j} - \sum_{j=1}^n Y_j \left(\frac{\partial f}{\partial x_j} X_i + f \frac{\partial X_i}{\partial x_j} \right)$$

$$= f\left(\sum_{j=1}^n X_j \frac{\partial Y_i}{\partial x_j} - \sum_{j=1}^n Y_j \frac{\partial X_i}{\partial x_j} \right) - \left(\sum_{j=1}^n Y_j \frac{\partial f}{\partial x_j} \right) X_i$$

$$= f\,[X,Y] - Y(f)\,X.$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] =$$

$$= \sum_{j=1}^n \left(X_j \frac{\partial [Y,Z]_i}{\partial x_j} - [Y,Z]_j \frac{\partial X_i}{\partial x_j} \right)$$

$$+ \sum_{j=1}^n \left(Y_j \frac{\partial [Z,X]_i}{\partial x_j} - [Z,X]_j \frac{\partial Y_i}{\partial x_j} \right)$$

$$+ \sum_{j=1}^n \left(Z_j \frac{\partial [X,Y]_i}{\partial x_j} - [X,Y]_j \frac{\partial Z_i}{\partial x_j} \right)$$

Bezeichne mit $-//-_{YZX}$ jeweils den gleichen Term mit Ersetzung $(X,Y,Z) \longrightarrow (Y,Z,X)$. Analog $-//-_{ZXY}$

$$= \sum_{j=1}^n \Bigg\{ X_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^n Y_k \frac{\partial Z_i}{\partial x_k} - \sum_{k=1}^n Z_k \frac{\partial Y_i}{\partial x_k} \right) - \frac{\partial X_i}{\partial x_j} \left(\sum_{k=1}^n Y_k \frac{\partial Z_j}{\partial x_k} - \sum_{k=1}^n Z_k \frac{\partial Y_j}{\partial x_k} \right) \Bigg\}$$

$$+ -//-_{YZX} + -//-_{ZXY}$$

$$= \sum_{j,k=1}^n \Bigg\{ X_j Y_k \frac{\partial^2 Z_i}{\partial x_j \partial x_k} - X_j Z_k \frac{\partial^2 Y_i}{\partial x_j \partial x_k}$$

$$+ X_j \frac{\partial Y_k}{\partial x_j} \frac{\partial Z_i}{\partial x_k} - X_j \frac{\partial Z_k}{\partial x_j} \frac{\partial Y_i}{\partial x_k}$$

$$- Y_k \frac{\partial X_i}{\partial x_j} \frac{\partial Z_j}{\partial x_k} + Z_k \frac{\partial X_i}{\partial x_j} \frac{\partial Y_j}{\partial x_k} \Bigg] + -//-_{YZX} + -//-_{ZXY}$$

$$= \sum_{j,k=1}^{n} \Big\{ x_j y_k \cancel{\frac{\partial^2 z_i}{\partial x_j \partial x_k}} - x_j z_k \cancel{\frac{\partial^2 y_i}{\partial x_j \partial x_k}} + z_j x_k \cancel{\frac{\partial^2 y_i}{\partial x_j \partial x_k}} - z_j y_k \frac{\partial^2 x_i}{\partial x_j \partial x_k}$$

$$+ y_j z_k \frac{\partial^2 x_i}{\partial x_j \partial x_k} - y_j x_k \cancel{\frac{\partial^2 z_i}{\partial x_j \partial x_k}} \Big\}$$

$$+ \sum_{j,k=1}^{n} \Big\{ x_j \cancel{\left(\frac{\partial y_k}{\partial x_j} \frac{\partial z_i}{\partial x_k} - \frac{\partial z_k}{\partial x_j} \frac{\partial y_i}{\partial x_k} \right)} - y_k \cancel{\frac{\partial x_i}{\partial x_j} \frac{\partial z_i}{\partial x_k}} + \cancel{z_k \frac{\partial x_i}{\partial x_j} \frac{\partial y_i}{\partial x_k}}$$

$$+ y_j \cancel{\left(\frac{\partial z_j}{\partial x_j} \frac{\partial x_i}{\partial x_k} - \frac{\partial x_k}{\partial x_j} \frac{\partial z_i}{\partial x_k} \right)} - z_k \cancel{\frac{\partial y_i}{\partial x_j} \frac{\partial x_j}{\partial x_k}} + \cancel{x_k \frac{\partial y_i}{\partial x_j} \frac{\partial z_i}{\partial x_k}}$$

$$+ z_j \cancel{\left(\frac{\partial x_k}{\partial x_j} \frac{\partial y_i}{\partial x_k} - \frac{\partial y_k}{\partial x_j} \frac{\partial z_i}{\partial x_k} \right)} - x_k \cancel{\frac{\partial z_i}{\partial x_j} \frac{\partial y_j}{\partial x_k}} + y_k \cancel{\frac{\partial z_i}{\partial x_j} \frac{\partial x_j}{\partial x_k}} \Big\}$$

$= 0$, wobei wir Summanden wegkürzen, die nach Vertauschung $(j \leftrightarrow k)$ ineinander übergehen (erlaubt, da j,k beide von 1 bis n summiert werden).

(b) Zeigen:

- $D_1 \circ D_2 - D_2 \circ D_1$ ist eine Derivation.

Es gilt: $(D_1 \circ D_2 - D_2 \circ D_1)(f \cdot g)$

$$= D_1\big((D_2 f)g + f(D_2 g)\big) - D_2\big((D_1 f)g + f(D_1 g)\big)$$

$$= ((D_1 \circ D_2)f)g + \cancel{(D_2 f)(D_1 g)} + \cancel{(D_1 f)(D_2 g)} + f((D_1 \circ D_2)g)$$

$$- ((D_2 \circ D_1)f)g - \cancel{(D_1 f)(D_2 g)} - \cancel{(D_2 f)(D_1 g)} - f((D_2 \circ D_1)g)$$

$$= ((D_1 \circ D_2 - D_2 \circ D_1)f)g + ((D_1 \circ D_2 - D_2 \circ D_1)g)f,$$ w.z.b.w.

- Falls X_1 bzw. X_2 zu D_1 bzw. D_2 korrespondierendes VF ist, dann entspricht $D_1 \circ D_2 - D_2 \circ D_1$ das VF $[X_1, X_2]$.

Es gelte also ④

$D_2(f) = X_2(f)$, $D_2(f) = X_2(f)$ $\forall f$.

$\Rightarrow (D_2 \circ D_2 - D_2 \circ D_1)f$

$= X_2(X_2(f)) - X_2(X_1(f)) = \sum_{j=1}^{n} \left\{ (X_1)_j \frac{\partial}{\partial x_j}(X_2(f)) - (X_2)_j \frac{\partial}{\partial x_j}(X_1(f)) \right\}$

$= \sum_{j,k=1}^{n} \left\{ (X_1)_j \frac{\partial}{\partial x_j}\left((X_2)_k \frac{\partial f}{\partial x_k} \right) - (X_2)_j \frac{\partial}{\partial x_j}\left((X_1)_k \frac{\partial f}{\partial x_k} \right) \right\}$

$= \sum_{j,k=1}^{n} \left\{ (X_1)_j \frac{\partial (X_2)_k}{\partial x_j} - (X_2)_j \frac{\partial (X_1)_k}{\partial x_j} \right] \frac{\partial f}{\partial x_k} + \underbrace{\sum_{j,k=1}^{n} (X_1)_j (X_2)_k \left(\frac{\partial^2 f}{\partial x_j \partial x_k} - \frac{\partial^2 f}{\partial x_j \partial x_k} \right)}_{=0}$

$\underset{\text{Def v. } [X,Y]}{=} [X,Y](f).$

(c) $\frac{d}{dt}\Big|_{t=0} e^{-tA} B e^{tA} = \left(\frac{d}{dt}\Big|_{t=0} e^{-tA} \right) \cdot B \cdot (e^{tA})|_{t=0}$

$+ (e^{-tA})_{t=0} \cdot B \cdot \frac{d}{dt}\Big|_{t=0} (e^{tA}) = (-Ae^{-tA})|_{t=0} \cdot B \cdot \mathbb{1}$

$+ \mathbb{1} \cdot B \cdot (A \cdot e^{tA})_{t=0} = -A \cdot B + B \cdot A.$

$[X_A, X_B](f) \overset{b)}{=} X_A(X_B(f)) - X_B(X_A(f))$

$= A \cdot B \cdot f - B \cdot A \cdot f = (A \cdot B - B \cdot A) \cdot f$

für jedes Vektorfeld f auf $\mathbb{R}^n$.

Hieraus folgt $[X_A, X_B] = X_{A \cdot B - B \cdot A}$.