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Homework Problems 12

Analysis and Geometry on Manifolds WS 06/07

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Problem 1
(1) Let $M \subset N$ be a differentiable submanifold of a Riemannian manifold (N, g). Show that the Levi–Civita connection ∇^M on M of the Riemannian metric induced by g and the Levi–Civita connection ∇^N are related by

$$\nabla^M_X Y = \mathrm{proj}^{\perp}_M(\nabla^N_X Y)$$

for tangent vector $X \in T_pM$ and a vector field $Y \in \mathcal{C}(M)$.
(2) Let $M \subset \mathbb{R}^n$ be a submanifold and the Riemannian metric on M induced by the euclidean metric. Show that normalized geodesics $\gamma = \gamma(t)$ on (M, g) are characterized by $\ddot{\gamma}(t) \perp T_{\gamma(t)}M \subset \mathbb{R}^N$ (Homework Set 11, Problem 2).

Problem 2
Let $\{B_p : T_pM \times T_pM \times ... \times T_pM \to \mathbb{R}\}$ be a differentiable family of k–linear forms and ∇ be a covariant derivative. For $X \in T_pM$ define $\nabla_X B$ by

$$(\nabla_X B)(X_1, ..., X_k) := X(B(X_1, ..., X_k)) - (B(\nabla_X X_1, ..., X_k) + ... + B(X_1, ..., \nabla_X X_k))$$

for vector fields $X_1, ..., X_k$.
(1) Show that $\nabla_X B$ is again a differentiable k–linear form, i.e. depends only on the values of $X_1, ..., X_k$ at p.
(2) Let g be a Riemannian structure on M. Show that a covariant derivative ∇ is metric if and only if $\nabla g = 0$, i.e. $\nabla_X g = 0$ for all $p \in M$ and $X \in T_pM$.
(3) Suppose that M in (2) is orientable and let $dM \in \Omega^n(M)$ be the volume form. Show that $\nabla(dM) = 0$.

Problem 3
Show that the volume form, dM, of a closed n–dimensional oriented Riemannian manifold M is *not* the exterior differential of an $(n-1)$–form on M. ("closed" is short for "compact and without boundary").

The following problems will be discussed in the tutorials:

Problem 4
Show that a covariant derivative on a differentiable manifold is metric if and only if the corresponding parallel transports are isometries.

Problem 5
Show that the definition of the covariant derivative along a differentiable map $u : F \to M$ induced by some covariant derivative ∇ on M defined with respect to a parameterization is independent of the parameterization.