Stochastic Analysis

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Chapter 1

Construction and properties of Brownian motion

1.1 Motivation

Why is the Brownian motion the central object of stochastic analysis?

Scaling limit of random walks

$(X_k)_{k\geq 1}$ i.i.d. random variables with $\mathbb{E}[X_k] = 0$, $\sigma^2 = \mathrm{Var}(X_k) < \infty$. Put $S_0 := 0$, $S_n := \sum_{k=1}^n X_k$, $n \geq 1$. Zooming out (rescale time): $Y_{n/N}^{(N)} := S_n$, for $n = 0, 1, \dots, N$. Then $\mathbb{E}[Y_{n/N}^{(N)}] = 0$, $\mathrm{Var}(Y_1^{(N)}) = \mathrm{Var}(S_N) = N\sigma^2$. Standardise (rescale space): $Z_{n/N}^{(N)} = \frac{1}{\sigma\sqrt{n}} Y_{n/N}^{(N)}$ for $n = 0, 1, \dots, N$. Then $\mathbb{E}[Z_{n/N}^{(N)}] = 0$, $\mathrm{Var}(Z_1^{(N)}) = 1$. Use linear interpolation to define $Z_t^{(N)}$, $t \in [0,1]$. Asymptotics $N \to \infty$: $(Z_t^{(N)}, t \in [0,1]) \xrightarrow{d} (B_t, t \in [0,1])$, where $\xrightarrow{d}$ means convergence in distribution on $C([0,1])$. This is Donsker's invariance principle (functional CLT).

Anti-derivative of "white noise"

Physicists and engineers often model random perturbations by a white noise process $(\Gamma_t, t \geqslant 0)$. They postulate: Γ is a Gaussian process (all $(\Gamma_{t_1}, \dots, \Gamma_{t_n})$ are Gaussian) with $\mathbb{E}[\Gamma_t] = 0$, $\mathrm{Cov}(\Gamma_t, \Gamma_s) = \delta(t-s)$ with "δ-function" defined by $\delta(x) = 0$ for $x \neq 0$ and $\int_{-\varepsilon}^{\varepsilon} \delta(x)\,dx = 1$ for all $\varepsilon > 0$. The idea of the covariance structure is that for $f \in L^2([0,1])$: the linear functional $\int_0^1 f(t)\Gamma_t\,dt$ is Gaussian with mean 0 and

$$\begin{aligned}
\mathrm{Var}\left(\int_0^1 f(t)\,\Gamma_t dt\right) &= \mathbb{E}\left[\int_0^1 f(t)\,\Gamma_t\,dt \cdot \int_0^1 f(s)\,\Gamma_s\,ds\right] \\
&\stackrel{?}{=} \int_0^1 \int_0^1 f(t)\,f(s)\;\mathbb{E}\Big[\Gamma_t \Gamma_s\Big]\,dt\,ds = \int_0^1 f^2(s)\;ds = \|f\|_{L^2}^2.
\end{aligned}$$

This will be made mathematically correct via the stochastic integral: $\int_0^1 f(t)\Gamma_t\,dt \rightsquigarrow \int_0^1 f(t)\,dB_t$ (Wiener's stochastic integral). White noise itself is difficult to define properly, but the stochastic integration theory is well developed. As we shall see, Brownian motion can be seen as the anti-derivative of white noise.

Why “white” noise? Fourier coefficients: for $k \geq 1$

$$\begin{aligned} C_k = &\ \textstyle\int_0^{2\pi} \Gamma_t \frac{1}{\sqrt{\pi}} \cos(kt)\ dt &\sim N(0,1)\,, \\ D_k = &\ \textstyle\int_0^{2\pi} \Gamma_t \frac{1}{\sqrt{\pi}} \sin(kt)\ dt &\sim N(0,1)\,, \end{aligned}$$

and for $k = 0$

$$C_0 = \frac{1}{\sqrt{2\pi}} \int_0^1 \Gamma_t \, dt \sim N(0,1)\,.$$

By polarisation we obtain:

$$\begin{aligned} \mathbb{E}[C_k D_l] &\overset{\text{pol.}}{=} \frac{1}{4}\left(\mathbb{E}\left[(C_k + D_l)^2\right] - \mathbb{E}\left[(C_k - D_l)^2\right]\right) \\ &\overset{\text{from above}}{=} \frac{1}{4}\left(\left\| \frac{1}{\sqrt{\pi}} (\cos(k\cdot) + \sin(l\cdot)) \right\|_{L^2}^2 - \left\| ... - ... \right\|_{L^2}^2\right) \\ &\overset{\text{pol.}}{=} \left\langle \frac{1}{\sqrt{\pi}} \cos(k\cdot)\,, \frac{1}{\sqrt{\pi}} \sin(l\cdot) \right\rangle_{L^2} \\ &= 0. \end{aligned}$$

Then C_k, D_l are uncorrelated. Equally, we can show that the entire set $\{C_k, k \geq 0\} \cup \{D_l, l \geq 0\}$ consists of independent (!) $N(0,1)$-distributed random variables. This gives formally

$$\Gamma_t = \sum_{k=1}^{\infty} \left(C_k \frac{1}{\sqrt{\pi}} \cos(kt) + D_k \frac{1}{\sqrt{\pi}} \sin(kt) \right) + C_0 \frac{1}{\sqrt{2\pi}}$$

for $t \in [0, 2\pi]$. Hence, Brownian motion should be

$$B_t = \sum_{k=1}^{\infty} \left(C_k \frac{1}{\sqrt{\pi} k} \sin(kt) - D_k \frac{1}{\sqrt{\pi} k} \cos(kt) \right) + C_0 \frac{t}{\sqrt{2\pi}}.$$

Continuous martingales

Let $(M_t, t \geq 0)$ be a continuous martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with *filtration* $(\mathcal{F}_t)_{t\geq 0}$ ($\forall\, 0 \leq t \leq s$ $\mathcal{F}_t \subseteq \mathcal{F}_s$ sub-σ-algebras of $\mathcal{F}$), i.e.

(i) $M_t \in L^1$,

(ii) M_t is $\mathcal{F}_t$-measurable (“adapted”),

(iii) $\forall 0 \leq t \leq s : \mathbb{E}[M_s | \mathcal{F}_t] = M_t$ a.s.,

(iv) $t \mapsto M_t(\omega)$ is continuous for almost all (a.a.) $\omega \in \Omega$.

They form basic stochastic objects! Fundamental results:

(a) M can be obtained by a (random) time shift of a Brownian motion B

$$M_t = B_{\tau(t)} + M_0, \quad t \geq 0.$$

(b) M can be obtained by averaging weighted Brownian increments (as a stochastic integral):

$$M_t = M_0 + \int_0^t H_s \, dB_s, \quad t \geq 0,$$

where B is a Brownian motion and H is a suitable (random) integrand.

Understanding B means understanding continuous martingales!

Diffusion, Laplace operator, physical Brownian motion

$(B_t, t \geq 0)$ should be a continuous Markov ("memoryless") process in $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be some physical quantity $f(y)$ at some point y (e.g. temperature). Consider a diffusion equation for some "density" $\varphi(y, t)$:

$$\frac{\partial}{\partial t}\varphi(y,t) + div\left(\vec{j}(y,t)\right) = 0,$$

where $\vec{j}(y,t)$ is the flux in y at time t. Usually, $\vec{j}(y,t)$ is proportional to $-grad\,\varphi(y,t) = -\nabla\varphi(y,t)$. Then

$$\frac{\partial}{\partial t}\varphi(y,t) - \sigma^2 \triangle\varphi(y,t) = 0$$

(here $\triangle = \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$). This is a *diffusion equation.* We suggest as solutions $\varphi(x,t) := \mathbb{E}_x[f(B_t)]$, $B_0 = x$ (Brownian motion starting in x), for all "nice" f. Let $p(x,y,t)$ be the transition density of B such that $\varphi(x,t) = \int_{\mathbb{R}^d} f(y)p(x,y,t)\,dy$. This gives a PDE for p:

$$\frac{\partial}{\partial t}p(x,y,t) = \sigma^2 \triangle_x p(x,y,t) \quad \text{(master/heat equation).} \tag{1.1.1}$$

The solution with $p(x,y,t) = \delta(x-y)$ is $p(x,y,t) = (2\pi)^{-d/2}\sigma^{-d}t^{-1/2}e^{-|x-y|^2/2t\sigma^2}$ (this is the Gaussian density!). This yields the mathematical Brownian motion (up to a factor $2\sigma^2$).

1.2 Approaches to construct Brownian motion

Definition 1.1. A stochastic process $(B_t, t \geq 0)$ is called *Brownian motion* (or *Wiener process*), if

(i) $B_0 = 0$ a.s.,

(ii) B has independent increments: $\forall n \geq 1$, $0 \leq t_0 < \cdots < t_n : (B_{t_i} - B_{t_{i-1}})_{1 \leq i \leq n}$ are independent random variables,

(iii) $B_t - B_s \sim N(0, t-s)$ for all $t > s \geq 0$ (stationary increments),

(iv) $t \mapsto B_t(\omega)$ is continuous for a.a. $\omega \in \Omega$ (continuous trajectories/paths).

Let $T > 0$. The *Wiener measure* $\mathbb{P}^W$ on $(C([0,T], \mathscr{B}_{C([0,T])})$, where $\mathscr{B}_{C([0,T])}$ is the Borel-σ-algebra on $C([0,T])$ (if $T < \infty$ this is induced by the sup-norm, if $T = \infty$ it is induced by a special metric inducing the topology of uniform convergence on compact sets) is given by the image measure induced by a Brownian motion B: $\mathbb{P}^W(A) = P(B \in A)$.

Remark 1.2.

(i) Given $\mathbb{P}^W$ the coordinate process $\pi_t : C([0,T]) \to \mathbb{R}, \pi_t(f) = f(t)$, $T < \infty$, defines a Brownian motion $(\pi_t, t \in [0,T])$ on $[0,T]$ (check via Stochastic processes I). Here, $(C([0,T]), \mathscr{B}_{C([0,T])})$ is called *(canonical) path space.*

(ii) $(B_t)_{t \geq 0} = (B_t, t \geq 0)$ is a centred Gaussian process with covariance function $c(t,s) := \mathrm{Cov}(B_t, B_s) = t \wedge s$ (recall: B is a Gaussian process $:\Leftrightarrow \forall n, 0 \leq t_1 < \cdots < t_n$: $(B_{t_1}, \ldots, B_{t_n})$ is Gaussian). The Gaussianity of B follows by the independence and normality of increments. With respect to the covariance function let $s \leq t$ such that

$$\begin{aligned}\mathrm{Cov}(B_t, B_s) &= \mathbb{E}[B_t B_s] = \mathbb{E}\Big[\big(B_s + \underbrace{B_t - B_s}_{\text{indep. of } B_s \text{by (ii)}}\big)B_s\Big] \\ &= \mathbb{E}\left[B_s^2\right] + \mathbb{E}[B_t - B_s]\,\mathbb{E}[B_s] = s.\end{aligned}$$

Hence, $\forall s,t \geq 0$ Cov $(B_t, B_s) = s \wedge t$. How many Brownian motions are there? Since the cylinder sets

$$A_{t_1,\ldots,t_n;C} := \{f \in C([0,T]) : (f(t_1), \ldots, f(t_n)) \in C\}$$

for $0 \leq t_1 < t_2 < \ldots, t_n \leq T < \infty$, $C \in \mathscr{B}_{\mathbb{R}^n}$, form an $\cap$-stable generator of $\mathscr{B}_{C([0,T])}$, the Wiener measure $\mathbb{P}^W$ is uniquely (!) defined by these Gaussian properties.

(iii) The *existence* of a Brownian motion is much less evident. The main difficulty is the continuity of paths (see exercises). Since $(t,s) \mapsto C(t,s) = t \wedge s$ is a positive semi-definite function, a Gaussian process with mean 0 and covariance function $C(t,s)$ always exists by Kolmogorov's consistency theorem (as a limit of a projective family of distributions) on $(\mathbb{R}^{\mathbb{R}_+}, \mathscr{B}_{\mathbb{R}}^{\otimes \mathbb{R}_+})$. Hence, a process $(\overline{B}_t, t \geq 0)$ satisfying properties (i)-(iii) of a Brownian motion always exists.

Donsker's invariance principle

Show existence by tightness and Prokhorov's theorem (see Stochastic processes I).

Kolmogorov/Chentsov: continuous version

Definition 1.3. A process $(\tilde{X}_t, t \in T)$ is a *version* of $(X_t, t \in T)$ if $\forall t \in T : \mathbb{P}(\tilde{X}_t = X_t) = 1$. $\tilde{X}$ and X are called *indistinguishable* if $\mathbb{P}(\forall t \in T : X_t = \tilde{X}_t) = 1$.

Note that a version $\tilde{X}$ of X has the same finite-dimensional distributions, i.e. $(\tilde{X}_{t_1}, \ldots, \tilde{X}_{t_n}) \stackrel{d}{=} (X_{t_1}, \ldots, X_{t_n})$ which means $\mathbb{P}^{(\tilde{X}_{t_1},\ldots,\tilde{X}_{t_n})} = \mathbb{P}^{(X_{t_1},\ldots,X_{t_n})}$. We shall show that a process $\overline{B}$ with properties (i)-(iii) of a Brownian motion has a continuous version B, which then satisfies (iv) as well (surely!).

Example 1.4. Suppose $(X_t, t \in [0,1])$ is a continuous process. Then we can define a version $(\tilde{X}_t, t \in [0,1])$ with discontinuous trajectories by

$$\tilde{X}_t = \begin{cases} X_t, & t \neq \tau \\ X_t + 1, & t = \tau \end{cases}$$

where $\tau \sim U([0,1])$ is independent of X on $(\Omega, \mathcal{F}, \mathbb{P})$. This follows from $\mathbb{P}(X_t = \tilde{X}_t) = \mathbb{P}(\tau \neq t) = 1$. Note that $\mathbb{P}(\forall t \in [0,1] : X_t = \tilde{X}_t) = 0$. Hence, $(\tilde{X}_t, t \in [0,1])$ is a version of $(X_t, t \in [0,1])$ but they are not indistinguishable.

Theorem 1.5 (Kolmogorov-Chentov)**.** *Let $(X_t, 0 \leq t \leq 1)$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. If there are $C > 0, \alpha, \beta > 0$ such that*

$$\forall s,t \in [0,1] : \mathbb{E}[|X_t - X_s|^\alpha] \leq C|t-s|^{1+\beta},$$

then X has a continuous version $\tilde{X}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. The paths $t \mapsto \tilde{X}_t(\omega)$ are even Hölder continuous of regularity $\gamma \in (0,1]$ for any $\gamma < \beta/\alpha$, i.e. $\exists L(\omega) \forall t,s \in [0,1] : |\tilde{X}_t(\omega) - \tilde{X}_s(\omega)| \leq L(\omega)|t-s|^\gamma$.

Proof. 1. Stochastic continuity: By Markov's inequality

$$\mathbb{P}(|X_t - X_s| \geq \varepsilon) \leq \varepsilon^{-\alpha}\mathbb{E}[|X_t - X_s|^\alpha] \leq C\varepsilon^{-\alpha}|t-s|^{1+\beta}.$$

For sequences $s_n \to t$ we have $X_{s_n} \xrightarrow{\mathbb{P}} X_t$ (stochastic continuity, necessary for a.s. continuity).

2. Control of increments along $D_n := \{k \cdot 2^{-n} : k = 0, \dots, 2^n\}$: Let $0 < \gamma < \beta/\alpha$. Then

$$\mathbb{P}\left(\left|X_{k2^{-n}} - X_{(k-1)2^{-n}}\right| \geq 2^{-\gamma n}\right) \overset{(*)}{\leq} C \cdot 2^{\gamma n \alpha} 2^{-n(1+\beta)} = C \cdot 2^{-n(1+\beta-\alpha\gamma)}.$$

By a union bound

$$\begin{aligned} \mathbb{P}\left(\max_{k=1,\dots,2^n} \left|X_{k2^{-n}} - X_{(k-1)2^{-n}}\right| \geq 2^{-\gamma n}\right) &\leq \sum_{k=1}^{2^n} \mathbb{P}\left(\left|X_{k2^{-n}} - X_{(k-1)2^{-n}}\right| \geq 2^{-\gamma n}\right) \\ &\leq C \cdot 2^{-n(\beta-\alpha\gamma)}. \end{aligned}$$

By the Borel-Cantelli-Lemma $\exists \Omega^* \in \mathcal{F}, \mathbb{P}(\Omega^*) = 1$ such that $\forall \omega \in \Omega^* \exists n^*(\omega) \forall n \geq n^*(\omega) : \max_{k=1,\dots,2^n} |X_{k2^{-n}} - X_{(k-1)2^{-n}}| < 2^{-\gamma n}$.

3. Beyond neighbours in D_n: Let $w \in \Omega^*$, $n \geq n^*(\omega)$. We show $\forall m > n \forall s, t \in D_m, 0 < |t-s| < 2^{-n} : |X_t(\omega) - X_s(\omega)| \leq 2 \cdot \sum_{j=n+1}^{m} 2^{-\gamma j}$ by induction on m. For $m = n+1$ and $s, t \in D_m$, $|t-s| < 2^{-n}$ we find that $|t-s| = 2^{-m} = 2^{-(n+1)}$. Apply 2. for $n+1$. For the induction step assume that the statement holds for $m-1$. With respect to m assume $\exists \tilde{s}, \tilde{t} \in D_{m-1}$ such that $|\tilde{t} - \tilde{s}| \leq |t-s|$, $|X_{\tilde{s}} - X_s| \leq 2^{-\gamma m}$, $|X_{\tilde{t}} - X_t| \leq 2^{-\gamma m}$. The induction hypothesis implies then that

$$|X_t - X_s| \leq |X_t - X_{\tilde{t}}| + |X_{\tilde{t}} - X_{\tilde{s}}| + |X_{\tilde{s}} - X_s| \leq 2 \cdot 2^{-\gamma m} + 2 \cdot \sum_{j=n+1}^{m-1} 2^{-\gamma j} = 2 \cdot \sum_{j=n+1}^{m} 2^{-\gamma j}.$$

4. Hölder continuity on $D := \bigcup_{m \geq 1} D_m$: For $s, t \in D, 0 < |t-s| < 2^{-n^*(\omega)}$ and $n \geq n^*(\omega)$ with $2^{-(n+1)} \leq |t-s| < 2^{-n}$ we have

$$|X_t(\omega) - X_s(\omega)| \leq 2 \cdot \sum_{j=n+1}^{\infty} 2^{-\gamma j} = \frac{2}{1 - 2^{-\gamma}} \cdot 2^{-\gamma(n+1)} \leq C\, |t-s|^{\gamma}.$$

5. Extension from D to $[0,1]$: Now define

$$\tilde{X}_t(\omega) := \begin{cases} 0, & \omega \notin \Omega^*, \\ X_t(\omega), & \omega \in \Omega^*, t \in D \\ \lim_{s \to t, s \in D} X_s(\omega), & \omega \in \Omega^*, t \notin D \end{cases}$$

Then $t \mapsto \tilde{X}_t(\omega)$ is continuous in t (and well defined, topology result) and measurable in ω. Even more: for $u \in D, t \notin D, (s_n) \subseteq D, s_n \to t, \omega \in \Omega^*, 0 < |t-u| < 2^{-n^*(\omega)}$

$$\left|\tilde{X}_t(\omega) - \tilde{X}_u(\omega)\right| = \lim_{s_n \to t} |X_{s_n}(\omega) - X_u(\omega)| \overset{4.}{\leq} \limsup_{n \to \infty} C \cdot |s_n - u|^{\gamma} = C \cdot |t-u|^{\gamma}$$

and similarly for $u, t \notin D$. For $2^{-n^*(\omega)} \leq |t-u|$ we can write $|t-u| \leq \sum_{k=1}^{2^{n^*(\omega)}} |t_k - t_{k-1}|$ with $t_0 = u, t_{2^{n^*(\omega)}} = t$ and $|t_k - t_{k-1}| \leq 2^{-n^*(\omega)}$ such that

$$\begin{aligned} \left|\tilde{X}_t(\omega) - \tilde{X}_u(\omega)\right| &\leq C \cdot \sum_{k=1}^{2^{n^*(\omega)}} \left|\tilde{X}_{t_k}(\omega) - \tilde{X}_{t_{k-1}}(\omega)\right| \\ &\leq C \cdot \sum_{k=1}^{2^{n^*(\omega)}} |t_k - t_{k-1}|^{\gamma} \\ &\leq C \cdot 2^{n^*(\omega)} |t-u|^{\gamma}. \end{aligned}$$

Then $\forall\omega \in \Omega^*$, $t, u \in [0,1]$:

$$\begin{aligned}\left|\tilde{X}_t(\omega) - \tilde{X}_u(\omega)\right| &\leq \begin{cases} C \cdot |t-u|^{\gamma}, & |t-u| \leq 2^{-n^*(\omega)}, \\ C \cdot 2^{n^*(\omega)} |t-u|^{\gamma}, & |t-u| > 2^{-n^*(\omega)}, \end{cases} \\ &\leq \underbrace{C \cdot 2^{n^*(\omega)}}_{L(\omega)} |t-u|^{\gamma}.\end{aligned}$$

6. $\tilde{X}$ is a version of X: By 1. for $s_n \in D, s_n \to t : X_{s_n} \xrightarrow{\mathbb{P}} X_t$ and there exists $\exists(n_k)$ such that $X_{s_{n_k}} \xrightarrow{\mathbb{P}-a.s.} X_t$. By construction, $\mathbb{P}(X_t = \tilde{X}_t) = \mathbb{P}(X_t = \lim_{k\to\infty} X_{s_{n_k}}) = 1$. □

Remark 1.6. Compare this to the very similar moment criterion for tightness (see Stochastic processes I).

Corollary 1.7. *Brownian motion exists and has a.s. γ-Hölder continuous paths for any $\gamma \in (0, 1/2)$.*

Proof. The process $\overline{B}$ satisfying properties (i)-(iii) of a Brownian motion fulfills $\overline{B}_t - \overline{B}_s \sim N(0, t-s)$ for $t \geq s$. Then $\forall m \in \mathbb{N}$:

$$\mathbb{E}\left[\left(\bar{B}_t - \bar{B}_s\right)^{2m}\right] = \mathbb{E}\left[\left(\sqrt{t-s}Z\right)^{2m}\right] = (t-s)^m \mathbb{E}\left[Z^{2m}\right] = (t-s)^m (2m-1)(2m-3)\cdots 1$$

for $Z \sim N(0,1)$. With respect to the conditions in the theorem of Kolmogorov-Chentsov we observe:

$$\begin{aligned} m = 1: &\quad \beta = 0 &&\text{(not yet...)}, \\ m = 2: &\quad \beta = 1 &&(\text{yes}, \beta > 0) \Rightarrow \gamma < 1/4 \, \text{(not enough regularity)}, \\ m \in \mathbb{N}: &\quad \beta = m-1 &&\Rightarrow \gamma < \frac{m-1}{2m}. \end{aligned}$$

Since m is arbitrary, there is for each $\gamma < \sup_{m\geq 1} \frac{m-1}{2m} = \frac{1}{2}$ a version $\tilde{B}$ of $\overline{B}$ with γ-Hölder continuous paths on $[0,1]$. Having constructed $(\tilde{B}_t, 0 \leq t \leq 1)$, we can take independent copies $(\tilde{B}_t^{(n)}, t \in [0,1])_{n\geq 1}$, i.e. $\tilde{B}^{(n)} \overset{d}{=} \tilde{B}$ and all independent, e.g. on a product space. Define $B_t = \sum_{n=1}^{\lfloor t \rfloor} \tilde{B}_1^{(n)} + \tilde{B}_{t-\lfloor t \rfloor}^{(\lfloor t \rfloor + 1)}$. It is easy to check that $(B_t, t \geq 0)$ is then a Brownian motion. □

Approach by Wiener-Lévy, Cisielski, Itô-Nisio

Idea: "white noise" $\Gamma_t(\omega) := \sum_{k=1}^{\infty} Y_k(\omega)\varphi_k(t)$ for $Y_k \overset{\text{iid}}{\sim} N(0,1)$ and a complete orthonormal system ("basis") $(\varphi_k)_{k\geq 1}$ of $L^2([0,1])$ (see exercises). The anti-derivative should define a Brownian motion

$$B_t(\omega) := \sum_{k=1}^{\infty} Y_k(\omega)\Phi_k(t) \tag{1.2.1}$$

with $\Phi_k(t) = \int_0^t \varphi_k(s)\, ds$.

Theorem 1.8. *(1.2.1) defines a Brownian motion on $[0,1]$ where the sum converges uniformly in probability, i.e.*

$$\forall \varepsilon > 0 : \lim_{n\to\infty} \mathbb{P}\left(\sup_{N\geq n} \sup_{t\in[0,1]} \left|\sum_{k=n+1}^{N} Y_k \Phi_k(t)\right| > \varepsilon\right) = 0.$$

Proof. 1. Pointwise for $t \in [0,1]$*:* Set $M_n(\omega) := \sum_{k=1}^n Y_k(\omega)\Phi_k(t)$, $\mathcal{F}_n = \sigma(Y_1,\dots,Y_n)$. Then $(M_n, \mathcal{F}_n)$ is a martingale, because

$$\mathbb{E}\left[M_{n+1}|\,\mathcal{F}_n\right] = M_n + \mathbb{E}\left[Y_{n+1}|\,\mathcal{F}_n\right]\Phi_{n+1}(t) = M_n + 0 = M_n.$$

(M_n) is L^2-bounded:

$$\begin{aligned}
\mathbb{E}\left[M_n^2\right] &= \sum_{k=1}^n \mathbb{E}\left[Y_k^2\right]\Phi_k^2(t) \\
&= \sum_{k=1}^n \langle \mathbf{1}_{[0,t]}, \varphi_k\rangle_{L^2}^2 \\
&\le \sum_{k=1}^\infty < \mathbf{1}_{[0,t]}, \varphi_k >^2 \\
&\overset{\text{Parseval identity}}{=} \|\mathbf{1}_{[0,t]}\|_{L^2}^2 \\
&= t \\
&< \infty.
\end{aligned}$$

By the 2nd martingale convergence theorem (M_n) converges almost surely and in L^2 to some $M_\infty = B_t \in L^2$. We know that $M_n \sim N(0, \sum_{k=1}^n \Phi_k^2(t))$. Since $\sum_{k=1}^n \Phi_k^2(t) \xrightarrow{n\to\infty} t$ (the φ_k form an orthonormal basis, see above), we have $M_n \xrightarrow{d} N(0,t)$, implying $B_t \sim N(0,t)$.

2. Independent and stationary increments: For $0 \le t_0 < t_1 < \dots < t_m \le 1$ it holds

$$\begin{aligned}
&\sum_{k=1}^n Y_k \underbrace{\left(\Phi_k(t_1) - \Phi_k(t_0), \dots, \Phi_k(t_m) - \Phi_k(t_{m-1})\right)}_{\mathbb{R}^m} \\
&\sim N\left(0, \left(\sum_{k=1}^n \left(\Phi_k(t_i) - \Phi_k(t_{i-1})\right)\cdot\left(\Phi_k(t_j) - \Phi_k(t_{j-1})\right)\right)_{1\le i,j\le m}\right).
\end{aligned}$$

Noting that $\Phi_k(t_i) - \Phi_k(t_{i-1}) = \int_{t_{i-1}}^{t_i} \varphi_k\,ds = \langle \mathbf{1}_{[t_{i-1},t_i]}, \varphi_k\rangle_{L^2}$ and $\langle f,g\rangle_{L^2} = \sum_{k=1}^\infty \langle f,\varphi_k\rangle_{L^2}\langle g,\varphi_k\rangle_{L^2}$ (polarisation of Parseval identity) we see

$$\lim_{n\to\infty}\sum_{k=1}^n \left(\Phi_k(t_i) - \Phi_k(t_{i-1})\right)\left(\Phi_k(t_j) - \Phi_k(t_{j-1})\right) = \langle \mathbf{1}_{[t_{i-1},t_i]}, \mathbf{1}_{[t_{j-1},t_j]}\rangle_{L^2} = \delta_{i,j}.$$

As above $(B_{t_1} - B_{t_0}, \dots, B_{t_m} - B_{t_{m-1}}) \sim N(0, \operatorname{diag}(t_i - t_{i-1})_{1\le i\le m})$.

3. Proof of continuity for the case of Haar basis: For the Haar basis consider double indices (j,k) with $j \in \mathbb{N}_0$, $k = 0,\dots,2^j-1$, and functions

$$\begin{aligned}
\varphi_0(t) &= \mathbf{1}_{[0,1]}, \\
\psi_{0,0}(t) &= \mathbf{1}_{[0,1/2]} - \mathbf{1}_{(1/2,1]}, \\
\psi_{j,k}(t) &= 2^{j/2}\psi_{0,0}\left(2^j t - k\right).
\end{aligned}$$

Then $(\psi_{j,k}) \cup \{\varphi_0\}$ is a complete orthonormal system in $L^2([0,1])$. The anti-derivatives are

$$\begin{aligned}
\Phi_0(t) &= t, \\
\Psi_{0,0}(t) &= t \wedge (1-t)
\end{aligned}$$

(the Ψ are "hat functions" or "linear B-splines"). Then

$$\Psi_{j,k}(t) = 2^{-j/2}\Psi_{0,0}\left(2^j t - k\right).$$

Consider

$$\triangle_j(\omega) := \sup_{0\le t\le 1}\left|\sum_{k=0}^{2^j-1} Y_{j,k}(\omega)\,\Psi_{j,k}(t)\right| \le \underbrace{\max_{k=0,\dots 2^j-1}|Y_{j,k}(\omega)|}_{=:M_j(\omega)}\cdot 2^{-(j+1)/2}.$$

Then

$$\begin{aligned}\mathbb{P}\left(\sup_{J'\ge J}\left|\sum_{j=J}^{J'}\sum_{k=0}^{2^j-1} Y_{j,k}\Psi_{j,k}(t)\right| > \varepsilon\right) &\le \mathbb{P}\left(\sum_{j=J}^{\infty} 2^{-(j+1)/2}M_j > \varepsilon\right)\\ &\le \mathbb{P}\left(\exists j\ge J : 2^{-(j+1)/2}M_j > 2^{-(j-J)/2}\left(1-\frac{1}{\sqrt{2}}\right)\varepsilon\right)\\ &\le \sum_{j\ge J}\mathbb{P}\left(M_j > 2^{J/2}\left(\sqrt{2}-1\right)\varepsilon\right).\end{aligned}$$

where we use that $\sum_{j\ge J} 2^{-(j-J)/2} = \frac{1}{1-2^{-1/2}}$. Now use for $Z\sim N(0,1)$ that $\mathbb{P}(|Z|>t)\le e^{-t^2/2}$ for any $t\ge 1$ (see Lemma 1.15) we obtain

$$\mathbb{P}\left(\sup_{J'\ge J}\left|\sum_{j=J}^{J'}\sum_{k=0}^{2^j-1} Y_{j,k}\Psi_{j,k}(t)\right| > \varepsilon\right) \le \sum_{j\ge J}\sum_{k=0}^{2^j-1}\exp(-2^J(\sqrt{2}-1)^2\varepsilon^2/2) \xrightarrow{J\to\infty} 0.$$

Hence, along a subsequence $J_n\to\infty$ we have a.s. convergence. Therefore, with probability 1 are the continuous functions

$$B_t^J(\omega) = Y_{0,0}(\omega)\Phi_0(t) + \sum_{j=0}^{J}\sum_{k=0}^{2^j-1} Y_{j,k}(\omega)\Psi_{j,k}(t)$$

converge uniformly to $B_t(\omega)$. □

Remark 1.9.

(i) For $\gamma\in(0,1/2)$ we even have $\sum_{i=J}^{\infty}\triangle_j(\omega)2^{\gamma j}\xrightarrow{\mathbb{P}} 0$. This implies (direct calculations or wavelet theory) also that B_t has γ-Hölder continuous paths.

(ii) This construction offers another way (beyond Donsker) to simulate Brownian motion by approximations $B_t^J(\omega)$ (dyadic refinements of Brownian motion).

1.3 Properties of Brownian sample paths

We start by considering the quadratic variation of Brownian paths. Let τ_n, $n\ge 1$, be a sequence of partitions of $[0,1]$, $\tau_n\subseteq\tau_{n+1}$, for all n and $\max_{t_i\in\tau_n}|t_{i+1}-t_i|\xrightarrow{n\to\infty} 0$. An example is $\tau_n = D_n$ from the previous proof.

Theorem 1.10. *For each $t\in[0,1]$ let $S_t^n := \sum_{t_i\in\tau_n, t_i\le t}(B_{t_{i+1}}-B_{t_i})^2$. Then we have*

$$\lim_{n\to\infty} S_t^n = t \quad \text{a.s. and in } L^2.$$

Remark 1.11. The limit is called *quadratic variation* in analogy to the variation of a function f:

$$V_{[0,t]}(f) = \sup_{\tau}\sum_{t_i\in\tau, t_i\le t}|f(t_{i+1})-f(t_i)|,$$

where the supremum ranges over all partitions of $[0, t]$. If $V_{[0,t]}(f) < \infty$ for all $t \geq 0$, then f is of finite/bounded variation. If f is continuous, it can be shown that

$$V_{[0,t]}(f) = \lim_{n\to\infty} \sum_{t_i \in \tau_n, t_i \leq t} |f(t_{i+1}) - f(t_i)|$$

holds for any sequence of partitions τ_n such that $\max_{t_i \in \tau_n} |t_{i+1} - t_i| \xrightarrow{n\to\infty} 0$.

Proof. 1. L^2-convergence: We have $\mathbb{E}[S_t^n] = \sum_{t_i \in \tau_n, t_i \leq t}(t_{i+1} - t_i) \to t$ and

$$\begin{aligned}
\operatorname{Var}(S_t^n) &= \sum_{t_i \in \tau_n, t_i \leq t} \operatorname{Var}\left(\left(B_{t_{i+1}} - B_{t_i}\right)^2\right) \\
&= 2 \sum_{t_i \in \tau_n, t_i \leq t} (t_{i+1} - t_i)^2 \\
&\leq 2 \max_{t_i \in \tau_n} |t_{i+1} - t_i| \underbrace{\sum_{t_i \in \tau_n} (t_{i+1} - t_i)}_{\to t} \\
&\to 0.
\end{aligned}$$

Hence, $S_t^n \xrightarrow{L^2} t$.

2. a.s. convergence for $\tau_n = D_n$: From 1. and $t_i \in \tau_n$ with $t_{i+1} - t_i = 2^{-n}$ we have $\mathbb{E}[(S_t^n - t)^2] \leq 2 \cdot 2^{-n}$, if $t \in D_n$. Hence, $\sum_{n\geq 1} \mathbb{E}[(S_t^n - t)^2] < \infty$. By Chebyshev inequality and Borel-Cantelli we obtain a.s. convergence (quick L^2-convergence implies a.s. convergence).

3. a.s. convergence for any (τ_n): Let $\mathscr{G}_n := \sigma((B_{t_{i+1}} - B_{t_i})^2, t_i \in \tau_m, m \geq n)$. Then $\mathscr{G}_n \supseteq \mathscr{G}_{n+1}$ holds. We show for $t \in \tau_n$: $S_t^n = \mathbb{E}[B_t^2 | \mathscr{G}_n]$. Interpreting "$n$" as "$-n$", this implies that $(S_t^n, \mathscr{G}_n)$ is a backwards martingale such that $S_t^n \xrightarrow{a.s.} \mathbb{E}[B_t^2 | \bigcap_{n\geq 1} \mathscr{G}_n]$. By 1. we must have have $\mathbb{E}[B_t^2 | \bigcap_{n\geq 1} \mathscr{G}_n] = t$. Hence, consider (wlog $t_1 = 0$)

$$\begin{aligned}
\mathbb{E}\left[B_t^2 \middle| \mathscr{G}_n\right] &= \mathbb{E}\left[\left(\sum_{t_i \in \tau_n, t_i \leq t} (B_{t_{i+1}} - B_{t_i})\right)^2 \middle| \mathscr{G}_n\right] \\
&= S_t^n + \sum_{i \neq j} \mathbb{E}\left[\left(B_{t_{i+1}} - B_{t_i}\right)\left(B_{t_{j+1}} - B_{t_j}\right) \middle| \mathscr{G}_n\right] \\
&= S_t^n + \sum_{i \neq j} \left|B_{t_{i+1}} - B_{t_i}\right| \cdot \left|B_{t_{j+1}} - B_{t_j}\right| \\
&\quad \cdot \mathbb{E}\left[\operatorname{sgn}\left(B_{t_{i+1}} - B_{t_i}\right) \operatorname{sgn}\left(B_{t_{i+1}} - B_{t_j}\right) \middle| \mathscr{G}_n\right] \\
&= S_t^n.
\end{aligned}$$

A precise argument for the conditional expectation uses that

$$\tilde{B}_t = \begin{cases} B_t, & t \leq t_i, \\ B_{t_i} - (B_t - B_{t_i}) & t > t_i, \end{cases}$$

is again a Brownian motion with $|\tilde{B}_{t_{i+1}} - \tilde{B}_{t_i}| = |B_{t_{i+1}} - B_{t_i}|$ but $\operatorname{sgn}(\tilde{B}_{t_{i+1}} - \tilde{B}_{t_i}) = -\operatorname{sgn}(B_{t_{i+1}} - B_{t_i})$. □

Remark 1.12. Even without the nestedness $\tau_n \subseteq \tau_{n+1}$ we have L^2-convergence, but not necessarily a.s. convergence.

Corollary 1.13. *A typical Brownian path is on no interval of finite variation, i.e.* $\mathbb{P}(\exists\, 0 \leq a \leq b \leq 1 : V_{[a,b]}(B) < \infty) = 0$. *In particular, Brownian motion is on no interval differentiable with probability one.*

Proof. If $V_{[a,b]}(B(\omega)) < \infty$, then

$$\sum_{t_i\in\tau_n, t_i\in[a,b]} \left(B_{t_{i+1}}(\omega) - B_{t_i}(\omega)\right)^2 \leq \underbrace{\max_{t_i\in\tau_n} \left|B_{t_{i+1}}(\omega) - B_{t_i}(\omega)\right|}_{\xrightarrow{n\to\infty} 0 \text{ (uniform continuity)}} \cdot \underbrace{\sum_{t_i\in\tau_n, t_i\in[a,b]} \left|B_{t_{i+1}}(\omega) - B_{t_i}(\omega)\right|}_{\xrightarrow{n\to\infty} V_{[a,b]}(B_\cdot(\omega))},$$

but the left hand side converges a.s. to $b - a > 0$. This is a contradiction! Finally, note that a differentiable function is of finite variation. □

Without proof let us state the much stronger result.

Theorem 1.14 (Paley, Wigner, Zygmund (1933))**.** *With probability one a Brownian path is nowhere differentiable.*

Proof. See Karatzas (1991). □

Lemma 1.15. *For $Z \sim N(0,1)$, $a > 0$, we have*

a) $\mathbb{P}(Z \geq a) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{a} e^{-a^2/2}$,

b) $\mathbb{P}(Z \geq a) \geq \frac{1}{\sqrt{2\pi}} \frac{1}{a+\frac{1}{a}} e^{-a^2/2}$.

Proof. a) use for $x \geq a$: $e^{-x^2/2} \leq \frac{x}{a} e^{-x^2/2}$, then integrate.

b) use for $x \geq a$: $e^{-x^2/2} \geq \frac{1}{1+1/a^2}(1 + \frac{1}{x^2})e^{-x^2/2} = \frac{1}{1+1/a^2}(-x^{-1}e^{-x^2/2})$ and integration. □

Theorem 1.16 (Law of iterated logarithm, Khinchine (1933))**.** *For a Brownian motion B and almost all $\omega \in \Omega$ we have*

a) $\limsup_{t\to 0} \frac{B(\omega)}{\sqrt{2t\log(\log(t^{-1}))}} = 1$,

b) $\liminf_{t\to 0} \frac{B_t(\omega)}{\sqrt{2t\log(\log(t^{-1}))}} = -1$,

c) $\limsup_{t\to\infty} \frac{B_t(\omega)}{\sqrt{2t\log(\log(t))}} = 1$,

d) $\liminf_{t\to\infty} \frac{B_t(\omega)}{\sqrt{2t\log(\log(t))}} = -1$.

Proof. By symmetry $-B_t$ is again a Brownian motion such that (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d). Moreover, by time inversion $X_t = t \cdot B_{1/t}$, $t > 0$, $X_0 = 0$, is also a Brownian motion (Stochastic processes I). We infer from (a) for X that

$$\limsup_{t\to 0} \frac{tB_{1/t}(\omega)}{\sqrt{2t\log(\log(t))}} = 1.$$

Letting $s = t^{-1}$ we obtain

$$\limsup_{s\to\infty} \frac{B_s(\omega)}{\sqrt{2s\log(\log(s))}} = 1,$$

which is (c). Hence, it suffices to prove (a).

Let $h(t) = \sqrt{2t\log(\log t^{-1})}$. The proof for

$$\limsup_{t\to 0} \frac{B_t}{h(t)} \le 1 \text{ a.s.} \tag{1.3.1}$$

will be given after Theorem 1.29. We show now that $\limsup_{t\to 0} \frac{B_t}{h(t)} \ge 1$ a.s. using the 2nd part of Borel-Cantelli. Fix $\vartheta \in (0,1)$ and set $A_n := \{B_{\vartheta^n} - B_{\vartheta^{n+1}} \ge \sqrt{1-\vartheta}h(\vartheta^n)\}$. By Lemma 1.15 we obtain for $x = \sqrt{2\log(n) + 2\log(\log(\vartheta^{-1}))}$

$$\mathbb{P}(A_n) = \mathbb{P}\left(\frac{B_{\vartheta^n} - B_{\vartheta^{n+1}}}{\sqrt{\vartheta^n - \vartheta^{n+1}}} \ge x\right) \overset{\text{Lemma}}{\ge} \frac{e^{-x^2/2}}{\sqrt{2\pi}\left(x + \frac{1}{x}\right)} \ge c \cdot \frac{1}{n\sqrt{\log n}}$$

for some constant $c > 0$ and $n > |1/\log\vartheta|$. Since $\sum_{n\ge 1} \frac{1}{n\sqrt{\log n}} = \infty$ and $(A_n)_{n\ge 1}$ are independent, Borel-Cantelli yields $\mathbb{P}(A_n \text{ infinitely often}) = 1$. The upper bound in (1.3.1) applied to $(-B_t)$ shows (with bounding small terms by 2 twice) that

$$-B_{\vartheta^{n+1}}(\omega) \le 2h\left(\vartheta^{n+1}\right) \le 4\vartheta^{1/2}h\left(\vartheta^n\right)$$

for all $n \ge N(\omega)$ and $\omega \in \Omega^*, \mathbb{P}(\Omega^*) = 1$. Hence, we have a.s.

$$\frac{B_{\vartheta^m}}{h(\vartheta^m)} = \frac{B_{\vartheta^m} - B_{\vartheta^{m+1}}}{h(\vartheta^m)} + \frac{B_{\vartheta^{m+1}}}{h(\vartheta^m)} \ge \sqrt{1-\vartheta} - 4\vartheta^{1/2}$$

holds for infinitely many $m \ge 1$. Therefore

$$\mathbb{P}\left(\limsup_{t\to 0} \frac{B_t}{h(t)} \ge \sqrt{1-\vartheta} - 4\vartheta^{1/2}\right) = 1$$

for any $\vartheta \in (0,1)$. Take $\vartheta_k \to 0$ to conclude

$$\mathbb{P}\left(\limsup_{t\to 0} \frac{B_t}{h(t)} \ge 1\right) = 1.$$

Except for the gap in 1.3.1 we are done. □

Without proof let us state the main result for the modulus of continuity.

Theorem 1.17 (Lévy, 1937). *It holds*

$$\mathbb{P}\left(\limsup_{\delta\to 0} \frac{1}{\sqrt{2\delta\log\delta^{-1}}} \max_{0\le s\le t\le 1, t-s\le\delta} |B_t - B_s| = 1\right) = 1.$$

Proof. See Karatzas (1991). □

1.4 Brownian motion as martingale and Markov process

Definition 1.18. A process $(X_t, t \ge 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called

a) *adapted* to a filtration $(\mathcal{F}_t)_{t\ge 0}$ if X_t is $\mathcal{F}_t$-measurable for all $t \ge 0$,

b) $(\mathcal{F}_t)_{t\ge 0}$-*martingale* if X is adapted, $X_t \in L^1(\mathbb{P})$ and $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for all $0 \le s \le t$.

c) $(\mathcal{F}_t)_{t\ge 0}$-*Brownian martingale* if X is continuous, adapted, $X_t - X_s \sim N(0, t-s)$ and if $X_t - X_s$ is independent of $\mathcal{F}_s$ (written $X_t - X_s \perp \mathcal{F}_s$) for all $s \le t$.

Remark 1.19. Any Brownian motion is also a Brownian motion with respect to its own filtration $\mathcal{F}_t^0 = \sigma(B_s, s \le t)$.

Proposition 1.20. *The following processes, derived from a Brownian motion B with respect to $(\mathcal{F}_t)_{t\geq 0}$, are $(\mathcal{F}_t)_{t\geq 0}$-martingales.*

a) $M_t = B_t, t \geq 0,$

b) $M_t = B_t^2 - t, t \geq 0,$

c) $M_t = \exp(\lambda B_t - \frac{\lambda^2}{2} t), t \geq 0,$ *for all* $\lambda \in \mathbb{R}$.

Proof. Adaptedness is clear in all cases. Just check the martingale property for $t > s$.

a) $\mathbb{E}[B_t|\mathcal{F}_s] = \mathbb{E}[B_s + (B_t - B_s)|\mathcal{F}_s] = B_s + \mathbb{E}[B_t - B_s] = B_s$.

b) $\mathbb{E}[B_t^2 - B_s^2|\mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^2 + 2(B_t - B_s)B_s|\mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^2] + 2B_s\mathbb{E}[B_t - B_s] = t - s$. Rearranging the terms proves the claim.

c) We have

$$\begin{aligned}
\mathbb{E}\left[\left.\frac{\exp\left(\lambda B_t - \frac{\lambda^2}{2}t\right)}{\exp\left(\lambda B_s - \frac{\lambda^2}{2}s\right)}\right| \mathcal{F}_s\right] &= \mathbb{E}\left[\left.\exp\left(\lambda\left(B_t - B_s\right) - \frac{\lambda^2}{2}\left(t-s\right)\right)\right| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[e^{\lambda(B_t - B_s)}\right] e^{-\frac{\lambda^2(t-s)}{2}} \\
&\overset{Z\sim N(0,1)}{=} \mathbb{E}\left[e^{\lambda\sqrt{t-s}Z}\right] e^{-\frac{\lambda^2(t-s)}{2}} \\
&= 1,
\end{aligned}$$

where we used that the moment generating function of a Gaussian satisfies $\mathbb{E}[e^{\lambda Z}] = e^{\frac{\lambda^2}{2}}$.

□

Theorem 1.21. *If $(B_t, t \geq 0)$ is a Brownian motion with respect to any filtration $(\mathcal{F}_t^0)_{t\geq 0}$, then also with respect to its right-continuous extension $\mathcal{F}_t := \mathcal{F}_{t+}^0 = \bigcap_{s>t} \mathcal{F}_s^0$.*

Proof. We show a little more general statement, i.e. we show

$$\mathbb{E}\left[f\left(B_{t+h} - B_t\right)\varphi_t\right] = \mathbb{E}\left[f\left(B_{t+h} - B_t\right)\right]\mathbb{E}\left[\varphi_t\right]$$

for $h > 0$, any bounded $\mathcal{F}_t$-measurable φ_t and any bounded Borel-measurable f (the statement follows then from choosing $f = \mathbf{1}_A$, $\varphi_t = \mathbf{1}_B$ for any $A \in \mathscr{B}_{\mathbb{R}}$, $B \in \mathcal{F}_t$). It suffices to consider $f \in \mathcal{C}_b(\mathbb{R})$ (approximate the open intervals in $\mathbb{R}$ by such functions and use the monotone class theorem). For $\varepsilon_n \to 0$

$$\begin{aligned}
\mathbb{E}\left[f\left(B_{t+h} - B_t\right)\varphi_t\right] &= \mathbb{E}\left[\lim_{n\to\infty} f\left(B_{t+h} - B_{t+\varepsilon_n}\right)\varphi_t\right] \\
&\overset{\text{Dom. conv.}}{=} \lim_{n\to\infty} \mathbb{E}\Big[\underbrace{f\left(B_{t+h} - B_{t+\varepsilon_n}\right)}_{\perp \mathcal{F}_{t+\varepsilon_n}^0 \supseteq \mathcal{F}_t}\underbrace{\varphi_t}_{\in \mathcal{F}_t}\Big] \\
&= \lim_{n\to\infty} \mathbb{E}\left[f\left(B_{t+h} - B_{t+\varepsilon_n}\right)\right]\mathbb{E}\left[\varphi_t\right] \\
&\overset{\text{Dom. conv.}}{=} \mathbb{E}\left[f\left(B_{t+h} - B_t\right)\right]\mathbb{E}\left[\varphi_t\right].
\end{aligned}$$

□

Remark 1.22.

a) $(\mathcal{F}_t)_{t\geq 0}$ is usually larger than $(\mathcal{F}_t^0)_{t\geq 0}$, admitting infinitesimal looks into the future. This allows larger classes of stopping times.

b) $(\mathcal{F}_t)_{t\geq 0}$ is itself right-continuous: $\mathcal{F}_t \subseteq \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} \subseteq \bigcap_{\varepsilon>0} \mathcal{F}^0_{t+2\varepsilon} = \mathcal{F}_t$.

Definition 1.23. For a filtration $(\mathcal{F}_t)_{t\geq 0}$ a random variable $\tau : \Omega \to [0,\infty]$ is called $(\mathcal{F}_t)_{t\geq 0}$-stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. We set $\mathcal{F}_\tau := \{A \in \mathcal{F} | A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$.

Example 1.24. Let $(X_t, t \geq 0)$ be adapted to a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. Is $\tau_A := \inf\{t \geq 0 : X_t \in A\}$ a stopping time for a Borel set A?

a) A open, (X_t) is right-continuous: $\{\tau_A < t\} = \bigcup_{r\in\mathbb{Q}, r<t} \underbrace{\{X_r \in A\}}_{\in \mathcal{F}_r \subseteq \mathcal{F}_t} \in \mathcal{F}_t$. Since $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, we have also $\{\tau_A \leq t\} \in \mathcal{F}_{t+} = \mathcal{F}_t$.

b) F closed, (X_t) continuous. Any open set O can be written as $O = \bigcup_{n\geq 1} F_n$ with F_n closed (e.g. $F_n = \overline{B(x_n, r_n)}$). Hence, any closed set F can be written as $F = \bigcap_{n\geq 1} U_n$, U_n open. Thus,
$$\{\tau_F \leq t\} = \bigcap_{n\geq 1} \{\tau_{U_n \leq t}\} \in \mathcal{F}_t.$$

c) Any Borel set A? That's very complicated...

The following facts are proved in the exercises.(the following is adapted from the lecture)

Theorem 1.25. *We have for a filtration $(\mathcal{F}_t)_{t\geq 0}$.*

a) $\mathcal{F}_\tau$ is a σ-algebra. If $\tau = t$ is deterministic, then $\mathcal{F}_\tau = \mathcal{F}_t$.

b) If τ is a bounded $(\mathcal{F}_t)_{t\geq 0}$-stopping time and $(X_t, t \geq 0)$ is a right-continuous process, then X_τ is well defined and $\mathcal{F}_\tau$-measurable. In particular, X_τ is $\mathcal{F}$-measurable.

c) If τ, σ are $(\mathcal{F}_t)_{t\geq 0}$-stopping times, then $\tau \wedge \sigma$ and $\tau \vee \sigma$ are stopping times, as well.

d) If τ, σ are $(\mathcal{F}_t)_{t\geq 0}$-stopping times and X is a $\mathcal{F}_\tau$-measurable random variable, then $\{\tau \leq \sigma\} \in \mathcal{F}_{\tau\wedge\sigma}$ and $X\mathbf{1}_{\{\tau\leq\sigma\}}$ is $\mathcal{F}_{\tau\wedge\sigma}$-measurable.

Proof. See exercises. □

Theorem 1.26. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and $(M_t, t \geq 0)$ a right-continuous $(\mathcal{F}_t)_{t\geq 0}$-adapted and integrable process. Then the following statements are equivalent:*

a) $(M_t, t \geq 0)$ is a $(\mathcal{F}_t)_{t\geq 0}$-martingale.

b) For all bounded $(\mathcal{F}_t)_{t\geq 0}$-stopping times τ holds $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$.

c) (Optional sampling) For all bounded $(\mathcal{F}_t)_{t\geq 0}$-stopping times $\sigma \leq \tau$ holds $\mathbb{E}\left[M_\tau | \mathcal{F}_\sigma\right] = M_\sigma$.

d) (Optional stopping) For all $(\mathcal{F}_t)_{t\geq 0}$-stopping times τ the stopped process $M_t^\tau := M_{\tau\wedge t}$, $t \geq 0$, is an $(\mathcal{F}_t)_{t\geq 0}$-martingale.

Proof. See exercises. □

Proposition 1.27. *Let $(M_t, t \geq 0)$ be a right-continuous martingale or a right-continuous non-negative submartingale and $\lambda > 0$. Then*

$$\mathbb{P}\left(\sup_{0\leq t\leq T}|M_t| \geq \lambda\right) \leq \frac{1}{\lambda}\mathbb{E}\left[|M_T|\right] \quad \textit{(}\text{Maximal inequality}\textit{)}$$

for all $T > 0$. Moreover, for $p > 1$ and $M_T \in L^p$ we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|M_t|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|M_T|^p\right] \quad \textit{(}\text{Doob's inequality}\textit{)}.$$

Proof. (based on Stochastic processes I. Set $S_n = \{k2^{-n}T : k = 0, \ldots, 2^n\}$ and consider the discrete-time martingale $(M_k, k \in S_n)$. Then by the discrete-time Doob's inequality

$$\mathbb{P}\left(\sup_{k\in S_n}|M_k| > \lambda\right) \leq \frac{\mathbb{E}\left[|M_T|\right]}{\lambda}.$$

Since $\mathbf{1}_{\{\sup_{k\in S_n}|M_k|>\lambda\}} \to \mathbf{1}_{\{\sup_{0\leq k\leq T}|M_k|>\lambda\}}$ as $n \to \infty$ by right-continuity, dominated convergence yields

$$\mathbb{P}\left(\sup_{0\leq t\leq T}|M_t| > \lambda\right) = \lim_{n\to\infty}\mathbb{P}\left(\sup_{k\in S_n}|M_k| > \lambda\right) \leq \frac{\mathbb{E}\left[|M_T|\right]}{\lambda}.$$

(adapted from the lecture) From this we deduce

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{0\leq t\leq T}|M_t| \geq \lambda + \frac{1}{n}\right) \leq \limsup_{n\to\infty}\frac{\mathbb{E}\left[|M_T|\right]}{\lambda+\frac{1}{n}} = \frac{\mathbb{E}\left[|M_T|\right]}{\lambda}$$

Doob's inequality follows in the same way from Stochastic processes I. □

1.4.1 Ruin problems

Let $\tau_{a,b} = \min\{t > 0 : B_t \notin (a, b)\}$ for $(a < 0 < b)$. $\tau_{a,b}$ is a stopping time (see above).

Theorem 1.28. *It holds*

a) $\mathbb{P}(B_{\tau_{a,b}} = b) = \frac{|a|}{|a|+b}$,

b) $\mathbb{P}(B_{\tau_{a,b}} = a) = \frac{b}{|a|+b}$,

c) $\mathbb{E}\left[\tau_{a,b}\right] = |a| \cdot b$.

Proof. $M_t = B_t^2 - t$ is a martingale. Hence, by the stopping theorem

$$\mathbb{E}\Big[\underbrace{B^2_{\tau_{a,b}\wedge m}}_{\leq(|a|+b)^2}\Big] = \mathbb{E}\Big[\underbrace{\tau_{a,b}\wedge m}_{\to\tau_{a,b}}\Big]$$

for any $m > 0$. The left hand side is bounded and the right hand side is monotone in m such that (adapted from the lecture)

$$\infty > \limsup_m \mathbb{E}\left[B^2_{\tau_{a,b}\wedge m}\right] = \limsup_m \mathbb{E}\left[\tau_{a,b}\wedge m\right] = \lim_m \mathbb{E}\left[\tau_{a,b}\wedge m\right] = \mathbb{E}\left[\tau_{a,b}\right].$$

Hence, $\tau_{a,b} < \infty$ $\mathbb{P}$-a.s. Using dominated convergence on the left and monotone convergence on the right as $m \to \infty$, we conclude

$$\mathbb{E}\left[B^2_{\tau_{a,b}}\right] = \mathbb{E}\left[\tau_{a,b}\right].$$

In particular, $\mathbb{P}(\tau_{a,b} < \infty) = 1$. Now use that $(B_{t\wedge\tau_{a,b}})_{t\geq 0}$ is a martingale such that $\mathbb{E}\left[B_{t\wedge\tau_{a,b}}\right] = 0$. We obtain $\mathbb{E}\left[B_{\tau_{a,b}}\right] = 0$ by proving that $(B_{m\wedge\tau_{a,b}})_{m\in\mathbb{N}}$ is uniformly integrable. This is done as in Stochastic processes I and is also called Wald identity. (or use dominated convergence, right?)Then

$$0 = \mathbb{E}\left[B_{\tau_{a,b}}\right] = \mathbb{P}\left(B_{\tau_{a,b}} = a\right) \cdot a + \left(1 - \mathbb{P}\left(B_{\tau_{a,b}} = a\right)\right) \cdot b$$

and thus

$$\mathbb{P}\left(B_{\tau_{a,b}} = a\right) = \frac{b}{|a| + b}.$$

Finally,

$$\mathbb{E}\left[B^2_{\tau_{a,b}}\right] = a^2\mathbb{P}\left(B_{\tau_{a,b}} = a\right) + b^2\left(1 - \mathbb{P}\left(B_{\tau_{a,b}} = a\right)\right) = |a| \cdot b.$$

□

Now consider $\tau_b = \inf\{t \geq 0 : B_t \geq b\}$ for $b > 0$. We have $\mathbb{P}(\tau_b < \infty) = 1$, but $\mathbb{E}[\tau_b] = \infty$, because

a) $\mathbb{P}\left(\tau_b < \infty\right) \geq \mathbb{P}\left(\tau_b = \tau_{a,b}\right) = \frac{|a|}{|a|+b} \to 1$, as $a \to -\infty$,

b) $\mathbb{E}[\tau_b] \geq \mathbb{E}[\tau_{a,b}] = |a| \cdot b \to \infty$, as $a \to -\infty$.

What about the exact law of τ_b?

Theorem 1.29 (Laplace transform of $\mathbb{P}^{\tau_b}$)**.** *We have for any $\lambda > 0$:*

$$\mathbb{E}\left[e^{-\lambda\tau_b}\right] = e^{-b\sqrt{2\lambda}}.$$

This means that τ_b has the Lebesgue density

$$f_b\left(t\right) = \frac{|b|}{\sqrt{2\pi t^3}}e^{-b^2/2t}\mathbf{1}_{\mathbb{R}_+}\left(t\right) \quad (\frac{1}{2}\text{-stable distribution}).$$

Proof. $M_t = e^{\alpha B_t - \frac{\alpha^2}{2}t}$ is a martingale and we have $0 \leq M_{t\wedge\tau_b} \leqslant e^{\alpha b}$, $\tau_b < \infty$ a.s. Thus we have for any t and by dominated convergence, as well as setting finally $\lambda = \frac{\alpha^2}{2}$, that

$$1 = \mathbb{E}\left[M_0\right] = \mathbb{E}\left[M_{t\wedge\tau_b}\right] = \mathbb{E}\left[M_{\tau_b}\right] = \mathbb{E}\left[e^{\alpha b - \frac{\alpha^2}{2}\tau_b}\right] = \mathbb{E}\left[e^{-\lambda\tau_b}\right]e^{b\sqrt{2\lambda}}.$$

Calculating the Laplace transform of f_b yields the density. □

Proof of the law of the iterated logarithm completed. It remains to show

$$\mathbb{P}\left(\limsup_{t\to 0}\frac{B_t}{h\left(t\right)} \leq 1\right) = 1$$

for $h(t) = \sqrt{2t\log\log t^{-1}}$. Consider $M_t = \exp(\lambda B_t - \frac{\lambda^2}{2}t)$ and the maximal inequality for M for any $\beta > 0$:

$$\begin{aligned}\mathbb{P}\left(\max_{0\leq s\leq t}\left(B_s - \frac{\lambda}{2}s\right) \geq \beta\right) &= \mathbb{P}\left(\max_{0\leq s\leq t}M_s \geq e^{\lambda\beta}\right) \\ &\leq \frac{\mathbb{E}\left[M_t\right]}{e^{\lambda\beta}} \\ &= e^{-\lambda\beta}.\end{aligned}$$

Take $\vartheta, \delta \in (0,1)$, $\lambda = (1+\delta)\vartheta^{-n}h(\vartheta^n)$, $\beta = \frac{1}{2}h(\vartheta^n)$:

$$\begin{aligned}\mathbb{P}\left(\max_{0\le s\le\vartheta^n}\left(B_s - \frac{\lambda}{2}s\right) \ge \beta\right) &\le \exp\left(-\left(1+\delta\right)\vartheta^{-n}h\left(\vartheta^n\right)^2/2\right)\\ &= \left(n\log\left(\vartheta^{-1}\right)\right)^{-(1+\delta)},\end{aligned}$$

which is summable in n. By Borel-Cantelli there exists $\Omega_{\vartheta,\delta}$ with $\mathbb{P}\left(\Omega_{\vartheta,\delta}\right) = 1$, $N_{\vartheta,\delta}(\omega)$ such that for all $\omega \in \Omega_{\vartheta,\delta}$ and all $n \ge N_{\vartheta,\delta}(\omega)$:

$$\max_{0\le s\le\vartheta^n}\left(B_s\left(\omega\right) - \frac{1+\delta}{2}s\vartheta^{-n}h\left(\vartheta^n\right)\right) < \frac{1}{2}h\left(\vartheta^n\right).$$

Then

$$\begin{aligned}\sup_{t\in(\vartheta^{n-1},\vartheta^n]}\frac{B_t\left(\omega\right)}{h\left(t\right)} &\le \sup_{t\in(\vartheta^{n-1},\vartheta^n]}\frac{\max_{\vartheta^{n-1}\le s\le\vartheta^n}B_s\left(\omega\right)}{h\left(t\right)}\\ &\le \left(1+\frac{\delta}{2}\right)\sup_{t\in(\vartheta^{n-1},\vartheta^n]}\frac{h\left(\vartheta^n\right)}{h\left(t\right)}\\ &\le \left(1+\frac{\delta}{2}\right)\vartheta^{-1/2}.\end{aligned}$$

As $n \to \infty$ we therefore obtain

$$\limsup_{t\to 0}\frac{B_t\left(\omega\right)}{h\left(t\right)} \le \left(1+\frac{\delta}{2}\right)\vartheta^{-1/2}$$

for all $\omega \in \Omega_{\delta,\vartheta}$. Taking $\delta_n \to 0$, $\vartheta_n \to 1$ rational we conclude

$$\mathbb{P}\left(\limsup_{t\to 0}\frac{B_t}{h\left(t\right)} \le 1\right) = 1.$$

□

1.4.2 Strong Markov property and the reflection principle

We shall now study the Markov property of Brownian motion. It is even a strong Markov process (see below) which is not true in general for continuous time processes.

Theorem 1.30. *Let B be a Brownian motion with respect to $(\mathcal{F}_t)_{t\ge0}$ and τ an a.s. finite $(\mathcal{F}_t)_{t\ge0}$-stopping time. Then $\tilde{B}_t := B_{\tau+t} - B_\tau, t \ge 0$, is again a Brownian motion independent of $\mathcal{F}_\tau$, i.e. a Brownian motion has the strong Markov property.*

Proof. We show for $\varphi : \Omega \to \mathbb{R}$ $\mathcal{F}_\tau$-measurable, bounded and $F : C([0,\infty)) \to \mathbb{R}$ Borel-measurable bounded

$$\mathbb{E}\left[\varphi F\left(\left(\tilde{B}_t, t\ge 0\right)\right)\right] = \mathbb{E}\left[\varphi\right]\int F\, d\mathbb{P}^*,$$

where $\mathbb{P}^*$ is the Wiener measure on $C([0,\infty))$. It suffices again to consider $F \in C_b(C([0,\infty)))$ (approximation argument in Polish spaces). Let τ^n be the nth dyadic approximation of τ, i.e. $\tau^n(\omega) \in \{k2^{-n} : k \in \mathbb{N}_0\}$, $\tau^n(\omega) \to \tau(\omega)$. Set $\tilde{B}^n_t(\omega) = B_{\tau_n(\omega)+t}(\omega) - B_{\tau_n(\omega)}(\omega)$. Then

$$\begin{aligned}
\mathbb{E}\left[\varphi F\left(\tilde{B}^n\right)\right] &= \sum_{k\geq 0}\mathbb{E}\left[\varphi F\left(\underbrace{\tilde{B}^n}_{B_{k2^{-n}+t}-B_{k2^{-n}} \text{ again BM, indep. of } \mathcal{F}_{k2^{-n}}}\right)\mathbf{1}_{\{\tau^n = k2^{-n}\}}\right] \\
&= \sum_{k\geq 0}\mathbb{E}\left[\underbrace{\varphi \mathbf{1}_{\{\tau^n=k2^{-n}\}}}_{\mathcal{F}_{k2^{-n}}\text{-mb.}}\right]\int F\, d\mathbb{P}^* \\
&= \mathbb{E}\left[\varphi\right]\int F\, d\mathbb{P}^*.
\end{aligned}$$

Since $F(\tilde{B}^n) \to F(\tilde{B})$, $D \subseteq T$ yields

$$\begin{aligned}
\mathbb{E}\left[\varphi F\left(\tilde{B}\right)\right] &= \lim_{n\to\infty}\mathbb{E}\left[\varphi F\left(\tilde{B}^n\right)\right] \\
&= \mathbb{E}\left[\varphi\right]\int F\, d\mathbb{P}^*.
\end{aligned}$$

□

We apply this to obtain the *reflection principle* (Bachelier 1900).

Theorem 1.31. *It holds* $\mathbb{P}(\tau_b \leq t) = 2\mathbb{P}(B_t \geq b) = \mathbb{P}(|B_t| \geq b)$ *for* $b > 0$.

Proof. We have $\mathbb{P}(B_t \geq b) = \mathbb{P}(B_t \geq b, \tau_b \leq t)$. Writing $B_t - b = B_t - B_{\tau_b} = \tilde{B}_{t-\tau_b}$ this yields

$$\mathbb{P}\left(B_t \geq b\right) = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{[0,\infty)}\left(\tilde{B}_{t-\tau_b}\right)\Big|\, \mathcal{F}_{\tau_b}\right]\mathbf{1}_{\{\tau_b \leq t\}}\right]$$

and by symmetrie (the probability for $\tilde{B}_{t-\tau_b}$ being positive is the same as being negative we have

$$\mathbb{P}\left(B_t \geq b\right) = \frac{1}{2}\mathbb{P}\left(\tau_b \leq t\right).$$

□

Corollary 1.32. *The random variables* $M_t = \max_{0\leq s\leq t} B_s$, $|B_t|$, $M_t - B_t$ *all have the same law.*

Proof. For the first two random variables observe that $\mathbb{P}(M_t \geq b) = \mathbb{P}(\tau_b \leq t) \overset{\text{see above}}{=} \mathbb{P}(|B_t| \geq b)$ for all $b \geq 0$. With respect to the third random variable we use time inversion: $\tilde{B}_s = B_{t-s} - B_t$, $0 \leq s \leq t$, is again a Brownian motion. Then

$$M_t - B_t = \max_{0\leq s\leq t}\left(B_s - B_t\right) = \max_{0\leq u\leq t}\left(B_{t-u} - B_t\right) = \max_{0\leq u\leq t}\tilde{B}_u =: \tilde{M}_t.$$

Since $\tilde{M}_t \overset{d}{=} M_t$ (same law), we also have $M_t - B_t \overset{d}{=} M_t$. □

Remark 1.33 (Lévy). $(M_t - B_t, t \geq 0)$ and $(|B_t|, t \geq 0)$ have the same law on $C[0,\infty)$.

Theorem 1.34 (First Arcsine law)**.** *For Brownian motion the random time* $\tau_M = \operatorname{argmax}_{t\in[0,1]} B_t$ *is a.s. unique and satisfies*

$$\mathbb{P}\left(\tau_M \leq t\right) = \frac{2}{\pi}\arcsin\left(\sqrt{t}\right), \quad t \in [0,1],$$

i.e. it has density $f_{\tau_M}(t) = \frac{1}{\pi\sqrt{t(1-t)}}$.

Proof. Let $M := \max_{0\le t\le 1} B_t$. Then

$$\begin{aligned}\mathbb{P}\left(\exists t \le s : B_t = M\right) &= \mathbb{P}\left(\max_{0\le u\le s} B_u \ge \max_{s\le v\le 1} B_v\right) \\ &= \mathbb{P}\left(\max_{0\le \tilde{u}\le s} (B_{s-\tilde{u}} - B_s) \ge \max_{s\le v\le 1} (B_v - B_s)\right).\end{aligned}$$

The processes $(B_{s-\tilde{u}} - B_s, 0 \le \tilde{u} \le s)$ and $(B_v - B_s, s \le v \le 1)$ are independent Brownian motions. Thus

$$\mathbb{P}\left(\exists t \le s : B_t = M\right) = \mathbb{P}\left(\underbrace{\sqrt{s}\,|Z_1|}_{\overset{d}{=}|B_s|} \ge \underbrace{\sqrt{1-s}\,|Z_2|}_{\overset{d}{=}|B_{1-s}|}\right)$$

for $Z_1, Z_2 \overset{\text{iid}}{\sim} N(0,1)$ such that rearranging the terms yields

$$\begin{aligned} &= \mathbb{P}\left(\frac{|Z_2|}{\sqrt{Z_1^2 + Z_2^2}} \le \sqrt{s}\right) \\ &\overset{\text{polar coordinates}}{=} \mathbb{P}\left(|\sin\vartheta| \le \sqrt{s}\right),\end{aligned}$$

where we use that $(R\cos\vartheta, R\sin\vartheta) \sim N(0, E_2)$ with $R^2 \sim exp(\frac{1}{2})$, $\vartheta \sim U[0, 2\pi]$, where E_2 is two dimensional identity matrix. By symmetrie considerations we have

$$\begin{aligned}\mathbb{P}\left(|\sin\vartheta| \le \sqrt{s}\right) &= \mathbb{P}\left(|\sin\vartheta| \le \sqrt{s}, 0 \le \vartheta \le \frac{\pi}{2}\right) + \mathbb{P}\left(|\sin\vartheta| \le \sqrt{s}, \frac{\pi}{2} \le \vartheta \le \pi\right) \\ &\quad + \mathbb{P}\left(|\sin\vartheta| \le \sqrt{s}, \pi \le \vartheta \le \frac{3}{2}\pi\right) + \mathbb{P}\left(|\sin\vartheta| \le \sqrt{s}, \frac{3}{2}\pi \le \vartheta \le 2\pi\right) \\ &= 4\mathbb{P}\left(\sin\vartheta \le \sqrt{s}\right) \\ &= \frac{2}{\pi}\arcsin\sqrt{s}.\end{aligned}$$

The calculation also shows

$$\mathbb{P}\left(\max_{0\le u\le s} B_u = \max_{s\le v\le 1} B_v\right) = \mathbb{P}\left(\sqrt{s}\,|Z_1| = \sqrt{1-s}\,|Z_2|\right) = 0.$$

Hence,

$$\mathbb{P}\left(\exists s \in \mathbb{Q} \cap [0,1] : \max_{0\le u\le s} B_u = \max_{s\le v\le 1} B_v\right) = 0$$

and therefore

$$\mathbb{P}\left(\text{there are } t_1 \ne t_2 \text{such that } B_{t_1} = B_{t_2} = M\right) = 0.$$

With probability one the argmax is unique and well-defined. □

Chapter 2

Continuous martingales and stochastic integration

2.1 Continuous (local) martingales

Definition 2.1. $(M_t, t \geq 0)$ is called $(\mathcal{F}_t)_{t\geq 0}$-*local martingale* if

- (adapted from the lecture)M is $(\mathcal{F}_t)_{t\geq 0}$-adapted,
- there are $(\mathcal{F}_t)_{t\geq 0}$-stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n(\omega) \to \infty$ a.s.,
- the stopped processes $M_t^{\tau_n}(\omega) := M_{\tau_n(\omega)\wedge t}(\omega)$, $t \geq 0$, are (added this) uniformly integrable $(\mathcal{F}_t)_{t\geq 0}$-martingales for all $n \geq 1$.

$(\tau_n)_{n\geq 1}$ is called *localising sequence* of stopping times for M.

Example 2.2.

a) Each right-continuous martingale is a local martingale by optional stopping.

b) Let A be a non-negative random variable with $\mathbb{E}[A] = \infty$, independent of a Brownian motion B. Then $M_t(\omega) := A(\omega)B_t(\omega)$, $t \geq 0$, is NOT a martingale, because for all $t > 0$ $\mathbb{E}[|M_t|] = \infty$, i.e. $M_t \notin L^1$. Put $\tau_n := \inf\{t \geq 0 : |M_t| \geq n\}$. Then $\tau_n, n \geq 1$, are stopping times with respect to $\mathcal{F}_t = \sigma(A, B_s, s \leq t)$, increasing in n and $\lim_{n\to\infty} \tau(\omega) = \infty$ a.s. (because M is continuous and thus locally bounded). We have

- (adapted from the lecture) $\mathbb{E}[|M_t^{\tau_n}|] = \mathbb{E}[\underbrace{|M_0|\mathbf{1}_{\{\tau_n=0\}}}_{=0} + \underbrace{|M_t^{\tau_n}|\mathbf{1}_{\{\tau_n>0\}}}_{\leq n}] \leq n < \infty$,
- M^{τ_n} is $(\mathcal{F}_t)_{t\geq 0}$-adapted (by definition of $(\mathcal{F}_t)_{t\geqslant 0}$),
- $s < t$:(adapted from the lecture!)$|A\mathbb{E}[B_{t\wedge\tau_n}|\mathcal{F}_s]| = |AB_{s\wedge\tau_n}| \leq n$ is integrable such that

$$\begin{aligned}\mathbb{E}[M_t^{\tau_n}|\mathcal{F}_s] &= A\mathbb{E}[B_{t\wedge\tau_n}|\mathcal{F}_s] \\ &\overset{\text{opt. stopp.}}{=} AB_{s\wedge\tau_n} \\ &= M_s^{\tau_n}.\end{aligned}$$

Recall the martingale transform or discrete stochastic integral from Stochastic processes I: If $(X_n)_{n\in\mathbb{N}}$ is predictable, bounded, $(M_n)_{n\in\mathbb{N}}$ a martingale, then

$$(X\circ M)_n=\sum_{k=1}^{n}\underbrace{X_k}_{\mathcal{F}_{k-1}\text{-mb.}}\left(M_k-M_{k-1}\right)$$

is again a martingale. Interpretation in finance as value of a portfolio (X_k number of stocks in period k, M_k price of stock in period k).

Definition 2.3. A process $(X_t,t\geq 0)$ of the form

$$X_t\left(\omega\right)=\sum_{k=0}^{\infty}\xi_k\left(\omega\right)\mathbf{1}_{\left(\tau_k(\omega),\tau_{k+1}(\omega)\right]}\left(t\right)$$

with $0=\tau_0<\tau_1<\cdots\to\infty$ a sequence of $(\mathcal{F}_t)_{t\geq 0}$-stopping times and ξ_k are $\mathcal{F}_{\tau_k}$-measurable random variables is called *simple*. For another $(\mathcal{F}_t)_{t\geq 0}$-adapted process Y then define

$$\left(X\circ Y\right)_t\left(\omega\right):=\sum_{k=0}^{\infty}\xi_k\left(\omega\right)\left(Y_{t\wedge\tau_{k+1}(\omega)}\left(\omega\right)-Y_{t\wedge\tau_k(\omega)}\left(\omega\right)\right).$$

This is called the *stochastic integral* and is sometimes denoted as $\int_0^t X_s\,dY_s$. We set $\mathcal{E}:=\left\{(X_t,t\geq 0):X\text{ simple and bounded}\right\}$.

Proposition 2.4. *(adapted: added linearity, stopping)Let X and Y be simple processes. We have the following properties of $(X\circ M)$:*

a) If M is a continuous L^2-martingale and X is bounded, then $(X\circ M)$ is again an L^2-martingale.

b) If M is a local continuous martingale, then $(X\circ M)$ is again a local martingale.

c) (Linearity) If M is a local continuous martingale, then $\forall\alpha,\beta\in\mathbb{R}$: $((\alpha X+\beta Y)\circ M)=\alpha\left(X\circ M\right)+\beta\left(Y\circ M\right)$.

Proof. See exercises. □

Lemma 2.5. *If X is a simple, bounded process and M is a continuous L^2-martingale, then*

$$\mathbb{E}\left[(X\circ M)_t^2\right]=\mathbb{E}\left[\sum_{k\geq 0}\xi_k^2\left(\mathbb{E}\left[M^2_{\tau_{k+1}\wedge t}\middle|\mathcal{F}_{\tau_k\wedge t}\right]-M^2_{\tau_k\wedge t}\right)\right]\leq C^2\mathbb{E}\left[M_t^2\right]$$

holds, where $||X(\omega)||_\infty\leq C$ a.s. for a deterministic constant $C>0$.

Proof. (adapted from the lecture!)Observe from Theorem 1.25 that $\xi_k\mathbf{1}_{\{\tau_k\leq t\}}$ is $\mathcal{F}_{\tau_k\wedge t}$-measurable. For $n\leq m$ we have

$$\begin{aligned}&\left(\sum_{k=n}^{m}\xi_k\left(M_{\tau_{k+1}\wedge t}-M_{\tau_k\wedge t}\right)\right)^2\\&=\left(\sum_{k=n}^{m}\xi_k\mathbf{1}_{\{\tau_k\leq t\}}\left(M_{\tau_{k+1}\wedge t}-M_{\tau_k\wedge t}\right)\right)^2\\&=2\sum_{n\leq k<j}^{m}\xi_k\xi_j\mathbf{1}_{\{\tau_k\leq t\}}\mathbf{1}_{\{\tau_j\leq t\}}\left(M_{\tau_{k+1}\wedge t}-M_{\tau_k\wedge t}\right)\left(M_{\tau_{j+1}\wedge t}-M_{\tau_j\wedge t}\right)\\&+\sum_{k=n}^{m}\xi_k^2\mathbf{1}_{\{\tau_k\leq t\}}\left(M_{\tau_{k+1}\wedge t}-M_{\tau_k\wedge t}\right)^2\end{aligned}$$

such that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=n}^{m}\xi_k\left(M_{\tau_{k+1}\wedge t}-M_{\tau_k\wedge t}\right)\right)^2\right]\\
&=\left(2\sum_{n\le k<j}^{m}\mathbb{E}\Bigg[\xi_k\mathbf{1}_{\{\tau_k\le t\}}\Big(M_{\tau_{k+1}\wedge t}-M_{\tau_k\wedge t}\Big)\underbrace{\mathbb{E}\Big[\xi_j\mathbf{1}_{\{\tau_j\le t\}}\Big(M_{\tau_{j+1}\wedge t}-M_{\tau_j\wedge t}\Big)\Big|\mathcal{F}_{\tau_j\wedge t}\Big]}_{=0}\Bigg]\right.\\
&\left.+\,\mathbb{E}\left[\sum_{k=n}^{m}\xi_k^2\mathbf{1}_{\{\tau_k\le t\}}\left(\mathbb{E}\left[M^2_{\tau_{k+1}\wedge t}\Big|\mathcal{F}_{\tau_k\wedge t}\right]-M^2_{\tau_k\wedge t}\right)\right]\right)\\
&=\mathbb{E}\left[\sum_{k=n}^{m}\xi_k^2\mathbf{1}_{\{\tau_k\le t\}}\underbrace{\left(\mathbb{E}\left[M^2_{\tau_{k+1}\wedge t}\Big|\mathcal{F}_{\tau_k\wedge t}\right]-M^2_{\tau_k\wedge t}\right)}_{\ge 0\text{ because of Jensen}}\right]\Bigg)\\
&\le C^2\sum_{k=n}^{m}\mathbb{E}\left[M^2_{\tau_{k+1}\wedge t}-M^2_{\tau_k\wedge t}\right]\\
&=C^2\left(\mathbb{E}\left[M^2_{\tau_{m+1}\wedge t}-M^2_{\tau_n\wedge t}\right]\right)\\
&\le C^2\left(\mathbb{E}\left[M_t^2\right]-\mathbb{E}\left[M^2_{\tau_n\wedge t}\right]\right),
\end{aligned}
$$

where the last inequality follows from Jensen's inequality and optional stopping (Theorem 1.26). Because M is uniformly integrable on $[0,t]$ and continuous, the last term converges to 0 as $n\to\infty$. In particular, $(\sum_{k=1}^{n}\xi_k(M_{\tau_{k+1}\wedge t}-M_{\tau_k\wedge t}))_{n\ge 1}$ is an $L^2(\mathbb{P})$-Cauchy sequence. Observing that $(X\circ M)_t(\omega)$ is a finite sum for all ω, i.e. $(X\circ M)_t(\omega)=\lim_{n\to\infty}(\sum_{k=1}^{n}\xi_k(\omega)(M_{\tau_{k+1}\wedge t}(\omega)-M_{\tau_k\wedge t}(\omega))$, this implies

$$
\begin{aligned}
\mathbb{E}\left[(X\circ M)_t^2\right] &= \lim_{n\to\infty}\mathbb{E}\left[\left(\sum_{k=1}^{n}\xi_k\left(M_{\tau_{k+1}\wedge t}-M_{\tau_k\wedge t}\right)\right)^2\right]\\
&= \mathbb{E}\left[\sum_{k\ge 0}\xi_k^2\mathbf{1}_{\{\tau_k\le t\}}\left(\mathbb{E}\left[M^2_{\tau_{k+1}\wedge t}\Big|\mathcal{F}_{\tau_k\wedge t}\right]-M^2_{\tau_k\wedge t}\right)\right]\\
&\le C^2\left(\mathbb{E}\left[M_t^2\right]-\mathbb{E}\left[M_0^2\right]\right)\\
&= C^2\mathbb{E}\left[M_t^2\right].
\end{aligned}
$$

□

In the sequel we fix a filtered probability space $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t))$ where $(\mathcal{F}_t)_{t\ge 0}$ is complete, i.e. each $N\in\mathcal{F}$ with $\mathbb{P}(N)=0$ is already in $\mathcal{F}_0\subseteq\mathcal{F}_t$ $\forall t\ge 0$. (added remark)This implies that a process which is indistinguishable of an adapted process is again adapted.

Definition 2.6. By $\mathcal{M}_c^2$ we denote the set of all $(\mathcal{F}_t)_{t\ge 0}$-martingales $(M_t, t\ge 0)$ with $M_0=0$, $M_t\in L^2$ for all $t\ge 0$ and with continuous paths. We put $\|M\|_{\mathcal{M}_c^2}=\sum_{n=1}^{\infty}2^{-n}(\|M_n\|_{L^2}\wedge 1)$ for $M=(M_t,t\ge 0)\in\mathcal{M}_c^2$.

Lemma 2.7. *$\mathcal{M}_c^2$ is a vector space and $d(M,N)=\|M-N\|_{\mathcal{M}_c^2}$ is a metric on $\mathcal{M}_c^2$ when identifying indistinguishable martingales.*

Proof. Vector space properties are easily checked. d is well-defined (i.e. $d(M,N)<\infty$ for all $M,N\in\mathcal{M}_c^2$), clearly symmetric, non-negative and satisfies the triangle inequality (cf. the metric on $C(\mathbb{R}^+)$). Moreover,

$$d(M,N)=0\Leftrightarrow\forall n\ge 1:\|M_n-N_n\|_{L^2}=0. \tag{2.1.1}$$

Furthermore, if M, N are indistinguishable, then $M_t - N_t = 0$ for all t $\mathbb{P}$-a.s. and therefore $d(M,N) = 0$. If, on the other hand, $M - N \in \mathcal{M}_c^2$ and $d(M,N) = 0$, then $(M_t - N_t)^2$ is a submartingale and for all $t > 0$

$$\mathbb{E}\left[(M_t - N_t)^2\right] \leq \mathbb{E}\left[\left(M_{\lfloor t\rfloor+1} - N_{\lfloor t\rfloor+1}\right)^2\right] = \|(M-N)_{\lfloor t\rfloor+1}\|_{L^2}^2 = 0$$

by (2.1.1). Hence, $\mathbb{P}(M_t = N_t) = 1$ for all $t \geq 0$ and therefore M, N indistinguishable, as they are also continuous (cf. exercises). □

Proposition 2.8. *$(\mathcal{M}_c^2, d)$ is a complete space.*

Proof. Let $(M^{(n)})_{n\in\mathbb{N}} \subseteq \mathcal{M}_c^2$ be a Cauchy sequence, i.e. $\lim_{m,n\to\infty}\|M^{(m)} - M^{(n)}\|_{\mathcal{M}_c^2} = 0$. (suggestion: (suggestion: Then Then for all $t \geq 0$ $(M_t^{(n)})_{n\geq 1}$ is a Cauchy sequence in $L^2(\mathcal{F}_t)$ (see the submartingale argument from above). Because $L^2(\mathcal{F}_t)$ is complete, there exist $M_t \in L^2(\mathcal{F}_t)$ such that $M_t^{(n)} \xrightarrow{L^2} M_t$. We claim that $(M_t, t \geq 0)$ is a martingale. Indeed, $M_t \in L^2$ and adaptedness are clear.Then for all $t \geq 0$ $(M_t^{(n)})_{n\geq 1}$ is a Cauchy sequence in L^2 (see the submartingale argument from above). Because L^2 is complete, there exist $M_t \in L^2$ such that $M_t^{(n)} \xrightarrow{L^2} M_t$. We claim that $(M_t, t \geq 0)$ is an L^2-martingale. Indeed, $M_t \in L^2$ is clear. Moreover, for all $t > 0$ there exists a subsequence $M_t^{(n_k)} \xrightarrow{a.s.} M_t$ and all $M_t^{(n_k)}$ are $\mathcal{F}_t$-measurable. Because all nullsets are already in $\mathcal{F}_t$, M_t is $\mathcal{F}_t$-measurable. For $s < t$ and $A \in \mathcal{F}_s$ we have then

$$\left|\mathbb{E}\left[M_t \mathbf{1}_A\right] - \mathbb{E}\left[M_t^{(n)}\mathbf{1}_A\right]\right| = \left|\langle M_t - M_t^{(n)}, \mathbf{1}_A\rangle_{L^2}\right| \xrightarrow{n\to\infty} \langle 0, \mathbf{1}_A\rangle_{L^2} = 0.$$

Hence,

$$\begin{aligned}\mathbb{E}\left[M_t\mathbf{1}_A\right] &= \lim_{n\to\infty}\mathbb{E}\left[M_t^{(n)}\mathbf{1}_A\right]\\ &\overset{M^n \text{ mart.}}{=} \lim_{n\to\infty}\mathbb{E}\left[M_s^{(n)}\mathbf{1}_A\right]\\ &= \mathbb{E}\left[M_s\mathbf{1}_A\right]\end{aligned}$$

such that $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$. We still have to show that M is continuous. By Proposition 1.27 (Doob's inequality) (adapted from the lecture)

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|M_t^{(m)} - M_t^{(n)}\right|^2\right] \leq 4\mathbb{E}\left[\left|M_T^{(m)} - M_T^{(n)}\right|^2\right] \to 0$$

as $m, n \to \infty$ for all $T > 0$. We can then select a subsequence $M^{(n_k)}$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|M_t^{(n_{k+1})} - M_t^{(n_k)}\right|^2\right] \leq 2^{-k}$$

for all $k \in \mathbb{N}$. Hence, Borel-Cantelli implies for almost all ω that $(M^{(n_k)}(\omega))_{k\in\mathbb{N}}$ is a $(C([0,T]), \|\cdot\|_\infty)$-Cauchy sequence. By completeness of this space and because $M_t^{(n_k)} \xrightarrow{L^2} M_t$ we see that M is a.s. continuous on $[0,T]$. We obtain that M is a.s. continuous on $\bigcup_{T\in\mathbb{N}}[0,T] = \mathbb{R}^+$. Because the filtration is complete, we can find a process $\tilde{M} \in \mathcal{M}_c^2$ which is indistinguishable of M. In particular, $\|M^{(n)} - \tilde{M}\|_{\mathcal{M}_c^2} = \|M^{(n)} - M\|_{\mathcal{M}_c^2} \to 0$. □

Remark 2.9. If we restrict the martingales in $\mathcal{M}_c^2$ to the time interval $[0,T]$ for some $T > 0$, then $\mathcal{M}_c^2|_{[0,T]}$ with $\|M\| := \|M_T\|_{L^2}^2$ is even a Hilbert space (cf. exercises).

Theorem 2.10. *If $M \in \mathcal{M}_c^2$ has finite variation on $[0,T]$, i.e. $V_T(M(\omega)) < \infty$ a.s., then M is a.s. constant on $[0,T]$ (i.e. equal to 0).*

Proof. Let $\pi = \{0 = t_0 < t_1 < \cdots < t_m = T\}$ be a partition of $[0,T]$. Then

$$\mathbb{E}\left[M_T^2\right] = \mathbb{E}\left[\sum_{k=0}^{m-1}\left(M_{t_{k+1}}^2 - M_{t_k}^2\right)\right] \overset{M\text{ martingale}}{=} \mathbb{E}\left[\sum_{k=0}^{m-1}\left(M_{t_{k+1}} - M_{t_k}\right)^2\right].$$

Assume first $\exists K > 0 \forall \omega \in \Omega$ such that $V_T(M(\omega)) \le K < \infty$. Then

$$\mathbb{E}\left[M_T^2\right] \le \mathbb{E}\left[\max_{0\le k\le m}\left|M_{t_{k+1}} - M_{t_k}\right| \cdot \underbrace{\sum_{k=0}^{m-1}\left|M_{t_{k+1}} - M_{t_k}\right|}_{\le V_T(M)\le K}\right] \le K\cdot\mathbb{E}\left[\max_{0\le k\le m}\left|M_{t_{k+1}} - M_{t_k}\right|\right].$$

For partitions $\pi^{(n)}$ such that $\max_k |t_{k+1}^{(n)} - t_k^{(n)}| \to 0$, uniform continuity of M on $[0,T]$ yields $\max_{0\le k\le m}|M_{t_{k+1}^{(n)}} - M_{t_k^{(n)}}| \xrightarrow{a.s.} 0$. Since

$$|M_t(\omega)| \le \underbrace{|M_0(\omega)|}_{=0} + V_T(M(\omega)) \le K,$$

we have $|M_{t_{k+1}} - M_{t_k}| \le 2K$ and dominated convergence implies

$$\mathbb{E}\left[\max_{0\le k\le m}\left|M_{t_{k+1}^{(n)}} - M_{t_k^{(n)}}\right|\right] \xrightarrow{n\to\infty} 0$$

(independent of the sequence of partitions). Hence, $\mathbb{E}\left[M_T^2\right] = 0$ and because M^2 is a submartingale, we also have $\mathbb{E}[M_t^2] = 0$ for all $t \in [0,T]$. This implies $M_t = 0$ for all $t \in [0,T]$ a.s. by continuity.
Let now $M \in \mathcal{M}_c^2$ and put $\tau_n = \inf\{t > 0 : V_t(M) \ge n\}$ (observe that $V_t(M)$ is continuous, increasing in t and $(\mathcal{F}_t)_{t\ge0}$-adapted). Then τ_n is a stopping time and the stopped martingale $M_t^{\tau_n} = M_{t\wedge\tau_n}$ satisfies by the first part above (note: $V_T(M^{\tau_n}) \le n$) for all $0 \le t \le T$ that $M_{t\wedge\tau_n} = 0$ a.s. More precisely, it holds $\mathbb{P}(\forall 0 \le t \le T : M_{t\wedge\tau_n} = 0) = 1$. Since $V_T(M) < \infty$ a.s., we have $\tau_n \to \infty$ a.s. and thus $\mathbb{P}(\forall n \ge 1, 0 \le t \le T : M_{t\wedge\tau_n} = 0) = 1$ and thus $\mathbb{P}(\forall 0 \le t \le T : M_t = 0) = 1$. ☐

Corollary 2.11. *Any non-trivial (non-constant) continuous L^2-martingale has indefinite variation on every interval $[s,t]$, in particular, is non-differentiable there.*

Proof. Immediate consequence of the previous theorem. ☐

Remark 2.12.

a) This holds more generally for any continuous local martingale.

b) There are of course many discontinuous martingales of finite variation, e.g. $M_t = N_t - \lambda_t, t \ge 0$, with N_t Poisson process of intensity λt (on $[0,T]$).

Theorem 2.13. *(adapted from lecture: we don't need $M_0 = 0$ here)Every continuous bounded martingale M posesses a unique (up to indistuinguishability) continuous (added adapted)adapted increasing process $(\langle M\rangle_t, t \ge 0)$ with $\langle M\rangle_0 = 0$ such that $(M_t^2 - \langle M\rangle_t, t \ge 0)$ is a martingale.*

Proof. We first show existence of $\langle M\rangle$. For all $n \ge 1$ introduce the stopping times $\tau_0^n(\omega) = 0$, $\tau_{k+1}^n(\omega) = \inf\{t > 0 : |M_{t+\tau_k^n(\omega)}(\omega) - M_{\tau_k^n(\omega)}(\omega)| \ge 2^{-n}\}$. Let us write $t_k^n = t \wedge \tau_k^n$ and note

$\lim_{k\to\infty}\tau_k^n(\omega)=\infty$, because M is uniformly continuous on each compact $[0,T]$. The main point is

$$\begin{aligned} M_t^2 &= \sum_{k=1}^{\infty}\left(M^2_{t^n_k}-M^2_{t^n_{k-1}}\right) \\ &= \underbrace{\sum_{k=1}^{\infty}\left(M_{t^n_k}-M_{t^n_{k-1}}\right)^2}_{=:A^n_t} + \underbrace{2\sum_{k=1}^{\infty}M_{t^n_{k-1}}\left(M_{t^n_k}-M_{t^n_{k-1}}\right)}_{=2\cdot(H^n\cdot M)_t} \end{aligned}$$

with $H^n_t=\sum_{k=1}^{\infty}M_{\tau^n_{k-1}}\mathbf{1}_{(\tau^n_{k-1},\tau^n_k]}(t)$ simple and bounded. The following properties are easily checked:

i. $J_n(\omega):=\{\tau^n_k(\omega):k\geq 0\}\subseteq J_{n+1}(\omega)$,

ii. $\sup_{t\geq 0}|H^n_t-H^{n-1}_t|\leq 2^{-(n+1)}$, $\sup_{t\geq 0}|H^n_t-M_t|\leq 2^{-n}$,

iii. $A^n_{\tau^n_k}\leq A^n_{\tau^n_{k+1}}$.

For all $t>0$ we have by linearity of the stochastic integral for simple processes (Proposition 2.4) and Lemma 2.5

$$\begin{aligned} \mathbb{E}\left[\left((H^n\circ M)_t-\left(H^{n+1}\circ M\right)_t\right)^2\right] &= \mathbb{E}\left[\left(\left(H^n-H^{n+1}\right)\circ M\right)_t^2\right] \\ &\leq 4^{-(n+1)}\mathbb{E}\left[M_t^2\right] \\ &\overset{M\text{ bounded}}{\leq} \underbrace{C\cdot 4^{-(n+1)}}_{\text{summable!}}. \end{aligned}$$

Hence, $((H^n\circ M))_{n\geq 1}$ converges in $\mathcal{M}^2_c$ to some continuous martingale $N\in\mathcal{M}^2_c$ (by completeness of $\mathcal{M}^2_c$ and by completeness of the underlying filtration). Therefore $(M^2_t-A^n_t,t\geq 0)_n$ converges in $\mathcal{M}^2_c$ to $2\cdot N$ and thus, A^n_t converges in L^2 better: in $L^2(\mathcal{F}_t)$ to some A_t for each t , i.e. A is adapted.. Moreover, convergence in $\mathcal{M}^2_c$ ensures even uniform convergence on compacts such that for a subsequence (n_k) we have

$$\mathbb{P}\left(A^{n_k}_t\to A_t\text{ uniformly on }[0,T]\right)=1$$

for all $T\in\mathbb{N}$ (cf. proof of Proposition 2.8), i.e. A is a.s. continuous. (ii) and (iii) yield that (A_t) is increasing on $J_\infty(\omega)=\bigcup_{n\geq 1}J_n(\omega)$. Suppose $I\subseteq J_\infty(\omega)^c$ is an open interval. Then $\forall n,k\ \tau^n_k(\omega)\notin I$ implies $M_t(\omega)$ is constant on I, i.e. $A_t(\omega)$ is constant on I (since each A^n_t is so). In all, we obtain that $A_t(\omega)$ is increasing on $[0,\infty)$ globally, i.e. A is an increasing, adapted, a.s. continuous process with $A_0=0$, $M^2_t-A_t=2N_t$, which is a continuous martingale. So, existence is proven if we choose $\langle M\rangle_t=A_t$.suggestion: if we choose a continuous indistinguishable version $\hat{A}$ of A which still satisfies these properties and set $\langle M\rangle_t=\hat{A}_t$.

With respect to uniqueness, suppose that $\tilde{A}$ is another such process with $M^2_t-\tilde{A}_t=\tilde{N}_t$, where $\tilde{N}$ is a continuous martingale. Then $A_t-\tilde{A}_t=\tilde{N}_t-2N_t$ is also a continuous martingale with $A_0-\tilde{A}_0=0$ and is of finite variation as difference of two increasing functions for each ω. By Theorem 2.10 we have $A_t-\tilde{A}_t=0$ for all $t\geq 0$ a.s. and therefore $A,\tilde{A}$ are indistinguishable.

□

Remark 2.14.

a) This is the analogue of the Doob composition of (M_n^2) in discrete time. There the *compensator* A_n of the submartingale M_n^2 satisfied

$$A_n = \sum_{k=1}^{m} \mathbb{E}\left[(M_k - M_{k-1})^2 \middle| \mathcal{F}_{k-1} \right] =: \langle M \rangle_n ,$$

where A_n is predictible (i.e. $\mathcal{F}_{n-1}$-measurable). In continuous time the predictability is replaced by the continuity requirement of A.

b) One can prove for partitions $\pi^{(n)}$ with $|\pi^{(n)}| = \max |t_{k+1}^{(n)} - t_k^{(n)}| \to 0$ that

$$A_t = \lim_{n\to\infty} \sum_k \left(M_{t_{k+1}^{(n)} \wedge t} - M_{t_k^{(n)} \wedge t} \right)^2$$

in probability, i.e. M has finite quadratic variation (cf. the Brownian motion case and Corollary 2.38 below).

Corollary 2.15. *(adapted from the lecture: added 0 in 0)For every continuous local martingale M there exists a unique (up to indistinguishability) increasing, continuous process $\langle M \rangle$ such that $\langle M \rangle_0 = 0$ and $M_t^2 - \langle M \rangle_t$ is a local martingale.*

Proof. Use stopping times and apply the previous theorem. See exercises. □

Example 2.16. For Brownian motion B we have $\langle B \rangle_t = t$ (deterministic!), because $(B_t^2 - t, t \geq 0)$ is a martingale and $f(t) = t$ is increasing, continuous and $f(0) = 0$.

2.2 Stochastic integration

Recall

A simple process X has the form $X_t(\omega) = \sum_{k=1}^{\infty} \xi_k(\omega) \mathbf{1}_{(\tau_{k-1}(\omega), \tau_k(\omega)]}(t)$, ξ_k is $\mathcal{F}_{\tau_{k-1}}$-measurable. For simple, bounded X, $M \in \mathcal{M}_c^2$ we defined the *stochastic integral*

$$\int_0^t X_s \, dM_s := (X \circ M)_t = \sum_{k=1}^{\infty} \xi_k \left(M_{\tau_k \wedge t} - M_{\tau_{k-1} \wedge t} \right) \in \mathcal{M}_c^2.$$

Can we extend this to more general integrands X? To put it differently: Which processes X can we approximate by simple, bounded processes $X^{(n)}$ such that $(X^{(n)} \cdot M)_{n \geq 1}$ converges in $\mathcal{M}_c^2$ (which is complete)?

Lemma 2.17. *Let τ be a bounded stopping time. For simple, bounded X and $M \in \mathcal{M}_c^2$ we have*

$$\langle X \circ M \rangle_\tau = \int_0^\tau X_s^2 \, d\langle M \rangle_s .$$

In particular,

$$\mathbb{E}\left[(X \circ M)_\tau^2 \right] = \mathbb{E}\left[\int_0^\tau X_s^2 \, d\langle M \rangle_s \right].$$

Proof. (adapted from the lecture!)The martingale property according to Lemma 2.5 ensures $\mathbb{E}\left[(X \circ M)_\tau \right] = 0$ and

$$\mathbb{E}\left[(X \circ M)_\tau^2 \right] \overset{\text{Lemma 2.5}}{=} \sum_k \mathbb{E}\left[\xi_k^2 \mathbf{1}_{\{\tau_k \leq \tau\}} \left(M_{\tau_k \wedge \tau}^2 - M_{\tau_{k-1} \wedge \tau}^2 \right) \right]$$

(check that the proof still works if we use τ instead of t using Theorem 1.25). Observe now that $N_t := M_t^2 - \langle M \rangle_t$ is a martingale such that

$$\begin{aligned}
&\mathbb{E}\left[\xi_k^2 \mathbf{1}_{\{\tau_{k-1}\le\tau\}}\left(M^2_{\tau_k\wedge\tau} - M^2_{\tau_{k-1}\wedge\tau}\right)\right]\\
&= \mathbb{E}\left[\xi_k^2 \mathbf{1}_{\{\tau_{k-1}\le\tau\}}\left(\langle M\rangle_{\tau_k\wedge\tau} - \langle M\rangle_{\tau_{k-1}\wedge\tau} + N_{\tau_{k-1}\wedge\tau} - N_{\tau_k\wedge\tau}\right)\right]\\
&= \mathbb{E}\left[\xi_k^2 \mathbf{1}_{\{\tau_{k-1}\le\tau\}}\left(\langle M\rangle_{\tau_k\wedge\tau} - \langle M\rangle_{\tau_{k-1}\wedge\tau}\right)\right]\\
&+ \mathbb{E}\Big[\underbrace{\xi_k^2 \mathbf{1}_{\{\tau_{k-1}\le\tau\}}}_{\in\mathcal{F}_{\tau_{k-1}\wedge\tau}}\underbrace{\mathbb{E}\left[N_{\tau_{k-1}\wedge\tau} - N_{\tau_k\wedge\tau}\,\middle|\,\mathcal{F}_{\tau_{k-1}\wedge\tau}\right]}_{=0}\Big]\\
&\overset{\text{opt. stopp.}}{=} \mathbb{E}\left[\xi_k^2 \mathbf{1}_{\{\tau_{k-1}\le\tau\}}\left(\langle M\rangle_{\tau_k\wedge\tau} - \langle M\rangle_{\tau_{k-1}\wedge\tau}\right)\right]\\
&= \mathbb{E}\left[\xi_k^2 \left(\langle M\rangle_{\tau_k\wedge\tau} - \langle M\rangle_{\tau_{k-1}\wedge\tau}\right)\right].
\end{aligned}$$

Thus,

$$\mathbb{E}\left[(X\circ M)_\tau^2\right] = \mathbb{E}\left[\int_0^\tau X_u^2\, d\langle M\rangle_u\right],$$

where the integral in the last line is just a usual Lebesgue-Stieltjes integral. In particular,

$$\mathbb{E}\left[(X\circ M)_\tau^2 - \int_0^\tau X_u^2\, d\langle M\rangle_u\right] = 0 = \mathbb{E}\left[(X\circ M)_0^2 - \int_0^0 X_u^2\, d\langle M\rangle_u\right].$$

Theorem 1.26 (part (b)) yields the claim. □

Remark 2.18. The last identity will be seen as a major tool in the construction of the stochastic integral and is called *Itô isometry* (for simple integrands).

Definition 2.19. A process $(X_t, t \ge 0)$ is called *progressively measurable* with respect to $(\mathcal{F}_t)_{t\ge0}$ if X is $(\mathcal{F}_t)$-adapted and $(\omega, s) \mapsto X_s(\omega)$ on $\Omega\times[0,t]$ is $\mathcal{F}_t\otimes\mathscr{B}_{[0,t]}$-measurable for all $t \ge 0$.

Lemma 2.20. *Every adapted and left- or right-continuous process is progressively measurable.*

Proof. We consider only the left-continuous case. Write $X_s^n := X_{(k-1)t/n}$ for $s \in [\frac{(k-1)t}{n}, \frac{kt}{n})$ and $X_t^n := X_t$. By left-continuity, $X_s^n \xrightarrow{a.s.} X_s$ for each $s\in[0,t]$. For all $A \in \mathscr{B}_\mathbb{R}$ we have

$$\begin{aligned}
&\{(w,s)\in\Omega\times[0,t] : X_s^n(\omega)\in A\}\\
&= \{X_t\in A\}\times\{t\}\cup\bigcup_{k=1}^n\left\{X_{\frac{(k-1)t}{n}}\in A\right\}\times[\frac{(k-1)}{n}t, \frac{k}{n}t)\in\mathcal{F}_t\otimes\mathscr{B}_{[0,t]}.
\end{aligned}$$

Therefore $(\omega,s)\mapsto X_s^n(\omega)$ is $\mathcal{F}_t\otimes\mathscr{B}_{[0,t]}$-measurable and thus also $(\omega,s)\mapsto X_s(\omega)$. □

Remark 2.21. The white noise process (cf. exercises) is NOT jointly measurable in $\Omega\times[0,t]$ for any $t\ge0$ and thus not progressively measurable.

Definition 2.22. For $M \in \mathcal{M}_c^2$ introduce the space of "integrands"

$$\mathscr{L}(M) := \left\{(X_t, t\ge0) \text{ progressively measurable process } : \forall t\ge0 : \mathbb{E}\left[\int_0^t X_s^2\, d\langle M\rangle_s\right] < \infty\right\}$$

and endow it with the (semi-)metric

$$d_M(X,Y) = \sum_{n=1}^\infty 2^{-n}\left(\|X-Y\|_{M,n}\wedge 1\right),$$

where

$$\|X\|_{M,n}^2 = \mathbb{E}\left[\int_0^n X_t^2 \, d\langle M\rangle_t\right].$$

(adapted: moved out the definition of $\mathcal{E}$ to the definition of simple processes above)

Lemma 2.23. *$(\mathscr{L}(M), d_M)$ is a complete metric space, if we identify any two elements with distance 0 with respect to d_M (quotient space with respect to the kernel of the metric).*

Proof. Use completeness of the restrictions in $\mathscr{L}(M)$ to $[0,n]$ under the semi-norm $\|\cdot\|_{M,n}$ (which is a norm after identificiation of elements in the kernel of d_M) and proceed as for the completeness proof of $\mathcal{M}_c^2$. □

Theorem 2.24. *$\mathcal{E}$ is dense in $\mathscr{L}(M)$.*

Proof. The proof relies on the fact that a subspace L of a Hilbert space H is dense if its orthogonal complement $L^\perp = \{h \in H : \forall l \in L \ \langle l, h\rangle = 0\}$ is trivial, i.e. $L^\perp = \{0\}$. Note that $d_M(X,Y) \leq \varepsilon$ if $N \in \mathbb{N}$ such that $\sum_{n=N+1}^\infty 2^{-n} \cdot 1 = 2^{-N} < \varepsilon/2$ and (changed $\varepsilon/2$ to $\varepsilon/4$) $\|X - Y\|_{M,N} \leq \varepsilon/4$. Therefore it suffices to show that $\mathcal{E}$ is dense with respect to $\|\cdot\|_{M,T}$ for all $T > 0$ (restricting to $[0,T]$). Then we have a Hilbert space $(\mathscr{L}(M)|_{[0,T]}, \|\cdot\|_{M,T})$, cf. exercises, i.e.

$$\langle X, Y\rangle_{M,T} = \mathbb{E}\left[\int_0^T X_s Y_s \, d\langle M\rangle_s\right].$$

Now suppose $Z \in \mathscr{L}(M)|_{[0,T]}$ satisfies $\mathbb{E}[\int_0^T X_t Z_t \, d\langle M\rangle_t] = 0$ for all $X \in \mathcal{E}$. For $X_t = \xi \cdot \mathbf{1}_{(s,u]}(t)$, ξ $\mathcal{F}_s$-measurable, bounded, $0 \leq s \leq u \leq T$, this means

$$\mathbb{E}\left[\int_s^u \xi \cdot Z_t \, d\langle M\rangle_t\right] = 0.$$

Therefore,

$$\mathbb{E}\left[\xi \cdot \mathbb{E}\left[\int_s^u Z_t \, d\langle M\rangle_t \middle| \mathcal{F}_s\right]\right] = 0$$

for all bounded $\mathcal{F}_s$-measurable ξ such that

$$\mathbb{E}\left[\int_s^u Z_t \, d\langle M\rangle_t \middle| \mathcal{F}_s\right] = 0 \ \text{ a.s.}$$

Hence, $(\int_0^u Z_t \, d\langle M\rangle_t, u \geq 0)$ is a martingale. Since $t \mapsto \langle M\rangle_t$ is continuous, so is $u \mapsto \int_0^u Z_t \, d\langle M\rangle_t$. Moreover, $u \mapsto \int_0^u Z_t \, d\langle M\rangle_t$ has finite variation. Indeed, for a partition $\pi = \{t_k\}$ of $[0,T]$ we have

$$\sum_k \left|\int_{t_{k-1}}^{t_k} Z_t \, d\langle M\rangle_t\right| \leq \sum_k \int_{t_{k-1}}^{t_k} |Z_t| \, d\langle M\rangle_t = \int_0^T |Z_t| \, d\langle M\rangle_t < \infty \ \text{ a.s.}$$

(as a proof for the last step consider e.g. $\mathbb{E}[\int_0^T 1 \cdot |Z_t| \, d\langle M\rangle_t]^2 \leq \mathbb{E}[\int_0^T Z_t^2 \, d\langle M\rangle_t] \cdot \mathbb{E}[\langle M\rangle_T] < \infty$ by Cauchy-Schwarz). Theorem 2.10 then yields $\int_0^u Z_t \, d\langle M\rangle_t$ is a.s. constant in u, i.e. $Z_t(\omega) = 0$ for almost all $\omega \in \Omega$ and $d\langle M(\omega)\rangle$-almost all $t \geq 0$. This in turn implies that $\int_0^T Z_t^2 \, d\langle M\rangle_t = 0$ a.s. for all $T > 0$ and thus $\|Z\|_{M,T} = 0$. This shows that the orthogonal complement is trivial (on the quotient space as $\|\cdot\|_{M,T}$ is only a semi-norm). □

Remark 2.25.

a) $\mathcal{E} \subseteq \mathscr{L}(M)$ holds because each $X \in \mathcal{E}$ is $(\mathcal{F}_t)$-adapted and left-continuous, thus by the lemma progressively measurable. Moreover,

$$\mathbb{E}\left[\int_0^t \underbrace{X_s^2}_{\leq C^2} d\langle M\rangle_s\right] \leq \mathbb{E}\left[\int_0^t C^2\, d\langle M\rangle_s\right] = C^2\mathbb{E}\left[\langle M\rangle_t\right] = C^2\left[M_t^2\right] < \infty.$$

b) The proof above is "algebraic". A more constructive approximation argument can also be used, but is challenging if $t \mapsto \langle M\rangle_t$ is not absolutely continuous (cf. Karatzas (1991)).

Now we are able to define the stochastic integral $\int_0^t X_s\, dM_s$ for all $X \in \mathscr{L}(M)$ by approximation. Choose $X^{(n)} \in \mathcal{E}$ such that $d_M(X^{(n)}, X) \to 0$ (by density always possible) and infer that

$$\left(\int_0^t X_s^{(n)}\, dM_s, t \geq 0\right)_{n\geq 1} \subseteq \mathcal{M}_c^2$$

converges in $\mathcal{M}_c^2$. By completeness, the limit is what we want:

$$\int_0^t X_s\, dM_s = \lim_{n\to\infty} \int_0^t X_s^n dM_s.$$

The convergence of $(\int_0^\cdot X^{(n)}\, dM)_{n\geq 1}$ in $\mathcal{M}_c^2$ follows easily by isometry (Lemma 2.17):

$$\begin{aligned}
&d_{\mathcal{M}_c^2}\left(\int_0^\cdot X_s^{(n)}\, dM_s, \int_0^\cdot X_s^{(m)} dM_s\right)\\
&= \sum_{k=1}^\infty 2^{-k}\left(\mathbb{E}\left[\left(\int_0^k \left(X_s^{(n)} - X_s^{(m)}\right) dM_s\right)^2\right]^{1/2} \wedge 1\right)\\
&= \sum_{k=1}^\infty 2^{-k}\left(\mathbb{E}\left[\int_0^k \left(X_s^{(n)} - X_s^{(m)}\right)^2 d\langle M\rangle_s\right]^{1/2} \wedge 1\right)\\
&= d_M\left(X^{(n)}, X^{(m)}\right).
\end{aligned}$$

This means that $X \mapsto \int_0^\cdot X_s\, dM_s$ is an isometry from $(\mathcal{E}, d_M)$ to $(\mathcal{M}_c^2, d_{\mathcal{M}_c^2})$. This extends to its closure $\overline{\mathcal{E}} = \mathscr{L}(M)$ by continuity.

Definition 2.26. For $X \in \mathscr{L}(M)$ define $(\int_0^t X_s\, dM_s, t \geq 0)$ as the element of $\mathcal{M}_c^2$ obtained by extending the isometry $X \mapsto \int_0^\cdot X_s\, dM_s$ from $\mathcal{E}$ to its closure $\mathscr{L}(M)$.

Example 2.27. Let M be a bounded continuous martingale, $M_0 = 0$ (i.e. $M \in \mathcal{M}_c^2$). We want to study $\int_0^t M_s\, dM_s$. First note

$$\mathbb{E}\left[\int_0^t M_s^2\, d\langle M\rangle_s\right] \leq C^2\mathbb{E}\left[\langle M\rangle_t\right] \leq C^4 < \infty$$

and $M \in \mathscr{L}(M)$. For a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_m = T\}$, $M_t^\pi := \sum_{k=1}^m M_{t_{k-1}} \mathbf{1}_{(t_{k-1},t_k]}(t) \in \mathcal{E}$. As M is continuous, this implies $M_t^\pi \xrightarrow{a.s.} M_t$ when $|\pi| = \sup_k |t_k - t_{k-1}| \to 0$. Dominated convergence (M is bounded!) yields

$$\mathbb{E}\left[\int_0^T \left(\underbrace{M_t^\pi - M_t}_{\to 0}\right)^2 d\langle M\rangle_t\right] \to 0$$

as $|\pi| \to 0$. Thus, $\int_0^T M_t\, dM_t = \lim_{|\pi|\to 0} \int_0^T M_t^\pi\, dM_t$ (in $\mathcal{M}_c^2$). Now note that

$$\begin{aligned}\int_0^t M_s^\pi\, dM_s &= \sum_{k=1}^m M_{t_{k-1}\wedge t}\left(M_{t_k\wedge t} - M_{t_{k-1}\wedge t}\right)\\ &= \frac{1}{2}\Bigg(M_t^2 - \underbrace{M_0^2}_{=0}\Bigg) - \frac{1}{2}\underbrace{\sum_{k=1}^m \left(M_{t_k\wedge t} - M_{t_{k-1}\wedge t}\right)^2}_{\text{converges in } L^2\text{,limit is increasing, continuous process}}.\end{aligned}$$

This means that $\sum_{k=1}^m \left(M_{t_k\wedge t} - M_{t_{k-1}\wedge t}\right)^2 \xrightarrow{L^2} \langle M\rangle_t$ (whenever $|\pi| \to 0$), cf. Brownian motion case. Furthermore, we have $\int_0^t M_s\, dM_s = \frac{1}{2}(M_t^2 - \langle M\rangle_t)$.

Remark 2.28. For $f \in C^1$: $\int_0^t f(s)\, df(s) = \int_0^t f(s)f'(s)\, ds = \frac{1}{2}(f(t)^2 - f(0)^2) = \frac{1}{2}f(t)^2$ if $f(0) = 0$.

Theorem 2.29. *(added properties from tutorial)For $M \in \mathcal{M}_c^2$ and $X, Y \in \mathscr{L}(M)$ the stochastic integral has the following properties:*

a) (linearity) $\forall \alpha, \beta \in \mathbb{R}$: $\int_0^\cdot (\alpha X + \beta Y)_s\, dM_s = \alpha \int_0^\cdot X_s\, dM_s + \beta \int_0^\cdot Y_s\, dM_s$,

b) (Itô-isometry) $\mathbb{E}\big[\big(\int_0^t X_s\, dM_s\big)^2\big] = \mathbb{E}\big[\int_0^t X_s^2\, d\langle M\rangle_s\big] = \|X\|_{M,t}^2$ *and* $\|\int_0^\cdot X_s\, dM_s\|_{\mathcal{M}_c^2} = \|X\|_M$,

c) (quadratic variation) $\left\langle \int_0^\cdot X_s\, dM_s\right\rangle_t = \int_0^t X_s^2\, d\langle M\rangle_s$, $t \geq 0$.

Proof. Show by approximation with simple and bounded processes (cf. exercises). □

Lemma 2.30. *For $M \in \mathcal{M}_c^2$, $X \in \mathscr{L}(M)$, τ stopping time (all with respect to some filtration $(\mathcal{F}_t)_{t\geq 0}$) we have*

$$(X \circ M)_{t\wedge\tau} = (X \circ M^\tau)_t = \left(\left(X\mathbf{1}_{[0,\tau]}\right) \circ M\right)_t$$

Proof. 1. for $X_t = \sum_{k=1}^\infty \xi_k \mathbf{1}_{(\tau_{k-1},\tau_k]}(t)$ simple and bounded. (adapted from the lecture) The first equality follows directly from

$$\begin{aligned}(X \circ M)_{\tau\wedge t} &= \sum_{k=1}^\infty \xi_k \left(M_{\tau_k\wedge\tau\wedge t} - M_{\tau_{k-1}\wedge\tau\wedge t}\right)\\ &= \sum_{k=1}^\infty \xi_k \left(M_{\tau_k\wedge t}^\tau - M_{\tau_{k-1}\wedge t}^\tau\right).\end{aligned}$$

For the second equality note that $X\mathbf{1}_{[0,\tau]} \in \mathscr{L}(M)$, because $\mathbf{1}_{[0,\tau]}$ adapted and left-continuous. Therefore the second equality is clear for $X_t^{(n)} = \sum_{k=1}^n \xi_k \mathbf{1}_{(\tau_{k-1},\tau_k]}(t)$, because

$$\begin{aligned}X_t^{(n)}\mathbf{1}_{[0,\tau]}(t) &= \sum_{k=1}^n \xi_k \mathbf{1}_{[0,\tau]}(t)\mathbf{1}_{(\tau_{k-1},\tau_k]}(t)\\ &= \sum_{k=1}^n \underbrace{\xi_k \mathbf{1}_{\{\tau_{k-1}\leq\tau\}}}_{\mathcal{F}_{\tau_{k-1}\wedge\tau}-mb.}\mathbf{1}_{(\tau_{k-1}\wedge\tau,\tau_k\wedge\tau]}(t)\end{aligned}$$

is simple and bounded by Thoerem 1.25 such that

$$\begin{aligned}\left(\left(X^{(n)}\mathbf{1}_{[0,\tau]}\right)\circ M\right)_{\tau\wedge t} &= \sum_{k=1}^{n}\xi_k\mathbf{1}_{\{\tau_{k-1}\le\tau\}}\left(M_{\tau_k\wedge\tau\wedge t}-M_{\tau_{k-1}\wedge\tau\wedge t}\right)\\ &= \sum_{k=1}^{n}\xi_k\left(M_{\tau_k\wedge\tau\wedge t}-M_{\tau_{k-1}\wedge\tau\wedge t}\right)\\ &= \left(X^{(n)}\circ M\right)_{\tau\wedge t}.\end{aligned}$$

Because $X^{(n)}\xrightarrow{d_M}X$ and $X^{(n)}\mathbf{1}_{[0,\tau]}\xrightarrow{d_M}X\mathbf{1}_{[0,\tau]}$, we obtain therefore

$$(X\circ M)_{t\wedge\tau}=\lim_{n\to\infty}\left(X^{(n)}\circ M\right)_{t\wedge\tau}=\lim_{n\to\infty}\left(\left(X^{(n)}\mathbf{1}_{[0,\tau]}\right)\circ M\right)=\left(\left(X\mathbf{1}_{[0,\tau]}\right)\circ M\right)$$

in $L^2(\mathbb{P})$.

2. for general $X\in\mathscr{L}(M)$: For $T>0$ we have $\int_0^T X_t^2\mathbf{1}_{[0,\tau]}(t)d\langle M\rangle_t\le\int_0^T X_t^2 d\langle M\rangle_t$. Moreover, we also have $X\in\mathscr{L}(M^\tau)$ because

$$\begin{aligned}\int_0^T X_t^2\,d\langle M^\tau\rangle_t &= \int_0^T X_t^2\,d\langle M\rangle_{\tau\wedge t}\\ &= \int_0^{T\wedge\tau} X_t^2\,d\langle M\rangle_t\\ &\le \int_0^T X_t^2\,d\langle M\rangle_t\,.\end{aligned}$$

(adapted from the lecture: have to argue why $\langle M^\tau\rangle=\langle M\rangle_{\tau\wedge\cdot}$ and why the second equality holds) For the first equality we use that $M^2_{\tau\wedge t}-\langle M\rangle_{\tau\wedge t}$ and $\left(M_t^\tau\right)^2-\langle M^\tau\rangle_t$ are martingales by optional stopping (Theorem 1.26) such that uniqueness of the quadratic variation shows $\langle M^\tau\rangle_t=\langle M\rangle_{\tau\wedge t}$ a.s. for all $t\ge0$. For the second equality we use that the measure $d\langle M\rangle_{\tau\wedge\cdot}$, which is induced by the map $t\mapsto\langle M\rangle_{\tau\wedge t}$, is supported on $[0,\tau]$. Now take simple processes $X^{(n)}\xrightarrow{d_M}X$ and use $X^{(n)}\mathbf{1}_{[0,\tau]}\xrightarrow{d_M}X\mathbf{1}_{[0,\tau]}$ as well as $X^{(n)}\xrightarrow{d_{M^\tau}}X$. Then the result is obtained by identifying the limits as $n\to\infty$. □

Remark 2.31. From now on we can just write $\int_0^{\tau\wedge t}X_s\,dM_s$ to mean one of the three stochastic integrals. If the limit $t\to\infty$ exists, we just write $\int_0^\tau X_s\,dM_s$. Similarly we write $\int_a^b X_s\,dM_s=\int_0^b X_s\,dM_s-\int_0^a X_s\,dM_s$ for $0\le a<b$, i.e. $\int_a^b X_s dM_s=\int_0^\infty X_s\mathbf{1}_{[a,b]}(s)\,dM_s$.

Definition 2.32. For a continuous local martingale M with $M_0=0$ we set

$$\mathscr{L}_{loc}(M)=\left\{(X_t,t\ge0):X\text{ progr. mb},\forall T>0:\mathbb{P}\left(\int_0^T X_t^2\,d\langle M\rangle_t<\infty\right)=1\right\}.$$

Let σ_n be the localizing sequence of M such that M^{σ_n} is a bounded martingale and let $\rho_n:=\inf\{t>0:\int_0^t(X_s^2+1)\,d\langle M\rangle_s\ge n\}$, $n\ge1$, for $X\in\mathscr{L}_{loc}(M)$ be stopping times. Set $\tau_n=\sigma_n\wedge\rho_n$ such that $\tau_n\to\infty$ a.s. and M^{τ_n} is still a bounded continuous martingale by Theorem 1.26. Then we define $(X\circ M)_t(\omega):=\left(\int_0^t X_s\,dM_s\right)(\omega)=\lim_{n\to\infty}\left(\int_0^t X_s\,dM_s^{\tau_n}\right)(\omega)$.

Remark 2.33.

a) Even for $M\in\mathcal{M}_c^2$ the space $\mathscr{L}_{loc}(M)$ is much larger than $\mathscr{L}(M)$. For Brownian motion, for instance, we have

$$\mathscr{L}_{loc}(B)=\left\{(X_t,t\ge0):X\text{ prog. mb. and }\int_0^T X_t^2\,dt<\infty\text{ a.s. for all }T>0\right\}$$

and any continuous, adapted process lies in $\mathscr{L}_{loc}(B)$ (no moment assumptions like $\mathbb{E}[X_T^2]<\infty$ for $T>0$).

b) If $M \in \mathcal{M}_c^2$ and $X \in \mathscr{L}(M)$, then by the lemma $\int_0^t X_s\, dM_s^{\tau_n} = \int_0^{t\wedge\tau_n} X_s\, dM_s$ which tends a.s. to $\int_0^t X_s\, dM_s$ as $n \to \infty$, since $\tau_n \to \infty$ a.s.

Theorem 2.34. *Let M be a continuous local martingale with $M_0 = 0$ and let $X \in \mathscr{L}_{loc}(M)$. Then:*

a) The stochastic integral $\int_0^t X_s\, dM_s$ is well-defined as an a.s. limit.

b) $\left(\int_0^t X_s\, dM_s, t \geq 0\right)$ is itself a continuous local martingale with quadratic variation $\left\langle \int_0^\cdot X_s\, dM_s \right\rangle_t = \int_0^t X_s^2\, d\langle M\rangle_s$.

c) (adapted from the lecture: added stopping from the lemma) For any stopping time τ which is a.s. finite we have $(X \circ M)_t^\tau = (X\mathbf{1}_{[0,\tau]} \circ M)_t = (X \circ M^\tau)_t$.

Proof. a) Let $(\tau_n)_{n\in\mathbb{N}}$ be as after Definition 2.32. Then $M^{\tau_n} \in \mathcal{M}_c^2$ and $X \in \mathscr{L}(M^{\tau_n})$, because

$$\mathbb{E}\left[\int_0^T X_s^2\, d\langle M^{\tau_n}\rangle_s\right] = \mathbb{E}\left[\int_0^T X_s^2\, d\langle M\rangle_{\tau_n\wedge s}\right] \leq \mathbb{E}\left[\int_0^{\tau_n} X_s^2\, d\langle M\rangle_s\right] \leq n < \infty,$$

(adapted from the lecture: have to argue why this holds) which follows as in the proof of Lemma 2.30. (adapted from the lecture: took out the first sentence)From Lemma 2.30 we have for $m \geq n \geq 1$ and $\tau_m \geq \tau_n$ that

$$\begin{aligned}\int_0^t X_s\, dM_s^{\tau_n} &= \int_0^t X_s\, dM_s^{\tau_n\wedge\tau_m} \\ &= \int_0^t X_s\, d\left(M^{\tau_m}\right)_s^{\tau_n} \\ &\overset{\text{Lemma 2.30}}{=} \int_0^{t\wedge\tau_n} X_s\, dM_s^{\tau_m}.\end{aligned}$$

Hence, taking $m = n$ we obtain $\int_0^t X_s\, dM_s^{\tau_n} = \int_0^{t\wedge\tau_n} X_s\, dM_s^{\tau_n}$ and therefore

$$\int_0^{t\wedge\tau_n} X_s\, dM_s^{\tau_n} = \int_0^{t\wedge\tau_n} X_s\, dM_s^{\tau_m} \tag{2.2.1}$$

In particular, $\int_0^t X_s\, dM_s^{\tau_n} = \int_0^t X_s\, dM_s^{\tau_m}$ a.s. on $\{t \leq \tau_n\}$. (adapted from the lecture: whole argument after this) This equality is satisfied for all $m \geq n$, so letting $m \to \infty$ this yields for $\omega \in \{t \leq \tau_n\}$ that

$$\left(\int_0^t X_s\, dM_s\right)(\omega) = \lim_{n\leq m\to\infty}\left(\int_0^t X_s\, dM_s^{\tau_m}\right)(\omega) = \left(\int_0^t X_s\, dM_s^{\tau_n}\right)(\omega) \tag{2.2.2}$$

Furthermore, since $\tau_n \to \infty$ a.s. for any $(\omega, t) \in (\Omega \times \mathbb{R}_+)$ we can find n_0 such that this is satisfied for all $n \geq n_0$. Thus the stochastic integral is well-defined.

b) (adapted from the lecture: whole argument after this)Setting $m = n$ in (2.2.1) we obtain with (2.2.2) that

$$(X \circ M)_{t\wedge\tau_n} = \int_0^{t\wedge\tau_n} X_s\, dM_s = \int_0^{t\wedge\tau_n} X_s\, dM_s^{\tau_n} = (X \circ M^{\tau_n})_{t\wedge\tau_n}$$

a.s. and the right-hand side is in $\mathcal{M}_c^2$. Thus, $\left(\int_0^t X_s\, dM_s, t \geq 0\right)$ is a continuous local martingale with localising sequence (τ_n). The quadratic variation is

$$\begin{aligned}\left\langle \int_0^\cdot X_s\, dM_s \right\rangle_t &= \lim_{n\to\infty}\left\langle \int_0^\cdot X_s\, dM_s^{\tau_n}\right\rangle_t = \lim_{n\to\infty}\int_0^t X_s^2\, d\langle M^{\tau_n}\rangle_s \\ &= \lim_{n\to\infty}\int_0^t X_s^2\, d\langle M\rangle_{s\wedge\tau_n} = \lim_{n\to\infty}\int_0^{t\wedge\tau_n} X_s^2\, d\langle M\rangle_s = \int_0^t X_s^2\, d\langle M\rangle_s.\end{aligned}$$

c) (adapted from the lecture: added proof)Follows from Lemma 2.30:

$$(X \circ M)_t^\tau = (X \circ M)_{\tau \wedge t} = \lim_{n\to\infty} (X \circ M^{\tau_n})_{t\wedge\tau} = \lim_{n\to\infty} (X \circ (M^\tau)^{\tau_n})_t = (X \circ M^\tau)_t \,.$$

□

Remark 2.35. Note that $\mathbb{E}\left[\int_0^t X_s^2 \, d\langle M\rangle_s\right]$ may be infinite such that Itô isometry may not make sense.

Theorem 2.36. *Let M be a continuous local martingale, $M_0 = 0$ and let X be an adapted continuous process. Then $X \in \mathscr{L}_{loc}(M)$ and for partitions π_m of $[0,t]$ with $|\pi_m| = \max_{t_k\in\pi_m} |t_{k+1} - t_k| \to 0$ we have*

$$\sum_{t_k\in\pi_n} X_{t_{k-1}} \left(M_{t_k} - M_{t_{k-1}}\right) \xrightarrow{\mathbb{P}} \int_0^t X_s \, dM_s, \quad n \to \infty.$$

Proof. For (σ_n) a localising sequence of M we define stopping times $\tau_n = \sigma_n \wedge \inf\{t \geq 0 : |X_t| \geq n\}$. Because of Lemma 2.20, X is progressively measurable. Since a continuous function is bounded on any compact interval, we thus obtain $X \in \mathscr{L}(M)$. We have (adapted from the lecture: added measurability and comment on X simple and changed I_[0,τ_n](t_k-1) to the one below)

$$\begin{aligned}
\sum_{t_k\in\pi_m} X_{t_{k-1}} \left(M_{t_k}^{\tau_n} - M_{t_{k-1}}^{\tau_n}\right) &= \sum_{t_k\in\pi_m} \underbrace{X_{t_{k-1}} \mathbf{1}_{\{\tau_n \geq t_{k-1}\}}}_{\mathcal{F}_{t_{k-1}\wedge\tau_n}-\text{mb.}} \left(M_{t_k\wedge\tau_n} - M_{t_{k-1}\wedge\tau_n}\right) \\
&= \int_0^t \left(\sum_{t_k\in\pi_m} X_{t_{k-1}} \mathbf{1}_{\{\tau_n\geq t_{k-1}\}} \mathbf{1}_{(t_{k-1}\wedge\tau_n, t_k\wedge\tau_n]}(s)\right) dM_s^{\tau_n} \\
&= \int_0^t \left(\sum_{t_k\in\pi_m} X_{t_{k-1}} \mathbf{1}_{(t_{k-1}\wedge\tau_n, t_k\wedge\tau_n]}(s)\right) dM_s^{\tau_n},
\end{aligned}$$

because

$$\sum_{t_k\in\pi_m} X_{t_{k-1}} \mathbf{1}_{\{\tau_n\geq t_{k-1}\}} \mathbf{1}_{(t_{k-1}\wedge\tau_n, t_k\wedge\tau_n]}(s) = \sum_{t_k\in\pi_m} X_{t_{k-1}} \mathbf{1}_{(t_{k-1}\wedge\tau_n, t_k\wedge\tau_n]}(s)$$

is simple. Observe (adapted from the lecture: changed arguments after this) that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \left(\sum_{t_k\in\pi_m} X_{t_{k-1}} \mathbf{1}_{\{\tau_n\geq t_{k-1}\}} \mathbf{1}_{(t_{k-1}\wedge\tau_n, t_k\wedge\tau_n]}(s) - X_s\right)^2 d\langle M^{\tau_n}\rangle_s\right] \\
&= \sum_{t_k\in\pi_m} \mathbb{E}\left[\int_{t_{k-1}\wedge\tau_n}^{t_k\wedge\tau_n} \underbrace{\left(X_{t_{k-1}} - X_s\right)^2}_{\to 0 \text{ by continuity of } X} d\langle M^{\tau_n}\rangle_s\right].
\end{aligned}$$

This, however, converges to 0 by dominated convergence (observe that $\sum_{t_k\in\pi_m} \int_{t_{k-1}\wedge\tau_n}^{t_k\wedge\tau_n} (X_{t_{k-1}} - X_s)^2 \, d\langle M^{\tau_n}\rangle_s \leq 4n^2 \langle M^{\tau_n}\rangle_t$). We have by Itô isometry (or convergence wrt. $d_{M^{\tau_n}}$) that

$$\sum_{t_k\in\pi_m} X_{t_{k-1}} (M_{t_k}^{\tau_n} - M_{t_{k-1}}^{\tau_n}) \xrightarrow{L^2(\mathbb{P})} \int_0^t X_s \, dM_s^{\tau_n} \overset{\text{Thm. 2.34}}{=} \int_0^{t\wedge\tau_n} X_s \, dM_s.$$

Let $Z_m := \sum_{t_k\in\pi_m} X_{t_{k-1}} (M_{t_k} - M_{t_{k-1}})$ and $\Omega_n := \{t \leq \tau_n\}$ such that $\Omega_n \subseteq \Omega_{n+1}$ and $\mathbb{P}(\bigcup_n \Omega_n) = 1$. We know that $Z_m \mathbf{1}_{\Omega_n} \xrightarrow{\mathbb{P}} Z\mathbf{1}_{\Omega_n}$ as $m \to \infty$ where $Z = \int_0^t X_s \, dM_s$, (adapted from the lecture: add because... because

$$\begin{aligned} Z_m \mathbf{1}_{\Omega_n} &= \left(\sum_{t_k \in \pi_m} X_{t_{k-1}} (M_{t_k} - M_{t_{k-1}}) \right) \mathbf{1}_{\{t \le \tau_n\}} \\ &= \left(\sum_{t_k \in \pi_m} X_{t_{k-1}} (M_{t_k}^{\tau_n} - M_{t_{k-1}}^{\tau_n}) \right) \mathbf{1}_{\{t \le \tau_n\}}, \\ Z \mathbf{1}_{\Omega_n} &= \left(\int_0^{\tau_n \wedge t} X_s \, dM_s \right) \mathbf{1}_{\{t \le \tau_n\}} \end{aligned}$$

and because of the Tschebycheff inequality. (adapted from the lecture: changed argument)For $\varepsilon > 0$ and $\delta > 0$ let n and m_0 large enough such that $\mathbb{P}(\Omega_n^c) \le \frac{\delta}{2}$ and for all $m \ge m_0$

$$\mathbb{P} \left(|Z_m - Z| \, \mathbf{1}_{\Omega_n} > \varepsilon \right) \le \frac{\delta}{2}$$

This implies

$$\mathbb{P} \left(|Z - Z_m| > \varepsilon \right) \le \mathbb{P} \left(\{ |Z - Z_m| > \varepsilon \} \cap \Omega_n \right) + \mathbb{P} \left(\Omega_n^c \right) \le \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

□

Remark 2.37. This is a Riemann-type approximation of $\int_0^t X_s \, dM_s$, but it is important to use $X_{t_{k-1}}$ and not any X_s for $s \in [t_{k-1}, t_k]$ in the sum to guarantee the martingale properties. Note that this gives a concrete approximation method for the stochastic integral. Form this result only, however, one cannot deduce the properties of $\left(\int_0^t X_s \, dM_s, t \ge 0 \right)$ as a process like being a local martingale, being continuous or calculating its quadratic variation.

Corollary 2.38. *If M is a continuous local martingale, $M_0 = 0$, then for partitions π_m of $[0, T]$ with $|\pi_m| \to 0$ as $m \to \infty$ we have for all $t \in [0, T]$:*

$$\begin{aligned} \sum_{t_k \in \pi_m} \left(M_{t_k \wedge t} - M_{t_{k-1} \wedge t} \right)^2 &\xrightarrow{\mathbb{P}} \langle M \rangle_t \, , \\ \int_0^t M_s \, dM_s = \frac{1}{2} M_t^2 - \frac{1}{2} \langle M \rangle_t \, . \end{aligned} \tag{2.2.3}$$

Proof. We write (always $t_0 = 0, \max_k t_k = T$)

$$\begin{aligned} M_t^2 &= \sum_{t_k \in \pi_m} M_{t \wedge t_k}^2 - M_{t \wedge t_{k-1}}^2 \\ &= \sum_{t_k \in \pi_m} \left(\left(M_{t \wedge t_k} - M_{t \wedge t_{k-1}} \right)^2 + 2 M_{t_{k-1}} \left(M_{t_k \wedge t} - M_{t_{k-1} \wedge t} \right) \right). \end{aligned}$$

By the theorem

$$\sum_{t_k \in \pi_m} M_{t_{k-1} \wedge t} \left(M_{t_k \wedge t} - M_{t_{k-1} \wedge t} \right) \xrightarrow{\mathbb{P}} \int_0^t M_s \, dM_s$$

such that

$$\sum_{t_k \in \pi_m} \left(M_{t_k \wedge t} - M_{t_{k-1} \wedge t} \right)^2 \xrightarrow{\mathbb{P}} M_t^2 - 2 \int_0^t M_s \, dM_s =: Q_t.$$

Since M and $\int_0^\cdot M_s \, dM_s$ are continuous, so is Q. The limit Q is independent of the choice of (π_m). We can thus consider refinements $\pi_m \subseteq \pi_{m+1}$ for all $m \ge 1$. We have for $m \ge n$ that $t \mapsto \sum_{t_k \in \pi_m} (M_{t_k \wedge t} - M_{t_{k-1} \wedge t})^2$ is increasing for $t \in \pi_m \supseteq \pi_n$. Hence, the limit Q is increasing a.s. on $\bigcup_{m \ge 1} \pi_m$. By continuity of Q and density of $\bigcup_{m \ge 1} \pi_m$ we conclude that Q_t is increasing on $[0, T]$. Observing finally that $M_t^2 - Q_t = 2 \int_0^t M_s \, dM_s$ is a continuous local martingale starting in 0, we see that $Q_t = \langle M \rangle_t$ a.s. for all $t \ge 0$ (by uniqueness of $\langle M \rangle_t$). □

Remark 2.39. Compare (2.2.3) to the standard equation for $f \in C^1$, $f(0) = 0$:

$$\int_0^t f(s)\, df(s) = \int_0^t f(s) f'(s)\, ds = \frac{1}{2} f^2(s).$$

Hence, the meaning of quadratic variation lies at the heart of the difference between stochastic and deterministic integration.

Chapter 3

Main theorems of stochastic analysis

3.1 Itô's formula

Definition 3.1. A *continuous semimartingale* $(X_t, t \geq 0)$ with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a continuous process which can be written as $X_t = M_t + A_t$ with a continuous local $(\mathcal{F}_t)_{t\geq 0}$-martingale M and an $(\mathcal{F}_t)_{t\geq 0}$-adapted, continuous process A with paths $t \mapsto A_t(\omega)$ of a.s. finite variation on compact intervals. Then we define for $t \geq 0$

$$\left(\int_0^t Y_s\, dX_s\right)(\omega) := \left(\int_0^t Y_s\, dM_s\right)(\omega) + \int_0^t Y_s(\omega)\, dA_s(\omega),$$

whenever the right-hand side is well-defined, i.e. $Y \in \mathscr{L}_{loc}(M)$ and $\int_0^t |Y_s|\, |dA_s| < \infty$ a.s. (here $|dA_s| = dA_s^+ + dA_s^-$ is the toal variation of the signed measure $dA_s = dA_s^+ - dA_s^-$). Moreover, we set

$$\langle X \rangle_t = \lim_{|\pi_m|\to 0} \sum_{t_k\in\pi_m} \left(X_{t_k\wedge t} - X_{t_{k-1}\wedge t}\right)^2,$$

whenever the limit exists in probability.

Definition 3.2. Let X, Y be continuous semimartingales. Then we define the *quadratic covariation* by *polarisation*:

$$\langle X, Y \rangle_t := \frac{1}{4}\left(\langle X+Y \rangle_t - \langle X-Y \rangle_t\right), \quad t \geq 0.$$

Proposition 3.3. *Let X, Y be continuous semimartingales.*

a) The quadratic covariation exists and satisfies $\langle X, Y\rangle_t = \lim_{|\pi|\to 0} \sum_{t_k\in\pi} \big(X_{t_k\wedge t} - X_{t_{k-1}\wedge t}\big)\big(Y_{t_k\wedge t} - Y_{t_{k-1}\wedge t}\big)$ in probability, where π is any partition of $[0,\infty)$.

b) a continuous semimartingale $(X_t, t \geq 0)$ with decomposition $X = M + A$ into a continuous local martingale and a continuous process of bounded variation on compacts A we have

$$\langle X \rangle_t = \langle M \rangle_t = \lim_{|\pi_m|\to 0} \sum_{t_k\in\pi_m} \left(X_{t_k\wedge t} - X_{t_{k-1}\wedge t}\right)^2.$$

Proof. See exercises. □

Theorem 3.4 (Partial integration)**.** *For continuous semimartingales X, Y we have*

$$X_tY_t = X_0Y_0 + \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \langle X, Y\rangle_t\,, \quad t \geq 0.$$

In particular,

$$X_t^2 = X_0^2 + 2\int_0^t X_s dX_s + \langle X\rangle_t\,, \quad t \geq 0.$$

Proof. By polarisation it suffices to prove the second identity. We have for any partition of $[0, T]$, $t \leq T$:

$$\begin{aligned}\sum_{t_k\in\pi}\left(X_{t_k\wedge t} - X_{t_{k-1}\wedge t}\right)^2 &= X_t^2 - X_0^2 - 2\sum_{t_k\in\pi} X_{t_{k-1}\wedge t}\left(X_{t_k\wedge t} - X_{t_{k-1}\wedge t}\right)\\ &= X_t^2 - X_0^2 - 2\sum_{t_k\in\pi} X_{t_{k-1}\wedge t}\left(M_{t_k\wedge t} - M_{t_{k-1}\wedge t}\right)\\ &\quad +2\sum_{t_k\in\pi} X_{t_{k-1}\wedge t}\left(A_{t_k\wedge t} - A_{t_{k-1}\wedge t}\right).\end{aligned}$$

The left-hand side converges in probability to $\langle X\rangle_t$ whereas the right-hand side converges in probability to

$$X_t^2 - X_0^2 - 2\int_0^t X_s\,dM_s - 2\int_0^t X_s\,dA_s = X_t^2 - X_0^2 - 2\int_0^t X_s\,dX_s$$

using Riemann-Stieltjes approximation (we even have $\sum_{t_k\in\pi} X_{t_{k-1}\wedge t}(A_{t_k\wedge t} - A_{t_{k-1}\wedge t}) \to \int_0^t X_s\,dA_s$ a.s.). □

Theorem 3.5 (Associativity of the stochastic integral)**.** *Let $M \in \mathcal{M}_c^2$, $X \in \mathscr{L}(M)$ and $Y \in \mathscr{L}(X\circ M)$ with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$. Then:*

a) $YX \in \mathscr{L}(M)$.

b) $(Y\circ(X\circ M)) = ((YX)\circ M)$, *a.s.*

Proof. See exercises. □

The main result of this section will be *Itô's formula* (a.k.a. *Itô's lemma*).

Theorem 3.6 (Itô's formula)**.** *For a continuous semimartingale X and $f \in C^2(\mathbb{R})$ the process $(f(X_t), t \geq 0)$ is again a continuous semimartingale and satisfies*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s\,, \quad t \geq 0.$$

Proof. There are two main proof strategies.

1. proof (sketch). Writing $f(X_t) - f(X_0)$ as a telescoping sum and Taylor expansions, we obtain

$$\begin{aligned}f(X_t) - f(X_0) &= \sum_{t_k\in\pi}\left(f(X_{t_k\wedge t}) - f(X_{t_{k-1}\wedge t}\right)\\ &\overset{\text{Taylor}}{=} \sum_{t_k\in\pi}\left(f'(X_{t_{k-1}\wedge t})(X_{t_k\wedge t} - X_{t_{k-1}\wedge t}\right)\\ &\quad +\frac{1}{2}f''(X_{t_{k-1}\wedge t})(X_{t_k\wedge t} - X_{t_{k-1}\wedge t})^2 + o\left((X_{t_k\wedge t} - X_{t_{k-1}\wedge t})^2\right)\\ &\overset{\mathbb{P}}{\to} \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,d\langle X\rangle_s\end{aligned}$$

as $|\pi| \to 0$, because the remainder terms converge to 0 in probability (not proven here, check literature).

2. proof. We first show that Itô's formula holds for polynomials f. We already know that it holds for $f(x) = x$ and $f(x) = x^2$ (by partial integration). By linearity it holds for polynomials f of maximal degree 2. We argue now inductively. Assume that the claim holds for polynomials of order of maximal degree $m-1$, i.e.

$$X_t^{m-1} = X_0^{m-1} + \int_0^t (m-1)\, X_s^{m-2}\, dX_s + \int_0^t \frac{(m-1)\,(m-2)}{2} X_s^{m-3}\, d\langle X\rangle_s\,.$$

By partial integration and associativity of the stochastic integral (Theorem 3.5) we then have

$$\begin{aligned} X_t^m &= X_t^{m-1} X_t \\ &= X_0^{m-1} X_0 + \int_0^t X_s^{m-1}\, dX_s + \int_0^t X_s\, dX_s^{m-1} + \langle X, X^{m-1}\rangle_t \\ &= X_0^m + \int_0^t X_s^{m-1}\, dX_s \\ &\quad + \left(\int_0^t X_s\,(m-1)\, X_s^{m-2}\, dX_s + \int_0^t X_s \frac{(m-1)\,(m-2)}{2} X_s^{m-3}\, d\langle X\rangle_s \right) \\ &\quad + \left\langle X, \int_0^\cdot (m-1)\, X_s^{m-2}\, dX_s + A \right\rangle_t \end{aligned}$$

for a finite variation process A. Therefore

$$\begin{aligned} X_t^m &= X_0^m + m \int_0^t X_s^{m-1}\, dX_s + \frac{(m-1)\,(m-2)}{2} \int_0^t X_s^{m-2}\, d\langle X\rangle_s \\ &\quad + \left\langle \int_0^\cdot 1\, dX_s, \int_0^\cdot (m-1)\, X_s^{m-2}\, dX_s \right\rangle_t . \end{aligned}$$

By polarisation we obtain

$$\begin{aligned} \left\langle \int_0^\cdot 1\, dX_s, \int_0^\cdot (m-1)\, X_s^{m-2}\, dX_s \right\rangle_t &= \left\langle \int_0^\cdot 1\, dM_s, \int_0^\cdot (m-1)\, X_s^{m-2}\, dM_s \right\rangle_t \\ &= \int_0^t (m-1)\, X_s^{m-2}\, d\langle M\rangle_s \\ &= \int_0^t (m-1)\, X_s^{m-2}\, d\langle X\rangle_s \end{aligned}$$

such that

$$X_t^m = X_0^m + m \int_0^t X_s^{m-1}\, dX_s + \frac{m\,(m-1)}{2} \int_0^t X_s^{m-2}\, d\langle X\rangle_s\,.$$

By linearity we thus have Itô's formula for all polynomials f of maximal degree m.

We now show Itô's formula for X taking values in the interval $[-K, K]$ for some $K > 0$. By Weierstraß's approximation theorem there are polynomials p_m such that

$$\begin{aligned} \sup_{x \in [-K,K]} |f''(x) - p_m''(x)| &\to 0, \\ \sup_{x \in [-K,K]} |f'(x) - p_m'(x)| &\to 0, \\ \sup_{x \in [-K,K]} |f(x) - p_m(x)| &\to 0 \end{aligned}$$

as $m \to \infty$. Therefore we have a.s. $p_m(X_t) \to f(X_t)$ and $p_m(X_0) \to f(X_0)$ and by the uniform convergences from above also $\int_0^t (f'(X_s) - p_m'(X_s))\, dX_s \xrightarrow{\mathbb{P}} 0$ and $\int_0^t (f''(X_s) - p_m''(X_s))\, d\langle X\rangle_s \to 0$ a.s. Since Itô's formula holds for each p_m by these convergences it also holds for f.

The last step in the proof is to show Itô's formula for general X and f. The formula holds for the stopped semi-martingales X^{τ_K} with $\tau_K = \inf\{t \geq 0 : |X_t| \geq K\}$:

$$\begin{aligned} f(X_{t\wedge\tau_K}) &= f(X_0) + \int_0^t f'(X_s^{\tau_K})\, dX_s^{\tau_K} + \frac{1}{2}\int_0^t f''(X_s^{\tau_K})\, d\langle X^{\tau_K}\rangle_s \\ &= f(X_0) + \int_0^t f'(X_s^{\tau_K})\, dM_s^{\tau_K} + \int_0^t f'(X_s^{\tau_K})\, dA_s^{\tau_K} + \frac{1}{2}\int_0^t f''(X_s^{\tau_K})\, d\langle M^{\tau_K}\rangle_s . \end{aligned}$$

By the stopping property of stochastic integrals in Theorem 2.34 and $\int_0^t f'(X_s^{\tau_K})\, dA_s^{\tau_K} = \int_0^{t\wedge\tau_K} f'(X_s^{\tau_K})\, dA_s$ as well as $\int_0^t f''(X_s^{\tau_K})\, d\langle M^{\tau_K}\rangle_s = \int_0^{t\wedge\tau_K} f''(X_s^{\tau_K}) d\langle M\rangle_s$ we obtain

$$f(X_{t\wedge\tau_K}) = f(X_0) + \int_0^{t\wedge\tau_K} f'(X_s)\, dX_s + \frac{1}{2}\int_0^{t\wedge\tau_K} f''(X_s)\, d\langle X\rangle_s .$$

Letting $K \to \infty$ we have $\tau_K \to \infty$ a.s. by continuity of X and thus $t \wedge \tau_K \to t$ and

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X\rangle_s .$$

□

Remark 3.7. Suppose $t \mapsto X_t$ is $C^1(\mathbb{R})$. Then $\langle X\rangle_t = 0$ and Itô's formula specialises to the f*undamental theorem of calculus*: $f(X_t) = f(X_0) + \int_0^t f'(X_s)X_s'\, ds$. Likewise, Itô's formula allows to calculate the stochastic integral $\int_0^t f'(X_s)\, dX_s$.

Example 3.8 (Geometric Brownian motion)**.** We want to solve the *stochastic differential equation*

$$dX_t = X_t\,(\mu\, dt + \sigma\, dB_t) = \mu X_t\, dt + \sigma X_t\, dB_t \tag{*}$$

with $X_0 = x_0$, i.e. we want to find a process $(X_t, t \geq 0)$ such that

$$X_t = X_0 + \int_0^t X_s \mu\, ds + \sigma \int_0^t X_s\, dB_s, \quad a.s.$$

Informally we consider $f(x) = \log x$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $x > 0$. If we assume that such a process X exists we have by Itô's formula

$$df(X_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, d\langle X\rangle_t ,$$

i.e.

$$\begin{aligned} \log(X_t) &= \log(X_0) + \int_0^t \frac{1}{X_s}\, dX_s + \frac{1}{2}\int_0^t \big(-\frac{1}{X_s^2}\big)\, d\langle X\rangle_s \\ &= \log(x_0) + \int_0^t (\mu\, ds + \sigma\, dB_s) - \frac{1}{2}\int_0^t \frac{1}{X_s^2}\sigma^2 X_s^2\, ds \\ &= \log(x_0) + \mu t + \sigma B_t - \frac{\sigma^2}{2} t. \end{aligned}$$

Applying now the exponential function we therefore get

$$X_t = x_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right).$$

Rigorously you can apply Itô's formula to the right-hand side and derive (*). What happens for $t \to \infty$? If $\mu > \frac{\sigma^2}{2}$, then $X_t \to \infty$ a.s. by the law of the iterated logarithm and similarly, if $\mu < \frac{\sigma^2}{2}$, then $X_t \to 0$ a.s. Finally, if $\mu = \frac{\sigma^2}{2}$, then $\limsup_{t\to\infty} X_t = \infty$ and $\liminf_{t\to\infty} X_t = 0$ a.s. If $\mu = 0$, then X is a martingale (see Proposition 1.20), but $\mu < \frac{\sigma^2}{2}$ such that $X_t \to 0$ such that X cannot be a uniformly integrable martingale.

Definition 3.9. A *d-dimensional continuous semimartingale* $X_t = (X_t^{(1)}, \dots, X_t^{(d)})^T$ is a vector of d one-dimensional continuous semimartingales $X^{(1)}, \dots, X^{(d)}$. A *d-dimensional Brownian motion* $B_t = (B_t^{(1)}, \dots, B_t^{(d)})^T$ consists of d independent Brownian motions $B^{(1)}, \dots, B^{(d)}$.

Remark 3.10. As usual we understand $\int_0^t (X_s^{(1)}, \dots, X_s^{(d)})^T \, dY_s := (\int_0^t X_s^{(1)} \, dY_s, \dots, \int_0^t X_s^{(d)} \, dY_s)$ for a continuous semimartingale Y. Moreover, if Y is a d-dimensional continuous semimartngale, then we define $\int_0^t \left\langle \left(X_s^{(1)}, \dots, X_s^{(d)}\right)^T, d\left(Y_s^{(1)}, \dots, Y_s^{(d)}\right)^T \right\rangle := \sum_{k=1}^d \int_0^t X_s^{(k)} \, dY_s^{(k)}$ etc.

Theorem 3.11. *Let X be a d-dimensional continuous semimartingale and*

$$\begin{aligned} f \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+) &= \Big\{ g : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}, (x,t) \mapsto g(x,t) : \frac{\partial^2 g}{\partial x_i x_j} \in C(\mathbb{R}^d \times \mathbb{R}^+) \\ &\qquad \text{for } 1 \le i,j \le d, \frac{\partial g}{\partial t} \in C(\mathbb{R}^d \times \mathbb{R}^+) \Big\}. \end{aligned}$$

Then $(f(X_t,t), t \ge 0)$ is a (one-dimensional) continuous semimartingale satisfying

$$\begin{aligned} f(X_t,t) &= f(X_0,0) + \sum_{k=1}^d \int_0^t \frac{\partial f}{\partial x_k}(X_k,s) \, dX_s^{(k)} + \int_0^t \frac{\partial f}{\partial t}(X_s,s) \, ds \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i x_j}(X_S,s) \, d\left\langle X^{(i)}, X^{(j)} \right\rangle_s \\ &= f(X_0,0) + \int_0^t \langle \nabla_x f(X_s,s), \, dX_s \rangle + \int_0^t \frac{\partial f}{\partial t}(X_s,s) \, ds \\ &\quad + \frac{1}{2} \int_0^t \langle \nabla_x^2 f(X_s,s), \, d\langle X \rangle_s \rangle_{HS(\mathbb{R}^{d\times d})}, \end{aligned}$$

where the Hilbert-Schmidt-norm on $\mathbb{R}^{d\times d}$ is induced by $\langle M, N \rangle_{HS} = \sum_{i,j=1}^d M_{ij} N_{ij} =$ $\mathrm{trace}(MN^T)$ for any $M, N \in \mathbb{R}^{d\times d}$.

Proof. Long and tedious analogue of the proof of Theorem 3.6. Check cited literature for details. □

Corollary 3.12. *For d-dimensional Brownian motion B and $f \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}^+)$ we have*

$$f(B_t,t) = f(0,0) + \int_0^t \langle \nabla_x f(B_s,s), \, dB_s \rangle + \int_0^t \frac{\partial}{\partial t} f(B_s,s) \, ds + \frac{1}{2} \int_0^t \triangle_x f(B_s,s) \, ds,$$

where $\triangle_x f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \dots + \frac{\partial^2 f}{\partial x_d}$ is the Laplace operator.

Proof. It remains to show $\left\langle B^{(i)}, B^{(j)} \right\rangle_t = t\delta_{ij}$. We know already $\left\langle B^{(i)}, B^{(i)} \right\rangle_t = \left\langle B^{(i)} \right\rangle_t = t$ and we must show for two independent Brownian motions $B^{(1)}, B^{(2)}$ that $\left\langle B^{(1)}, B^{(2)} \right\rangle_t = 0$ for all $t \ge 0$. We have by definition

$$\left\langle B^{(1)}, B^{(2)} \right\rangle_t = \frac{1}{4} \left(\left\langle B^{(1)} + B^{(2)} \right\rangle_t - \left\langle B^{(1)} - B^{(2)} \right\rangle_t \right),$$

but $\frac{1}{\sqrt{2}}\left(B^{(1)} \pm B^{(2)}\right)_t$ is again a Brownian motion such that $\left\langle B^{(1)} \pm B^{(2)}\right\rangle_t = 2t$, i.e. $\left\langle B^{(1)}, B^{(2)}\right\rangle_t = 0$. □

Corollary 3.13. *If $f \in C^{2,1}(\mathbb{R}^d \times \mathbb{R})$ satisfies $\frac{\partial f}{\partial t} = -\frac{1}{2}\triangle_x f$ (for all (x,t)), then $\big(f(B_t,t), t \geq 0\big)$ is a continuous local martingale.*

Proof. We have $f\big(B_t,t\big) = f\big(0,0\big) + \int_0^t \big\langle \nabla_x f\big(B_s,s\big), dB_s\big\rangle$ which is a sum of continuous local martingales and therefore itself again a local martingale. □

Remark 3.14. Functions $f \in C^{2,1}(\mathbb{R}^d \times \mathbb{R})$ as in the corollary satisfy the *heat equation* which is an important partial differential equation in applications. See also (1.1.1) in the introduction.

Example 3.15. Note that if $f : \mathbb{R}^d \to \mathbb{R}$ is harmonic, i.e. $\triangle f(x) = 0$ for all $x \in \mathbb{R}^d$. Then $\big(f(B_t), t \geq 0\big)$ is a continuous local martingale.

1. If $d = 2$, then for $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}, f(x) = \log|x|$ we calculate that $\frac{\partial f}{\partial x_i}(x) = \left(\frac{1}{2}\log(x_1^2 + x_2^2\right) = \frac{2x_i}{2\left(x_1^2+x_2^2\right)} = \frac{x_i}{|x|^2}$ and $\frac{\partial^2 f}{\partial x_i^2}(x) = \frac{|x|^2-2x_ix_i}{|x|^4} = \frac{1}{|x|^2} - \frac{2x_i^2}{|x|^4}$. Thus $\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} = \frac{2}{|x|^2} - 2\frac{x_1^2+x_2^2}{|x|^4} = 0$. Let now $D_{r,R} := \{x \in \mathbb{R}^2 : r < |x| < R\}$ for $0 < r < R$. Then $h(x) := \frac{\log R - \log|x|}{\log R - \log r}$ is harmonic on $D_{r,R}$, $h(x) = 0$ for $|x| = R$, $h(x) = 1$ for $|x| = r$. Define the stopping time $\tau = \inf\{t \geq 0 : |B_t + x| \in \{R, r\}\}$ for some $x \in D_{r,R}$. We have $\mathbb{P}(\tau < \infty) = 1$ because $\limsup_{t\to\infty}|B_t| = \infty$ a.s. (consider e.g. $\limsup_{t\to\infty}|B_t| \geq \limsup_{t\to\infty}|B_t^{(1)}| = \infty$). Moreover, $h(x + B_t), t \geq 0$, is bounded on $[0, \tau]$. Dominated convergence yields
$$\begin{aligned} h(x) &= \mathbb{E}\left[h(x+B_0)\right] \\ &\overset{\text{opt. sampling}}{=} \mathbb{E}\left[h(x+B_\tau)\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\{|x+B_\tau|=r\}}\right] \\ &= \mathbb{P}\left(|x+B_\tau| = r\right). \end{aligned}$$
Here we use that the above arguments yield that also $\big(h(x + B_{t\wedge\tau}), t \geq 0\big)$ is a continuous local martingale if $\triangle h = 0$. Since h is bounded on $[0,\tau]$, we have that $\big(h(x + B_{t\wedge\tau}, t \geq 0\big)$ is a martingale and thus $\mathbb{E}\left[h(x + B_{t\wedge\tau})\right] = h(x)$, take $t \to \infty$ by dominated convergence. For $R \to \infty$ we have $\tau_{r,R} \to \tau_{r,\infty} := \inf\{t \geq 0 : |B_t + x| = r\}$ such that
$$\mathbb{P}\left(\tau_{r,\infty} < \infty\right) = \lim_{R\to\infty}\frac{\log R - \log|x|}{\log R - \log r} = 1.$$
Hence, with probability one the 2-dimensional Brownian motion hits any disc in finite time. This is called *2D-Brownian motion is recurrent for discs.* This means that a.s. the trajectories of 2D-Brownian motion lie dense in $\mathbb{R}^2$, i.e. $\overline{\{|B_t|, t \geq 0\}} = \mathbb{R}^2$ (show this by considering any disc $D_{r,R}$ around any point $y \in \mathbb{R}^2$, not only around 0). By considering only rational coordinates and rational r, R the claim follows a.s.).

2. If $d = 3$, then for $f(x) = |x|^{2-d}$ is harmonic on $\mathbb{R}^d \setminus \{0\}$. With $D_{r,R}$ as before and $h(x) = \frac{R^{2-d} - |x|^{2-d}}{R^{2-d} - r^{2-d}}$ harmonic on $D_{r,R}$, $h(x) = 0$ for $|x| = R$, $h(x) = 1$ for $|x| = r$ you conclude similarly $\mathbb{P}(\tau_{r,\infty} < \infty) = \frac{|x|^{2-d}}{r^{2-d}} < 1$ (because of $R^{2-d} \to 0$ as $R \to \infty$). Hence, d-dimensional Brownian motion is transient for $d \geq 3$.

3.2 First consequences of Itô's formula

First we establish Lévy's characterisation of Brownian motion.

Theorem 3.16. *The following are equivalent:*

a) B is a Brownian motion (on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$.

b) B is a continuous local martingale with $B_0 = 0$, $\langle B \rangle_t = t$ for $t \geq 0$ on $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$.

Proof. (a) $\Rightarrow$ (b). Clear.
(b) $\Rightarrow$ (a). Let us show that $M_t = \exp(iuB_t + \frac{u^2 t}{2})$, $t \geq 0$, is for any $u \in \mathbb{R}$ a complex-valued martingale (i.e. real and imaginary parts are real-valued martingales). We apply Itô's formula (which equally holds for $\mathbb{C}$-valued functions f):

$$\begin{aligned} M_t &= M_0 + \int_0^t iuM_s\,dB_s + \underbrace{\frac{1}{2}\int_0^t (iu)^2 M_s\,d\underbrace{\langle B \rangle_s}_{=s} + \int_0^t \frac{1}{2}u^2 M_s\,ds}_{=0} \\ &= M_0 + \int_0^t iuM_s\,dB_s. \end{aligned}$$

Since $|M_s| \leq e^{u^2 s/2} < \infty$, we have $iuM \in \mathscr{L}(B)$ and $(\int_0^t iuM_s\,dB_s, t \geq 0)$ is in $\mathcal{M}_c^2$ (everything coordinatewise for complex-valued processes). Thus, for all $0 \leq s \leq t$ we have $\mathbb{E}[\frac{M_t}{M_s}|\mathcal{F}_s] = 1$ (note $M_s \neq 0$ a.s.) and $\mathbb{E}[\exp(iu(B_t - B_s)|\mathcal{F}_s] = \exp(-\frac{u^2(t-s)}{2})$. We obtain immediately from $\mathbb{E}[e^{iu(B_t-B_s)}] = e^{-\frac{u^2(t-s)}{2}}$ for all $u \in \mathbb{R}$ that $B_t - B_s \sim N(0, t-s)$. More precisely, for $A \in \mathcal{F}_s$ we get

$$\mathbb{E}\left[e^{iu(B_t-B_s)}\mathbf{1}_A\right] = e^{-\frac{u^2(t-s)}{2}}\underbrace{\mathbb{E}\left[\mathbf{1}_A\right]}_{=\mathbb{P}(A)}.$$

This shows that $B_t - B_s$ is independent of $\mathcal{F}_s$ (cf. exercises or argue directly that independence can be checked on a generator of the σ-algebras $\sigma(B_t - B_s)$ and $\mathcal{F}_s$ and use that the distribution of $B_t - B_s$ and therefore the independence of its generated σ-algebra of $\mathcal{F}_s$ is uniquely determined by its characteristic function). Putting things together we have shown that $B_0 = 0$ (by assumption), $t \mapsto B_t$ is continuous and $\mathcal{F}_t$-adapted (by assumption) and for all $0 \leq s \leq t$ $B_t - B_s$ is independent of $\mathcal{F}_s$ and $B_t - B_s \sim N(0, t-s)$. □

Consequences of this result are far reaching, see e.g. next section. Now we establish a very useful moment inequality.

Theorem 3.17 (Burkholder-Davis-Gundy inequality (BDG))**.** *For every $p > 0$ there are constants $c_p, C_p > 0$ such that for any continuous local martingale M with $M_0 = 0$ we have*

$$c_p\mathbb{E}\left[\langle M \rangle_\infty^{p/2}\right] \leq \mathbb{E}\left[(M_\infty^*)^p\right] \leq C_p\mathbb{E}\left[\langle M \rangle_\infty^{p/2}\right],$$

where $M_t^* = \max_{0 \leq s \leq t} |M_s|$.

Remark 3.18.

a) Since $t \mapsto \langle M \rangle_t$, $t \mapsto M_t^*$ are increasing, $\langle M \rangle_\infty$ and M_∞^* are well-defined in $[0, \infty]$.

b) Usually, we are interested in M_τ^* for a stopping (or deterministic) time τ. The BDG inequality applied to the stopped local martingale M^τ yields

$$c_p\mathbb{E}\left[\langle M \rangle_\tau^{p/2}\right] \leq \mathbb{E}\left[(M_\tau^*)^p\right] \leq C_p\mathbb{E}\left[\langle M \rangle_\tau^{p/2}\right].$$

Proof. For the lower bound see exercises. We prove the upper bound only for $p \geq 2$ (for the general case cf. Revuz and Yor (1999); Karatzas (1991)). We apply Itô's formula to $f(x) = |x|^p$ which is in C^2 for $p \geq 2$:

$$|M_t|^p = \underbrace{|M_0|^p}_{=0} + \int_0^t p\,|M_s|^{p-1}\,\mathrm{sgn}(M_s)\,dM_s + \frac{1}{2}\int_0^t p(p-1)\,|M_s|^{p-2}\,d\langle M\rangle_s\,.$$

Let us assume that M is a bounded martingale. Otherwise localise via the localising sequence of stopping times with the minimum of $\tau_n = \inf\{t > 0 : |M_t| \geq n\}$. For bounded $M \in \mathcal{M}_c^2$ we have $p|M_s|^{p-1}\,\mathrm{sgn}(M_s) \in \mathscr{L}(M)$ such that the stochastic integral is a true martingale with expectation zero. Therefore

$$\begin{aligned}
\mathbb{E}\left[|M_t|^p\right] &= \frac{p(p-1)}{2}\mathbb{E}\left[\int_0^t |M_s|^{p-2}\,d\langle M\rangle_s\right] \\
&\leq \frac{p(p-1)}{2}\mathbb{E}\left[(M_t^*)^{p-2}\,\langle M\rangle_t\right] \\
&\leqslant \frac{p(p-1)}{2}\mathbb{E}\left[(M_t^*)^p\right]^{\frac{p-2}{p}}\,\mathbb{E}\left[\langle M\rangle_t^{p/2}\right]^{2/p},
\end{aligned}$$

using Hölder inequality in the third line. By Doob's inequality (Proposition 1.27)

$$\mathbb{E}\left[(M_t^*)^p\right] \leq \left(\frac{p}{p-1}\right)^p \frac{p(p-1)}{2}\mathbb{E}\left[(M_t^*)^p\right]^{\frac{p-2}{p}}\,\mathbb{E}\left[\langle M\rangle_t^{p/2}\right]^{2/p}$$

for any $p > 1$ such that

$$\mathbb{E}\left[(M_t^*)^p\right]^{2/p} \leq \left(\frac{p}{p-1}\right)^p \frac{p(p-1)}{2}\mathbb{E}\left[\langle M\rangle_t^{p/2}\right]^{2/p}.$$

Hence, observing that $\langle M\rangle$ is a non-negative increasing process for all $t > 0$

$$\mathbb{E}\left[(M_t^*)^p\right] \leq C_p\mathbb{E}\left[\langle M\rangle_\infty^{p/2}\right].$$

Monotone convergence yields the assertion for $t \to \infty$. □

3.3 Martingale representation theorems

Theorem 3.19 (Doob 1953)**.** *Suppose M is a continuous local martingale with $M_0 = 0$ and an absolutely continuous quadratic variation process $t \mapsto \langle M\rangle_t$. Then there is a Brownian motion B (possibly defined on an extension of the original probability space) and a process $X \in \mathscr{L}_{loc}(B)$ such that*

$$M_t = \int_0^t X_s\,dB_s, \quad t \geq 0, a.s$$

Proof. 1. step. Write $\langle M\rangle_t(\omega) = \int_0^t G_s(\omega)\,ds$ and suppose $G_s(\omega) > 0$ a.s. and a.e. (almost everywhere). Put $B_t := \int_0^t G_s^{-1/2}\,dM_s$, $t \geq 0$. If well-defined, then B_t is a continuous local martingale with $B_0 = 0$ (as a stochastic integral) ad

$$\langle B\rangle_t = \int_0^t G_s^{-1}\,d\langle M\rangle_s = \int_0^t G_s^{-1}G_s\,ds = t.$$

By Theorem 3.16 B is a Brownian motion and by associativity of the stochastic integral (Theorem 3.5)

$$\int_0^t G_s^{1/2}\,dB_s = \int_0^t G_s^{1/2}G_s^{-1/2}\,dM_s = M_t.$$

Thus, we choose $X_s = G_s^{1/2}$. It remains to show $G_s^{-1/2} \in \mathscr{L}_{loc}(M)$. For a.a. s and a.a. ω we have $G_s(\omega) = \lim_{h\to 0} \frac{\langle M\rangle_s(\omega) - \langle M\rangle_{s-h}(\omega)}{h}$, from which we may conclude that G_s is progressively measurable. Moreover, we have for all $t \geq 0$

$$\int_0^t \left(G_s^{-1/2}\right)^2 d\langle M\rangle_s = \int_0^t G_s^{-1} \cdot G_s \, ds = t < \infty.$$

This implies $G^{-1/2} \in \mathscr{L}_{loc}(M)$.

2. step. If $G_s(\omega) > 0$ does not almost always hold, then construct a Brownian motion B' on some filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t), \mathbb{P}')$ and consider the product space $\left(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', (\mathcal{F}_t \otimes \mathcal{F}'_t)_{t\geq 0}, \mathbb{P}\right)$, where M and B' are still $(\mathcal{F}_t \otimes \mathcal{F}'_t)$-local martingales. Put

$$B_t := \int_0^t G_s^{-1/2} \mathbf{1}_{\{G_s>0\}} \, dM_s + \int_0^t \mathbf{1}_{\{G_s=0\}} \, dB'_s, \quad t \geq 0.$$

Then B is a continuous local martingale, $B_0 = 0$ and

$$\begin{aligned}
\langle B\rangle_t &= \left\langle \int_0^\cdot G_s^{-1/2}\mathbf{1}_{\{G_s>0\}}\, dM_s \right\rangle_t + \left\langle \int_0^\cdot \mathbf{1}_{\{G_s=0\}}\, dB'_s \right\rangle_t \\
&\quad + 2\underbrace{\left\langle \int_0^\cdot G_s^{-1/2}\mathbf{1}_{\{G_s>0\}}\, dM_s, \int_0^\cdot \mathbf{1}_{\{G_s=0\}}\, dB'_s \right\rangle_t}_{=:A} \\
&= \int_0^t G_s^{-1/2}\mathbf{1}_{\{G_s>0\}} \underbrace{d\langle M\rangle_s}_{=G_s\, ds} + \int_0^t \mathbf{1}_{\{G_s=0\}} \underbrace{d\langle B'\rangle_s}_{=ds} + 0 \\
&= \int_0^t \mathbf{1}_{\{G_s>0\}}\, ds + \int_0^t \mathbf{1}_{\{G_s=0\}}\, ds \\
&= t,
\end{aligned}$$

if we can show that $A = 0$. For this there are two possible arguments:

i) $\left\langle \int_0^\cdot X_s\, dM_s^1, \int_0^\cdot Y_s\, dM_s^2 \right\rangle_t = \int_0^t X_s Y_s \, d\left\langle M^1, M^2\right\rangle_s$ holds (use approximation by simple integrands, see e.g. Karatzas (1991)) such that

$$\left\langle \int_0^\cdot G^{-1/2}\mathbf{1}_{\{G>0\}}\, dM_s, \int_0^\cdot \mathbf{1}_{\{G=0\}}\, dB'_s \right\rangle_t = \int_0^t G^{-1/2} \cdot 0 \, d\langle M, B'\rangle_s = 0.$$

ii) The processes $((M_t, G_t, B'_t), t \geq 0)$ and $(M_t, G_t, -B'_t), t \geq 0)$ have exactly the same distribution such that

$$\left\langle \int_0^\cdot G^{-1/2}\mathbf{1}_{\{G>0\}}\, dM, \int_0^\cdot \mathbf{1}_{\{G=0\}}\, dB'_s \right\rangle_t = -\left\langle \int_0^\cdot G^{-1/2}\mathbf{1}_{\{G>0\}}\, dM_s, \int_0^\cdot \mathbf{1}_{\{G=0\}}\, dB'_s \right\rangle_t$$

such that they both equal zero.

This means again by Theorem 3.16 that B is a Brownian motion and as above $M_t = \int_0^t G_s^{1/2}\, dB_s$.

□

Remark.

a) A function $f : \mathbb{R}^+ \to \mathbb{R}$ is *absolutely continuous* if there is a function $g \in L^1([0,T])$ for all $T > 0$ such that $f(t) = f(0) + \int_0^t g(s)\, ds$. We have $g(s) = f'(s)$ for Lebesgue-a.a.

b) For general M_0 we then obtain $M_t = M_0 + \int_0^t X_s\, dB_s$ by considering $\tilde{M}_t = M_t - M_0$.

Theorem 3.20 (Brownian martingales)**.** *Let $(\mathcal{F}_t)_{t\geqslant 0}$ be the canonical filtration of a Brownian motion B, i.e. $\mathcal{F}_t^0 = \sigma(B_s, s \le t)$ completed by events of probability zero in $\sigma(B_t, t \ge 0)$. Then for each random variable $Z \in L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ there is a unique process $h \in \mathscr{L}(B)$ such that*

$$Z = \mathbb{E}\left[Z\right] + \int_0^\infty h_s\, dB_s,$$

where $\mathbb{E}[\int_0^\infty h_s^2\, ds] < \infty$. Moreover, for each martingale M bounded in L^2 (for each continuous local martingale, respectively) adapted to $(\mathcal{F}_t^0)_{t\ge 0}$, there is an $h \in \mathscr{L}(B)$ ($h \in \mathscr{L}_{loc}(B)$) and a constant $C > 0$ such that

$$M_t = C + \int_0^t h_s\, dB_s, \quad t \ge 0, a.s.$$

Remark 3.21. M is not assumed to be continuous a priori (see below).

We first need a Lemma.

Lemma 3.22. *The vector space generated by the random variables $\exp(i\sum_{j=1}^n \lambda_j(B_{t_j} - B_{t_{j-1}}))$ for $0 = t_0 < t_1 < \dots < t_n$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ is dense in $L^2_{\mathbb{C}}(\Omega, \mathcal{F}_\infty, \mathbb{P})$ of $\mathbb{C}$-valued L^2-random variables.*

Proof. We show that $Z \in L^2_{\mathbb{C}}(\Omega, \mathcal{F}_\infty, \mathbb{P})$ with

$$\left\langle Z, \exp\left(i\sum_{j=1}^n \lambda_j\left(B_{t_j} - B_{t_{j-1}}\right)\right)\right\rangle_{L^2} = \mathbb{E}\left[Z \exp\left(-i\sum_{j=1}^n \lambda_j\left(B_{t_j} - B_{t_{j-1}}\right)\right)\right] = 0 \qquad (*)$$

for all $n, (t_j), (\lambda_j)$ must satisfy $Z = 0$ a.s. For $F \in \mathscr{B}_{\mathbb{R}^n}$ we set

$$\mu(F) = \mathbb{E}\left[Z \mathbf{1}_F\left(B_{t_1} - B_{t_0}, \dots, B_{t_n} - B_{t_{n-1}}\right)\right].$$

Then μ is a complex-valued finite measure. Then (*) shows that the characteristic function of μ vanishes identically. By uniqueness of characteristic functions this means that $\mu = 0$ holds. Hence, $\mathbb{E}\left[Z\mathbf{1}_A\right] = 0$ holds for all $A \in \sigma(B_{t_1}, \dots, B_{t_n})$. By a monotone class argument (or measure-theoretic induction) this extends to $A \in \sigma(B_s, s \ge 0)$. Adding nullsets to A does not affect validity of $\mathbb{E}\left[Z\mathbf{1}_A\right] = 0$. Therefore $\mathbb{E}\left[Z\mathbf{1}_A\right] = 0$ for all $A \in \mathcal{F}_\infty$ and thus $Z = 0$ a.s. (consider for this $A = \{Z > 0\}$ and $A = \{Z < 0\}$).

Proof of Theorem 3.20. Let H be the vector space of all $Z \in L^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ with representation $Z = \mathbb{E}[Z] + \int_0^\infty h_s\, dB_s$. The $Z \in H$ the process h is unique because for h, h' $\int_0^\infty h_s\, dB_s = \int_0^\infty h'_s\, dB_s$ a.s. Then $\int_0^\infty (h_s - h'_s)\, dB_s = 0$ a.s. and by Itô's isometry

$$\begin{aligned} 0 &= \mathbb{E}\left[\left(\int_0^\infty \left(h_s - h'_s\right)\, dB_s\right)^2\right] \\ &= \mathbb{E}\left[\int_0^\infty \left(h_s - h'_s\right)^2\, ds\right]. \end{aligned}$$

Thus $h_s = h'_s$ a.s. for almost all s. (indistinguishable?) Moreover, for $Z \in H$ we have

$$\mathbb{E}\left[Z^2\right] = \mathbb{E}\left[\left(\mathbb{E}\left[Z\right] + \int_0^\infty h_s\, dB_s\right)^2\right] = \left(\mathbb{E}\left[Z\right]\right)^2 + \int_0^\infty \mathbb{E}\left[h_s^2\right]\, ds + 2 \cdot 0.$$

Using this formula for $\mathbb{E}\left[Z^2\right]$ we obtain directly that a sequence (Z_n) in H converging to $Z \in L^2$ has corresponding processes (h_n) which form a Cauchy sequence with respect to

$\|h\|^2 = \int_0^\infty \mathbb{E}\left[h_s^2\right] ds$. Then by the construction of the stochastisc integral this implies that $\int_0^\infty h_n \, dB_s$ converges in L^2 (Itô isometry on $[0, \infty)$). Since also $\mathbb{E}[Z_n] \to \mathbb{E}[Z]$ holds, we have

$$Z = \mathbb{E}[Z] + \int_0^\infty h_s \, dB_s,$$

where $h_s = \lim_{n\to\infty} h_{n,s}$. That's why H is closed. Now let us write $f(s) = \sum_{j=1}^n \lambda_j \mathbf{1}_{(t_{j-1},t_j]}(s)$ and consider

$$\mathcal{E}_t^f = \exp\Bigg(\underbrace{i\int_0^t f(s) \, dB_s}_{=:X_t} + \frac{1}{2}\int_0^t f^2(s) \, ds\Bigg).$$

By Itô's formula we get

$$\mathcal{E}_t^f = \mathcal{E}_0^f + \int_0^t i\mathcal{E}_s^f f(s) \, dB_s,$$

because the quadratic variation terms cancel (see proof of Theorem 3.16). Then

$$\exp\left(i\sum_{j=1}^n \lambda_j \left(B_{t_j} - B_{t_{j-1}}\right) + \frac{1}{2}\sum_{j=1}^n \lambda_j^2 (t_j - t_{j-1})\right) = 1 + \int_0^\infty \mathcal{E}_s^f f(s) \, dB_s$$

and both the left-hand side and therefore also $\exp(i\sum_{j=1}^n \lambda_j(B_{t_j} - B_{t_{j-1}}))$ is in H. By the lemma, linear combinations of the latter random variables are dense in L^2. Therefore H is dense in L^2. Since H is closed, we must have $H = L^2$.

For the second part we know by the martingale convergence theorem (reference?)(e.g. from Stochastic processes I) that if M is an L^2-bounded martingale, then there exists an $\mathcal{F}_\infty$-measurable random variable M_∞ such that $M_t \xrightarrow{L^2, a.s.} M_\infty$. From the first part we therefore find $h \in \mathscr{L}(B)$ with $M_\infty = \mathbb{E}[M_\infty] + \int_0^\infty h_s \, dB_s$ and thus for all $t \geq 0$ we have $M_t = \mathbb{E}[M_\infty | \mathcal{F}_t] = \mathbb{E}[M_\infty] + \int_0^t h_s \, dB_s$, a.s. In particular, M has a continuous version (namely the right-hand side of this equality).

If M is only a local martingale, but continuous, with associated stopping times (τ_n), then the stopped processes $N = M^{\tau_n}$ are uniformly integrable martingales. By the martingale convergence theorem (reference?) there exists an $\mathcal{F}_\infty$-measurable random variable $M_\infty \in L^1$. Since L^2 is dense in L^1, we find $\mathcal{F}_\infty$-measurable random variables $M_\infty^{(n)} \in L^2$ such that $M_\infty^{(n)} \xrightarrow{L^1} M_\infty$ as $n \to \infty$. By the first part we can associate with each $M_\infty^{(n)}$ a continuous L^2-bounded martingale $M_t^{(n)} = \mathbb{E}[M_\infty^{(n)}] + \int_0^t h_s^{(n)} \, dB_s$ for processes $h^{(n)} \in \mathscr{L}(B)$. By Doob's maximal inequality we find then for $\varepsilon > 0$ that

$$\mathbb{P}\left(\sup_{0\leq t\leq\infty} \left|M_t - M_t^{(n)}\right|\right)$$

define further stopping times $\sigma_n = \tau_n \wedge \inf\{t \geq 0 : |M_t| \geq n\}$. Since M_0 is $\mathcal{F}_0$-measurable, we have that M_0 is constant a.s. $(\sigma(B_0) = \{\emptyset, \Omega\}$. Then there exist processes $h_n \in \mathscr{L}(B)$ such that $M_{t\wedge\sigma_n} = M_0 + \int_0^t h_{n,s} \, dB_s$, $t \geq 0$. By uniqueness of h_n, we have $h_{m,s} = h_{n,s} \cdot \mathbf{1}_{[0,\sigma_m]}(s)$ for $m < n$ ($\mathbb{P}$-a.s., λ-a.e.). So, since $\sigma_n \to \infty$ a.s. we can define a process h_s such that $h_{m,s} = h_s \mathbf{1}_{[0,\sigma_m]}(s)$ for all $m \geq 1$. Hence, we have a.s.

$$M_t = \lim_{n\to\infty} M_{t\wedge\sigma_n} = M_0 + \lim_{n\to\infty} \int_0^{t\wedge\sigma_n} h_s \, dB_s = M_0 + \int_0^t h_s \, dB_s.$$

□

□

Corollary 3.23. *Every local martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$ as above has a continuous version.*

Proof. $t \mapsto \int_0^t h_s\,dB_s$ is continuous such that $M_t = C + \int_0^t h_s\,dB_s$ a.s. and the right-hand side is a.s. continuous in t. □

Corollary 3.24. $(\mathcal{F}_t)_{t\geq 0}$*, the completion of the canonical Brownian filtration, is right-continuous, i.e.* $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$*,* $t \geq 0$*.*

Proof. Let Z be an $\mathcal{F}_{t+}$-measurable bounded random variable. Then there is an h such that $Z = \mathbb{E}[Z] + \int_0^\infty h_s\,dB_s$ by the theorem. Since Z is $\mathcal{F}_{t+\varepsilon}$-measurable for any $\varepsilon > 0$, we have

$$Z = \mathbb{E}\left[Z|\,\mathcal{F}_{t+\varepsilon}\right] = \mathbb{E}\left[Z\right] + \int_0^{t+\varepsilon} h_s\,dB_s.$$

By uniqueness of h, we derive that $h_s = 0$ a.s. for *a.a.* $s \in [t+\varepsilon, \infty)$. Use $\varepsilon_n \to 0$ such that $h = h\mathbf{1}_{[0,t]}$ up to indistinguishability and $Z = \mathbb{E}[Z] + \int_0^t h_s\,dB_s$ is $\mathcal{F}_t$-measurable. Therefore $\mathcal{F}_{t+} = \mathcal{F}_t$. □

Theorem 3.25. *If M is a continuous local martingale, then there exists a Brownian motion B and a family of stopping times τ_t such that $M_t = B_{\tau_t}$ (="random change of Brownian motion").*

3.4 The Girsanov theorem

3.4.1 Motivation

Let $\mathbb{P}^W$ be the Wiener measure (i.e. the law of Brownian motion) on $(C([0,1]), \mathscr{B}_{C([0,1])})$. Which probability measures $\mathbb{Q}$ on the same space are equivalent/absolutely continuous with respect to $\mathbb{P}^W$ (i.e. $\mathbb{Q} \sim \mathbb{P}^W$, $\mathbb{Q} \ll \mathbb{P}^W$) and what are the corresponding Radon-Nikodym densities? As motivation consider first the finite-dimensional case, i.e. let $X_1, \ldots X_n \sim N(0,1)$ be iid random variables and let $\mathbb{P}_n$ be the law on $(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n})$. We realize $X_1, \ldots, X_n$ as the coordinate projections on $\mathbb{R}^n$. Consider

$$Z_n(x_1, \ldots, x_n) = \exp\left(\sum_{k=1}^n \mu_k x_k - \frac{1}{2}\sum_{k=1}^n \mu_k^2\right)$$

for some $\mu_k \in \mathbb{R}$. Then by independence

$$\begin{aligned}
\mathbb{E}\left[Z_n(X_1, \ldots, X_n)\right] &= \mathbb{E}\left[\exp\left(\sum_{k=1}^n \mu_k X_k\right)\right] e^{-\frac{1}{2}\sum_{k=1}^n \mu_k^2} \\
&= \left(\prod_{k=1}^n \mathbb{E}\left[\exp\left(\mu_k X_k\right)\right]\right) e^{-\frac{1}{2}\sum_{k=1}^n \mu_k^2} \\
&= \left(\prod_{k=1}^n e^{\mu_k^2/2}\right) e^{-\frac{1}{2}\sum_{k=1}^n \mu_k^2} \\
&= 1.
\end{aligned}$$

This means that $\int_{\mathbb{R}^n} Z_n(x_1, \ldots, x_n)\,d\mathbb{P}_n(x_1, \ldots, x_n) = 1$ and we have $Z_n > 0$. This means Z_n is a density with respect to $\mathbb{P}_n$. Hence, we can define a probability measure $\mathbb{Q}_n$ on $(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n})$ via $\frac{d\mathbb{Q}_n}{d\mathbb{P}_n} = Z_n$, i.e. $\mathbb{Q}_n(A) = \int_A Z_n\,d\mathbb{P}_n$, $A \in \mathscr{B}_{\mathbb{R}^n}$. What is the law of the coordinate projections $X_1, \ldots, X_n$ under $\mathbb{Q}_n$? With λ_n denoting the n-dimensional Lebesgue-measure, we have

$$\begin{aligned}\frac{d\mathbb{Q}_n}{d\lambda_n}(x) &= \frac{d\mathbb{Q}_n}{d\mathbb{P}_n}(x) \cdot \frac{d\mathbb{P}_n}{d\lambda_n}(x) = Z_n(x) \cdot (2\pi)^{-n/2} e^{-|x|^2/2} \\ &= (2\pi)^{-n/2} \exp\left(\sum_{k=1}^n \mu_k x_k - \frac{1}{2}\sum_{k=1}^n \mu_k^2 - \frac{1}{2}\sum_{k=1}^n x_k^2\right) \\ &= (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\sum_{k=1}^n (\mu_k - x_k)^2\right).\end{aligned}$$

Thus, $(X_1, \ldots, X_n)$ are independent under $\mathbb{Q}_n$ and each X_k is $N(\mu_k, 1)$-distributed. In particular, $(\bar{X}_1, \ldots, \bar{X}_n)$ with $\bar{X}_k = X_k - \mu_k$ is iid $N(0,1)$-distributed under $\mathbb{Q}_n$. This is also true if $(X_1, \ldots, X_n)$ is defined on some abstract probability space $(\Omega, \mathcal{F}, \mathbb{P}_n)$ (not only on $\Omega = \mathbb{R}^n$). We shall exploit this to obtain an infinite-dimensional analogue. Suppose $h : [0,1] \to \mathbb{R}$ is given such that $h(t) = \int_0^t g(s)\,ds$ for some function $g \in L^2([0,1])$. Let $B = (B_t, t \geq 0)$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Then $X_k := \sqrt{n}(B_{k/n} - B_{(k-1)/n}) \overset{iid}{\sim} N(0,1)$, $k = 1, \ldots, n$. Putting $\mu_k = \sqrt{n}(h(\frac{k}{n}) - h(\frac{k-1}{n}))$ and

$$Z_n(\omega) = \exp\left(\sum_{k=1}^n \sqrt{n}\left(h\left(\frac{k}{n}\right) - h\left(\frac{k-1}{n}\right)\right)\sqrt{n}\left(B_{\frac{k}{n}}(\omega) - B_{\frac{k-1}{n}}(\omega)\right) - \frac{1}{2}\sum_{k=1}^n n\left(h\left(\frac{k}{n}\right) - h\left(\frac{k-1}{n}\right)\right)^2\right)$$

we define as above $\mathbb{Q}_n$ on $(\Omega, \mathcal{F})$ via $\frac{d\mathbb{Q}_n}{d\mathbb{P}} = Z_n$. Then we also have

$$\begin{aligned}\bar{X}_k &= \sqrt{n}\left(B_{\frac{k}{n}} - B_{\frac{k-1}{n}}\right) - \sqrt{n}\left(h\left(\frac{k}{n}\right) - h\left(\frac{k-1}{n}\right)\right) \\ &= \sqrt{n}\left(\left(B_{\frac{k}{n}} - h\left(\frac{k}{n}\right)\right) - \left(B_{\frac{k-1}{n}} - h\left(\frac{k-1}{n}\right)\right)\right)\end{aligned}$$

and $(\bar{X}_1, \ldots, \bar{X}_n) \sim N(0, E_n)$ under $\mathbb{Q}_n$, where E_n is the n-dimensional unit matrix. Under $\mathbb{Q}_n$ we have that $\bar{X}_1, \ldots, \bar{X}_n$ is distributed like the increments of Brownian motion at k/n. We want to study now the asymptotic behaviour of $\mathbb{Q}_n$. For this take the dyadic grid with $n = 2^j$, $j \to \infty$. Let $\mathcal{F}_j = \sigma(B_{k2^{-j}}, k = 0, \ldots, 2^j)$, $j \geq 1$, be a filtration on $(\Omega, \mathcal{F})$. Then $(Z_{2^j})_{j\geq 1}$ is an $(\mathcal{F}_j)_{j\geq 1}$-martingale (cf. exercises). We use $h(t) = \int_0^t g(s)\,ds$ to obtain

$$Z_n = \exp\left(\sum_{k=1}^n \left(n\int_{\frac{k-1}{n}}^{\frac{k}{n}} g(s)\,ds\right)\left(B_{\frac{k}{n}} - B_{\frac{k-1}{n}}\right) - \frac{1}{2}\sum_{k=1}^n \left(n\int_{\frac{k-1}{n}}^{\frac{k}{n}} g(s)\,ds\right)^2 \frac{1}{n}\right).$$

Since $g_n(t) = \sum_{k=1}^n \left(n\int_{\frac{k-1}{n}}^{\frac{k}{n}} g(s)\,ds\right)\mathbf{1}_{[\frac{k-1}{n},\frac{k}{n})}(t)$ (Haar approximation) converges to g a.e. and in $L^2([0,1])$ we have

$$\begin{aligned}Z_n &= \exp\Big(\underbrace{\int_0^1 g_n(s)\,dB_s}_{\xrightarrow{L^2(\mathbb{P})} \int_0^1 g(s)\,dB_s} - \frac{1}{2}\underbrace{\int_0^1 g_n^2(s)\,ds}_{\to \int_0^1 g^2(s)\,ds}\Big) \\ &\xrightarrow{\mathbb{P}} Z_\infty = \exp\left(\int_0^1 g(s)\,dB_s - \frac{1}{2}\int_0^1 g^2(s)\,ds\right).\end{aligned}$$

One can show $Z_{2^j} \xrightarrow{j\to\infty} Z_\infty$ holds even in L^1. This is easily checked by showing $\mathbb{E}[Z_\infty|\mathcal{F}_j] = Z_{2^j}$, i.e. $(Z_{2^j}, j \geq 1)$ is a closable martingale (to be done precisely, see below). This implies in particular $\mathbb{E}_{\mathbb{P}}[Z_\infty] = \mathbb{E}_{\mathbb{P}}[Z_{2^j}] = 1$. Define $\mathbb{Q}_\infty$ on $(\Omega, \mathcal{F})$

via $\frac{d\mathbb{Q}_\infty}{d\mathbb{P}} = Z_\infty$. From exercises we obtain for any $n = 2^j$ that $\frac{1}{\sqrt{n}}\left(\bar{X}_1, \ldots, \bar{X}_n\right) = \left(\left(B_{\frac{1}{n}} - h\left(\frac{1}{n}\right)\right) - (B_0 - h(0)), \ldots, (B_1 - h(1)) - \left(B_{\frac{n-1}{n}} - h(\frac{n-1}{n})\right)\right)$ has law $N(0, \frac{1}{n}E_n)$ under $\mathbb{Q}_\infty$, i.e. the law of the increments of Brownian motion $(B_{\frac{1}{n}} - B_0, \ldots, B_1 - B_{\frac{n-1}{n}})$ under $\mathbb{P}$. Since $\mathscr{B}_{C([0,1])} = \sigma(\mathcal{F}_j, j \geq 1)$ holds and $(B_t - h(t), 0 \leq t \leq 1)$ is continuous. This implies $(B_{\frac{k}{n}} - h(\frac{k}{n}), k = 0, \ldots, n)$ under $\mathbb{Q}_\infty$ is thus distributed like $(B_{\frac{k}{n}}, k = 0, \ldots, n)$ under $\mathbb{P}$ and $\bar{B}_t := B_t - h(t)$ is a Brownian motion under $\mathbb{Q}_\infty$ using that $\mathscr{B}_{C([0,1])} = \sigma(\mathcal{F}_j, j \geq 1)$ and the definition of Brownian motion. This is the Cameron-Martin theorem.

3.4.2 The Girsanov and the Cameron-Martin theorem

Lemma 3.26. *Suppose $(Z_t, 0 \leq t \leq T)$ is a non-negative martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $\mathbb{E}[Z_T] = 1$. Define $\mathbb{Q}_T$ on $(\Omega, \mathcal{F}_T)$ via $\frac{d\mathbb{Q}_T}{d\mathbb{P}} = Z_T$. Then for any $Y \in L^1(\mathbb{Q}_T)$ we have for all $0 \leq s \leq t \leq T$*

$$\mathbb{E}_{\mathbb{Q}_T}\left[Y \middle| \mathcal{F}_s\right] = \frac{1}{Z_s}\mathbb{E}_{\mathbb{P}}\left[Y Z_t \middle| \mathcal{F}_t\right], \quad \mathbb{P}\text{-a.s.}, \ \mathbb{Q}_T\text{-a.s.}$$

Proof. See exercises. □

Corollary 3.27. *If $(\bar{M}_t Z_t, 0 \leq t \leq T)$ is a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ for some $\mathcal{F}_t$-adapted process $\bar{M}$, then $\bar{M} = (\bar{M}_t, 0 \leq t \leq T)$ is a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q}_T)$ (with the notation from above).*

Proof. See exercises. □

Now let us recall the stochastic exponential

$$Z_t = \exp\left(\int_0^t X_s \, dB_s - \frac{1}{2}\int_0^t X_s^2 \, ds\right), \quad t \geq 0,$$

for some $X \in \mathscr{L}_{loc}(B)$. This is a non-negative local martingale due to Itô's formula:

$$\begin{aligned} Z_t &= 1 + \int_0^t Z_s X_s \, dB_s + \frac{1}{2}\int_0^t Z_s X_s^2 \, ds - \frac{1}{2}\int_0^t Z_s X_s^2 \, ds \\ &= 1 + \int_0^t Z_s X_s \, dB_s. \end{aligned}$$

We have that $(Z_t, 0 \leq t \leqslant T)$ is a martingale if (and only if) $\mathbb{E}[Z_T] = 1$.

Theorem 3.28 (Girsanov, 1960)**.** *If $(Z_t, 0 \leq t \leq T)$ with $Z_t = \exp(\int_0^t X_s \, dB_s - \frac{1}{2}\int_0^t X_s^2 \, ds)$ is a martingale on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), \mathbb{P})$, then*

$$\bar{B}_t := B_t - \int_0^t X_s \, ds, \quad 0 \leq t \leq T,$$

defines a Brownian motion with respect to $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q}_T)$ with $\frac{d\mathbb{Q}_T}{d\mathbb{P}} := Z_T$.

Proof. The key idea is to apply Levy's characterisation of Brownian motion (Theorem 3.16). *1. $(\bar{B}_t Z_t, 0 \leq t \leq T)$ is a continuous local $\mathbb{P}$-martingale.* Integration by parts (under $\mathbb{P}$) yields indeed:

$$\begin{aligned} \bar{B}_t Z_t &= \bar{B}_0 Z_0 + \int_0^t B_s \, dZ_s + \int_0^t Z_s \, d\bar{B}_s + \langle \bar{B}, Z \rangle_t \\ &= 0 + \int_0^t \bar{B}_s Z_s X_s \, d\bar{B}_s + \int_0^t Z_s \, d\bar{B}_s - \int_0^t Z_s X_s \, ds + \int_0^t 1 \cdot Z_s X_s \, ds \\ &= \int_0^t (\bar{B}_s X_s + 1) Z_s \, dB_s \end{aligned}$$

which is a continuous local $\mathbb{P}$-martingale.

2. $(\bar{B}_t, 0 \le t \le T)$ *is a local* $\mathbb{Q}_T$*-martingale.* Let $(\bar{B}Z)_{t\wedge\tau_n}$, $0 \le t \le T$, be $\mathbb{P}$-martingales for suitable stopping times $\tau_n \to \infty$. By the first step and Corollary 3.27 we see that $\bar{B}_{t\wedge\tau_n}$ is a continuous $\mathbb{Q}_T$-martingale with respect to $(\mathcal{F}_{t\wedge\tau_n})_{0\le t\le T}$ and thus $(\bar{B}_t, 0 \le t \leqslant T)$ is a continuous local martingale with respect to $\mathbb{Q}_T$.

3. $\bar{B}$ *has quadratic variation* $\langle \bar{B}\rangle_t = t$ *under* $\mathbb{Q}_T$. We need to show that $(\bar{B}_t^2 - t, 0 \le t \le T)$ is a continuous local martingale under $\mathbb{Q}_T$. Equivalently (see below) it is enough to show that $(Z_t(\bar{B}_t^2 - t), 0 \le t \le T)$ is a continuous local martingale under $\mathbb{P}$. Continuity is obvious. Use (under $\mathbb{P}$) $\langle \bar{B}\rangle_t = \langle B - \int_0^\cdot X_s\,ds\rangle_t = \langle B\rangle_t = t$ such that

$$\bar{B}_t^2 = 2\int_0^t \bar{B}_t\,d\bar{B}_t + \langle \bar{B}\rangle_t = 2\int_0^t \bar{B}_t\,d\bar{B}_t + t$$

and by partial integration

$$\begin{aligned} Z_t(\bar{B}_t^2 - t) &= \int_0^t Z_s 2\bar{B}_s\,d\bar{B}_s + \int_0^t (\bar{B}_s^2 - s)Z_s X_s\,dB_s + \int_0^t Z_s X_s 2\bar{B}_s\,ds \\ &= 2\int_0^t (Z_s\bar{B}_s + (\bar{B}_s^2 - s)Z_s X_s\,dB_s \end{aligned}$$

which is a local $\mathbb{P}$-martingale. This yields directly the claim. □

We are now interested in the support of Wiener measure $\mathbb{P}^W$ on $(C([0,1]), \mathscr{B}_{C([0,1])})$.

Trivial question: Suppose $U \sim U([0,1])$. Which of the outcomes $U = 0.5$, $U = 0$, $U = -1$ is typical? We can argue that 0 and 0.5 are typical outcomes, because any open interval around 0 and 0.5 has positive probability while this is not true for -1.

Definition 3.29. The support of a probability measure $\mathbb{P}$ on a metric space S equipped with its Borel-σ-algebra is the smallest closed set A such that $\mathbb{P}(A) = 1$ holds, i.e. $A = \bigcap_{F \text{ closed}, \mathbb{P}(F)=1}$. The set

$$\mathscr{H} = \left\{ f \in C([0,1]) : \exists g \in L^2([0,1])\, \forall t \in [0,1]\;\, f(t) = \int_0^t g(s)\,ds \right\}$$

is called *Cameron-Martin space.*

Remark. $\mathscr{H}$ is the space of all weakly differentiable (= absolutely continuous) functions f with $f(0) = 0$, $f' \in L^2$, i.e. $\mathscr{H} = H^1 \cap \{f \in C([0,1]) : f(0) = 0\}$ with the L^2-Sobolev space H^1.

The Girsanov theorem yields a very interesting shift property of Wiener measure.

Proposition 3.30. *For all* $h \in \mathscr{H}$ *the laws of Brownian motion* $(B_t, t \in [0,1])$ *and Brownian motion with drift h, i.e.* $(B_t + h(t), t \in [0,1])$*, are equivalent on* $(C([0,1]), \mathscr{B}_{C([0,1])})$.

Proof. For $g \in L^2([0,1])$ with $h(t) = \int_0^t g(s)\,ds$ we consider $Z_t = \exp(\int_0^t g(s)\,dB_s - \frac{1}{2}\int_0^t g(s)^2\,ds)$, $0 \le t \le 1$. Since $g \in L^2$ is deterministic, $g \in \mathscr{L}(B)$ and Z_t is well-defined. We have that $\int_0^1 g(s)\,dB_s$ is normally distributed (via Gaussian approximations, cf. exercises) with $\mathbb{E}[\int_0^1 g(s)\,dB_s] = 0$, $\mathbb{E}[(\int_0^1 g(s)\,dB_s)^2] = \int_0^1 g^2(s)\,ds$, i.e. $\int_0^1 g(s)\,dB_s \sim \|g\|_{L^2}^2 U$, where $U \sim N(0,1)$. Z is a martingale, because

$$\mathbb{E}\left[Z_1\right] = \mathbb{E}\left[e^{\int_0^1 g(s)\,dB_s}\right] e^{-\frac{1}{2}\int_0^1 g^2\,ds} = e^{\frac{1}{2}\|g\|_{L^2}^2 - \frac{1}{2}\|g\|_{L^2}^2} = 1.$$

By Girsanov $\bar{B}_t = B_t - \int_0^t g(s)\,ds = B_t - h(t)$ is a $\mathbb{Q}_1$-Brownian motion with $\frac{d\mathbb{Q}_1}{d\mathbb{P}} = Z_1$ on $\mathcal{F}_1$ such that $B_t = \bar{B}_t + h(t)$ is a Brownian motion with drift under $\mathbb{Q}_1$. Since by definition

$\mathbb{Q}_1 \ll \mathbb{P}$ on $(\Omega, \mathcal{F}_1)$ and the density Z_1 is strictly positive, $\mathbb{Q}_1$ and $\mathbb{P}$ are equivalent measures and so are their image measures $\mathbb{Q}_1^B$, $\mathbb{P}^B$ under B on $(C([0,1]), \mathscr{B}_{C([0,1])})$. Hence, the law of Brownian motion $\mathbb{P}^B$ and the law of Brownian motion $\mathbb{P}^B$ and the law of Brownian motion with drift h $\mathbb{Q}_1^B$ are equivalent. □

Corollary 3.31. *The support of* $\mathbb{P}^W$ *on* $(C([0,1]), \mathscr{B}_{C([0,1])})$ *is given by* $\overline{\mathscr{H}} = \{f \in C([0,1]) : f(0) = 0\}$.

Proof. For any $h \in \mathscr{H}$, $\varepsilon > 0$, we have ($\|f\|_\infty = \sup_{0\le t\le 1} |f(t)|$)

$$\begin{aligned}
\mathbb{P}(\|B+h\|_\infty \le \varepsilon) &= \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\|B+h\|_\infty \le \varepsilon\}}\right] \\
&= \mathbb{E}_{\mathbb{Q}_1}\left[\mathbf{1}_{\left\{\|\tilde{B}+h\|_\infty \le \varepsilon\right\}}\right] \\
&= \mathbb{E}_{\mathbb{Q}_1}\left[\mathbf{1}_{\{\|B\|_\infty \le \varepsilon\}}\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\|B\|_\infty \le \varepsilon\}Z_1}\right] \\
&> 0,
\end{aligned}$$

because $Z_1 > 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(\|B\|_\infty \le \varepsilon) > 0$ such that $\mathbf{1}_{\{\|B\|_\infty \le \varepsilon\}} Z_1 > 0$ on a set of positive $\mathbb{P}$-measure (note: we have proved

$$\mathbb{P}\left(\sup_{0\le t\le 1} B_t \le \varepsilon\right) = \mathbb{P}(|Z| \le \varepsilon) = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx > 0$$

for all $\varepsilon > 0$; it is also possible to show that $\mathbb{P}(\sup_{0\le t\le 1} |B_t| \le \varepsilon) > 0$ for all $\varepsilon > 0$ ("small ball property of Brownian motion")). This means all $\|\cdot\|_\infty$-balls around $h \in \mathscr{H}$ of radius $\varepsilon > 0$ are *charged* (i.e. have positive probability) by $\mathbb{P}^W$. For any open set O with $O \cap \mathscr{H} \neq \emptyset$ we have $\mathbb{P}^W(O) > 0$. Hence, O open with $\mathbb{P}^W(O) = 0$ must satisfy $O \cap \mathscr{H} = \emptyset$, i.e. $O \subseteq \mathscr{H}^c$ such that $\bigcup_{O \text{ open}, \mathbb{P}^W(0)=0} O \subseteq \mathscr{H}^c$. Taking complements this means the support of $\mathbb{P}^W$ contains $\mathscr{H}$ and thus $\overline{\mathscr{H}}$. Because $B_0 = 0$ a.s. the support of $\mathbb{P}^W$ is exactly $\overline{\mathscr{H}}$. □

So far it remained open how to check in general whether $Z_t = \exp\left(\int_0^t X_s\, dB_s - \frac{1}{2}\int_0^t X_s^2\, ds\right)$ is a martingale. There are two useful sufficient conditions for that, namely the *Kazamaki* and the *Novikov* condition, see. e.g. Revuz and Yor (1999). Here we merely prove an ε-weaker version of Novikov's condition.

Theorem 3.32 (Weak Novikov condition)**.** *Let M be a continuous local martingale, $M_0 = 0$ and $Z_t = e^{M_t - \frac{1}{2}\langle M\rangle_t}$. Then $\mathbb{E}[Z_T] = 1$ holds (and $(Z_t, 0 \le t \le T)$ is a martingale) if for some $\varepsilon > 0$*

$$\mathbb{E}\left[\exp\left(\left(\frac{1}{2} + \varepsilon\right)\langle M\rangle_T\right)\right] < \infty.$$

Proof. Suppose $\tau_n \to \infty$ are stopping times such that $(Z_{t\wedge\tau_n}, 0 \le t \le T)$ are martingales (cf. exercises). We show that $(Z_{T\wedge\tau_n})_{n\ge 1}$ are uniformly integrable. Then

$$\mathbb{E}[Z_T] = \lim_{n\to\infty} \underbrace{\mathbb{E}[Z_{T\wedge\tau_n}]}_{=\mathbb{E}[Z_0]} = 1.$$

For this we prove that $\sup_{n\ge 1} \mathbb{E}[Z_{T\wedge\tau_n}^r] < \infty$ for some $r > 1$. For any $p > 1$ we have

$$\begin{aligned}
Z_t^r &= \exp\left(rM_t - \frac{1}{2} r\langle M\rangle_t\right) \\
&= \exp\left(rM_t - \frac{p}{2}\langle rM\rangle_t + \frac{1}{2}\left(pr^2 - r\right)\langle M\rangle_t\right)
\end{aligned}$$

such that by Hölder-inequality

$$\mathbb{E}\left[Z^r_{T\wedge\tau_n}\right] \le \underbrace{\mathbb{E}\left[\exp\left(prM_{T\wedge\tau_n} - \frac{p^2}{2}\langle rM\rangle_{T\wedge\tau_n}\right)\right]^{1/p}}_{=\mathbb{E}\left[\exp\left(prM_{T\wedge\tau_n}-\frac{1}{2}\langle prM\rangle_{T\wedge\tau_n}\right)\right]^{1/p}} \mathbb{E}\left[\exp\left(\frac{q}{2}\left(pr^2-r\right)\langle M\rangle_{T\wedge\tau_n}\right)\right]^{1/q}.$$

Observe that prM_t is a local martingale such that $\exp(\dots)$ is a stopped non-negative local martingale and, in particular, a submartingale. Hence, by Fatou's lemma and because $\langle M\rangle_{T\wedge\tau_n} \le \langle M\rangle_T$

$$\mathbb{E}\left[Z^r_{T\wedge\tau_n}\right] \le 1^{1/p}\mathbb{E}\left[\exp\left(\frac{p}{2(p-1)}\left(pr^2-r\right)\langle M\rangle_T\right)\right]^{1/q},$$

where we use that $q = \frac{p}{p-1}$ by Hölder inequality. This is finite if $\frac{p}{2(p-1)}(pr^2-r) \le \frac{1}{2}+\varepsilon$. The left-hand side for $r\to 1$ converges to $\frac{p}{2}$ which for $p\to 1$ in turn converges to $\frac{1}{2}$. By continuity there are $r,p>1$ such that it is smaller than $\frac{1}{2}+\varepsilon$. □

Corollary 3.33. *The previous proposition still holds if there are times $0=t_0<t_1<\dots<t_n=T$ such that*

$$\mathbb{E}\left[\exp\left(\left(\frac{1}{2}+\varepsilon\right)\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}\right)\right] < \infty$$

holds for $k=1,\dots,n$.

Proof. We have

$$\begin{aligned}\mathbb{E}\left[Z_T\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(M_T - \frac{1}{2}\langle M\rangle_T\right)\Big|\mathcal{F}_{t_{n-1}}\right]\right]\\ &= \mathbb{E}\left[\exp\left(M_{t_{n-1}} - \frac{1}{2}\langle M\rangle_{t_{n-1}}\right)\mathbb{E}\left[\exp\left(\left(M_{t_n}-M_{t_{n-1}}\right) - \frac{1}{2}\left(\langle M\rangle_{t_n} - \langle M\rangle_{t_{n-1}}\right)\right)\Big|\mathcal{F}_{t_{n-1}}\right]\right].\end{aligned}$$

Now $(M_t - M_{t_{n-1}}, t\in[t_{n-1},t_n])$ is also a continuous local martingale starting at $t=t_{n-1}$ in zero with quadratic variation $\langle M\rangle_t - \langle M\rangle_{t_{n-1}}$. Thus, the previous argument (for Novikov's condition) applied to $(M_t - M_{t_{n-1}}, t\in[t_{n-1},t_n])$ and conditional on $\mathcal{F}_{t_{n-1}}$ yields that $\mathbb{E}[\exp(M_{t_n} - M_{t_{n-1}} - \frac{1}{2}(\langle M\rangle_{t_n} - \langle M\rangle_{t_{n-1}})|\mathcal{F}_{t_{n-1}}] = 1$, using $\mathbb{E}[\exp((\frac{1}{2}+\varepsilon)(\langle M\rangle_{t_n} - \langle M\rangle_{t_{n-1}}))] < \infty$. We obtain the claim by applying this argument for $t_{n-1}, t_{n-2},\dots$ yields $\mathbb{E}[Z_T]=1$. □

Remark 3.34. A different proof is given in Karatzas (1991) using a multi-dimensional version of Novikov's condition.

3.4.3 Maximum-Likelihood estimation for Ornstein-Uhlenbeck processes

Consider the Ornstein-Uhlenbeck-process X solving the SDE $dX_t = -aX_t\,dt + dB_t$, $X_0=0$, where $a\in\mathbb{R}$ is a parameter. A solution is given by $X_t = \int_0^t e^{-a(t-s)}\,dB_s$. $a=0$ corresponds to Brownian motion. Our aim is to estimate $a\in\mathbb{R}$ from the observation of one trajectory $(X_t, t\in[0,T])$. The Maximum-Likelihood approach is then the following, first in a more general setting: Suppose we observe Y, a random variable (even function valued), with density p_a where $a\in A$, the parameter set, is an unknown parameter. Here, we assume that all densities p_a are taken with respect to one dominating measure (e.g. Lebesgue-measure or the Wiener-measure). Then the Maximum-Likelihood estimator $\hat{a}$ is the value of a which maximises $p_a(y)$ over $a\in A$ where y is a realisation of Y. Formally, $\hat{a} = \arg\max_{a\in A} p_a(y)$. Here, we shall use the Girsanov Theorem to determine the density of $(X_t, t\in[0,T])$ with respect to the Wiener measure on $C([0,T], \mathscr{B}_{C([0,T])})$. We want that $(X_t, t \in [0,T])$ is an Ornstein-Uhlenbeck-process with parameter a under $\mathbb{Q}^{(a)}$. For this write $dX_t = -aX_t + d\bar{B}_t$, $X_0 = 0$ with a $\mathbb{Q}^{(a)}$-Brownian motion $\bar{B}$, whereas under $\mathbb{Q}^{(0)} = \mathbb{P}$ X should be just a Brownian motion. So, we have $\bar{B}_t = X_t - \left(-a \int_0^t X_s\, ds\right)$, which is indeed a $\mathbb{Q}^{(a)}$-Brownian motion for

$$\frac{d\mathbb{Q}^{(a)}}{d\mathbb{P}} = Z_T = \exp\left(\int_0^T (-aX_s dX_s - \frac{1}{2}\int_0^T a^2 X_s^2\, ds\right)$$

by Girsanov's Theorem. In order to apply Girsanov, we must make sure that

$$\mathbb{E}\left[Z_T\right] = \mathbb{E}\left[\exp\left(\int_0^T (-aX_s\, dX_s - \frac{1}{2}\int_0^T a^2 X_s^2\, ds\right)\right] = 1$$

holds. By the corollary to Novikov's condition it suffices to show

$$\mathbb{E}\left[\exp\left(\left(\frac{1}{2}+\varepsilon\right) a^2 \int_{t_{k-1}}^{t_k} B_s^2\, ds\right)\right] < \infty$$

for suitable $0 = t_0 < t_1 < \cdots < t_n = T$ (observe for $Z \sim N(0,1)$ that $\mathbb{E}[e^{cZ^2}] < \infty$ if and only if $c < \frac{1}{2}$). For interval lengths $t_k - t_{k-1}$ such that $\left(\frac{1}{2}+\varepsilon\right) a^2 \left(t_k - t_{k-1}\right)$ this expectation is indeed finite (check!). Hence,

$$\frac{d\mathbb{Q}^{(a)}}{d\mathbb{Q}^{(0)}} = \exp\left(-a\int_0^T X_s\, dX_s - \frac{1}{2}\int_0^T X_s^2\, ds\right)$$

is the density of the law of the Ornstein-Uhlenbeck-process on $[0,T]$ with respect to the law of Brownian motion. Thus, the Maximum-Likelihood-estimator is given by

$$\begin{aligned}
\hat{a} &= \arg\max_{a\in\mathbb{R}} \exp\left(-a\int_0^T X_s\, dX_s - \frac{a^2}{2}\int_0^T X_s^2\, ds\right) \\
&= \arg\max_{a\in\mathbb{R}} \left(-a\int_0^T X_s\, dX_s - \frac{a^2}{2}\int_0^T X_s^2\, ds\right) \\
&= \frac{-\int_0^T X_s\, dX_s}{\int_0^T X_s^2\, ds} \\
&= \frac{-\frac{1}{2}\left(X_T^2 - T\right)}{\int_0^T X_s^2\, ds}.
\end{aligned}$$

If our observations $(X_t, t \in [0,T])$ are generated under $\mathbb{Q}^{(a_0)}$ for some $a_0 \in \mathbb{R}$, then

$$\hat{a} = \frac{-\int_0^T X_s(-a_0 X_s\, ds + d\bar{B}_s)}{\int_0^T X_s^2\, ds} = a_0 - \frac{\int_0^T X_s\, d\bar{B}_s}{\int_0^T X_s^2\, ds} = a_0 - \frac{M_T}{\langle M\rangle_T}$$

for the martingale $M_t = \int_0^t X_s\, d\bar{B}_s$. We always have $\langle M\rangle_T = \int_0^T X_s^2\, ds \xrightarrow{a.s.} \infty$ for $T \to \infty$ such that the law of large numbers for martinges (cf.) yields $\hat{a}_T \xrightarrow{\mathbb{Q}^{(a)}\text{-a.s.}} a_0$ as $T \to \infty$ (consistent estimator). If $a_0 > 0$ ('asymptotically stationary case'), then a central limit theorem holds:

$$\sqrt{T}\left(\hat{a}_T - a_0\right) \xrightarrow{T\to\infty} N(0, \frac{1}{2a_0})$$

under $\mathbb{Q}^{(a_0)}$. For $a_0 < 0$ we even have an exponentially fast convergence in $e^{cT}(\hat{a}_T - a_0) \xrightarrow{a.s.} 0$ for some $c > 0$.

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