Markus Reiß

Course *Stochastic Processes*
Winter 2016/17
Humboldt-Universität zu Berlin

Exercises: sheet 1

1. Flies and wasps land on your dinner plate in the manner of independent Poisson processes with respective intensities μ and λ. Show that the arrival of flying beasts forms a Poisson process of intensity $\lambda + \mu$ (*superposition*). The probability that an arriving fly is a blow-fly is p. Does the arrival of blow-flies also form a Poisson process? (*thinning*)

2. Let $(N_t,\, t \geqslant 0)$ be a Poisson process of intensity $\lambda > 0$ and let $(Y_k)_{k \geqslant 1}$ be a sequence of i.i.d. random variables, independent of N. Then $X_t := \sum_{k=1}^{N_t} Y_k$, $t \geqslant 0$, is called *compound Poisson process* ($X_t := 0$ if $N_t = 0$).

   (a) Show that $(X_t,\, t \geqslant 0)$ has independent and stationary increments.

   (b) Determine the expectation of X_t in the case $Y_k \in L^1$.

   (c) Introduce the *Lévy measure* $\nu(B) := \lambda P(Y_1 \in B)$, $B \in \mathfrak{B}_{\mathbb{R}}$. Show that X_t has characteristic function

   $$\varphi_t(u) = \mathbb{E}[e^{iuX_t}] = \exp\Big(t \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx)\Big).$$

   (d) Find a sequence of compound Poisson processes $(X_t^{(n)}, t \geqslant 0)$ with Lévy measures ν_n such that $X_t^{(n)} \xrightarrow{d} N(0,1)$ as $n \to \infty$ for some fixed $t > 0$. Describe heuristically how the sample paths evolve.

   (e*) Characterize all sequences $(\nu_n)_{n \geqslant 1}$ with $X_1^{(n)} \xrightarrow{d} N(0,1)$ in (c).

3. The number of busses that arrive until time t at a bus stop follows a Poisson process with intensity $\lambda > 0$ (in our model). Adam and Berta arrive together at time $t_0 > 0$ at the bus stop and discuss how long they have to wait in the mean for the next bus.
   Adam: Since the waiting times are $\mathrm{Exp}(\lambda)$-distributed and the exponential distribution is memoryless, the mean is λ^{-1}.
   Berta: The time between the arrival of two busses is $\mathrm{Exp}(\lambda)$-distributed and has mean λ^{-1}. Since on average the same time elapses before our arrival and after our arrival, we obtain the mean waiting time $\frac{1}{2}\lambda^{-1}$ (at least assuming that at least one bus had arrived before time t_0).
   What is the correct answer to this *waiting time paradoxon*?

4. Let $C([0,\infty))$ be equipped with the topology of uniform convergence on compacts using the metric $d(f,g) := \sum_{k\geqslant 1} 2^{-k}(\sup_{t\in[0,k]}|f(t)-g(t)|\wedge 1)$. Prove:

   (a) $(C([0,\infty)), d)$ is Polish.

   (b) The Borel σ-algebra is the smallest σ-algebra such that all coordinate projections $\pi_t : C([0,\infty)) \to \mathbb{R}$, $t \geqslant 0$, are measurable.

   (c) For any continuous stochastic process $(X_t,\, t\geqslant 0)$ on $(\Omega, \mathscr{F}, \mathbb{P})$ the mapping $\bar{X} : \Omega \to C([0,\infty))$ with $\bar{X}(\omega)_t := X_t(\omega)$ is Borel-measurable.

   (d) The law of $\bar{X}$ is uniquely determined by the finite-dimensional distributions of X.

Submit before the lecture on Thursday, 3 November 2016. Put solutions to each problem on separate sheets.
Written Exam on Friday, 17 February 2017, 9 a.m.

Markus Reiß

Course *Stochastic Processes*
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Humboldt-Universität zu Berlin

Exercises: sheet 2

1. Prove the regularity lemma: Let P be a probability measure on the Borel σ-algebra $\mathfrak{B}$ of any metric space. Then

$$\mathcal{D} := \Big\{ B \in \mathfrak{B} \,\Big|\, P(B) = \sup_{K \subseteq B \text{ compact}} P(K) = \inf_{O \supseteq B \text{ open}} P(O) \Big\}$$

is closed under set differences and countable unions ($\mathcal{D}$ is a σ-*ring*).
Conclude for a Polish space, using the lecture results, that $\mathcal{D}$ is a σ-algebra and $\mathcal{D} = \mathfrak{B}$.

2. A discrete-time *Markov process* with general state space $(S, \mathscr{S})$ is specified by an initial distribution μ^0 on $(S, \mathscr{S})$ and a *transition kernel* $P : S \times \mathscr{S} \to [0,1]$ (i.e. $B \mapsto P(x,B)$ is a probability measure for all $x \in S$ and $x \mapsto P(x,B)$ is measurable for all $B \in \mathscr{S}$). Show:

   (a) If we put iteratively $P^n(x,B) := \int_S P^{n-1}(y,B)\, P(x,dy)$ for $n \geqslant 2$ and $P^1 := P$, then each P^n is again a transition kernel.

   (b) Put for all $n \geqslant 0$, $A \in \mathscr{S}^{\otimes(n+1)}$

$$Q_n(A) := \int_{S^{n+1}} \mathbf{1}_A(x_0, x_1, \ldots, x_n)\mu^0(dx_0)P(x_0, dx_1)\cdots P(x_{n-1}, dx_n).$$

   Then $(Q_n)_{n\geqslant 0}$ induces a projective family on $S^{\mathbb{N}_0}$.

   (c) Let $(S, \mathscr{S})$ be Polish. Then for each initial distribution μ_0 and each transition kernel P there exists a stochastic process $(X_n,\, n \geqslant 0)$ satisfying $\mathbb{P}^{X_0} = \mu_0$ and $\mathbb{P}^{(X_0,\ldots,X_n)} = Q_n$, $n \geqslant 0$ (the Markov process).

3. Let (X, Y) be a two-dimensional random vector with Lebesgue density $f^{X,Y}$.

   (a) For $x \in \mathbb{R}$ with $f^X(x) > 0$ ($f^X(x) = \int f^{X,Y}(x, \eta)\, d\eta$) consider the *conditional density* $f^{Y \mid X=x}(y) := f^{X,Y}(x, y)/f^X(x)$. Which condition on $f^{X,Y}$ ensures for any Borel set B

   $$\lim_{h \downarrow 0} \mathbb{P}(Y \in B \mid X \in [x, x+h]) = \int_B f^{Y \mid X=x}(y)\, dy \quad ?$$

   (b) Show that for $Y \in L^2$ (without any condition on $f^{X,Y}$) the function

   $$\varphi_Y(x) := \begin{cases} \int y f^{Y \mid X=x}(y)\, dy, & \text{if } f^X(x) > 0 \\ 0, & \text{otherwise} \end{cases}$$

   minimizes the L^2-distance $\mathbb{E}[(Y - \varphi(X))^2]$ over all measurable functions φ. We write $\mathbb{E}[Y \mid X = x] := \varphi_Y(x)$ and $\mathbb{E}[Y \mid X] := \varphi_Y(X)$.

   (c) Prove that φ_Y is $\mathbb{P}^X$-a.s. uniquely characterized by solving

   $$\forall A \in \mathfrak{B}_{\mathbb{R}} : \mathbb{E}[\varphi(X)\mathbf{1}_A(X)] = \mathbb{E}[Y\mathbf{1}_A(X)]$$

   among all $\varphi : \mathbb{R} \to \mathbb{R}$ measurable.

4. In the situation of exercise 3 prove the following properties:

   (a) $\mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[Y]$;

   (b) if X and Y are independent, then $\mathbb{E}[Y \mid X] = \mathbb{E}[Y]$ holds a.s.;

   (c) if $Y \geqslant 0$ a.s., then $\mathbb{E}[Y \mid X] \geqslant 0$ a.s.;

   (d) for all $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ we have $\mathbb{E}[\alpha Y + \beta \mid X] = \alpha\, \mathbb{E}[Y \mid X] + \beta$ a.s.;

   (e) if $\varphi : \mathbb{R} \to \mathbb{R}$ is such that $(x, y) \mapsto (x, y\varphi(x))$ is a diffeomorphism and $Y\varphi(X) \in L^2$, then $\mathbb{E}[Y\varphi(X) \mid X] = \mathbb{E}[Y \mid X]\varphi(X)$ a.s.

Submit before the lecture on Thursday, 10 November 2016.
Put solutions to each problem on separate sheets.
Übung Friday 11:00-12:30 in room 1.115, RUD 25.
Corrector Paul Bach: Friday 12:30-13:00, room 1.1.04, RUD 25.

Markus Reiß

Course *Stochastic Processes*
Winter 2016/17
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Exercises: sheet 3

1. Let $\Omega = \bigcup_{n\in\mathbb{N}} B_n$, $B_m \cap B_n = \varnothing$ for $m \neq n$, be a measurable, countable partition for given $(\Omega, \mathcal{F}, \mathbb{P})$ and put $\mathcal{B} := \sigma(B_n,\, n \in \mathbb{N})$. Show:
   (a) Every $\mathcal{B}$-measurable random variable X can be written as $X = \sum_n \alpha_n \mathbf{1}_{B_n}$ with suitable $\alpha_n \in \mathbb{R}$. For $Y \in L^1$ we have $\mathbb{E}[Y \,|\, \mathcal{B}] = \sum_{n:\, \mathbb{P}(B_n)>0} \Big(\frac{1}{\mathbb{P}(B_n)} \int_{B_n} Y \, d\,\mathbb{P}\Big) \mathbf{1}_{B_n}$ $\mathbb{P}$-a.s.
   (b) Specify $\Omega = [0,1)$ with Borel σ-algebra and $\mathbb{P} = U([0,1))$, the uniform distribution. For $Y(\omega) := \omega$, $\omega \in [0,1)$, determine $\mathbb{E}[Y \,|\, \sigma([(k-1)/n, k/n),\, k = 1, \ldots, n)]$. For $n = 1, 3, 5, 10$ plot the conditional expectations and Y itself as functions on Ω.

2. Let (X, Y) be a two-dimensional $N(\mu, \Sigma)$-random vector.
   (a) For which $\alpha \in \mathbb{R}$ are X and $Y - \alpha X$ uncorrelated?
   (b) Conclude that X and $Y - (\alpha X + \beta)$ are independent for these values α and for arbitrary $\beta \in \mathbb{R}$ such that $\mathbb{E}[Y|X] = \alpha X + \beta$ with suitable $\beta \in \mathbb{R}$.

3. For $Y \in L^2$ define the *conditional variance* of Y given X by
$$\operatorname{Var}(Y|X) := \mathbb{E}[(Y - \mathbb{E}[Y \,|\, X])^2 \,|\, X].$$
   (a) Why is $\operatorname{Var}(Y|X)$ well defined?
   (b) Show $\operatorname{Var}(Y) = \operatorname{Var}(\mathbb{E}[Y \,|\, X]) + \mathbb{E}[\operatorname{Var}(Y|X)]$.
   (c) Use (b) to prove for independent random variables $(Z_k)_{k\geqslant 1}$ and N in L^2 with (Z_k) identically distributed and N $\mathbb{N}$-valued:
$$\operatorname{Var}\Big(\sum_{k=1}^{N} Z_k\Big) = \mathbb{E}[Z_1]^2 \operatorname{Var}(N) + \mathbb{E}[N] \operatorname{Var}(Z_1).$$

4. Let $I \subseteq \mathbb{R}$ be an open interval and $\varphi : I \to \mathbb{R}$ be a convex function. Recall from Stochastics I that
$$\forall x, y \in I :\; \varphi(y) \geqslant \varphi(x) + \varphi'(x+)(y - x)$$
with the right derivative $\varphi'(x+)$ of φ at x. Let Y be an I-valued random variable with $Y, \varphi(Y) \in L^1$. Prove Jensen's inequality for conditional expectations:
$$\mathbb{E}[\varphi(Y) \,|\, \mathcal{G}] \geqslant \varphi(\mathbb{E}[Y \,|\, \mathcal{G}]).$$

Submit before the lecture on Thursday, 17 November 2016.

Markus Reiß

Course *Stochastic Processes*
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Exercises: sheet 4

1. Consider $Y \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ and a $(S, \mathscr{S})$-valued random variable X on $(\Omega, \mathscr{F}, \mathbb{P})$. Show:

   (a) The conditional expected value $\mathbb{E}[Y \mid X = x]$ is $\mathbb{P}^X$-almost surely unique, meaning that for measurable $\varphi, \psi : S \to \mathbb{R}$ with $\mathbb{E}[Y \mid X] = \varphi(X) = \psi(X)$ $\mathbb{P}$-a.s. we have $\varphi = \psi$ $\mathbb{P}^X$-a.s.

   (b) For $x \in S$ with $\{x\} \in \mathscr{S}$, $\mathbb{P}(X = x) > 0$ we have

   $$\mathbb{E}[Y \mid X = x] = \int_\Omega Y(\omega)\, \mathbb{P}^{X=x}(d\omega)$$

   with $\mathbb{P}^{X=x}(A) = \mathbb{P}(A \mid \{X = x\})$, $A \in \mathscr{F}$.

   (c) Suppose Y is $\mathrm{Exp}(\lambda)$-distributed and $X = Y \wedge R := \min(Y, R)$ for some $R > 0$. Determine first $\mathbb{E}[Y \mid X = R]$ and then $\mathbb{E}[Y \mid X = r]$ for all $r \in \mathbb{R}$. State explicitly in what sense different versions of $\mathbb{E}[Y \mid X]$ may differ.

2. Let X, Y be independent real-valued random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ Borel-measurable with $\mathbb{E}[|\varphi(X, Y)|] < \infty$. Prove

   $$\mathbb{E}[\varphi(X, Y) \mid X = x] = \mathbb{E}[\varphi(x, Y)] = \int_\mathbb{R} \varphi(x, y)\, \mathbb{P}^Y(dy) \quad \text{for } \mathbb{P}^X\text{-almost all } x.$$

   Use $\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(X, Y) \mid X]]$ to calculate $\mathbb{E}[\max(X, Y)]$ for $X, Y \sim \mathrm{Exp}(\lambda)$ independent.

3. Let $Y, Y_n \in \mathcal{M}^+(\Omega, \mathscr{F})$, $n \geqslant 1$, and let $\mathscr{G} \subseteq \mathscr{F}$ be a sub-σ-algebra. Deduce from the characterisation of conditional expectations and the monotone convergence theorem for expectations: $Y_n \uparrow Y$ implies $\mathbb{E}[Y_n \mid \mathscr{G}] \uparrow \mathbb{E}[Y \mid \mathscr{G}]$ $\mathbb{P}$-a.s.

   Prove further Fatou's Lemma and the dominated convergence theorem for conditional expectations, following the proofs for expectations.

4. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $\mathscr{G} \subseteq \mathscr{F}$ a sub-σ-algebra. Recall the definition of *conditional probability*:

$$\mathbb{P}\left(A|\mathscr{G}\right) = \mathbb{E}\left[\mathbf{1}_A \,|\, \mathscr{G}\right] .$$

Prove:

(a) For all $A \in \mathscr{F}, B \in \mathscr{G}$: $\mathbb{P}(A \cap B) = \int_B \mathbb{P}(A \,|\, \mathcal{G})\, d\,\mathbb{P}$.

(b) For all $A \in \mathscr{F}$: $0 \leqslant \mathbb{P}(A \,|\, \mathscr{G}) \leqslant 1$ $\mathbb{P}$-almost surely.

(c) $\mathbb{P}(\varnothing \,|\, \mathscr{G}) = 0$ and $\mathbb{P}(\Omega \,|\, \mathscr{G}) = 1$ $\mathbb{P}$-almost surely.

(d) For pairwise disjoint events $A_n \in \mathcal{F}$, $n \geqslant 1$, we have $\mathbb{P}$-almost surely:

$$\mathbb{P}\Big(\bigcup_{n \geqslant 1} A_n \,|\, \mathscr{G}\Big) = \sum_{n \geqslant 1} \mathbb{P}(A_n \,|\, \mathscr{G}).$$

Why does this not necessarily mean that $A \mapsto \mathbb{P}(A \,|\, \mathscr{G})(\omega)$ is a probability measure for $\mathbb{P}$-almost all $\omega \in \Omega$?

Submit before the lecture on Thursday, 24 November 2016.

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Exercises: sheet 5

1. *Doubling strategy*: In each round a fair coin is tossed, for *heads* the player receives his double stake, for *tails* he loses his stake. His initial capital is $K_0 = 0$. At game $n \geqslant 1$ his strategy is as follows: if *heads* has appeared before, his stake is zero (he stops playing); otherwise his stake is 2^{n-1} Euro.

   (a) Argue why his capital K_n after game n can be modeled with independent (X_i) such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$ via

   $$K_n = \begin{cases} -(2^n - 1), & \text{if } X_1 = \dots = X_n = -1, \\ 1, & \text{otherwise.} \end{cases}$$

   (b) Represent K_n as martingale transform.

   (c) Prove $\lim_{n\to\infty} K_n = 1$ a.s. although $\mathbb{E}[K_n] = 0$ for all $n \geqslant 0$ holds.

2. Let T be an $\mathbb{N}_0$-valued random variable and $S_n := \mathbf{1}_{\{n \geqslant T\}}$, $n \geqslant 0$. Show:

   (a) The natural filtration satisfies $\mathcal{F}_n^S = \sigma(\{T = k\}, k = 0, \dots, n)$.

   (b) (S_n) is a submartingale with respect to $(\mathcal{F}_n^S)$ and

   $$\mathbb{E}[S_{n+1} \,|\, \mathcal{F}_n^S] = \mathbf{1}_{\{S_n = 1\}} + \mathbb{P}(T = n+1 \,|\, T \geqslant n+1)\mathbf{1}_{\{S_n = 0\}} \text{ } \mathbb{P}\text{-a.s.}$$

   (c) Determine the Doob decomposition of (S_n). Sketch for geometrically distributed T ($\mathbb{P}(T = k) = (1-p)p^k$, $k \in \mathbb{N}_0$) the sample paths of (S_n), its compensator and their difference.

3. Prove the *Hoeffding inequality*: Let (M_n) be a martingale with $M_0 = 0$ and $|M_n(\omega) - M_{n-1}(\omega)| \leqslant K_n$ for all $\omega \in \Omega$, $n \geqslant 1$. Then:

$$\mathbb{P}(|M_n| \geqslant t) \leqslant 2\exp\Big(-\frac{t^2}{2\sum_{i=1}^n K_i^2}\Big), \quad t > 0.$$

Proceed stepwise:

(a) Use the convexity of the exponential function to show that

$$e^{\eta x} \leqslant \tfrac{1-x}{2}e^{-\eta} + \tfrac{1+x}{2}e^{\eta} \qquad \text{for all } \eta > 0, |x| \leq 1.$$

(b) Deduce that $\mathbb{E}[e^{\eta(M_n - M_{n-1})} \,|\, \mathcal{F}_{n-1}] \leqslant (e^{-\eta K_n} + e^{\eta K_n})/2 \leqslant e^{\eta^2 K_n^2/2}$.

(c) By iteration show that $\mathbb{E}[e^{\eta M_n}] \leqslant \exp(\eta^2 \sum_{i=1}^n K_i^2/2)$.

(d) Use the (generalized) Markov inequality and optimize over η to conclude.

4. Let $X_1, \ldots, X_n$ be i.i.d. real-valued random variables and let $g : \mathbb{R}^n \to \mathbb{R}$ be a Borel-measurable function, which changes at most by K_i if the ith argument changes, i.e.

$$|g(x_1, \ldots, x_n) - g(y_1, \ldots, y_n)| \leqslant \sum_{i=1}^n K_i \mathbf{1}(x_i \neq y_i).$$

(a) Consider $M_k = \mathbb{E}[g(X_1, \ldots, X_n) \,|\, \mathscr{F}_k] - \mathbb{E}[g(X_1, \ldots, X_n)]$ with $\mathscr{F}_0 = \{\varnothing, \Omega)$, $\mathscr{F}_k = \sigma(X_1, \ldots, X_k)$, $k = 1, \ldots, n$ (why is M_k well defined?). Use the Hoeffding inequality to derive the following *bounded difference or McDiarmid inequality* for $t > 0$:

$$\mathbb{P}\Big(|g(X_1, \ldots, X_n) - \mathbb{E}[g(X_1, \ldots, X_n)]| \geqslant t\Big) \leqslant 2\exp\Big(-\frac{t^2}{2\sum_{i=1}^n K_i^2}\Big).$$

(b) Let $\widehat{F}_n(x) := \frac{1}{n}\sum_{i=1}^n \mathbf{1}(X_i \leqslant x)$ be the empirical distribution function of $X_1, \ldots, X_n$ and $F(x) = \mathbb{P}(X_i \leqslant x)$ be the true distribution function. Consider $g(X_1, \ldots, X_n) = \sup_{x\in\mathbb{R}}|\widehat{F}_n(x) - F(x)|$ and deduce for $t > 0$

$$\mathbb{P}\Big(\Big|\sup_{x\in\mathbb{R}}|\widehat{F}_n(x) - F(x)| - \mathbb{E}\Big[\sup_{x\in\mathbb{R}}|\widehat{F}_n(x) - F(x)|\Big]\Big| \geqslant \frac{t}{\sqrt{n}}\Big) \leqslant 2e^{-t^2/2}.$$

(c*) Extra (+2 points): For comparison, show that $\sqrt{n}(\widehat{F}_n(x) - F(x)) \xrightarrow{d} N(0, F(x)(1 - F(x)))$ as $n \to \infty$ for fixed $x \in \mathbb{R}$ and derive the exact asymptotics of the probability in (b) when there are no suprema inside, but $x \in \mathbb{R}$ is fixed.

Submit before the lecture on 1 December 2016

Markus Reiß

Course *Stochastic Processes*
Winter 2016/17
Humboldt-Universität zu Berlin

Exercises: sheet 6

1. Continuous-time martingales.

   (a) Find the definition of a (sub-/super-)martingale $(X_t, t \geqslant 0)$ in continuous time (state reference!).

   (b) Prove that an $(\mathscr{F}_t)$-adapted process $(X_t, t \geqslant 0)$ with $X_t \in L^1$, $t \geqslant 0$, independent increments (in the following filtration sense: $X_{t+h} - X_t$ is independent of $\mathscr{F}_t$ for all $t, h \geqslant 0$) and constant (increasing/decreasing) expectation function $t \mapsto \mu(t) = \mathbb{E}[X_t]$ is a martingale (sub-/super-martingale).

   (c) For the natural filtrations deduce that Brownian motion $(B_t, t \geqslant 0)$ is a martingale, the Poisson process $(N_t, t \geqslant 0)$ of intensity λ is a submartingale and the compensated Poisson process $(N_t - \lambda t, t \geqslant 0)$ is a martingale.

2. Let $(X_n)_{n \geqslant 0}$ be an $(\mathscr{F}_n)$-adapted family of random variables in L^1. Show that $(X_n)_{n \geqslant 0}$ is a martingale if and only if for all bounded $(\mathscr{F}_n)$-stopping times τ the identity $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$ holds.

3. Let $(\mathcal{F}_n^X)_{n \geqslant 0}$ be the natural filtration of a process $(X_n)_{n \geqslant 0}$ and consider a finite stopping time τ with respect to $(\mathcal{F}_n^X)$.

   (a) Prove $\mathcal{F}_\tau = \sigma(\tau,\, X_{\tau \wedge n},\, n \geqslant 0)$.
   Hint: for '$\subseteq$' write $A \in \mathcal{F}_\tau$ as $A = \bigcup_n A \cap \{\tau = n\}$.

   (b) Show that even $\mathcal{F}_\tau = \sigma(X_{\tau \wedge n},\, n \geqslant 0)$ holds.

4. Let $(S_n)_{n\geqslant 0}$ be a simple random walk with $\mathbb{P}(S_n - S_{n-1} = 1) = p$, $\mathbb{P}(S_n - S_{n-1} = -1) = q = 1-p$, $p \in (0,1)$. Prove:

(a) With $M(\lambda) = pe^{\lambda} + qe^{-\lambda}$, $\lambda \in \mathbb{R}$ the process

$$Y_n^{\lambda} := e^{\lambda S_n} M(\lambda)^{-n}, \quad n \geqslant 0,$$

is a martingale (w.r.t. the natural filtration of (S_n)).

(b) For $a, b \in \mathbb{Z}$ with $a < 0 < b$ and the stopping time(!) $\tau := \inf\{n \geqslant 0 \,|\, S_n \in \{a, b\}\}$ we have

$$e^{a\lambda}\,\mathbb{E}[M(\lambda)^{-\tau}\mathbf{1}_{\{S_\tau = a\}}] + e^{b\lambda}\,\mathbb{E}[M(\lambda)^{-\tau}\mathbf{1}_{\{S_\tau = b\}}] = 1 \text{ if } M(\lambda) \geqslant 1.$$

(c) This implies for all $s \in (0,1]$ (solve $s = M(\lambda)^{-1}$)

$$\mathbb{E}[s^{\tau}\mathbf{1}_{\{S_\tau = a\}}] = \frac{\nu_+(s)^b - \nu_-(s)^b}{\nu_+(s)^b\nu_-(s)^a - \nu_+(s)^a\nu_-(s)^b},$$
$$\mathbb{E}[s^{\tau}\mathbf{1}_{\{S_\tau = b\}}] = \frac{\nu_-(s)^a - \nu_+(s)^a}{\nu_+(s)^b\nu_-(s)^a - \nu_+(s)^a\nu_-(s)^b}$$

with $\nu_{\pm}(s) = (1 \pm \sqrt{1-4pqs^2})/(2ps)$ and continuous extension in the case $\nu_+ = \nu_-$.

(d) Now let $a \downarrow -\infty$ and infer that the generating function of the first passage time $\tau_b := \inf\{n \geqslant 0 \,|\, S_n = b\}$ is given by

$$\varphi_{\tau_b}(s) := \mathbb{E}[s^{\tau_b}\mathbf{1}_{\{\tau_b < \infty\}}] = \Big(\frac{1-\sqrt{1-4pqs^2}}{2qs}\Big)^b, \quad s \in (0,1].$$

In particular, we have $\mathbb{P}(\tau_b < \infty) = \varphi_{\tau_b}(1) = \min(1, p/q)^b$.

Submit before the lecture on 8 December 2016.

Starting with this homework, you may hand in solutions jointly for two people.

Markus Reiß

Course *Stochastic Processes*
Winter 2013/14
Humboldt-Universität zu Berlin

Exercises: sheet 7

1. Prove that a family $(X_i)_{i\in I}$ of random variables is uniformly integrable if and only if $\sup_{i\in I}\|X_i\|_{L^1}<\infty$ holds as well as

$$\forall\,\varepsilon>0\;\exists\,\delta>0:\;\mathbb{P}(A)<\delta\Rightarrow\sup_{i\in I}\mathbb{E}[|X_i|\mathbf{1}_A]<\varepsilon.$$

2. Show for an L^p-bounded martingale (M_n) (i.e. $\sup_n\mathbb{E}[|M_n|^p]<\infty$) with $p\in(1,\infty)$:

   (a) (M_n) converges a.s. and in L^1 to some $M_\infty\in L^1$.

   (b) Use $|M_\infty|\leqslant\sup_{n\geqslant 0}|M_n|$ (why?) and Doob's inequality to infer $M_\infty\in L^p$.

   (c) Prove with dominated convergence that (M_n) converges to M_∞ in L^p.

3. Give a martingale proof of Kolmogorov's 0-1 law:

   (a) Let $(\mathscr{F}_n)$ be a filtration and $\mathscr{F}_\infty=\sigma(\mathscr{F}_n,n\geqslant 0)$. Then for $A\in\mathscr{F}_\infty$ we have $\lim_{n\to\infty}\mathbb{E}[\mathbf{1}_A\,|\,\mathscr{F}_n]=\mathbf{1}_A$ a.s.

   (b) For a sequence $(X_k)_{k\geqslant 1}$ of independent random variables consider the natural filtration $(\mathscr{F}_n)$ and the terminal σ-algebra $\mathscr{T}:=\bigcap_{n\geqslant 1}\sigma(X_k,\,k\geqslant n)$. Then for $A\in\mathscr{T}$ we deduce $\mathbb{P}(A)=\mathbb{E}[1_A\,|\,\mathscr{F}_n]\to\mathbf{1}_A$ a.s. such that $P(A)\in\{0,1\}$ holds.

4. A monkey types at random the 26 capital letters of the Latin alphabet. Let τ be the first time by which the monkey has completed the sequence ABRACADABRA. Prove that τ is almost surely finite and satisfies

$$\mathbb{E}[\tau]=26^{11}+26^4+26.$$

   How much time does it take on average if one letter is typed every second?
   Hint: You may look at a fair game with gamblers G_n arriving before times $n=1,2,\dots$ Then G_n bets 1 Euro on 'A' for letter n; if she wins, she puts 26 Euro on 'B' for letter $n+1$, otherwise she stops. If she wins again, she puts 26^2 Euro on 'R', otherwise she stops etc.

Submit before the lecture on **Friday**, 16 December 2016.

Markus Reiß

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Humboldt-Universität zu Berlin

Exercises: sheet 8

1. Prove in detail for probability measures $\mathbb{Q} \ll \mathbb{P}$, $Z = \frac{d\,\mathbb{Q}}{d\,\mathbb{P}}$ and $Y \in L^1(\mathbb{Q})$ that YZ is in $L^1(\mathbb{P})$ and that the identity

$$\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}_{\mathbb{P}}[YZ], \text{ i.e. } \int Y d\,\mathbb{Q} = \int Y \frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\, d\,\mathbb{P}$$

holds. Give an example where Y is in $L^1(\mathbb{Q})$, but not in $L^1(\mathbb{P})$.

2. Suppose $\mathbb{P}_0$, $\mathbb{P}_1$, $\mathbb{P}_2$ are probability measures on $(\Omega, \mathscr{F})$. Show:

   (a) If $\mathbb{P}_2 \ll \mathbb{P}_1 \ll \mathbb{P}_0$ holds, then $\frac{d\,\mathbb{P}_2}{d\,\mathbb{P}_0} = \frac{d\,\mathbb{P}_2}{d\,\mathbb{P}_1}\frac{d\,\mathbb{P}_1}{d\,\mathbb{P}_0}$ holds $\mathbb{P}_0$-a.s.

   (b) $\mathbb{P}_0$ and $\mathbb{P}_1$ are *equivalent* if and only if $\mathbb{P}_1 \ll \mathbb{P}_0$ and $\frac{d\,\mathbb{P}_1}{d\,\mathbb{P}_0} > 0$ holds $\mathbb{P}_0$-a.s. In that case we have $\frac{d\,\mathbb{P}_0}{d\,\mathbb{P}_1} = \big(\frac{d\,\mathbb{P}_1}{d\,\mathbb{P}_0}\big)^{-1}$ $\mathbb{P}_0$-a.s. and $\mathbb{P}_1$-a.s.

3. Let $(X_k)_{k\geqslant 1}$ be a sequence of i.i.d. $\{-1,+1\}$-valued random variables. Under the probability measure $\mathbb{P}_0$ (the null hypothesis H_0) we have $\mathbb{P}_0(X_k = +1) = p_0$ with $p_0 \in (0,1)$, while under $\mathbb{P}_1$ (the alternative H_1) we have $\mathbb{P}_1(X_k = +1) = p_1$ with $p_1 \in (0,1)$, $p_1 \neq p_0$.

   (a) Explain why the likelihood quotient $L_n = \frac{d(\otimes_{i=1}^n \mathbb{P}_1^{X_i})}{d(\otimes_{i=1}^n \mathbb{P}_0^{X_i})}$ after n observations $X_1, \ldots, X_n$ is given by

   $$L_n = \frac{p_1^{(n+S_n)/2}(1-p_1)^{(n-S_n)/2}}{p_0^{(n+S_n)/2}(1-p_0)^{(n-S_n)/2}} \text{ with } S_n = \textstyle\sum_{k=1}^n X_k.$$

   (b) Show that the *likelihood process* $(L_n)_{n\geqslant 0}$ (put $L_0 := 1$) forms a non-negative martingale under the hypothesis H_0 (i.e. under $\mathbb{P}_0$) with respect to its natural filtration.

   (c) A *sequential likelihood-quotient test*, based on $0 < A < B$ and the stopping time

   $$\tau_{A,B} := \inf\{n \geqslant 1 \,|\, L_n \geqslant B \text{ or } L_n \leqslant A\},$$

   rejects H_0 if $L_{\tau_{A,B}} \geqslant B$, and accepts H_0 if $L_{\tau_{A,B}} \leqslant A$. Determine the probability for errors of the first and second kind (i.e., $\mathbb{P}_0(L_{\tau_{A,B}} \geqslant B)$ and $\mathbb{P}_1(L_{\tau_{A,B}} \leqslant A)$) in the case $p_0 = 0.4$, $p_1 = 0.6$, $A = (2/3)^5$, $B = (3/2)^5$. Calculate $\mathbb{E}[\tau_{A,B}]$.

   (d*) Compare the error probabilities of this sequential test with those of the test which after $n = \lfloor \mathbb{E}[\tau_{A,B}] \rfloor$ observations rejects H_0 if $L_n \geqslant 1$ and accepts H_0 if $L_n < 1$.

4. Let $Z_n(x) = (3/2)^n \sum_{k\in\{0,2\}^n} \mathbf{1}_{I(k,n)}(x)$, $x \in [0,1]$, with intervals $I(k,n) := [\sum_{i=1}^n k_i 3^{-i}, \sum_{i=1}^n k_i 3^{-i} + 3^{-n}]$. Show:

   (a) $(Z_n)_{n\geqslant 0}$ with $Z_0 = 1$ forms a martingale on $([0,1], \mathfrak{B}_{[0,1]}, \lambda, (\mathscr{F}_n))$ with Lebesgue measure λ on $[0,1]$ and $\mathscr{F}_n := \sigma(I(k,n),\, k \in \{0,1,2\}^n)$.

   (b) (Z_n) converges λ-a.s., but not in $L^1([0,1], \mathfrak{B}_{[0,1]}, \lambda)$.

   (c) Interpret Z_n as the density of a probability measure $\mathbb{P}_n$ with respect to λ. Then $(\mathbb{P}_n)$ converges weakly to some probability measure $\mathbb{P}_\infty$ ($\mathbb{P}_\infty$ is called *Cantor measure*). Identify a Borel set $C \subseteq [0,1]$ with $\mathbb{P}_\infty(C) = 1$, $\lambda(C) = 0$.
   Hint: Consider the distribution functions.

Submit before the lecture on Thursday, 12 January 2017

Markus Reiß

Course *Stochastic Processes*
Winter 2016/17
Humboldt-Universität zu Berlin

Exercises: sheet 9

1. Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain on $\{1,2,3,4,5,6,7\}$ with arbitrary initial distribution μ and transition matrix

$$P = \begin{pmatrix} 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 & 0 & 0 \end{pmatrix}.$$

   Draw a graph representing the transition probabilities with states as vertices and edges for positive transition probabilities to clarify the structure of the Markov chain. Give the precise definition for recurrent, transient, closed and irreducible sets of states and determine them in this example.

2. Let $(X_n)_{n\geqslant 0}$ be a time-homogeneous Markov chain on $S=\{1,\dots,M\}$ with initial distribution μ and transition matrix $P\in\mathbb{R}^{M\times M}$. Show that the following are equivalent:

   (a) (X_n) is a stationary process;

   (b) μ is an *invariant* initial distribution, i.e. $\mathbb{P}_\mu(X_1\in B)=\mu(B)$ for all $B\subseteq S$;

   (c) μ interpreted as a vector $(\mu(\{i\}))_{i=1,\dots,M}\in\mathbb{R}^M$ solves the left-eigenvalue problem $\mu^\top P=\mu^\top$ for the eigenvalue 1.

   For Exercise 1 find at least two different invariant μ.
   (*) Determine all invariant μ.

3. Let $\mathscr{I}_T$ be the σ-algebra of invariant events for the measure-preserving map T on $(\Omega,\mathscr{F},\mathbb{P})$. Show:

   (a) A random variable Y is $\mathscr{I}_T$-measurable if and only if $Y\circ T=Y$ holds $\mathbb{P}$-a.s.

   (b) T is ergodic if and only if all bounded random variables Y with $Y\circ T=Y$ $\mathbb{P}$-a.s. are constant $\mathbb{P}$-a.s.

   (c) For all invariant events A there is a *strictly invariant* event B (i.e., $T^{-1}(B)=B$ holds) such that $\mathbb{P}(A\Delta B):=\mathbb{P}(A\setminus B\cup B\setminus A)=0$.

4. *Gelfand's Problem:* Does the decimal representation of 2^n ever start with the initial digit 7? Study this as follows:

   (a) Determine the relative frequencies of the initial digits of $(2^n)_{1 \leqslant n \leqslant 30}$.

   (b) Let $A \sim U([0,1])$. Prove that the relative frequency of the initial digit k in $(10^A 2^n)_{1 \leqslant n \leqslant m}$ converges as $m \to \infty$ a.s. to $\log_{10}(k+1) - \log_{10}(k)$ (consider $X_n = A + n \log_{10}(2) \mod 1$!).

   (c) Prove that the convergence in (b) even holds everywhere. In particular, the relative frequency of the initial digit 7 in the powers of 2 converges to $\log_{10}(8/7) \approx 0,058$.
   Hint: Show for trigonometric polynomials $p(a) = \sum_{|m| \leqslant M} c_m e^{2\pi i m a}$ that $\frac{1}{n} \sum_{k=0}^{n-1} p(a+k\eta) \to \int_0^1 p(x)dx$ holds for all $\eta \in \mathbb{R} \setminus \mathbb{Q}$, $a \in [0,1]$ (calculate explicitly for monomials!) and approximate.

Submit before the lecture on Thursday, 19 January 2017

Markus Reiß

Course *Stochastic Processes*
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Humboldt-Universität zu Berlin

Exercises: sheet 10

1. Give the missing details in the proof of von Neumann's ergodic theorem: For measure-preserving T and $X \in L^p$, $p \geqslant 1$, we have that $A_n := \frac{1}{n}\sum_{i=0}^{n-1} X \circ T^i$ converges to $\mathbb{E}[X \mid \mathscr{I}_T]$ in L^p.

2. Show that a measure-preserving map T on $(\Omega, \mathscr{F}, \mathbb{P})$ is ergodic if and only if for all $A, B \in \mathscr{F}$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{P}(A \cap T^{-k}B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

   Hint: For one direction apply an ergodic theorem to $\mathbf{1}_B$.
   (*) Extension: If even $\lim_{n\to\infty} \mathbb{P}(A \cap T^{-n}B) = \mathbb{P}(A)\,\mathbb{P}(B)$ holds, then T is called *mixing*. Show that T mixing implies T ergodic, but not conversely (e.g., consider rotation by an irrational angle).

3. Consider the Ehrenfest model for gas dynamics, i.e. a Markov chain on $S = \{0, 1, \dots, N\}$ with transition probabilities $p_{i,i+1} = (N-i)/N$, $p_{i,i-1} = i/N$.

   (a) Show that $\mu(\{i\}) = \binom{N}{i}2^{-N}$, $i \in S$, is an invariant initial distribution.

   (b) Is the Markov chain starting in μ ergodic?

   (*c) Simulate the Ehrenfest model with initial value $i_0 \in \{N/2; N\}$, $N = 100$ for $T \in \{100; 100,000\}$ time steps. Plot the relative frequencies of visits to each state in S and compare with μ. Explain what you see!

4. Let $\alpha \in (0,1)$. Choose $X_0 \in [0,1]$ and perform the following independent iterations for $n \in \mathbb{N}$: given $X_{n-1} \in [0,1]$, go with probability 1/2 left, setting $X_n = \alpha X_{n-1}$, and with probability 1/2 right, setting $X_n = (1-\alpha) + \alpha X_{n-1}$.

   (a) Write $X_n = \alpha X_{n-1} + (1-\alpha)Z_n$, $n \in \mathbb{N}$, with suitable i.i.d. random variables (Z_n). Interpret $(X_n, n \geqslant 0)$ as a Markov process on $([0,1], \mathfrak{B}_{[0,1]})$ in the sense of Exercise 2.2.

   (b) For $\alpha = 1/2$ and $\alpha = 1/3$ determine an invariant initial distribution μ such that $(X_n, n \geqslant 0)$ becomes stationary with $X_0 \sim \mu$.
   Hint: Represent $x \in [0,1]$ in a dyadic or triadic expansion.

   (c) Show that, whatever the initial distribution of X_0 is, we have $X_n \xrightarrow{d} \mu$ in (b). Conclude that with $X_0 \sim \mu$ the process $(X_n, n \geqslant 0)$ is ergodic.

   (*d) Extend the results to general α and probabilities different from 1/2. Is (X_n) mixing in the above sense? Simulate!

   (*e) Consider the triangle Δ spanned by the corner points $(0,0)$, (1,0), $(0,1)$ in $\mathbb{R}^2$. Perform iterations, where for given $X_{n-1} \in \Delta$ with probability 1/3 one of the corners is selected and X_n is obtained as middle point between that corner and X_{n-1}. Expand $x \in \Delta$ as $x = \sum_i b_i 2^{-i}$ with certain $b_i \in \{0,1\}^2$ and describe the unique invariant initial distribution μ. Plot the support set of μ approximately by simulating (X_n). Try to understand and explore further!

Submit before the lecture on Thursday, 26 January 2017

Markus Reiß

Course *Stochastic Processes*
Winter 2016/17
Humboldt-Universität zu Berlin

Exercises: sheet 11

1. For probability measures $\mathbb{P}$ and $\mathbb{Q}$ on a measurable space $(\Omega, \mathcal{F})$ their total variation distance is given by $\|\mathbb{P} - \mathbb{Q}\|_{TV} = \sup_{A \in \mathcal{F}} |\mathbb{P}(A) - \mathbb{Q}(A)|$. Decide whether for $n \to \infty$ the probabilities $\mathbb{P}_n$ with the following Lebesgue densities f_n on $\mathbb{R}$ converge in total variation distance, weakly or not at all:
$$f_n(x) = ne^{-nx}\mathbf{1}_{[0,\infty)}(x), \quad f_n(x) = \tfrac{n+1}{n} x^{1/n} \mathbf{1}_{[0,1]}(x), \quad f_n(x) = \tfrac{1}{n} \mathbf{1}_{[0,n]}(x).$$

2. For random variables X, Y with values in a Polish space (S, d) with Borel σ-algebra define $d_0(X, Y) = \mathbb{E}[d(X, Y) \wedge 1]$.
   (a) Show that $d_0(X, Y)$ is well-defined (measurability of $\omega \mapsto d(X(\omega), Y(\omega))$!) and defines a metric on the space $L^0(\Omega; S)$ of all S-valued random variables on $(\Omega, \mathscr{F}, \mathbb{P})$.
   (b) Prove $d_0(X_n, X) \to 0 \iff X_n \xrightarrow{\mathbb{P}} X$ (stochastic convergence).
   (c) Deduce that $X_n \xrightarrow{\mathbb{P}} X$ implies $X_n \xrightarrow{d} X$.
   (d) Prove that $X_n \xrightarrow{d} c$ for some constant $c \in S$ implies $X_n \xrightarrow{\mathbb{P}} c$.

3. For probability measures $\mathbb{P}, \mathbb{Q}$ on a metric space (S, d) with Borel σ-algebra define the *Bounded-Lipschitz metric*
$$d_{BL}(\mathbb{P}, \mathbb{Q}) = \sup\Big\{ \Big| \int_S f \, d\,\mathbb{P} - \int_S f \, d\,\mathbb{Q} \Big| \, \Big| \, f \in BL_1(S) \Big\}$$
with $BL_1(S) = \{f : S \to \mathbb{R} \mid \|f\|_\infty \leqslant 1, \forall x, y \in S : |f(x) - f(y)| \leqslant d(x, y)\}$. Prove that d_{BL} is indeed a metric and that $d_{BL}(\mathbb{P}_n, \mathbb{P}) \to 0 \Rightarrow \mathbb{P}_n \xrightarrow{w} \mathbb{P}$. For a compact space (S, d) use the Arzelà-Ascoli Theorem to prove
$$d_{BL}(\mathbb{P}_n, \mathbb{P}) \to 0 \iff \mathbb{P}_n \xrightarrow{w} \mathbb{P}.$$
Remark: This holds in fact on any Polish space (S, d), using tightness.

4. Let $(X_k)_{k \geqslant 1}$ be an i.i.d. sequence of random variables in L^2 with $\mu = \mathbb{E}[X_k]$. Introduce the sample mean $\bar{X}_n := \frac{1}{n} \sum_{k=1}^n X_k$ and the sample variance $\bar{\sigma}_n^2 := \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2$. Use Slutsky's Lemma to prove for $n \to \infty$
$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\bar{\sigma}_n} \xrightarrow{d} N(0, 1).$$
Determine approximately a real number $c > 0$ such that
$$\lim_{n \to \infty} \mathbb{P}\left(\mu \in \left[\bar{X} - c\tfrac{\bar{\sigma}_n}{\sqrt{n}}, \bar{X} + c\tfrac{\bar{\sigma}_n}{\sqrt{n}}\right] \right) = 0.95.$$

Submit before the lecture on Thursday, 2 February 2017

Markus Reiß

Course *Stochastic Processes*
Winter 2016/17
Humboldt-Universität zu Berlin

Exercises: sheet 12

1. We say that a family of real-valued random variables $(X_i)_{i\in I}$ is *stochastically bounded*, notation $X_i = O_{\mathbb{P}}(1)$, if $\lim_{R\to\infty} \sup_{i\in I} \mathbb{P}(|X_i| > R) = 0$.

   (a) Show $X_i = O_{\mathbb{P}}(1)$ if and only if the laws $(\mathbb{P}^{X_i})_{i\in I}$ are uniformly tight.

   (b) Prove that any L^p-bounded family of random variables, $p > 0$, is stochastically bounded, hence has uniformly tight laws.

   (c) If $X_n \xrightarrow{\mathbb{P}} 0$ holds, then we write $X_n = o_{\mathbb{P}}(1)$. Check the symbolic rules $O_{\mathbb{P}}(1) + O_{\mathbb{P}}(1) = O_{\mathbb{P}}(1)$ and $O_{\mathbb{P}}(1)o_{\mathbb{P}}(1) = o_{\mathbb{P}}(1)$.

2. Prove: Every relatively (weakly) compact family $(\mathbb{P}_i)_{i\in I}$ of probability measures on a Polish space $(S, \mathfrak{B}_S)$ is uniformly tight. Proceed as follows (cf. proof of Ulam's Theorem):

   (a) For $k \geqslant 1$ consider open balls $(A_{k,m})_{m\geqslant 1}$ of radius $1/k$ that cover S. If $\lim_{M\to\infty} \inf_i \mathbb{P}_i(\bigcup_{m=1}^M A_{k,m}) < 1$ were true, then by assumption and by the Portmanteau Theorem we would have $\lim_{M\to\infty} \mathbb{Q}(\bigcup_{m=1}^M A_{k,m}) < 1$ for some limiting probability measure $\mathbb{Q}$, which is contradictory.

   (b) Conclude that for any $\varepsilon > 0$, $k \geqslant 1$ there are indices $M_{k,\varepsilon} \geqslant 1$ such that $\inf_i \mathbb{P}_i(K) > 1 - \varepsilon$ holds with $K := \bigcap_{k\geqslant 1} \bigcup_{m=1}^{M_{k,\varepsilon}} A_{k,m}$. Moreover, K is relatively compact in S, which suffices.

*3. Arzelà-Ascoli Theorem.

   (a) Understand the proof of the Arzelà-Ascoli Theorem for $C([0,T])$ from the literature and present it in your own words.

   (b) Suppose $f_i \in C([0,T])$ is Lipschitz-continuous with Lipschitz constant L_i for $i \in I$. Verify that $(f_i)_{i\in I}$ is relatively compact if $\sup_i |f_i(0)| < \infty$ and $\sup_i L_i < \infty$ hold.

   (c) Prove that a subset $A \subseteq C(\mathbb{R}^+)$ is relatively compact if

      i. $\sup_{f\in A} |f(0)| < \infty$ and

      ii. $\lim_{\delta\to 0} \sup_{f\in A} \max\{|f(s) - f(t)| \mid s,t \in [0,T],\ |s-t| \leqslant \delta\} = 0$ holds for all $T > 0$.

   (d) Let $p_n(x) = \sum_{k=0}^m a_k^{(n)} x^k$ be real polynomials of maximal degree m with $a_k^{(n)} \to a_k$ as $n \to \infty$ for $k = 0, \dots, m$. Deduce $p_n \to p$ in $C(\mathbb{R}^+)$ with $p(x) = \sum_{k=0}^m a_k x^k$.

*4. Let $(S, \mathscr{S})$ be a measurable space, T an uncountable set.

(a) Show that for each $B \in \mathscr{S}^{\otimes T}$ there is a countable set $I \subseteq T$ such that

$$\forall x \in S^T,\, y \in B:\ (x(t) = y(t) \text{ for all } t \in I) \Rightarrow x \in B.$$

Hint: Check first that sets B with this property form a σ-algebra.

(b) Conclude for a metric space S with at least two elements that the set $C := \{f : [0,1] \to S \,|\, f \text{ continuous}\}$ is not product-measurable, i.e. $C \notin \mathscr{S}^{\otimes[0,1]}$.

Submit before the lecture on Thursday, 9 February 2017. Problems 3 and 4 are optional.

Markus Reiß

Course *Stochastic Processes*
Winter 2016/17
Humboldt-Universität zu Berlin

Probeklausur

1. Entscheiden Sie, ob die folgenden Aussagen wahr oder falsch sind. (5P)

   (a) Seien $(N_t)_{t\geqslant 0}$ ein Poissonprozess, $(Y_k)_{k\geqslant 1}$ eine Folge unabhängiger, identisch verteilter Zufallsvariablen und $X_t = \sum_{k=1}^{N_t} Y_k$, $t \geqslant 0$, der zugehörige zusammengesetzte Poissonprozess. Dann ist $(X_t)_{t\geqslant 0}$ stochastisch stetig.

   (b) Jedes Wahrscheinlichkeitsmaß auf $(\mathbb{R}^d, \mathscr{B}_{\mathbb{R}^d})$ ist straff und regulär.

   (c) Für unabhängige Zufallsvariablen X, Y mit $XY \in L^1$ gilt $\mathbb{E}[XY|X] = X\mathbb{E}[Y]$ f.s.

   (d) Aus $|Y| \leqslant c$ f.s. folgt $|\mathbb{E}[Y|\mathscr{G}]| \leqslant c$ f.s.

   (e) Ist $(M_n)_{n\geqslant 0}$ ein Submartingal, so ist $(M_n - \mathbb{E}[M_n])_{n\geqslant 0}$ ein Martingal.

   (f) Für Stoppzeiten σ, τ gilt $\mathscr{F}_{\sigma\vee\tau} \subseteq \mathscr{F}_\tau \subseteq \mathscr{F}_{\sigma\wedge\tau}$.

   (g) Ist $(X_k)_{k\geqslant 1}$ eine Folge unabhängiger, identisch verteilter L^2-Zufallsvariablen, so konvergiert $\sum_{n\geqslant 1} X_n/n^{2/3}$ f.s.

   (h) Sind ν und μ zwei Maße mit $\nu \ll \mu$ und $\nu \perp \mu$, so gilt $\mu = 0$.

   (i) Ist $(X_k)_{k\geqslant 0}$ eine zeitlich homogene, irreduzible Markovkette mit endlichem Zustandsraum und invarianter Anfangsverteilung μ, so gilt

   $$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) \to \mathbb{E}_\mu[f(X_0)] \qquad \mathbb{P}_\mu\text{-f.s.}$$

   für alle Funktionen $f : S \to \mathbb{R}$.

   (j) Sind $X, X_1, X_2, \ldots$ und $Y_1, Y_2, \ldots$ reelle Zufallsvariablen mit $X_n \xrightarrow{d} X$ und $Y_n \xrightarrow{d} c$, wobei $c \in \mathbb{R}$, so gilt $(X_n, Y_n) \xrightarrow{d} (X, c)$.

   (k) Sei (S, d) ein kompakter metrischer Raum mit Borel-σ-Algebra $\mathscr{B}_S$. Sind $\mathbb{P}_0, \mathbb{P}_n, \mathbb{P}_{nk}$ Wahrscheinlichkeitsmaße auf $(S, \mathscr{B}_S)$ mit $\mathbb{P}_n \xrightarrow{w} \mathbb{P}_0$ und $\mathbb{P}_{nk} \xrightarrow{w} \mathbb{P}_n$ für jedes n, so existiert eine Teilfolge $\mathbb{P}_{nk(n)}$ mit $\mathbb{P}_{nk(n)} \xrightarrow{w} \mathbb{P}_0$.

   (l) Sei $(B_t)_{t\geqslant 0}$ eine Brownsche Bewegung, $c > 0$ und $X_t = (1/\sqrt{c})B_{ct}$ für $t \geqslant 0$. Dann ist $(X_t)_{t\geqslant 0}$ eine Brownsche Bewegung.

2. Sei $(X_k)_{k\geqslant 1}$ eine Folge unabhängiger, identisch verteilter Zufallsvariablen mit $\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = 1/2$. Setze $S_0 = 0$ und $S_n = \sum_{k=1}^n X_k$ für $n \geqslant 1$. Desweiteren sei $(\mathscr{F}_n^S)$ die natürliche Filtration von (S_n).

   (a) Für $a \in \mathbb{Z}$ setze $\tau_a := \inf\{n \geqslant 1 \,|\, S_n = a\}$. Zeigen Sie, dass τ_a eine Stoppzeit bezüglich $(\mathscr{F}_n^S)$ ist. (2P)

(b) Begründen Sie, weshalb $\mathbb{P}(\tau_a < \infty) = 1$ gilt. Folgern Sie, dass $\mathbb{E}[S_{\tau_a}] = a \neq 0 = \mathbb{E}[S_0]$. Warum widerspricht dies nicht der Wald-Identität. (1P)

(c) Finden Sie eine Konstante c, so dass $\exp(S_n - cn)$ ein $(\mathscr{F}_n^S)$-Martingal ist. Konvergiert das resultierende Martingal f.s. bzw. in L^1? Geben Sie die quadratische Variation des resultierenden Martingals an. (2P)

3. L^p-Konvergenz.

(a) Formulieren Sie den Satz von Vitali. Erklären Sie weshalb dieser als Verallgemeinerung des Satzes von der dominierten Konvergenz angesehen werden kann. (1P)

(b) Zeigen Sie: Ist $p \geqslant 1$ und sind $X, X_1, X_2, \ldots \in L^p$ mit $(|X_n|^p)_{n\geqslant 1}$ gleichgradig integrierbar, so ist $(|X - X_n|^p)_{n\geqslant 1}$ gleichgradig integrierbar.

(c) Seien $X, X_1, X_2, \ldots \in L^p$ mit $X_n \xrightarrow{\mathbb{P}} X$, wobei $p \geqslant 1$. Schließen Sie aus (a) und (b), dass folgende Aussagen äquivalent sind: (2P)

(i) $(|X_n|^p)_{n\geqslant 1}$ ist gleichgradig integrierbar;
(ii) $X_n \to X$ in L^p;
(iii) $\mathbb{E}[|X_n|^p] \to \mathbb{E}[|X|^p]$.

4. Sei $(X_n)_{n\geqslant 0}$ eine Markovkette auf $\mathbb{N}_0$ mit Übergangswahrscheinlichkeiten $p_{0,1} = 1$ und $p_{i,i+1} = r$, $p_{i,i-1} = 1 - r$ für $i \geqslant 1$, wobei $r \in (0, 1)$.

(a) Skizzieren Sie Zustände und mögliche Übergänge als Graphen. Zeigen Sie, dass die Markovkette irreduzibel ist. (1P)

(b) Definieren Sie was eine invariante Anfangsverteilung ist. Zeigen Sie, dass für $r \in (0, 1/2)$ durch

$$\mu(\{0\}) = \frac{1/2 - r}{1 - r}, \qquad \mu(\{i\}) = \frac{1/2 - r}{1 - r}\frac{r^{i-1}}{(1-r)^i}, \qquad i \geqslant 1,$$

eine invariante Anfangsverteilung definiert ist. Ist diese eindeutig bestimmt? (2P)

(c) Ein *Bernoulli-Pfad* der Länge $2k$ ist eine Folge $(s_n)_{0\leqslant n\leqslant 2k}$ von natürlichen Zahlen mit $s_0 = 0$ und $|s_{n+1} - s_n| = 1$ für $n < 2k$. Sei C_k die Anzahl der nichtnegativen Bernoulli-Pfade der Länge $2k$ mit $s_{2k} = 0$. Für $i \in \mathbb{N}_0$ setze $T_i = \inf\{n \geqslant 1 : X_n = i\}$. Zeigen Sie: (1P)

$$\mathbb{P}_0(T_0 < \infty) = \sum_{k=1}^{\infty} \mathbb{P}_0(T_0 = 2k) = (1 - r) \sum_{k=1}^{\infty} C_{k-1}(r(1 - r))^{k-1}.$$

(d) Beweisen Sie die folgenden Gleichungen: (1P)

$$C_0 = 1, \qquad C_k = \binom{2k}{k} - \binom{2k}{k-1} = \frac{1}{k+1}\binom{2k}{k}, \qquad k \geqslant 1.$$

(*Hinweis*: Man kann zeigen, dass die Anzahl der Bernoulli-Pfade der Länge $2k$ mit $s_{2k} = 0$ und $s_n < 0$ für ein $n < 2k$ gleich ist mit der Anzahl der Bernoulli-Pfade der Länge $2k$ mit $s_{2k} = -2$, indem man die Pfade beim ersten Eintreffen in -1 reflektiert.)

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Course *Stochastic Processes*
Winter 2016/17
Humboldt-Universität zu Berlin

List of exam-related questions

1. Formulate and give main steps in the proof: Ulam's Theorem, Kolmogorov's consistency theorem, Factorisation Lemma, existence and properties of conditional expectations, Doob decomposition and quadratic variation, optional stopping and optional sampling theorems, Wald identity, Martingale inequalities, 1st martingale convergence theorem, Vitali's Theorem, 2nd martingale convergence theorem, strong law for L^2-martingales, backward martingale convergence theorem, Radon-Nikodym theorem, Lebesgue decomposition, Kakutani's Theorem, Birkhoff's ergodic theorem, sufficient condition for ergodicity of Markov chains, Continuous Mapping Theorem, Portmanteau Lemma, Slutsky-Lemma, Kolmogorov-Centsov criterion for weak convergence in $C([0,T])$, Donsker Theorem for random walks.

2. Which ways exist to construct a Poisson process? What is the Markov property? Why can a stochastic process be considered as a $(S^T, \mathscr{S}^{\otimes T})$-valued random variable? How is the conditional expectation in L^2 constructed? What is the meaning of $\mathbb{E}[Y \,|\, X = x]$? What is $(X \bullet M)_n$ for X predictable, M martingale and what are its properties? When does $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$ hold for M martingale, τ stopping time? What are sufficient conditions for uniform integrability? Does uniform integrability imply tightness of the laws? How many ways do you know to prove the classical strong law of large numbers? What are equivalent characterisations for T being ergodic? What do irreducible, recurrent, transient, positive-recurrent mean for a Markov chain or its states? What is the implication of the Portmanteau Lemma for distribution functions of real-valued random variables? What is the relationship between $\mathbb{P}_n \xrightarrow{w} \mathbb{P}$ for laws on $C([0,T])$ and convergence of the finite-dimensional distributions? How can we prove existence of Brownian motion?

3. Give examples and counter-examples for: Polish spaces, (sub-/super-)martingales, predictable processes, stopping times, uniformly integrable random variables, absolutely continuous and singular measures, (weakly) stationary and ergodic processes, irreducible Markov chains, recurrent, transient, positive-recurrent states, invariant initial distributions, weak convergence, tight laws.

4. Solve the exercise problems again.

5. For each result in point 1 find examples and possibly counter-examples where assertions do not hold. Where do the conditions in the theorems enter in the proof?