

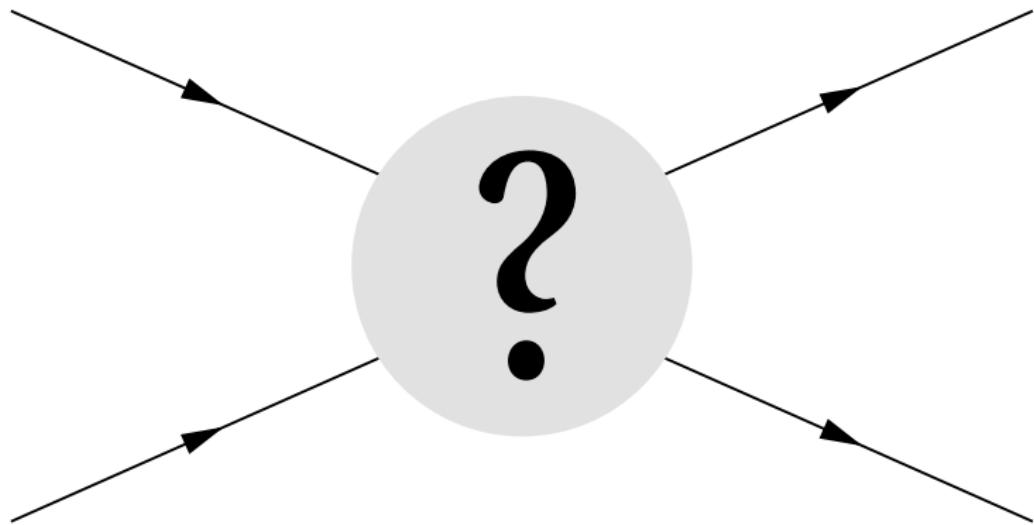
# Feynman integrals, graph polynomials and zeta values

Erik Panzer

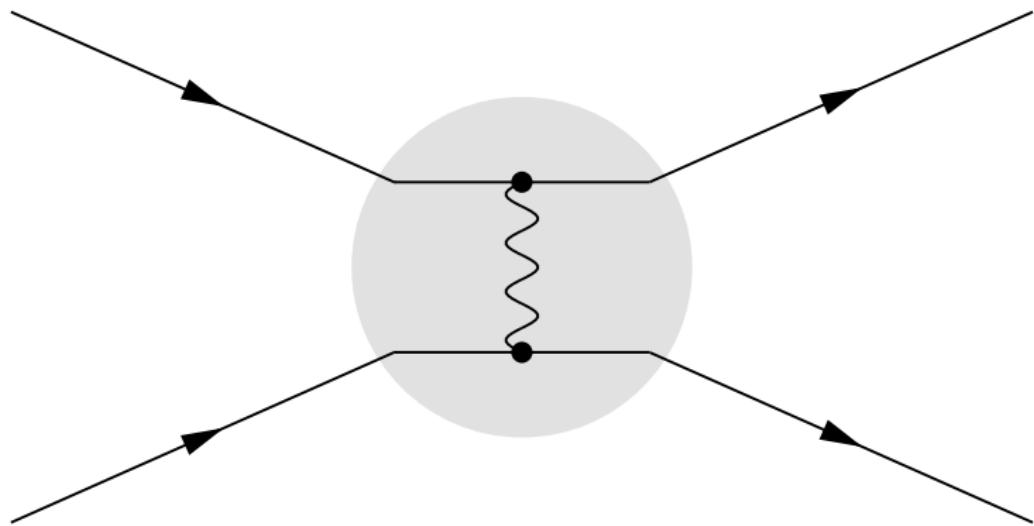
All Souls College

North meets South Colloquium  
May 26th  
Oxford

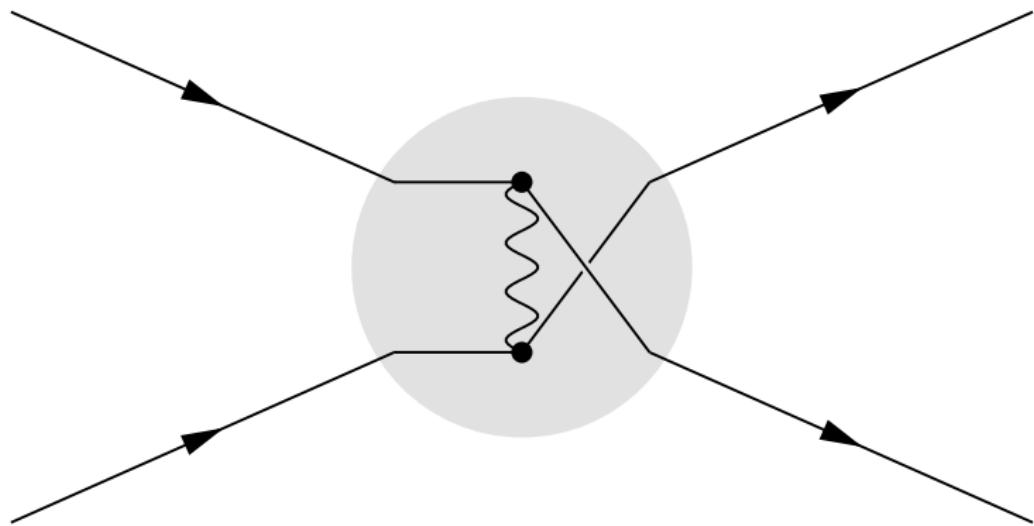
# Perturbative Quantum Field Theory (**QFT**)



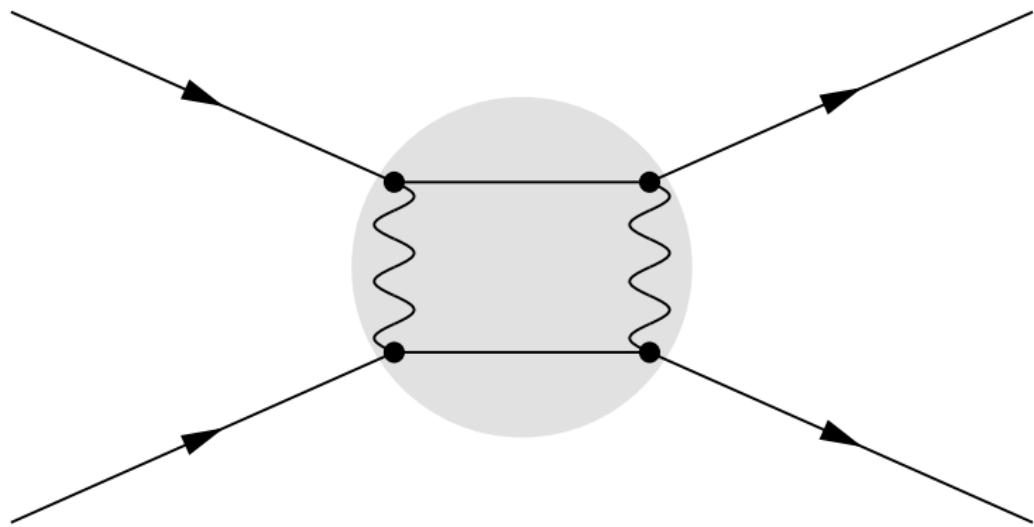
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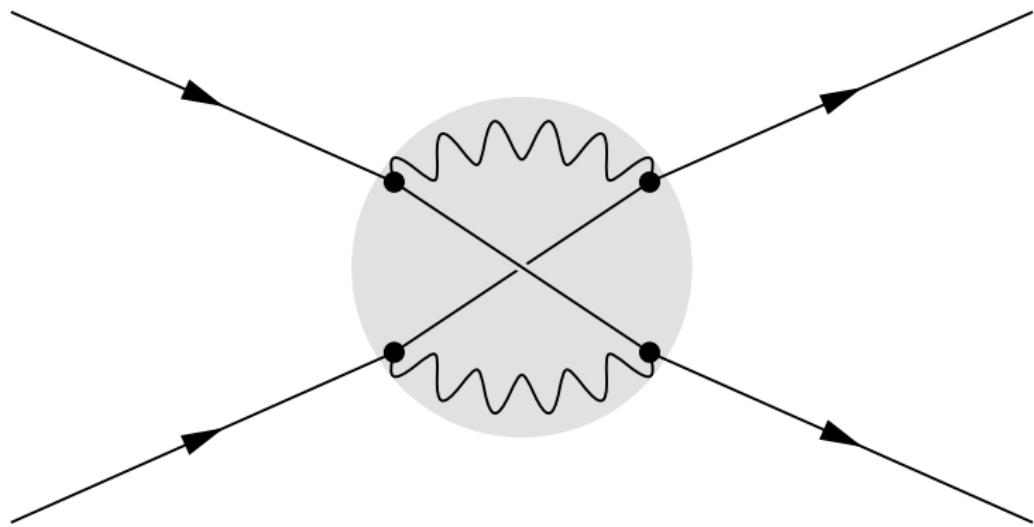
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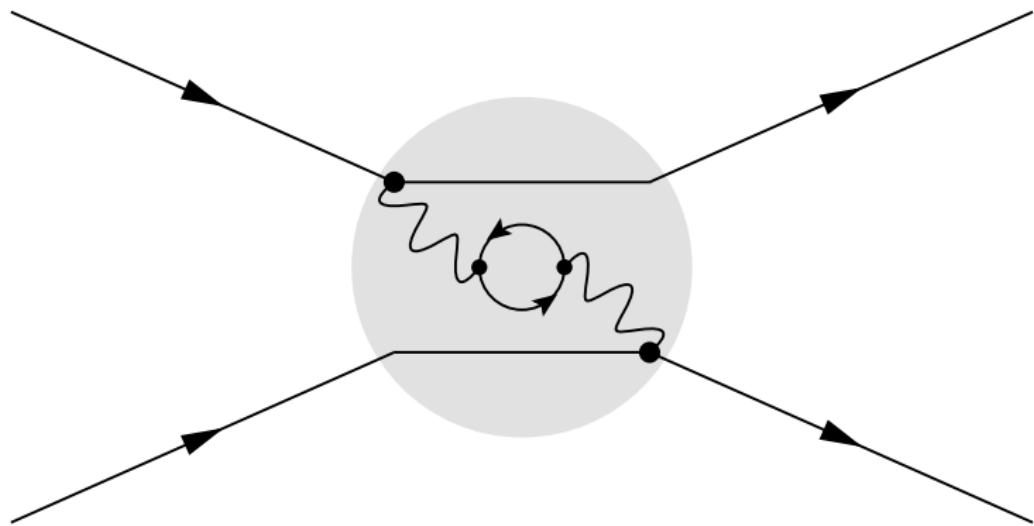
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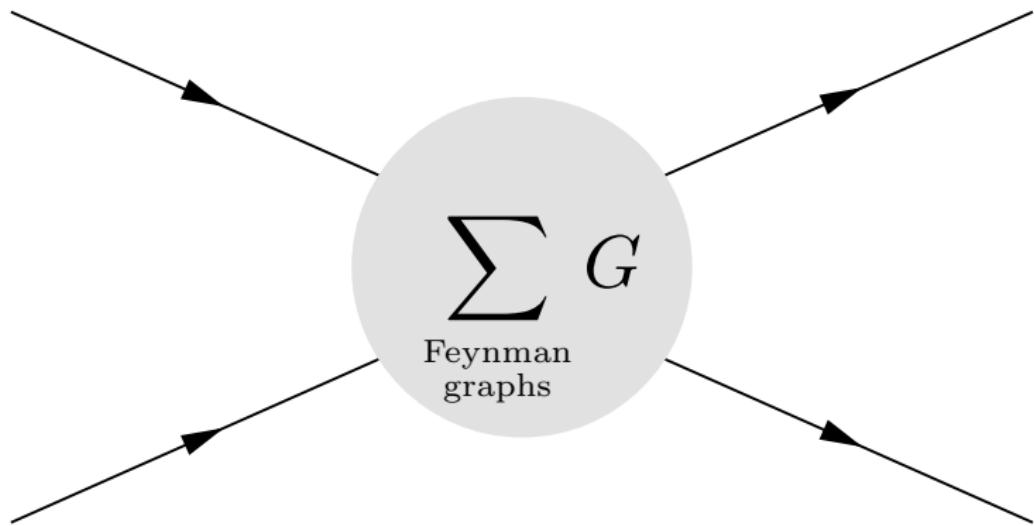
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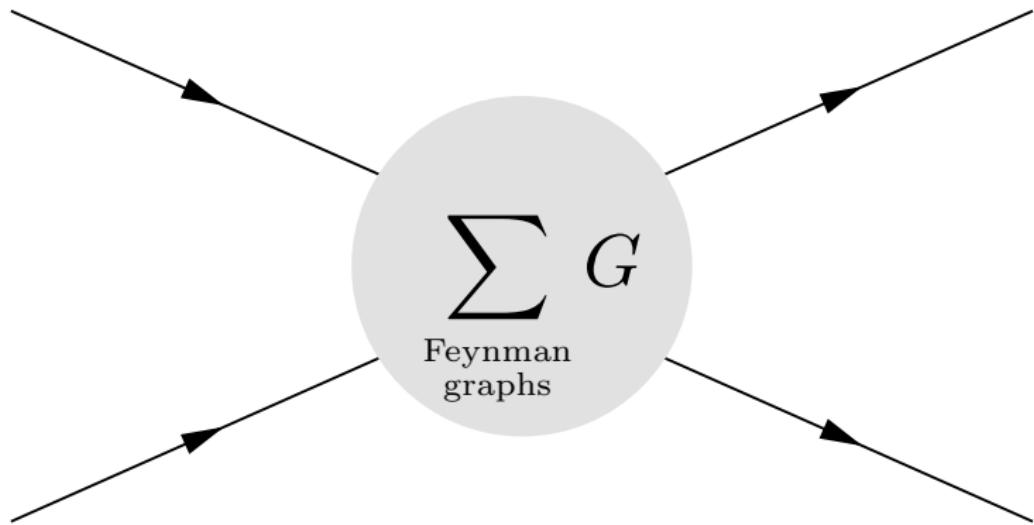
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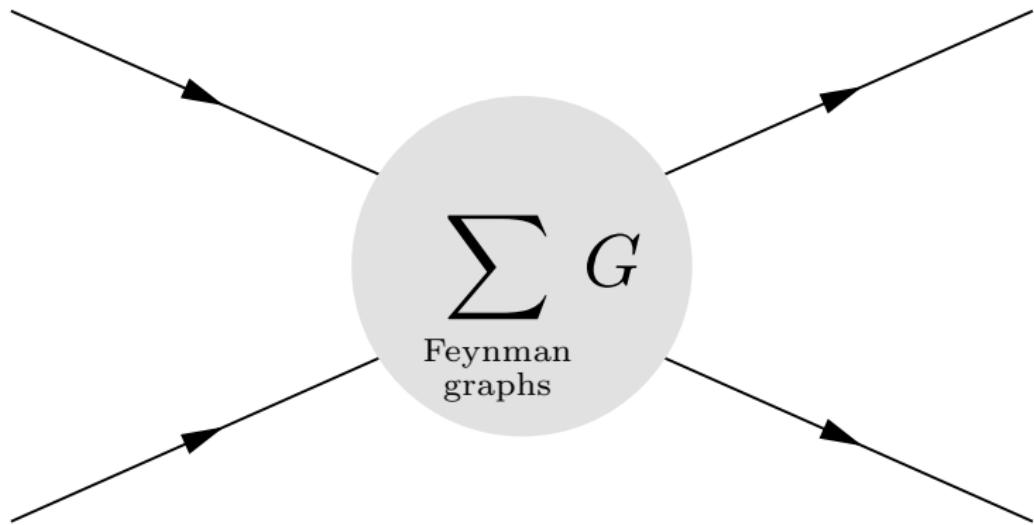


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- each Feynman graph represents a **Feynman integral**  $\Phi(G)$
- truncated sum  $\sum_G \Phi(G)$  approximates the process

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- each Feynman graph represents a **Feynman integral**  $\Phi(G)$
- truncated sum  $\sum_G \Phi(G)$  approximates the process
- very accurate measurements demand precise theory predictions  
⇒ many graphs have to be included

## Example

$$\Phi \left( \begin{array}{c} \text{Diagram of two circles} \\ \text{with 4 vertices and 6 edges} \end{array} \right) = 6\zeta_3 \quad \Phi \left( \begin{array}{c} \text{Diagram of a hexagon with internal diagonals} \\ \text{and 6 vertices and 15 edges} \end{array} \right) = 20\zeta_5$$

Riemann zeta function:  $\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}$

## Example

$$\Phi \left( \begin{array}{c} \text{Diagram of two circles} \\ \text{with 4 points on the left and 2 on the right} \end{array} \right) = 6\zeta_3 \quad \Phi \left( \begin{array}{c} \text{Diagram of a hexagon with internal lines} \\ \text{forming a central hexagon and a large outer hexagon} \end{array} \right) = 20\zeta_5$$

Riemann zeta function:  $\zeta_n = \sum_{0 < k} \frac{1}{k^n}$   $\left( \zeta_3 \notin \mathbb{Q} \text{ [Ap\'ery]} \right)$

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Riemann zeta function:

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## Example

$$\Phi \left( \begin{array}{c} \text{Diagram of a triangle with 3 vertices} \\ \text{and 3 edges, each with a double arrow} \end{array} \right) = \frac{2 \operatorname{Im} [\operatorname{Li}_2(z) + \log(1-z) \log |z|]}{\operatorname{Im} z} = \frac{2D_2(z)}{\operatorname{Im} z}$$

Polylogarithms:

$$\operatorname{Li}_n = \sum_{0 < k} \frac{z^k}{k^n} \quad \Rightarrow \quad \zeta_n = \operatorname{Li}_n(1)$$

## Example

$$\Phi \left( \begin{array}{c} \text{Diagram: A hexagon with vertices at } (-1,0), (1,0), (0,\sqrt{3}), (0,-\sqrt{3}), (-1,0), (1,0) \\ \text{and internal edges connecting } (-1,0) \text{ to } (0,\sqrt{3}) \text{ and } (0,-\sqrt{3}), \\ \text{ and } (1,0) \text{ to } (0,\sqrt{3}) \text{ and } (0,-\sqrt{3}). \end{array} \right) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{1044}{5}\zeta_8$$

Double zeta value:

$$\zeta_{3,5} = \sum_{0 < k < m} \frac{1}{k^3 m^5}$$

## Example

$$\Phi \left( \begin{array}{c} \text{Diagram: A hexagon with internal diagonals forming a star-like shape} \end{array} \right) = 252\zeta_3\zeta_5 + \frac{432}{5}\zeta_{3,5} - \frac{1044}{5}\zeta_8$$

$$\begin{aligned} \Phi \left( \begin{array}{c} \text{Diagram: A hexagon with internal diagonals forming a more complex star-like shape} \end{array} \right) &= \frac{92943}{160}\zeta_{11} + \frac{3381}{20}(\zeta_{3,5,3} - \zeta_3\zeta_{3,5}) - \frac{1155}{4}\zeta_3^2\zeta_5 \\ &\quad + 896\zeta_3 \left( \frac{27}{80}\zeta_{3,5} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8 \right) \end{aligned}$$

Multiple zeta values:  $\zeta_{n_1, \dots, n_d} = \sum_{0 < k_1 < \dots < k_d} \frac{1}{k_1^{n_1} \cdots k_d^{n_d}}$

(one) definition of the Feynman integrals

Introduce variables  $\alpha_e$  for each edge  $e$ , and let  $sdd = |E(G)| - 2 \cdot \text{loops}(G)$ :

$$\Phi(G) = \int_{(0,\infty)^E} \frac{\Omega}{\psi^{2-sdd} \varphi^{sdd}}, \quad \Omega = \delta(1 - \alpha_N) \prod_{e \in E} d\alpha_e$$

Graph polynomials:

$$\psi = \sum_T \prod_{e \notin T} \alpha_e \quad \varphi = \sum_{F=T_1 \dot{\cup} T_2} q^2(T_1) \prod_{e \notin F} \alpha_e$$

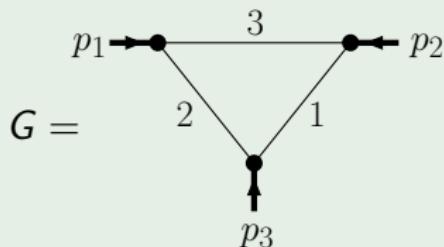
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## Example



$$\psi =$$

$$\varphi =$$

$$\Phi(G) = \iint \frac{d\alpha_2 \, d\alpha_3}{\psi \varphi} \Big|_{\alpha_1=1}$$

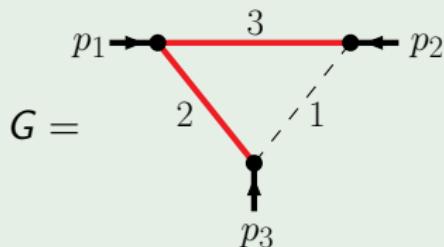
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## Example



$$\psi = \alpha_1 + \alpha_2 + \alpha_3$$

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## Example

$$G = \begin{array}{c} p_1 \xleftarrow{} \bullet \xrightarrow[3]{} \bullet \xleftarrow{} p_2 \\ \backslash \quad \diagdown \quad \backslash \\ \bullet \xrightarrow[2]{} \bullet \xrightarrow[1]{} \end{array}$$

$\psi = \alpha_1 + \color{red}\alpha_2 + \alpha_3$

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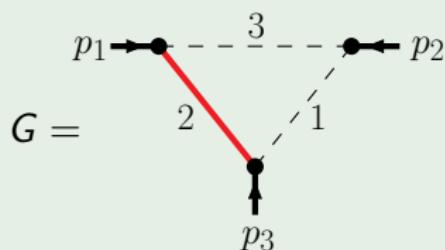
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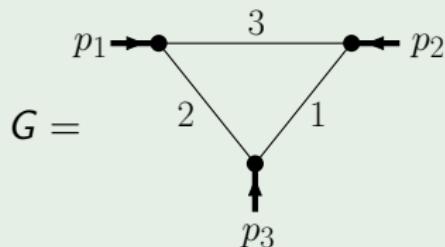
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### Example



$$\psi = \alpha_1 + \alpha_2 + \alpha_3$$

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$$\Phi(G) = \iint \frac{d\alpha_2 \ d\alpha_3}{\psi \varphi} \Big|_{\alpha_1=1}$$

# Comments

- Feynman integrals are periods à la [Kontsevich,Zagier]
- underlying geometry:

$$\Phi(G) = \int_{\sigma} \omega$$

$$\sigma = \{[\alpha_1, \dots, \alpha_E] : \alpha_i \geq 0\} \in H_{E-1}(\mathbb{P}^{E-1}(\mathbb{R}), \cup \{\alpha_i = 0, \infty\})$$

$$\omega = \frac{\Omega}{\psi^{2-\text{sdd}} \varphi^{\text{sdd}}} \in H_{\text{dR}}^{E-1}(\mathbb{P}^{E-1} \setminus \{\psi \cdot \varphi = 0\})$$

- after desingularization, Feynman integrals become *motivic periods* [Francis Brown]
- graph hypersurfaces like  $\{\psi = 0\}$  tend to be very singular

## Example: massless triangle

$$\Phi \left( \begin{array}{c} \text{---} \bullet \text{---} \\ | \quad \backslash \\ \bullet \end{array} \right) = \int \int \frac{d\alpha_2 d\alpha_3}{(1 + \alpha_2 + \alpha_3)(\alpha_2 \alpha_3 + z \bar{z} \alpha_3 + (1 - z)(1 - \bar{z}) \alpha_2)}$$

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Polylogarithms are **iterated integrals**:

$$\text{Li}_1(z) = \sum_{0 < k} \frac{z^k}{k} = -\log(1 - z) = \int_0^z \frac{dt}{1 - t}$$

$$\text{Li}_2(z) = \sum_{0 < k} \frac{z^k}{k^2} = \int_0^z \frac{dt}{t} \text{Li}_1(t)$$

⋮

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$$\text{Li}_n(z) = \sum_{0 < k} \frac{z^k}{k^n} = \int_0^z \frac{dt}{t} \text{Li}_{n-1}(t)$$

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Insert these equations into each other:

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Insert these equations into each other:

$$\text{Li}_n(z) = \int_0^z d \log(t_1) \int_0^{t_1} d \log(t_2) \cdots \int_0^{t_{n-1}} d \log(t_n - 1)$$

## Definition (Hyperlogarithms)

$$G(\underbrace{\sigma_1, \dots, \sigma_w}_{\vec{\sigma}}; z) := \int_0^z \frac{dz_1}{z_1 - \sigma_1} \int_0^{z_1} \frac{dz_2}{z_2 - \sigma_2} \cdots \int_0^{z_{w-1}} \frac{dz_w}{z_w - \sigma_w}$$

- The space of  $\mathbb{Q}(z)$ -linear combinations of  $G(w; z)$ 's is closed under  $\partial_z$  and  $\int dz$ .

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- The space of  $\mathbb{Q}(z)$ -linear combinations of  $G(w; z)$ 's is closed under  $\partial_z$  and  $\int dz$ .
- **Shuffle product:**  $G(\vec{\sigma}; z) \cdot G(\vec{\tau}; z) = G(\vec{\sigma} \sqcup \vec{\tau}; z)$

## Example

$$G(\sigma_3; z) \cdot G(\sigma_2, \sigma_1; z) =$$

$$\{t_3\} \times \{t_1 \leq t_2\} = \{t_1 \leq t_2 \leq t_3\} \cup \{t_1 \leq t_3 \leq t_2\} \cup \{t_3 \leq t_1 \leq t_2\}$$

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$$G(\underbrace{\sigma_1, \dots, \sigma_w}_{\vec{\sigma}}; z) := \int_0^z \frac{dz_1}{z_1 - \sigma_1} \int_0^{z_1} \frac{dz_2}{z_2 - \sigma_2} \cdots \int_0^{z_{w-1}} \frac{dz_w}{z_w - \sigma_w}$$

- The space of  $\mathbb{Q}(z)$ -linear combinations of  $G(w; z)$ 's is closed under  $\partial_z$  and  $\int dz$ .
- **Shuffle product:**  $G(\vec{\sigma}; z) \cdot G(\vec{\tau}; z) = G(\vec{\sigma} \sqcup \vec{\tau}; z)$

## Example

$$G(\sigma_3; z) \cdot G(\sigma_2, \sigma_1; z) = G(\sigma_3, \sigma_2, \sigma_1; z) + G(\sigma_2, \sigma_3, \sigma_1; z) + G(\sigma_2, \sigma_1, \sigma_3; z)$$

$$\{t_3\} \times \{t_1 \leq t_2\} = \{t_1 \leq t_2 \leq t_3\} \cup \{t_1 \leq t_3 \leq t_2\} \cup \{t_3 \leq t_1 \leq t_2\}$$

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- multivalued, monodromies, path concatenation
- algebraic description  $G : T(\Sigma) \rightarrow \{\text{transcendental functions}\}$

Tensor algebra     $T(\Sigma) := \mathbb{Q}\langle \omega_\sigma : \sigma \in \Sigma \rangle = \text{lin}_{\mathbb{Q}} \Sigma^*$

# Linearly reducible families (infinite)

- 3-constructible graphs (3-point functions) [Brown, Schnetz, Panzer]



## Example

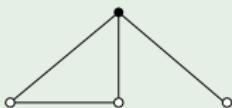


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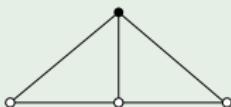


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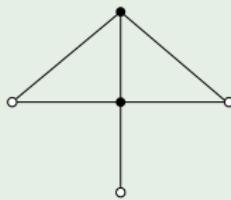


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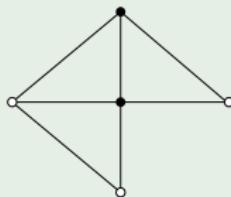


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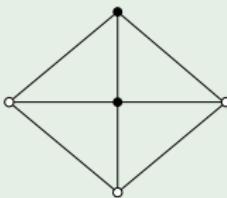


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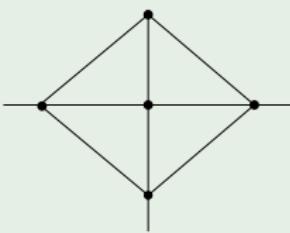


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## Example



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Theorem (Panzer)

All such Feynman integrals are MPL over the alphabet  
 $\{z, \bar{z}, 1 - z, 1 - \bar{z}, z - \bar{z}, 1 - z\bar{z}, 1 - z - \bar{z}, z\bar{z} - z - \bar{z}\}$ .

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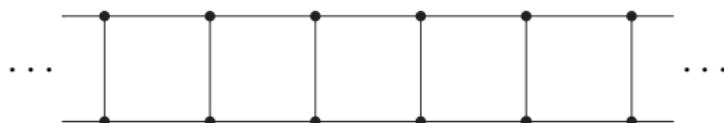
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- minors of ladder-boxes (up to 2 legs off-shell)



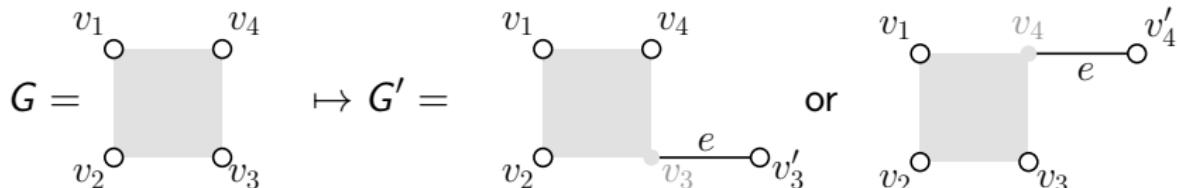
## Theorem (Panzer)

These are MPL with alphabet  $\{x, 1 + x\}$  for  $x = s/t$ .

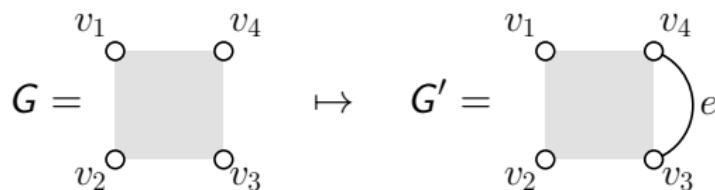
## 4-point recursions

Start with the box and repeat, in any order:

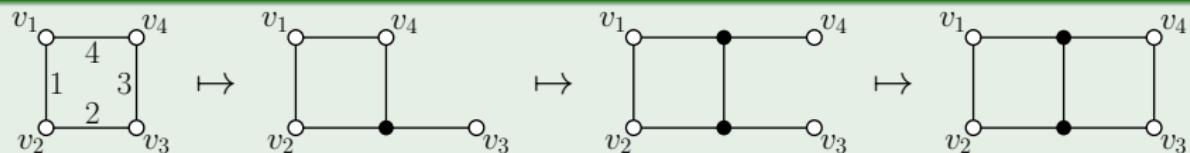
- Appending a vertex:



- Adding an edge:



### Example

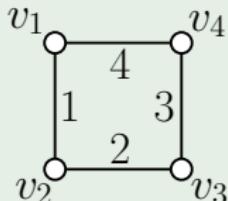


# Forest polynomials

Let  $f_3$ ,  $f_4$ ,  $f_{12}$  and  $f_{14}$  denote the **spanning forest polynomials** such that

$$\varphi = \mathcal{F} = (p_1 + p_2)^2 f_{12} + (p_1 + p_4)^2 f_{14} + p_3^2 f_3 + p_4^2 f_4$$

## Example



$$\psi = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$f_{12} = \alpha_2 \alpha_4$$

$$f_{14} = \alpha_1 \alpha_3$$

$$f_3 = \alpha_2 \alpha_3$$

$$f_4 = \alpha_3 \alpha_4$$

# Forest polynomials

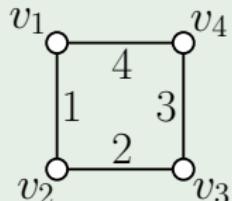
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## Definition

$$F(G; z) := \int_{\mathbb{R}_+^E} \psi_G^{-D/2} \cdot \delta^{(4)} \left( \frac{f}{\psi} - z \right) \prod_{e \in E} d\alpha_e^{a_e-1} \alpha_e \quad (\mathbb{R}_+^4 \longrightarrow \mathbb{R}_+)$$

## Example



$$\begin{aligned} \psi &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & f_{12} &= \alpha_2 \alpha_4 & f_3 &= \alpha_2 \alpha_3 \\ && f_{14} &= \alpha_1 \alpha_3 & f_4 &= \alpha_3 \alpha_4 \end{aligned}$$

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## Example

$$F \left( \begin{array}{ccccc} v_1 & & v_4 & & \\ \circ & & \circ & & \\ & 4 & & & \\ & | & & & \\ & 1 & & & \\ & | & & & \\ v_2 & & v_3 & & \end{array}; z \right) = \begin{cases} \frac{1}{z_3 z_4} & (D = 4) \\ \frac{z_{12}}{\underbrace{[z_{12}(z_{14} + z_3 + z_4) + z_3 z_4]^2}_Q} & (D = 6) \end{cases}$$

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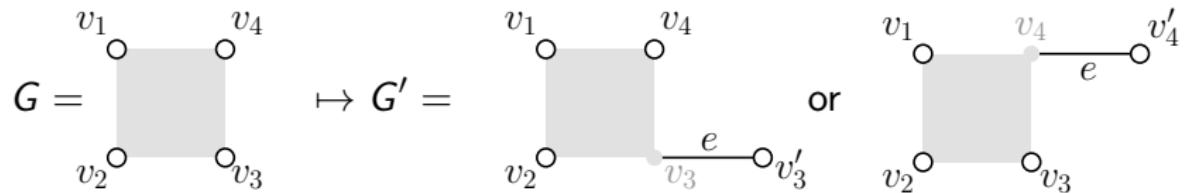
## Definition

$$\Phi(G) = \frac{\Gamma(\text{sdd})}{\prod_e \Gamma(a_e)} \int_0^\infty \frac{F(G; z) \Omega}{[(p_1 + p_2)^2 z_{12} + (p_1 + p_4)^2 z_{14} + p_3^2 z_3 + p_4^2 z_4]^{\text{sdd}}}$$

## Example

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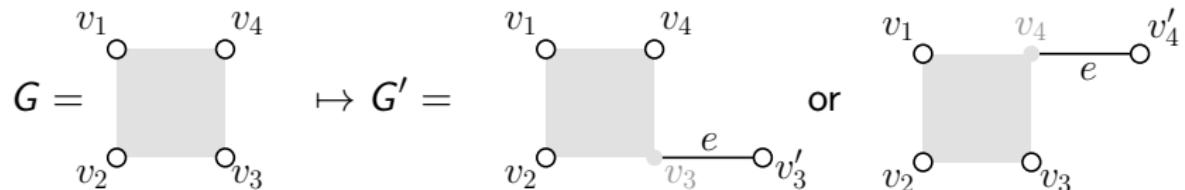
## Appending a vertex



Using  $(f'_{12}, f'_{14}, f'_3, f'_4, \psi') = (f_{12}, f_{14}, f_3, f_4 + \alpha_e \psi, \psi)$ ,

$$F(G'; z) = \int_0^{z_4} F(G; z_{12}, z_{14}, z_3, z_4 - \alpha_e) \alpha_e^{a_e - 1} d\alpha_e$$

# Appending a vertex



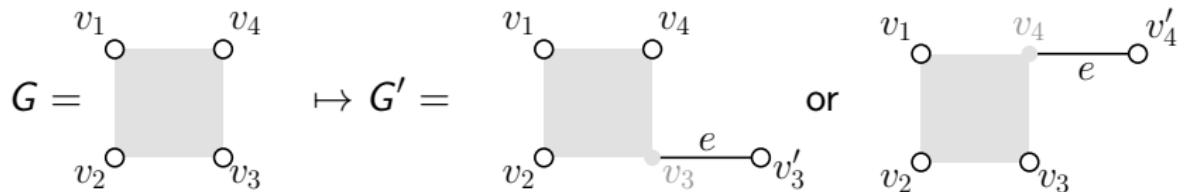
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Example ( $D = 6$  and  $a_e = 1$ )

$$F \left( \begin{array}{ccccc} v_1 & & v_4 & & \\ \circ & & \circ & & \\ \text{---} & & \text{---} & & \\ v_2 & & v_3 & & \end{array}; z \right) = \int_0^{z_3} F \left( \begin{array}{ccccc} v_1 & & v_4 & & \\ \circ & & \circ & & \\ \text{---} & & \text{---} & & \\ v_2 & & v_3 & & \\ | & & | & & \\ 1 & & 3 & & \\ | & & | & & \\ 2 & & 4 & & \end{array}; z_{12}, z_{14}, z'_3, z_4 \right) dz'_3$$

# Appending a vertex



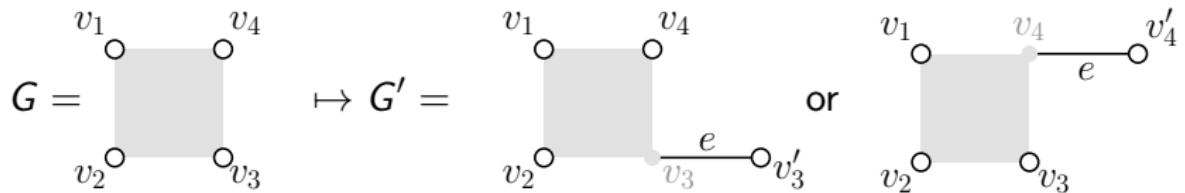
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# Appending a vertex



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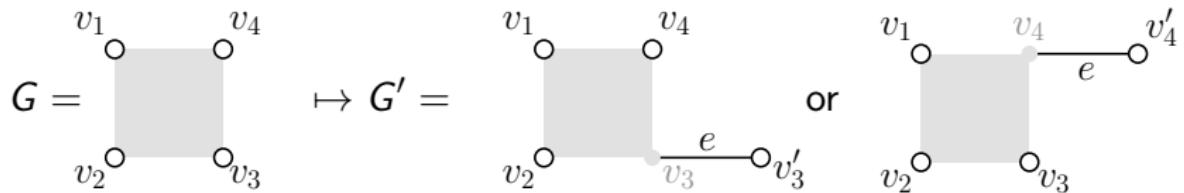
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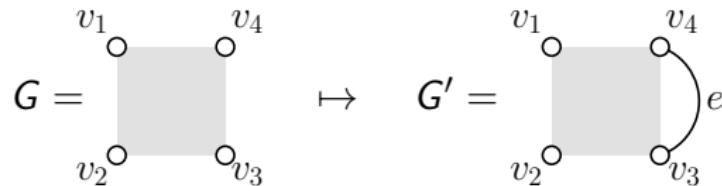
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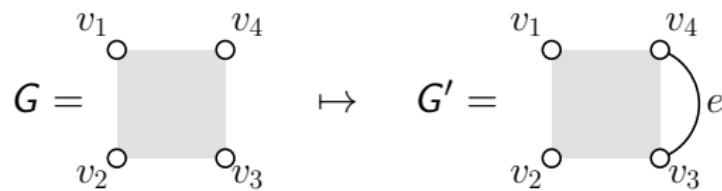
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## Adding an edge



$$F_{G'}(z) = Q^{a_e + \text{sdd} - D} \int_0^{z_{12}} x^{D/2-2} \left[ Q^{D/2-\text{sdd}} \cdot F_G \right]_{z_{12}=z_{12}-x} dx$$

## Adding an edge



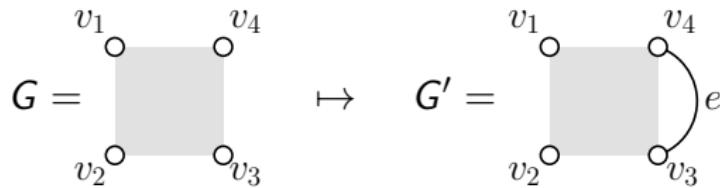
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Example ( $D = 6$  and  $a_e = 1$ )

$$F\left(\begin{array}{ccccc} v_1 & & v_4 \\ \circ & \bullet & \circ \\ & \text{---} & \\ & \bullet & \circ \\ v_2 & & v_3 \end{array}; z\right) = \frac{1}{Q^2} \int_0^{z_{12}} F\left(\begin{array}{ccccc} v_1 & & v_4 \\ \circ & \bullet & \circ \\ & \text{---} & \\ & \bullet & \circ \\ v_2 & & v_3 \end{array}; z_{12} - x, z_{14}, z_3, z_4\right) x dx$$

The equation shows the function \$F\$ for a specific graph configuration. The graph has four vertices \$v\_1, v\_2, v\_3, v\_4\$ arranged in a rectangle. Vertices \$v\_1\$ and \$v\_4\$ are at the top, \$v\_2\$ and \$v\_3\$ are at the bottom. There are two internal black dots: one on the top edge between \$v\_1\$ and \$v\_4\$, and another on the bottom edge between \$v\_2\$ and \$v\_3\$. The function \$F\$ is expressed as an integral of another function \$F\$ over the interval \$[0, z\_{12}]\$, where the argument of the inner function is modified by subtracting \$x\$ from \$z\_{12}\$ and adding \$z\_{14}\$, \$z\_3\$, and \$z\_4\$.

# Adding an edge

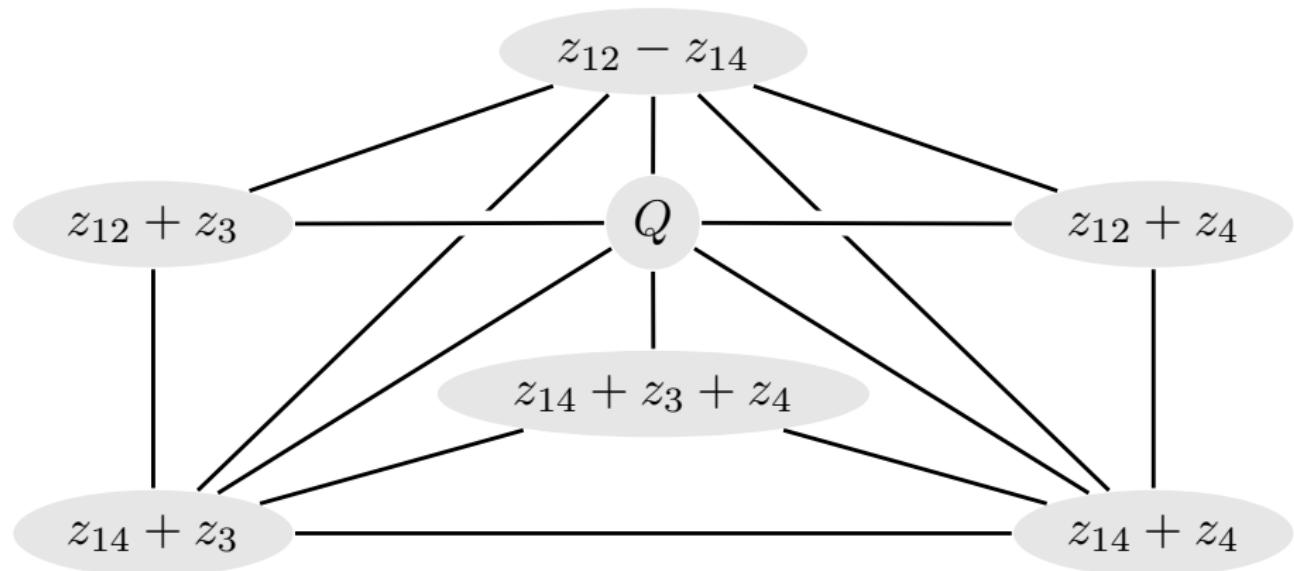


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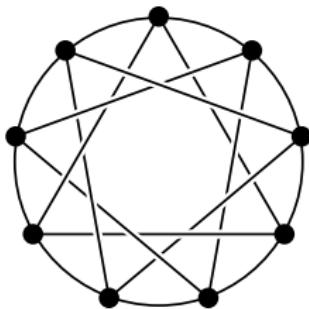
Example (\$D = 6\$ and \$a\_e = 1\$)

$$\begin{aligned} F\left(\begin{array}{ccccc} v_1 & & v_4 \\ \circ & \bullet & \circ \\ & \text{---} & \\ & & \\ v_2 & & v_3 \end{array}; z\right) &= \frac{1}{Q^2} \int_0^{z_{12}} F\left(\begin{array}{ccccc} v_1 & & v_4 \\ \circ & \bullet & \circ \\ & \text{---} & \\ & & \\ v_2 & & v_3 \end{array}; z_{12}-x, z_{14}, z_3, z_4\right) x dx \\ &= \frac{z_{12} - z_{14}}{Q^2} \left[ \ln \frac{Q}{z_3 z_4} \ln \frac{(z_{14} + z_3)(z_{14} + z_4)}{z_{14}(z_{14} + z_3 + z_4)} - \text{Li}_2\left(\frac{z_3 z_4 (z_{14} - z_{12})}{z_{14} Q}\right) \right] \\ &+ \frac{z_{12} - z_{14}}{Q^2} \text{Li}_2\left(\frac{z_3 z_4}{Q}\right) + \frac{z_{12}}{Q^2} \ln \frac{z_{14} z_3 z_4}{z_{12}(z_{14} + z_3)(z_{14} + z_4)} - \frac{\ln(z_3 z_4 / Q)}{Q(z_{14} + z_3 + z_4)} \end{aligned}$$

# Compatibility graph of box-ladders



You made it! The talk is over!



Thank you for your attention!

A Galois coaction on  $\phi^4$  periods?

## Theorem (Deligne)

For  $N \in \{1, 2, 3, 4, 6, 8\}$ , the algebra of motivic MPL at  $N$ th roots of unity is isomorphic to a freely generated shuffle algebra. Example:

$$MZV \cong \mathbb{Q}[\pi^2] \otimes \mathbb{Q}\langle f_3, f_5, f_7, \dots \rangle \quad MDV \cong \mathbb{Q}[i\pi] \otimes \mathbb{Q}\langle f_2, f_3, f_4, f_5, \dots \rangle$$

## Example

$$\zeta_{2n+1} \mapsto f_{2n+1} \quad \zeta_{3,5} \mapsto -5f_5f_3$$

**Message:** periods are not just numbers, but have a structure! Consider the map  $\delta_k$  which clips off the first letter:

$$\delta_k(f_{n_1} \dots f_{n_r}) := \begin{cases} f_{n_2} \dots f_{n_r} & \text{if } k = n_1 \\ 0 & \text{else} \end{cases}$$

## Example

$$\delta_3(\zeta_3) = 1 \quad \delta_3(\zeta_{3,5}) = 0 \quad \delta_5(\zeta_{3,5}) = -5\zeta_3 \quad \delta_k \zeta_{2n} = 0$$

## Coaction conjecture (O. Schnetz)

The periods of primitive log.-div.  $\phi^4$  graphs are closed under the action of the operators  $\delta_k$ .

Other words: The cosmic Galois group acts on  $\phi^4$  periods.

### Example

$$\begin{aligned} P_{7,11} = & -\frac{332262}{43} f_8 f_3 + \frac{54918}{55} f_6 f_5 + \frac{1134}{13} f_4 f_7 - \frac{1874502}{3485} f_2 f_9 \\ & + 5670 f_2 f_3 f_3 f_3 - \frac{3216912825399005402331281812377062149}{14080217073343074027422017273458000} \left(\frac{\pi}{\sqrt{3}}\right)^{11}. \end{aligned}$$

Note: After  $\delta_{2k}$ , only odd letters survive  $\Rightarrow$  MZV, in  $\phi^4$ .

- highly non-trivial constraint on  $\phi^4$  periods
- proven for generalized periods [F. Brown]