# EIGENVALUE BRACKETING FOR DISCRETE AND METRIC GRAPHS

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ABSTRACT. We develop eigenvalue estimates for the Laplacians on discrete and metric graphs using different types of boundary conditions at the vertices of the metric graph. Via an explicit correspondence of the equilateral metric and discrete graph spectrum (also in the "exceptional" values of the metric graph corresponding to the Dirichlet spectrum) we carry over these estimates from the metric graph Laplacian to the discrete case. We apply the results to covering graphs and present examples where the covering graph Laplacians have spectral gaps.

## 1. INTRODUCTION

Analysis on graphs is an area of current research in mathematics with many applications e.g. in network theory, nanotechnology, optics, chemistry and medicine. In this context one studies different kinds of linear operators, typically Laplacians, on a graphs. From the spectral properties of these operators one may infer relevant information of the corresponding model. For example, the tight binding model in physics describes atoms and molecules by a nearest neighbour model closely related to the discrete graph Laplacian. Moreover, network properties like connectivity can be described with spectral graph theory. In applications, the spectrum may encode transport properties of the medium. We will call an interval *disjoint* from the spectrum a *spectral gap*. In applications, a spectral gap may describe a set of wave-lengths for which no transport is permitted through the media.

There are basically two ways to give a "natural" definition of the Laplace operator on graphs: first, on discrete graphs, the operator acts on functions on the *vertices* as *difference* operator. Here, edges play a secondary role as labels that connect the vertices. Second, one can consider the graph as a (non-discrete) metric space consisting of vertices and edges as *one-dimensional* spaces. In this context one defines *differential* operators acting on functions on the edges. Laplacians are second order operators with suitable boundary conditions on the vertices chosen in such a way that the operator is self-adjoint in the corresponding $\mathsf{L}_2$-space. One usually refers to a metric graph together with a self-adjoint differential operator as a *quantum graph*. Recent interesting reviews on discrete geometric analysis and quantum graphs can be found in [Sun08] resp. [Kuc08] (see also references therein).

The aim of the present paper is to use spectral results for the metric graph to obtain spectral information of the discrete Laplacian. In particular, we will obtain results on the spectrum of infinite discrete covering graphs. This partially answers a question of Sunada concerning the spectrum of infinite discrete graphs [Sun07, p. 64]. In particular, we generalise the so-called *Neumann-Dirichlet bracketing* (see below) to the Laplacian acting on a metric graph, where the lower bound estimate of the Neumann eigenvalue is replaced by the Kirchhoff condition. Due to an explicit relation between the spectrum of the discrete and (equilateral) metric Laplacian, we can carry over the eigenvalue estimates to the discrete case. We also treat the exceptional eigenvalues in this relation (usually due to the Dirichlet spectrum of a single edge), and relate them with relative homology of the graph and its boundary. This gives a complete relation between the discrete and metric spectra (see Theorem A below).

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**The basic idea of the eigenvalue bracketing.** Our basic technique is to localise the eigenvalues within suitable closed intervals which we can control. We call this process *bracketing*. Since this technique is crucial for our analysis, we will briefly recall the main idea here.

Dirichlet-Neumann bracketing is a tool usually available for differential operators like Schrödinger operators or Laplacians on manifolds. The simplest example is provided by the operator $\Delta f = -f''$ on the interval $[0,1]$. In order to obtain a self-adjoint operator in $\mathsf{L}_2(0,1)$ one has to fix boundary conditions at 0 and 1. A very elegant way to provide such conditions is to define the Laplacian via an associated quadratic form

$$\mathfrak{h}(f) := \int_0^1 |f'(x)|^2 \,\mathrm{d}x, \quad f \in \operatorname{dom}\mathfrak{h} \qquad \text{related by} \quad \langle f, \Delta f\rangle = \mathfrak{h}(f).$$

The quadratic form domain is a closed subspace of the Sobolev space $\mathsf{H}^1(0,1)$. The two extremal cases are

(i) the *Dirichlet* boundary condition, $\operatorname{dom}\mathfrak{h}^{\mathrm{D}} := \{\, f \in \mathsf{H}^1(0,1) \mid f(0) = 0,\ f(1) = 0\,\}$.
(ii) the *Neumann* boundary condition, $\operatorname{dom}\mathfrak{h}^{\mathrm{N}} := \mathsf{H}^1(0,1)$.

Note that the usual Neumann conditions $f'(0) = 0$ and $f'(1) = 0$ only enter in the *operator* domain by requiring the boundary terms to vanish which appear after partial integration. For details, we refer to [RS80, Sec. VIII.6] and [RS78, Sec. XIII.15] or [D95]. Any other (linear) boundary condition, like e.g. the $\vartheta$-equivariant condition $f(1) = \mathrm{e}^{\mathrm{i}\vartheta} f(0)$ leads to a space $\operatorname{dom}\mathfrak{h}^{\vartheta}$ between $\operatorname{dom}\mathfrak{h}^{\mathrm{D}}$ and $\operatorname{dom}\mathfrak{h}^{\mathrm{N}}$ (the action of $\mathfrak{h}^{\vartheta}$ being the same, namely $\mathfrak{h}^{\vartheta}(f) = \|f'\|^2$). Floquet theory implies that the spectrum of the corresponding ($\mathbb{Z}$-periodic) Laplacian $\Delta_{\mathbb{R}}$ on $\mathbb{R}$ is given by $\{\, \lambda_k^{\vartheta} \mid k \in \mathbb{N}, \vartheta \in [0, 2\pi]\,\} = [0,\infty)$. The variational characterisation of the associated eigenvalues is given by

$$\lambda_k^{\bullet} = \inf_{D \subset \operatorname{dom}\mathfrak{h}^{\bullet}} \sup_{f \in D} \frac{\mathfrak{h}^{\bullet}(f)}{\|f\|^2} \tag{1.1}$$

where $D$ runs through all $k$-dimensional subspaces and the dot $^\bullet$ is a placeholder for the labels N, D, $\vartheta$. Extending the non-negative forms $\mathfrak{h}^{\bullet}$ naturally to the whole Hilbert space by $\mathfrak{h}^{\bullet}(f) := \infty$ if $f \notin \operatorname{dom}\mathfrak{h}^{\bullet}$, the extended forms become monotone in the obvious sense, i.e. $\mathfrak{h}^{\mathrm{N}}(f) \le \mathfrak{h}^{\vartheta}(f) \le \mathfrak{h}^{\mathrm{D}}(f)$ for all $f \in \mathsf{L}_2(0,1)$ (opposite to the inclusion of the domains). It follows now from Eq. (1.1) that

$$\lambda_k^{\mathrm{N}} \le \lambda_k^{\vartheta} \le \lambda^{\mathrm{D}},$$

to what we will refer to as *bracketing*. In this simple example the bracketing does not imply the existence of spectral gaps of $\Delta_{\mathbb{R}}$ inside $[0,\infty)$, since $\lambda_k^{\mathrm{N}} = \pi^2(k-1)^2$ and $\lambda_k^{\mathrm{D}} = \pi^2 k^2$, $k = 1, 2, \dots$, and therefore the intervals $I_k := [\lambda_k^{\mathrm{N}}, \lambda_k^{\mathrm{D}}]$ cover already $[0,\infty)$. Of course, we do not expect gaps here since $\sigma(\Delta_{\mathbb{R}}) = [0,\infty)$.

The strength of this bracketing method can be seen in Proposition 7.2 where we use the same idea for eigenvalues of equilateral metric graph Laplacians and *arbitrary* finite-dimensional unitary representations $\rho$. Proposition 7.2 may be seen as the core of our analysis. Its proof is amazingly simple, namely, it is a vector-valued generalisation of the above argument.

**Main results.** Let us briefly describe our main results: Denote by $\mathscr{N}(\lambda)$ the eigenspace of the standard (Kirchhoff) metric Laplacian, and by $\check{\mathscr{N}}(\mu)$ the eigenspace of the standard discrete Laplacian (for precise definitions, see Sections 2 and 3).

The following theorem resumes Propositions 4.1, 4.7 and 5.2, where the precise statements can be found. The first statement for equilateral graphs (i.e., metric graphs with constant length function, say, $\ell_e = 1$) is standard (see e.g. [vB85, E97, Ca97, Pa06, P07a, BGP08]) and only mentioned for completeness:

**Main Theorem A** (Propositions 4.1, 4.7 and 5.2)**.** *Let $X$ be a compact, connected and equilateral metric graph and set $\mu(\lambda) := 1 - \cos(\sqrt{\lambda})$.*

(i) *If* $\lambda \notin \{ n^2\pi^2 \mid n = 1, 2, \dots \}$, *then there is an isomorphism* $\Phi_\lambda \colon \check{\mathscr{N}}(\mu(\lambda)) \longrightarrow \mathscr{N}(\lambda)$. *The corresponding metric eigenfunctions are called* vertex based.

(ii) *If* $\lambda_n = n^2\pi^2$ *and* $n$ *even, then there is an injective homomorphism* $\Psi_n \colon H_1(X) \longrightarrow \mathscr{N}(\lambda_n)$, *where* $H_1(X)$ *is the first homology group. The range of* $\Psi_n$ *consists of functions vanishing on all vertices ("Dirichlet eigenfunctions"), called* edge-based *or* topological *eigenfunctions of the metric graph. The orthogonal complement of the range of* $\Psi_n$ *contains an additional eigenfunction* $\varphi_n$ *which is constant as function restricted to the set of vertices, called* trivial vertex based.

For shortness, we omit the case $n$ odd, in which a similar statement with $H_1(X)$ replaced by the "unoriented" homology group $\bar{H}_1(X)$ holds. In this case, one has to distinguish whether $G$ is bipartite or not. In the former case, the orthogonal complement of the range of $\Psi_n$ contains the additional eigenfunction $\varphi_n$ related to the discrete bipartite eigenfunction. In the latter case, $\Psi_n$ is already an isomorphism. Moreover, a similar result holds when we consider Laplacians with Dirichlet conditions on a subset $\partial V$ of the vertices. In this case, the *relative* homology group $H^1(X, \partial V)$ enters. The multiplicities of the eigenvalues were already calculated in [vB85] by a direct proof without using the homology groups. The advantage of using homology groups is that is can be generalised to other types of vertex boundary conditions (like Dirichlet and equivariant) in a natural way, see e.g. Remark 7.4).

Let now $X \to X_0$ be a covering of metric graphs (i.e., a covering respecting the combinatorial graph structure and the length function). For the next statement, the metric graph need not to be equilateral.

**Main Theorem B** (Theorem 8.5)**.** *Let* $X \to X_0$ *be a covering of metric graphs with compact quotient and residually finite covering group* $\Gamma$ *and denote by* $\Delta_X$ *the Kirchhoff Laplacian. Then*

$$\sigma(\Delta_X) \subset \bigcup_{k \in \mathbb{N}} I_k, \qquad I_k = \bigl[\lambda_k, \lambda_k^{\partial V}\bigr],$$

*where* $\lambda_k$ *and* $\lambda_k^{\partial V}$ *are the eigenvalues of the Kirchhoff and Kirchhoff-Dirichlet Laplacian on a fundamental domain* $Y \subset X$. *In particular, for any subset* $M \subset [0, \infty)$ *such that* $M \cap \bigcup_k I_k = \emptyset$, *then* $M \cap \sigma(\Delta_X) = \emptyset$.

Abelian groups, finite extensions of Abelian groups (so-called *type-I-groups*) and free groups are examples of the large class of residually finite groups (see [LP08] for more details). For Abelian groups, the Floquet-Bloch decomposition can be used in order to calculate the spectrum of the operator on the covering, leading to a detailed analysis in certain models, see e.g. [KP07] for hexagonal lattices (modeling carbon nano-structures).

We refer to the intervals $I_k = I_k(Y, \partial V)$ as *Kirchhoff-Dirichlet (KD) intervals.* Note that they depend usually on the fundamental domain. The Kirchhoff condition plays the role of the Neumann condition in the usual *Dirichlet-Neumann* bracketing. Note that the Kirchhoff condition is optimal in a sense made precise in Remark 9.6, namely that a symmetrised version of the KD intervals (explained below) give the exact spectrum of the corresponding (Abelian) covering Laplacian.

We call the set $M$ also a *spectral gap.* Note that we do not assume that the spectral gap is *maximal*, i.e., if we state that the spectrum has two disjoint gaps $(a_1, b_1)$ and $(a_2, b_2)$ with $b_1 \le a_2$ we do not make a statement about the existence of spectrum inside $[b_1, a_2]$. In certain situations (e.g. if $\Gamma$ is amenable), we can assure the existence of spectrum between the gaps, and therefore have a lower bound on the number of components of $\sigma(\Delta_X)$ in terms of the components of $\bigcup_k I_k$ (see Theorem 8.7).

For an equilateral metric graph, we can combine the last two theorems and obtain the following *discrete Kirchhoff-Dirichlet bracketing.* Let $G \to G_0$ be a covering of discrete graphs with fundamental domain $H$ (being a subgraph of $G$ with vertex set $V(H)$ and boundary $\partial V$):

**Main Theorem C** (Theorem 8.6)**.** *Assume that the covering group is residually finite, then*

$$\sigma(\check{\Delta}_G) \subset \bigcup_{k=1}^{|V(H)|} J_k, \qquad J_k = [\mu_k, \mu_k^{\partial V}],$$

*where $\mu_k$ and $\mu_k^{\partial V}$ are the eigenvalues of the discrete Laplacians on the fundamental domain $H$ with Dirichlet condition on $\partial V$ in the latter case. In particular, for any subset $M \subset [0, \infty)$ such that $M \cap \bigcup_k J_k = \emptyset$, then $M \cap \sigma(\check{\Delta}_X) = \emptyset$.*

We refer to the intervals $J_k = J_k(H, \partial V)$ as the *discrete Kirchhoff-Dirichlet intervals.* Note that this method allows to determine in a very easy way whether a set $M$ is not contained in the spectrum of the covering Laplacian. The only step to be done is to calculate the eigenvalues $\mu_k$ and $\mu_k^{\partial V}$ (which give immediately the corresponding metric eigenvalues for equilateral graphs) and check whether neighbouring KD intervals $J_k$ have empty intersection. We will see in Section 9, that in simple examples, only the first KD intervals do not overlap. As in the case of manifolds and Schrödinger operators (see e.g. [HP03, LP07, LP08]) we expect that the number of gaps should be large if the fundamental domain has "small" boundary $\partial V$ compared to the number of vertices $V(H)$ and edges $E(H)$ inside. In other words, a "high contrast" between the different copies of a suitable fundamental domain is necessary in order that our method works.

It is a priori not clear how the eigenvalue bracketing can be seen directly for discrete Laplacians, so our analysis may serve as an example of how to use metric graphs to obtain results for discrete graphs.

**Structure of the article.** The structure of the paper is as follows: in the following two sections we present the basic definitions and results for various Laplacians on discrete and metric graphs. Sections 4 and 5 contains the complete relation of discrete and equilateral metric graphs and in particular Main Theorem A. Details on the different homologies needed can be found in Section 5. Section 6 is devoted to the definition of the metric and discrete Kirchhoff-Dirichlet intervals and contains a careful analysis of the metric eigenvalues including multiplicities. Section 7 contains relevant information on equivariant Laplacians and the basic idea of decoupling an equivariant Laplacian via Dirichlet and Kirchhoff Laplacians (see Propositions 7.2 and 7.5). In Section 8 we combine the results on equilateral Laplacians and KD intervals in order to prove our Main Theorems B and C. The last section provides several examples of graphs with spectral gaps.

## 2. Discrete graphs

Let $G = (V, E, \partial)$ be a (connected) discrete graph, i.e., $V = V(G)$ is the set of vertices, $E = E(G)$ the set of edges and $\partial = \partial_G \colon E \longrightarrow V \times V$ the connection map, $\partial e = (\partial_- e, \partial_+ e)$ is the pair of the initial and terminal vertex, respectively. Clearly, $\partial_\pm e$ fixes an orientation of the edge $e$. We prefer to consider $E$ and $V$ as independent sets (and not the edge sets as pairs of vertices), in order to treat easily multiple edges (i.e., edges $e_1$, $e_2$ with $\{\partial_- e_1, \partial_+ e_1\} = \{\partial_- e_2, \partial_+ e_2\}$) and self-loops (i.e., edges with $\partial_- e = \partial_+ e$). For two subsets $A, B \subset V$ we denote by

$$E^+(A, B) := \{\, e \in E \,|\, \partial_- e \in A,\ \partial_+ e \in B \,\}$$

the set of edges with terminal vertex in $A$ and initial vertex in $B$, and similarly, $E^-(A, B) := E^+(B, A)$. Moreover we let $E(A, B) := E^+(A, B) \mathbin{\dot\cup} E^-(A, B)$ be the *disjoint* union of all edges between $A$ and $B$. Due to the disjoint union, a self-loop at a vertex $v \in A \cap B$ is counted twice in $E(A, B)$. In particular, $E(v, w)$ is the set of all edges between the vertices $v$ and $w$; and

$$E_v^\pm := E^\pm(V, v) = \{\, e \in E \,|\, \partial_\pm e = v \,\}$$

is the set of edges terminating (+) and starting (−) at $v$. Similarly, $E_v = E_v^+ \mathbin{\dot\cup} E_v^-$ is the set of all edges at $v$. We call

$$\deg v := |E_v|$$

the *degree* of the vertex $v$ in the graph $G$. Note that a self-loop at the vertex $v$ increases the degree by 2.

A graph is called *bipartite* if there is a disjoint decomposition $V = A \mathbin{\dot\cup} B$ such that $E = E(A, B)$, i.e., if each edge has exactly one end-point in $A$ and the other in $B$.

We will use frequently the following elementary fact about reordering a sum over edges and vertices, namely

$$\sum_{e \in E} F(\partial_\pm e, e) = \sum_{v \in V} \sum_{e \in E_v^\pm} F(v, e) \tag{2.1}$$

for a function $(v, e) \mapsto F(v, e)$ depending on $v$ and $e \in E_v$ with the convention that a sum over the empty set is 0. Note that this equation is also valid for self-loops and multiple edges. The reordering is a bijection since the union $E = \dot\bigcup_{v \in V} E_v^\pm$ is *disjoint.*

We start with a more general setting, namely with *weighted* graphs, i.e., we assume that there are two functions $m = m_V : V \longrightarrow (0, \infty)$ and $m = m_E : E \longrightarrow (0, \infty)$ (mostly denoted by the same symbol $m$) associating to a vertex $v$ its weight $m(v)$ and to an edge $e$ its weight $m_e$. We will call $(G, m)$ a *weighted* discrete graph. The basic Hilbert spaces associated with $(G, m)$ are

$$\begin{aligned} \ell_2(V, m) &:= \bigl\{ F : V \longrightarrow \mathbb{C} \bigm| \|F\|_{V,m}^2 := \sum_{v \in V} |F(v)|^2 m(v) < \infty \bigr\}, \\ \ell_2(E, m) &:= \bigl\{ \eta : V \longrightarrow \mathbb{C} \bigm| \|f\|_{V,m}^2 := \sum_{e \in E} |\eta_e|^2 m_e < \infty \bigr\}. \end{aligned}$$

We define the *discrete exterior derivative* $d$ as

$$d : \ell_2(V, m) \longrightarrow \ell_2(E, m), \qquad (dF)_e = F(\partial_+ e) - F(\partial_- e).$$

We define the *relative* weight $\rho : V \longrightarrow (0, \infty)$ as

$$\rho(v) := \frac{1}{m(v)} \sum_{e \in E_v} m_e \tag{2.2}$$

and we will assume throughout this article that

$$\rho_\infty := \sup_{v \in V} \rho(v) < \infty, \tag{2.3}$$

i.e., that the relative weight is uniformly bounded. We will call the weights *normalised* if $\rho(v) = 1$ for all vertices. A straightforward calculation using (2.1) shows that $d$ is an operator with norm bounded by $(2\rho_\infty)^{1/2}$. Similarly, one can calculate the adjoint $d^* : \ell_2(E, m) \longrightarrow \ell_2(V, m)$ and one gets

$$(d^* F)(v) = \frac{1}{m(v)} \sum_{e \in E_v} m_e \widehat{\eta}_e(v),$$

where

$$\widehat{\eta}_e(v) = \eta_e \quad \text{if } v = \partial_+ e \qquad \text{and} \qquad \widehat{\eta}_e(v) = -\eta_e \quad \text{if } v = \partial_- e. \tag{2.4}$$

The *discrete Laplacian* is now defined as

$$\check\Delta = \check\Delta_{(G,m)} := d^* d : \ell_2(V, m) \longrightarrow \ell_2(V, m) \tag{2.5}$$

and acts as

$$(\check\Delta_{(G,m)} F)(v) = \rho(v) F(v) - \frac{1}{m(v)} \sum_{e \in E_v} m_e F(v_e), \tag{2.6}$$

where $v_e$ denotes the vertex on the edge $e \in E_v$ opposite to $v$. If no confusion arises we also denote the Laplacian simply by $\check\Delta$. The *standard discrete Laplacian* is the Laplacian associated with the weights $m(v) = \deg v$ and $m_e = 1$. We will often refer to the standard weighted graph as $(G, \deg)$ or simply as $G$. Note that these weights are normalised, i.e., that $\rho(v) = 1$.

*Remark* 2.1. Note that as second order difference operator, the Laplacian does not see the orientation of the graph, whereas the discrete exterior derivative as first order operator depend on the orientation. We will define below an *unoriented* version of the exterior derivative $\bar{d}$ that does not see the orientation. The corresponding (co-)homologies for $d$ and $\bar{d}$ will be useful in order to analyse exceptional metric graph eigenfunctions composed of antisymmetric and symmetric Dirichlet eigenfunctions on a single edge (see Section 5).

A graph without multiple edges (i.e., $|E(v,w)| \le 1$ for all $v, w \in E$ is called *simple*. In particular, $\partial$ is injective and we can consider $E$ as a subset of $V \times V$. In this case, we also write $v \sim w$ if $v$ and $w$ are connected by an edge.

One reason for considering graphs with arbitrary weights is the fact that one can express the standard Laplacian on a graph with multiple edges and self-loops equivalently by a Laplacian on a simple graph by changing the weights. We will use multiple edges and self-loops in Examples 9.3–9.4 in order to generate gaps. Note that the corresponding discrete exterior derivatives will of course differ, as well as the topology of the graph. Nevertheless, the reduction to simple graphs is more convenient when calculating the spectrum of the Laplacian.

2.1. **Multiple edges.** Assume that $G$ is a graph with the standard weights $m(v) = \deg v$, $m_e = 1$ and that $G$ has multiple edges. We can pass to a graph $\widetilde{G}$ having the same set of vertices as $G$ but only simple edges. The multiple edges $e \in E(v,w)$ in $G$ are replaced by a single edge $(v,w)$ (not taking care about the original orientation) in $\widetilde{G}$. Note that for the degree $\deg_{\widetilde{G}} v \le \deg_G v$ where $\deg_{\widetilde{G}} v$ denotes the degree of $v$ in the simple graph $\widetilde{G}$. We define

$$\widetilde{m}(v) := \deg_G v \qquad \text{and} \qquad \widetilde{m}_{(v,w)} := |E(v,w)|,$$

where $\deg_G v$ is the degree in the original graph. Now, the relative weight $\widetilde{\rho}$ is still normalised, since

$$\widetilde{\rho}(v) = \frac{1}{\widetilde{m}(v)} \sum_{w \sim v} \widetilde{m}_{(v,w)} = \frac{1}{\deg_G v} \sum_{w \sim v} |E(v,w)| = 1.$$

Note that the Laplacians on $(\widetilde{G}, \widetilde{m})$ and $(G, \deg)$ agree.

2.2. **Self-loops.** Assume that $G$ is a graph with a self-loop $e$, i.e., $\partial_+ e = \partial_- e = v$. Obviously, for such an edge, we have $(dF)_e = 0$, i.e., we can eliminate this edge from $E$. We define a new graph $\widetilde{G}$ having again the same vertex set as $G$ and where the edge set $\widetilde{E}$ is the original edge set without self-loops. The degree in the new graph is given by $\deg_{\widetilde{G}} v = \deg_G v - |E(v,v)|$, i.e., the original degree minus *twice* the number of self-loops removed (remember that $E(v,v)$ was defined as the formal disjoint union of $E^+(v,v)$ and $E^-(v,v)$). We set

$$\widetilde{m}(v) := \deg_G v \qquad \text{and} \qquad \widetilde{m}_e = 1,$$

so that the relative weight $\widetilde{\rho}$ satisfies

$$\widetilde{\rho}(v) = \frac{1}{\widetilde{m}(v)} \sum_{e \in \widetilde{E}_v} 1 = \frac{\deg_G v - |E(v,v)|}{\deg_G v} < 1$$

provided there was a self-loop at $v$. Again, the corresponding Laplacians on $(G, \deg)$ and $(\widetilde{G}, \widetilde{m})$ agree.

2.3. **Matrix representation of the Laplacian.** For concrete computations of the eigenvalues of the weighted Laplacian, it is convenient to have the associated matrix at hand. Let $\{\varphi_v\}_v$ be the standard orthonormal basis of $\ell_2(V,m)$, where $\varphi_v(w) := m(v)^{-1/2}$ if $v = w$ and $\varphi_v(w) = 0$

otherwise. Then the matrix $L$ associated to the Laplacian $\check{\Delta} = \check{\Delta}_{(G,m)}$ is given as

$$\check{\Delta}_{v,w} := \langle \varphi_v, \check{\Delta}\varphi_w \rangle = \begin{cases} \rho(v) - \dfrac{1}{m(v)} \sum_{e \in E(v,v)} m_e & \text{if } v = w \\ -\dfrac{1}{(m(v)m(w))^{1/2}} \sum_{e \in E(v,w)} m_e & \text{if } v \sim w,\ v \neq w, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

If the graph has the standard weights, then we obtain

$$\check{\Delta}_{v,w} := \langle \varphi_v, \check{\Delta}\varphi_w \rangle = \begin{cases} \dfrac{\deg(v) - |E(v,v)|}{\deg v} & \text{if } v = w \\ -\dfrac{|E(v,w)|}{(\deg v \deg w)^{1/2}} & \text{if } v \sim w,\ v \neq w, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Note that the latter expression also applies for graphs with multiple edges and loops, inserting as degree function the degree of the original (non-simple) graph.

2.4. **Discrete Dirichlet Laplacians.** A *boundary* of $G$ is a subset $\partial V$ of $V$. We denote by $\mathring{V} := V \setminus \partial V$ its complement, the *inner* vertices. We set

$$\ell_2^{\partial V}(V, m) := \{\, F \in \ell_2(V, m) \,|\, F\restriction_{\partial V} = 0 \,\}$$

and define the *Dirichlet* discrete exterior derivative $d_0$ as the restriction of $d$ to $\ell_2^{\partial V}(V, m)$. Formally, we can write $d_0 := d \circ \iota$, where $\iota$ is the canonical embedding of $\ell_2^{\partial V}(V, m)$ into $\ell_2(V, m)$. The adjoint of $d_0$ is $d_0^* = \iota^* \circ d^*$, i.e.,

$$d_0^* \eta = (d^* \eta)\restriction_{\mathring{V}},$$

since $\iota^* F$ is the restriction of $F$ onto the inner vertices $\mathring{V}$.

The *discrete Dirichlet Laplacian* is defined as

$$\check{\Delta}^{\partial V} = \check{\Delta}^{\partial V}_{(G,m)} := d_0^* d_0$$

and acts as in (2.6), but only for $v \in \mathring{V}$.

*Remark* 2.2. One can give an equivalent definition of the Dirichlet Laplacian as a discrete Laplacian on the graph $\mathring{G}$ with vertex set $\mathring{V}$ and edge set $\mathring{E} := E \setminus E(V, \partial V)$ (removing the edges to the boundary or inside the boundary). Again, this leads to a weighted Laplacian: If for instance, $\check{\Delta}_G^{\partial V}$ is the Dirichlet Laplacian with standard weights, we define

$$\mathring{m}(v) := \deg_G v \qquad \text{and} \qquad \mathring{m}_e := 1$$

having again a non-normalised relative weight $\rho(v) < 1$ provided $v$ is joined with $\partial V$ by an edge in the original graph $G$.

2.5. **Bipartiteness and the spectrum.** Let us recall the following spectral characterisation of bipartiteness of a graph:

**Proposition 2.3.** *Let $(G, m)$ be a weighted, connected graph with normalised weights (i.e., $\rho = 1$). Assume in addition that $G$ has finite mass $m(V) = \sum_{v \in V} m(v) < \infty$ (e.g. that $G$ is finite). Then the following assertions are equivalent:*

(i) *The graph $G$ is bipartite*
(ii) *If $\mu \in \sigma(\check{\Delta}_{(G,m)})$ then $2 - \mu \in \sigma(\check{\Delta}_{(G,m)})$. For short, we write $\sigma(\check{\Delta}_{(G,m)}) = 2 - \sigma(\check{\Delta}_{(G,m)})$. The multiplicity is preserved.*
(iii) *2 is an eigenvalue of $\check{\Delta}_{(G,m)}$.*

*Moreover, the implication* (i) ⇒ (ii) *holds for the discrete* Dirichlet *Laplacian* $\check{\Delta}^{\partial V}_{(G,m)}$. *Similarly, if* $m(V)$ *is infinite, then the implication* (i) ⇒ (ii) *is still valid.*

*Proof.* The proof of the equivalence for $\partial V = \emptyset$ and finite graphs can be found e.g. in [Ch97]; the case $\partial V \neq \emptyset$ follows similarly. If $G$ has finite mass, then the constant function $\mathbb{1}_V$ is in $\ell_2(V, m)$, and the argument for finite graphs carries over.

If $m(V)$ is infinite, then the spectral symmetry follows from the fact that

$$\check{\Delta} T = T(2 - \check{\Delta}), \qquad T := \mathbb{1}_A - \mathbb{1}_B \tag{2.9}$$

where $V = A \mathbin{\dot\cup} B$ is the bipartite partition. Here, $T$ is a unitary involution (i.e., $T = T^* = T^{-1} = T^2$) on $\ell_2(V, m)$. □

Note that in the finite mass case, $T$ interchanges the constant eigenfunction and the eigenfunction $\mathbb{1}_A - \mathbb{1}_B$ associated to the eigenvalue 2, also called the *bipartite eigenfunction*. Moreover, the condition (2.9) is equivalent to the fact that $T$ *anticommutes* with the *principal part* of the Laplacian $L := \mathrm{id} - \check{\Delta}$, i.e., that $\{L, T\} = LT + TL = 0$.

2.6. **Unoriented exterior derivatives.** We briefly describe another sort of discrete exterior derivative, this time an operator which does not see the orientation of the graph. More precisely, we define the *unoriented discrete exterior derivative* as

$$\bar{d} \colon \ell_2(V, m) \longrightarrow \ell_2(E, m), \qquad (dF)_e = F(\partial_+ e) + F(\partial_- e),$$

i.e., compared with the (oriented) version $d$, we only change the sign of the value of $F$ at the initial vertex. As a consequence, the corresponding adjoint is given by

$$(\bar{d}^* F)(v) = \frac{1}{m(v)} \sum_{e \in E_v} m_e \eta_e.$$

One can also define a Laplacian associated via $\bar{\check{\Delta}} := \bar{d}^* \bar{d}$, and the relation with the Laplacian $\check{\Delta} = d^* d$ is given by

$$\bar{\check{\Delta}} = 2\rho - \check{\Delta}, \tag{2.10}$$

where $\rho$ denotes the multiplication operator with the relative weight. We will need the operators $\bar{d}$ and $\bar{d}^*$ in Section 5. For more details and a general concept, in which the oriented and unoriented version of an exterior derivative embed naturally, we refer to [P07b] (see also [P07a, P07c]).

As for the oriented exterior derivative, we can also define a Dirichlet version of $\bar{d}$, namely,

$$\bar{d}_0 \colon \ell_2^{\partial V}(V, m) \longrightarrow \ell_2(E, m), \qquad \bar{d}_0 := \bar{d} \circ \iota.$$

As before, we have $\bar{d}_0^* \eta = (\bar{d}^* \eta) \restriction_{\mathring{V}}$ for the adjoint.

## 3. Metric graphs

Let $G = (V, E, \partial)$ be a discrete graph. A *topological graph* associated to $G$ is a CW complex $X$ containing only 0-cells and 1-cells. The 0-cells are the vertices $V$ and the 1-cells are labelled by the edge set $E$.

A *length function* $\ell \colon E \longrightarrow (0, \infty)$ of a graph $G$ is the inverse of an edge weight function $m$, i.e., $\ell_e = 1/m_e$. We will assume that the edge weight is bounded, i.e., that there exists $\ell_0 > 0$ such that

$$\ell_e \geq \ell_0, \qquad \forall\, e \in E. \tag{3.1}$$

The *metric graph* $X$ associated to a weighted discrete graph $(G, m)$ is a topological graph associated to $(V, E, \partial)$ such that for every edge $e \in E$ there is a continuous map $\Phi_e \colon \overline{I}_e \longrightarrow X$, $I_e := (0, \ell_e)$, whose image is the 1-cell corresponding to $e$, and the restriction $\Phi_e \colon I_e \longrightarrow \Phi(I_e) \subset X$ is a homeomorphism. The maps $\Phi_e$ induce a metric on $X$. In this way, $X$ becomes a metric space.

Given a weighted discrete graph, we can abstractly construct the associated metric graph as the disjoint union of the intervals $I_e$ for all $e \in E$ and together with a natural identification $\sim$ of the end-points of these intervals (according to the combinatorial structure of the graph), i.e.,

$$X = \dot{\bigcup_{e \in E}} \overline{I}_e / {\sim}. \tag{3.2}$$

We denote the union of the 0-cells and the (disjoint) union of the (open) 1-cells (edges) by $X^0$ and $X^1$, respectively, i.e.,

$$X^0 = V \hookrightarrow X, \qquad X^1 = \dot{\bigcup_{e \in E}} I_e \hookrightarrow X,$$

and both subspaces are canonically embedded in $X$.

The metric graph $X$ becomes canonically a *metric measure space* by defining the distance of two points to be the length of the shortest path in $X$, joining these points. We can think of the maps $\Phi_e \colon I_e \longrightarrow X$ as coordinate maps and the Lebesgue measures on the intervals $I_e$ induce a (Lebesgue) measure on the space $X$.

Since a metric graph is a topological space, and isometric to intervals outside the vertices, we can introduce the notion of measurability and differentiate function on the edges. We start with the basic Hilbert space

$$\mathsf{L}_2(X) := \bigoplus_{e \in E} \mathsf{L}_2(I_e), \qquad f = \{f_e\}_e \quad \text{with } f_e \in \mathsf{L}_2(I_e) \text{ and}$$
$$\|f\|^2 = \|f\|^2_{\mathsf{L}_2(X)} := \sum_{e \in E} \int_{I_e} |f_e(x)|^2 \, \mathrm{d}x.$$

We define several types of Sobolev spaces on $X$. The *maximal* Sobolev space of order $k$ is given by

$$\mathsf{H}^k_{\max}(X) := \bigoplus_{e \in E} \mathsf{H}^k(I_e)$$

together with its natural norm. The *standard* or *continuous* Sobolev space is given by

$$\mathsf{H}^1(X) := \mathsf{C}(X) \cap \mathsf{H}^1_{\max}(X).$$

It can be shown that $\mathsf{H}^1(X)$ is indeed a Hilbert space as closed subspace of the maximal Sobolev space using the length condition (3.1) (see e.g. [P07b, Lem. 5.2]). For a graph with boundary $\partial V$, we define

$$\mathsf{H}^1_{\partial V}(X) := \{ f \in \mathsf{H}^1(X) \mid f{\restriction}_{\partial V} = 0 \}$$

satisfying Dirichlet boundary conditions on $\partial V$. Again, $\mathsf{H}^1_{\partial V}(X)$ is closed in $\mathsf{H}^1_{\max}(X)$. Note that $\mathsf{H}^1_V(X) = \bigoplus_{e \in E} \mathring{\mathsf{H}}^1(I_e)$ is the *minimal* Sobolev space of order 1. We have the following inclusion of Sobolev spaces

$$\mathsf{H}^1_V(X) \subset \mathsf{H}^1_{\partial V}(X) \subset \mathsf{H}^1(X) \subset \mathsf{H}^1_{\max}(X). \tag{3.3}$$

We define quadratic forms in $\mathsf{L}_2(X)$ with domains

$$\operatorname{dom} \mathfrak{h}^{\partial V} := \mathsf{H}^1_{\partial V}(X), \qquad \operatorname{dom} \mathfrak{h} := \mathsf{H}^1(X) \quad \text{and} \quad \operatorname{dom} \mathfrak{h}^{\mathrm{N}} := \mathsf{H}^1_{\max}(X)$$

acting as $\mathfrak{h}^\bullet(f) = \|f'\|^2 = \sum_{e \in E} \int_{I_e} |f'_e|^2 \, \mathrm{d}x$ in all cases. Denote by $\Delta^{\partial V}_X$, $\Delta_X$ and $\Delta^{\mathrm{N}}_X$ the corresponding Laplacians, called *Dirichlet(-Kirchhoff)*, *Kirchhoff* and *fully decoupled Neumann* Laplacian. Note that $\Delta^{\emptyset}_X = \Delta_X$ and

$$\Delta^V_X = \bigoplus_{e \in E} \Delta^{\mathrm{D}}_{I_e} \quad \text{and} \quad \Delta^{\mathrm{N}}_X = \bigoplus_{e \in E} \Delta^{\mathrm{N}}_{I_e}$$

are *decoupled*, justifying the names *fully decoupled Dirichlet* resp. *Neumann Laplacian.*

A function $f$ is in the domain of the Dirichlet(-Kirchhoff) Laplacian $\Delta_X^{\partial V}$ if and only if $f \in \mathsf{H}^2_{\max}(X)$ and

$$f(v) = 0 \quad \forall\, v \in \partial V \tag{3.4a}$$

$$f \text{ is continuous at each vertex } v \in \mathring{V} = V \setminus \partial V \tag{3.4b}$$

$$\sum_{e \in E_v} f_e'(v) = 0 \quad \forall\, v \in \mathring{V} \tag{3.4c}$$

where $f_e'(v) = -f_e'(0)$ if $v = \partial_- e$ and $f_e'(v) = f_e'(\ell_e)$ denotes the *inwards* derivative of $f$ at the vertex $v$ along the edge $e$.

If $X$ is a compact metric graph, the spectrum of all these operators is purely discrete. We denote the eigenvalues by $\lambda_k^{\partial V}$, $\lambda_k$ and $\lambda_k^{\mathrm{N}}$, $k = 1, 2, \dots$, respectively. It is written in increasing order and respecting multiplicity. Using the variational characterisation of the eigenvalues, the *min-max principle* (1.1) (see e.g. [D95]), we obtain from the quadratic form inclusions (3.3) the reverse inequality for the corresponding eigenvalues, namely

$$\lambda_k^V \geq \lambda_k^{\partial V} \geq \lambda_k \geq \lambda_k^{\mathrm{N}}.$$

For an equilateral metric graph we obtain:

**Lemma 3.1.** *Assume that the metric graph $X$ is compact and all lengths $\ell_e$ are equal to 1, then*

$$(n+1)^2\pi^2 = \lambda_k^V \geq \lambda_k^{\partial V} \geq \lambda_k \geq \lambda_k^{\mathrm{N}} = n^2\pi^2$$

*for $k = 1 + n|E|, \dots, (n+1)|E|$, $n = 0, 1, \dots$ In particular, the eigenvalues of the Dirichlet resp. Kirchhoff Laplacian on $X$ group into sets of cardinality $|E|$ (respecting multiplicity) lying inside the intervals $K_n := [n^2\pi^2, (n+1)^2\pi^2]$.*

## 4. Spectral relation between discrete and equilateral metric graphs

In this section, we give a complete description of the spectrum of the standard discrete Laplacian and the Kirchhoff Laplacian (and the corresponding Dirichlet versions on the boundary). Outside the fully decoupled Dirichlet spectrum $\Sigma^{\mathrm{D}} := \{\, n^2\pi^2 \mid n = 1, 2, \dots \}$, the relation is well-known, and there exist more general results relating different spectral components also in the case of infinite graphs (see e.g. [vB85, E97, Ca97, Pa06, P07a, BGP08] and the references therein).

Throughout this section, $G$ will denote a finite weighted graph with standard weight $m(v) = \deg v$ and $m_e = 1$. Moreover, $X$ will be the associated compact metric graph with lengths $\ell_e = 1$. We will refer to such metric graphs also as *equilateral*. To avoid unnecessary exceptional cases, we assume that the graph is connected. Some results hold also for non-compact graphs, see Remark 5.4.

Denote by $\Delta_X^{\partial V}$ the metric graph Laplacian with Dirichlet boundary conditions on $\partial V$ and Kirchhoff conditions on $\mathring{V}$. Similarly, let $\check{\Delta}_G^{\partial V}$ be the discrete Dirichlet Laplacian associated to the underlying discrete graph $(G, \deg)$ with standard weights. We denote by

$$\check{\mathscr{N}}^{\partial V}(\eta) := \ker(\check{\Delta}_G^{\partial V} - \eta) \qquad \text{and} \qquad \mathscr{N}^{\partial V}(\lambda) := \ker(\Delta_X^{\partial V} - \lambda)$$

the corresponding eigenspaces.

**Proposition 4.1.** *Assume that the metric graph $X$ is compact and equilateral and set $\mu(\lambda) := 1 - \cos\sqrt{\lambda}$. Suppose in addition that $\lambda \notin \Sigma^{\mathrm{D}}$, i.e., $\mu(\lambda) \notin \{0, 2\}$. Then the map*

$$\Phi_\lambda \colon \check{\mathscr{N}}^{\partial V}(\mu(\lambda)) \longrightarrow \mathscr{N}^{\partial V}(\lambda), \qquad F \mapsto f = \Phi_\lambda F$$

*is an isomorphism where*

$$f_e(x) = F(\partial_- e)\frac{\sin\sqrt{\lambda}(1-x)}{\sin\sqrt{\lambda}} + F(\partial_+ e)\frac{\sin\sqrt{\lambda}x}{\sin\sqrt{\lambda}}, \qquad \lambda > 0,$$

*and* $f_e(x) = F(\partial_- e)(1-x) + F(\partial_+ e)x$ *for* $\lambda = 0$*. In particular,*

$$\lambda \in \sigma(\Delta_X^{\partial V}) \qquad \text{if and only if} \qquad \mu(\lambda) \in \sigma(\check{\Delta}_G^{\partial V})$$

*(preserving the multiplicities of the eigenvalues).*

The proof is straightforward. Note that it is the Kirchhoff boundary condition leading to the discrete Laplacian expression and vice versa. The continuity condition and the eigenvalue equation on the metric graph are automatically fulfilled by this Ansatz.

**Definition 4.2.** We refer to the eigenfunctions $f = \Phi_\lambda F$ on the metric graph as *(non-trivial) vertex-based* eigenfunctions, since they are completely determined by their values on the vertices and interpolated on the edges according to the solution of the differential equation.

4.1. **Spectral relation at the Dirichlet spectrum.** The aim of the present subsection is to give a complete analysis of the spectrum of $\Delta_X^{\partial V}$ at the exceptional values $\lambda_n = n^2\pi^2 \in \Sigma^{\mathrm{D}}$. The multiplicity of these eigenvalues was already calculated in [vB85] by a direct proof not using the homology groups introduced in the next section.

We will show in the next lemma that there are two types of eigenfunctions: the first type, vanishing at each vertex, is related with the (relative) homology of the graph; the second type does not vanish at any vertex and is related to the spectral points 0 and 2 of the discrete graph.

**Lemma 4.3.** *Assume that* $X$ *is a connected compact equilateral metric graph and that* $f \in \mathscr{N}^{\partial V}(\lambda_n)$*. Then*

(i) *either* $f(v) = 0$ *for all vertices* $v \in V$*,*
(ii) *or* $f(v) \neq 0$ *for all vertices* $v \in V$*. This case can only occur if there are no Dirichlet boundary conditions, i.e.,* $\partial V = \emptyset$*.*

*In the first case we have*

$$f_e(x) = \frac{f_e'(0)}{n\pi}\sin(n\pi x),$$

*and in the latter case,* $f$ *is constant in all vertices if* $n$ *is even, or* $f(\partial_+ e) = -f(\partial_- e)$ *if* $n$ *is odd and* $G$ *is bipartite.*

*Proof.* Since $-f_e'' = \lambda_n f_e$ on each edge, we must have

$$f_e(x) = \alpha_e \cos(n\pi x) + \eta_e \sin(n\pi x). \tag{4.1}$$

In particular, we have at a vertex $v = \partial_- e$ that $f(v) = f_e(0) = \alpha_e$ and $f(v_e) = f_e(1) = \alpha_e(-1)^n$ and similarly if $v = \partial_+ e$.

If $f(v) = 0$ for a vertex $v$ then $\alpha_e = 0$ hence also $f(v_e) = 0$. By the connectedness of the graph the first claim follows.

If $f(v) \neq 0$, then $\alpha_e \neq 0$ and therefore $f(v_e) = (-1)^n f(v)$. If $n$ is even, the second claim follows. The existence of a non-trivial function with alternating sign ($n$ odd) is an eigenfunction of the standard discrete Laplacian with eigenvalue 2 and therefore equivalent to the fact that the graph is bipartite (see Proposition 2.3). □

The previous lemma motivates the following definition:

**Definition 4.4.** For the exceptional value $\lambda_n := n^2\pi^2 \in \Sigma^{\mathrm{D}}$ $(n \geq 1)$ we denote by

$$\mathscr{N}_0^{\partial V}(\lambda_n) := \{\, f \in \mathscr{N}^{\partial V}(\lambda_n) \,|\, f(v) = 0 \quad \forall\, v \in V \,\}$$

the space of eigenfunctions vanishing at *all* vertices. We call these eigenfunctions *topological* or *edge-based.*

The name "topological" will be justified in Section 5, where we relate this space with certain first homology groups. Note that these eigenfunction still satisfy the Kirchhoff condition in the *inner* vertices which will give the relation with homology (see especially Proposition 5.2).

Let us state the following simple observation for general eigenfunctions associated to $\lambda_n$:

**Lemma 4.5.** *Assume that $f$ is written in the general form* (4.1). *Then $f$ fulfills the Kirchhoff conditions in all inner vertices $v \in \mathring{V}$ iff $\eta = \{\eta_e\}_e \in \ker d_0^*$ if $n$ is even resp. $\eta \in \ker \bar{d}_0^*$ if $n$ is odd.*

*Proof.* From the form of $f$ on each edge, it follows $f_e'(0) = n\pi\eta_e$ and $f_e'(1) = (-1)^n n\pi\eta_e$, i.e., the inwards derivative is given by $f_e'(v) = n\pi\widehat{\eta}_e(v)$ if $n$ is even and $f_e'(v) = -n\pi\eta_e$ if $n$ is odd (recall that $f_e'(v)$ denotes the inward derivative, see Section 3). Now the Kirchhoff condition at $v \in \mathring{V}$ is equivalent to $d_0^*\eta(v) = 0$ resp. $\bar{d}_0^*\eta(v) = 0$, since

$$\sum_{e \in E_v} f_e'(v) = n\pi \sum_{e \in E_v} \widehat{\eta}_e(v) \qquad \text{and} \qquad \sum_{e \in E_v} f_e'(v) = -n\pi \sum_{e \in E_v} \eta_e$$

if $n$ is even or $n$ is odd, respectively. □

**Definition 4.6.** Assume that the graph $X$ is connected. In case that $n$ is odd, we assume furthermore that the graph is bipartite with corresponding partition $V = A \,\dot\cup\, B$, and that the graph is oriented such that $E = E^+(A, B)$, i.e., all edges start in $A$ and end in $B$. If $n$ is even, we do not need such an assumption.

We call the function $\varphi_n = \{\varphi_{n,e}\}_e$ defined on each edge as

$$\varphi_{n,e}(x) = \cos(n\pi x)$$

the eigenfunction *corresponding to the constant eigenfunction* if $n$ is even and *corresponding to the bipartite eigenfunction* if $n$ is odd. In both cases, we refer to $\varphi_n$ as the *trivial vertex-based eigenfunction.*

The above names have the following justification: $\varphi_n$ obviously fulfills the eigenvalue equation for $\lambda_n$. Moreover, it fulfills the Kirchhoff condition, since $\varphi_{n,e}'(v) = 0$ for all $e \in E_v$. If $n$ is even, then $\varphi_{n,e}(0) = \varphi_{n,e}(1) = 1$, i.e., $\varphi$ restricted to the vertices is the *discrete constant* eigenfunction. If $n$ is odd, then the above defined function $\varphi$ is continuous at each vertex, namely, $\varphi_{n,e}(v)$ is independent of $e \in E_v$. Moreover, $\varphi_{n,e}(v) = \varphi_{n,e}(0) = 1$ if $v \in A$ and $\varphi_{n,e}(v) = \varphi_{n,e}(1) = -1$ if $v \in B$. In particular, $F(v) := \varphi_n(v)$ is the discrete bipartite eigenfunction. Note that $F$ can be properly defined only in the bipartite case. Again, the eigenfunction $\varphi$ arises from a discrete eigenfunction, and is interpolated on the edges, justifying the name "vertex-based" (cf. Definition 4.2).

We can express Lemma 4.3 in terms of spaces:

**Proposition 4.7.** *Assume that $X$ is a connected compact equilateral metric graph.*

(i) *If $\partial V \neq \emptyset$, then $\mathscr{N}^{\partial V}(\lambda_n) = \mathscr{N}_0^{\partial V}(\lambda_n)$.*

(ii) *If $\partial V = \emptyset$, then*

$$\mathscr{N}^{\partial V}(\lambda_n) = \begin{cases} \mathscr{N}_0(\lambda_n) & n \text{ odd and } G \text{ not bipartite,} \\ \mathscr{N}_0(\lambda_n) \oplus \mathbb{C}\varphi_n & \text{otherwise,} \end{cases}$$

*where $\varphi_n$ is defined in the previous definition.*

*Proof.* If $\partial V \neq \emptyset$ or if $\partial V = \emptyset$, $n$ is odd and $G$ is not bipartite, then Lemma 4.3 implies that $\mathscr{N}^{\partial V}(\lambda_n) = \mathscr{N}_0^{\partial V}(\lambda_n)$. This covers case (i) and the first part of (ii). In any other case there is, in addition to the space $\mathscr{N}_0^{\partial V}(\lambda_n)$, a trivial vertex-based eigenfunction $\varphi_n$. By the explicit characterisation of the elements in $\mathscr{N}_0^{\partial V}(\lambda_n)$ (cf. Definition 4.6 and Lemma 4.3) it is immediate that $\varphi_n$ is orthogonal to any function in $\mathscr{N}_0^{\partial V}(\lambda_n)$. This shows the first part in case (ii) and the proof is concluded. □

## 5. Homology on graphs

In order to understand the topological content of the eigenspace $\mathscr{N}_0^{\partial V}(\lambda_n)$, we introduce the concept of (relative) homology for both, the oriented exterior derivative $d$ as well as for the unoriented version $\bar{d}$. The main reason why we need both is the fact, that in the case of even $n$, the function $x \mapsto \sin(n\pi x)$ on an edge is *antisymmetric* (with respect to the middle point of $(0,1)$ and therefore encodes the orientation of the edge. For odd $n$, the function is *symmetric*, and the orientation of an edge is irrelevant. This material, in particular the computation of the corresponding Betti numbers, will be crucial for the eigenvalue bracketing in the next section and the relation between metric and discrete eigenvalues.

Let $X$ be the topological graph associated to the *finite* graph $G$, and set $X^0 = V$, $X^1 = X \setminus X^0$. Then $X^1$ contains $|E|$-many components homeomorphic to $(0,1)$ and labelled by $e \in E$. Let $C_p(X)$ be the group of $p$-chains with complex coefficients, i.e., the vector space of formal sums

$$C_0(X) = \sum_{v \in V} \mathbb{C} \cdot v \quad \text{and} \quad C_1(X) = \sum_{e \in E} \mathbb{C} \cdot e.$$

For a subset $\partial V$ of $V = X^0$ we define the group of *relative* $p$-chains as

$$C_p(X, \partial V) := C_p(X)/C_p(\partial V).$$

Note that since $\partial V$ consists only of points, we have the natural identifications

$$C_0(X, \partial V) = C_0(\mathring{V}) = \sum_{v \in \mathring{V}} \mathbb{C} \cdot v \quad \text{and} \quad C_1(X, \partial V) = C_1(X).$$

5.1. **Oriented homology.** The (oriented) boundary map $\partial \colon C_1(X) \longrightarrow C_0(X)$ is defined as $\partial e = \partial_+ e - \partial_- e$, i.e., the formal difference of the terminal minus the initial vertex of $e$. (We use the same symbol as in the definition of the discrete graph since no confusion is possible.) In particular, for $c = \sum_{e \in E} \eta_e \cdot e$ we have

$$\begin{aligned} \partial c = \sum_{e \in E} \eta_e \cdot (\partial_+ e - \partial_- e) &= \sum_{v \in V} \Big( \sum_{e \in E_v^+} \eta_e - \sum_{e \in E_v^-} \eta_e \Big) \cdot v \\ &= \sum_{v \in V} \Big( \sum_{e \in E_v} \widehat{\eta}_e(v) \Big) \cdot v = \sum_{v \in V} m(v)(d^* \eta)(v) \cdot v \end{aligned}$$

using (2.1) (recall that we assumed that $m_e = 1$). The definition of the corresponding boundary map $\partial_{\mathrm{r}}$ is naturally given by the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(\partial V) & \longrightarrow & C_0(X) & \longrightarrow & C_0(X, \partial V) & \longrightarrow & 0 \\ & & \uparrow {\scriptstyle 0} & & \uparrow {\scriptstyle \partial} & & \uparrow {\scriptstyle \partial_{\mathrm{r}}} & & \\ 0 & \longrightarrow & C_1(\partial V) & \longrightarrow & C_1(X) & \longrightarrow & C_1(X, \partial V) & \longrightarrow & 0. \end{array}$$

In particular, we have

$$\partial_{\mathrm{r}} e = \begin{cases} \partial_+ e - \partial_- e & \text{if } \partial_\pm e \in \mathring{V}, \\ \partial_+ e & \text{if } \partial_+ e \in \mathring{V},\ \partial_- e \in \partial V, \\ -\partial_- e & \text{if } \partial_- e \in \mathring{V},\ \partial_+ e \in \partial V \\ 0 & \text{if } \partial_\pm e \in \partial V. \end{cases}$$

Note that one can check as above that

$$\partial_{\mathrm{r}} c = \sum_{v \in \mathring{V}} m(v)(d_0^* \eta)(v) \cdot v. \tag{5.1}$$

The corresponding homologies resp. relative homologies are now defined as

$$\begin{aligned} H_0(X) &:= C_0(X)/\operatorname{ran} \partial, & H_0(X, \partial V) &:= C_0(X, \partial V)/\operatorname{ran} \partial_{\mathrm{r}} \\ H_1(X) &:= \ker \partial, & H_1(X, \partial V) &:= \ker \partial_{\mathrm{r}}. \end{aligned}$$

5.2. **Unoriented homology.** The unoriented boundary map $\bar{\partial}\colon C_1(X) \longrightarrow C_0(X)$ is defined similarly as $\bar{\partial} e = \partial_+ e + \partial_- e$, i.e., the formal *sum* of the terminal and initial vertex of $e$. As before, we see that

$$\bar{\partial} c = \sum_{v \in V} m(v)(\bar{d}^* \eta)(v) \cdot v.$$

The corresponding unoriented relative boundary map is given as before but just replacing $-\partial_- e$ by $+\partial_- e$. Similarly, we have

$$\bar{\partial}_{\mathrm{r}} c = \sum_{v \in \mathring{V}} m(v)(\bar{d}_0^* \eta)(v) \cdot v. \tag{5.2}$$

The corresponding homologies resp. relative homologies are now defined as

$$\bar{H}_0(X) := C_0(X)/\operatorname{ran} \bar{\partial}, \qquad \bar{H}_0(X, \partial V) := C_0(X, \partial V)/\operatorname{ran} \bar{\partial}_{\mathrm{r}}$$
$$\bar{H}_1(X) := \ker \bar{\partial}, \qquad \bar{H}_1(X, \partial V) := \ker \bar{\partial}_{\mathrm{r}}.$$

5.3. **Calculation of the Betti numbers.** Denote by $b_p = b_p(X) = \dim H_p(X)$ the (oriented) Betti-numbers, and similarly, $b_p^{\partial V} = b_p(X, \partial V) = \dim H_p(X, \partial V)$ the corresponding relative Betti-numbers. Moreover, the corresponding notation with a bar, e.g., $\bar{b}_p = \dim \bar{H}_p(X)$ refers to the unoriented homology. The result for the oriented Betti-numbers is standard. We include a short proof for the unoriented case.

**Lemma 5.1.** *Assume that the topological graph $X$ is compact and connected, and that $\partial V \neq \emptyset$. Then the oriented Betti numbers are given as*

$$b_0(X) = 1, \qquad b_0(X, \partial V) = 0,$$
$$b_1(X) = |E| - |V| + 1, \qquad b_1(X, \partial V) = |E| - |V| + |\partial V|.$$

*The unoriented Betti numbers are*

$$\bar{b}_0(X) = \beta \qquad \bar{b}_0(X, \partial V) = 0$$
$$\bar{b}_1(X) = |E| - |V| + \beta \qquad \bar{b}_1(X, \partial V) = |E| - |V| + |\partial V|$$

*where $\beta = 1$ if $X$ is bipartite and $0$ otherwise.*

*Proof.* We give the proof only for the unoriented case. It is more convenient to use the corresponding cohomologies, defined via

$$\bar{H}^0(X) := \ker \bar{d}, \qquad \bar{H}^1(X) := \ker \bar{d}^*$$

and using the natural Hilbert space structure of the $\ell_2$-spaces with the standard weights $m(v) = \deg v$ and $m_e = 1$. Similarly, the relative cohomologies are defined as kernels of $\ker \bar{d}_0$ and $\ker \bar{d}_0^*$. From (5.2), it is easy to see that the $p$-th relative homology and cohomology spaces are isomorphic, and similarly for the other cases.

Moreover, $F \in \ker \bar{d}$ is equivalent to $0 = \bar{\check{\Delta}} F$, and by (2.10), we conclude that $\check{\Delta} F = 2F$ for the "oriented" Laplacian $\check{\Delta} = d^* d$. Since 2 is an eigenvalue of $\check{\Delta}$ iff the graph is bipartite (cf. Proposition 2.3), it follows that $\bar{b}_0(X) = \beta$ (recall that the graph is connected). The Euler characteristic is the same for the oriented and unoriented homology (see e.g. [P07b]). Therefore $\bar{b}_1(X) = \bar{b}_0(X) - \chi(X) = |E| - |V| + \beta$.

The relative Betti number $\bar{b}_0(X, \partial V)$ is easily seen to vanish, since the graph is connected and the function (the bipartite eigenfunction $F \in \ker(\check{\Delta} - 2)$) is determined by its value at a single vertex. To compute $\bar{b}_1(X, \partial V)$ we have to analyse $\ker \bar{d}_0^*$, where $\bar{d}_0^* = \iota^* \circ \bar{d}^*$ is given in Section 2.6. Note that

$$\ker \bar{d}_0^* = \ker \bar{d}^* \oplus \{\, \eta \in (\ker \bar{d}_0^*)^\perp \mid \bar{d}^* \eta \restriction_{\mathring{V}} = 0 \,\}.$$

To compute the dimension of the second term of the previous equation note that

$$\dim\{\, \eta \in (\ker \bar{d}_0^*)^\perp \,|\, \bar{d}^*\eta\restriction_{\mathring{V}} = 0 \,\} = \dim\{\, F \in \ell_2(V) \,|\, F \in \operatorname{ran} \bar{d}^* = (\ker \bar{d})^\perp,\ \operatorname{supp} F \subset \partial V \,\} \\ = |\partial V| - \beta.$$

Altogether we have

$$\bar{b}_1(X, \partial V) = \bar{b}_1(X) + |\partial V| - \beta = |E| - |V| + |\partial V|$$

and the proof is concluded. □

5.4. **The topological eigenspaces.** We can now relate the eigenfunctions vanishing at all vertices with the homology. Recall that $\lambda_n := n^2\pi^2$.

**Proposition 5.2.** *For any 1-chain $c = \sum_{e\in E} \eta_e \cdot e$ define $f_c \in \mathsf{L}_2(X)$ by $f_{c,e}(x) := \eta_e \sin(n\pi x)$. Then the mappings*

$$\Psi_n \colon H_1(X, \partial V) \longrightarrow \mathscr{N}_0^{\partial V}(\lambda_n), \qquad n \neq 0 \text{ even, and}$$
$$\bar{\Psi}_n \colon \bar{H}_1(X, \partial V) \longrightarrow \mathscr{N}_0^{\partial V}(\lambda_n), \qquad n \text{ odd},$$

*given by $\Psi_n(c) := f_c$ and $\bar{\Psi}_n(c) := f_c$, respectively, are isomorphisms.*

*Proof.* We show first that $f_c \in \mathscr{N}_0^{\partial V}(\lambda_n)$. Note that, by construction $f_c\restriction_V = 0$ and that $f_c$ is continuous on each vertex. It remains to check the Kirchhoff condition at the inner vertices $v \in \mathring{V}$. Since $c = \sum_{e\in E} \eta_e \cdot e \in H_1(X, \partial V)$ we have that $\partial_{\mathrm{r}} c = 0$, hence $d_0^*\eta = 0$ with $\eta = (\eta_e)_e$ and $n \neq 0$ even. From Lemma 4.5 we have that $f_c$ satisfies the Kirchhoff condition at $\mathring{V}$. Finally we have to show that $\Psi_n$ is bijective. The injectivity of $\Psi_n$ is clear. In order to show the surjectivity, let $f \in \mathscr{N}_0^{\partial V}(\lambda_n)$ and put $\eta_e := f'(0)/(n\pi)$. Then $\Psi_n(c) = f$ by construction, and $d_0^*\eta = 0$. The case $n$ odd is done similarly. □

Note that for the topological eigenfunctions (or, what is the same, edge-based) it is again the Kirchhoff condition giving the relation with the discrete graph (or at least with its homology), as we have already noticed for the vertex-based eigenfunctions in Proposition 4.1.

*Remark* 5.3. Note that Cattaneo [Ca97] already calculated the spectrum of an equilateral (possibly infinite) graph (with $\partial V = \emptyset$) also for the exceptional values $\Sigma^{\mathrm{D}}$ without taking care about the multiplicities. She obtains the same result. Namely, if the graph has at least one even cycle (i.e., a closed path passing an even number of edges), then the first homology is non-trivial in the oriented and unoriented case ($b_1(X) \geq \bar{b}_1(X) > 0$), and $\lambda_n$ is in the spectrum of $\Delta_X$

If $n$ is odd and the graph has only one odd cycle, then Cattaneo uses the following characterisation: $\lambda_n \in \Delta_X$ iff the graph is transient. The transience is equivalent to the existence of a *flow* with finite energy and source $a$; in our notation, that there exists an element $\eta \in \ell_2(E)$ such that $d^*\eta = \delta_a$ ($\delta_a(v) = 1$ if $a = v$ and $\delta_a(v) = 0$ otherwise). The latter condition means that $\delta_a$ is in $\operatorname{ran} d^*$, i.e, orthogonal to $\ker d = \mathbb{C}\mathbb{1}_V$ if the graph is finite. But $\delta_a$ is never orthogonal to $\mathbb{1}_V$, so in this case, there are no eigenvalues, as we already conclude from $\bar{b}_1(X) = 0$ and Proposition 5.2.

Note that Cattaneo's primary interest are Laplacians on infinite metric graphs with weights defined in a slightly different way than our metric graph Laplacians, see [Ca97].

Moreover, von Below [vB85] already calculated the multiplicities of the exceptional eigenvalues $\lambda_n$ in the case $\partial V = \emptyset$, but without using homology groups.

Although non-compact graphs are not our main purpose here, let us make a few comment on this case. The non-compact case occurs in Sections 8 and 9 were we consider infinite covering graphs.

*Remark* 5.4. If $X$ is non-compact and connected, the spectral relation of Proposition 4.1 is still true, even more, one can show that all spectral types (discrete and essential, absolutely and singular continuous, (pure) point) are preserved, see [BGP08] for details. Moreover, $\mathscr{N}^{\partial V}(\lambda_n) = \mathscr{N}_0^{\partial V}(\lambda_n)$

($n \geq 1$) due to the fact that the trivial vertex based eigenfunctions $\varphi_n$ are no longer in $\mathsf{L}_2(X)$. Moreover, we can easily extend the above results to the infinite case. Namely, if $n$ is even, Proposition 5.2 extends to the assertion that

$$\Psi^n \colon H^1(X, \partial V) \longrightarrow \mathscr{N}^{\partial V}(\lambda_n), \qquad \eta \mapsto \sqrt{2} f_\eta, \qquad f_{\eta,e}(x) = \eta_e \sin(n\pi x)$$

is an isometric isomorphism using the corresponding $\ell_2$-cohomology $H^1(X, \partial) = \ker d_0^* \subset \ell_2(E)$. The case $n$ odd can be treated similarly.

## 6. Eigenvalue bracketing

6.1. **Eigenvalue counting for metric graphs.** Let us now combine the results of the previous sections. In particular, we will show how the $|V|$ eigenvalues $\mu_k$ of the discrete Laplacian are related with the $|E|$ eigenvalues $\lambda_k$ in $K_n = [n^2\pi^2, (n+1)^2\pi^2]$ of the Kirchhoff Laplacian. For the Dirichlet operators we relate the $|V| - |\partial V|$ discrete eigenvalues $\mu_k^{\partial V}$ with $|E|$ metric eigenvalues $\lambda_k^{\partial V} \in K_n$. In Figures 1 and 2 we illustrated the spectral relations for a bipartite and non-bipartite graph of Examples 9.1 and 9.2 (see Figures 3 and 4). Doing a neat bookkeeping one can check the different possibilities given in the tables below.

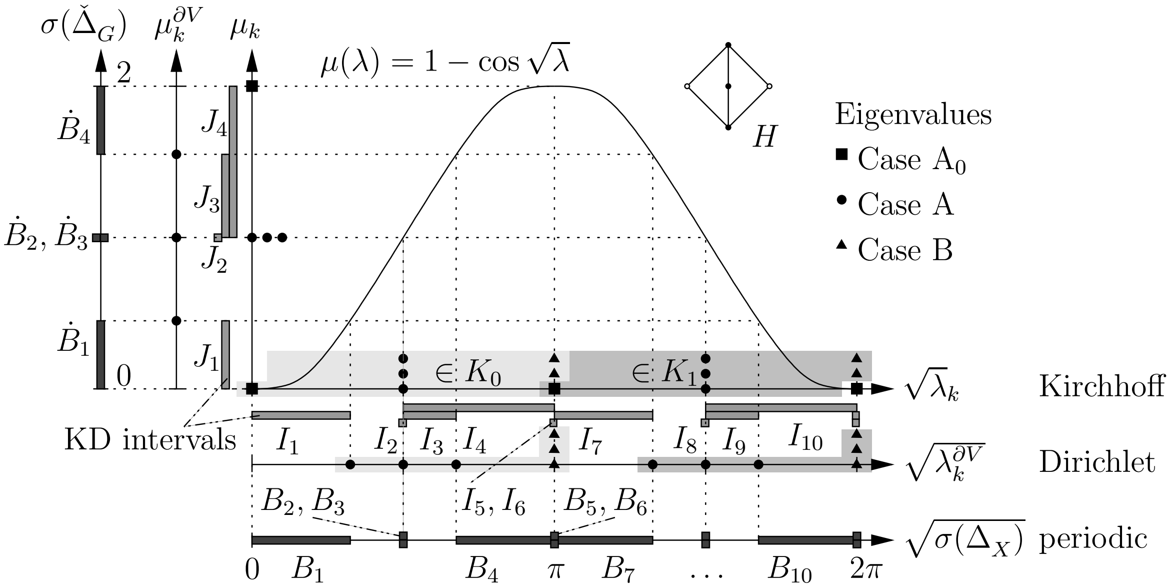


FIGURE 1. The various eigenvalues for the bipartite graph with fundamental domain $H$ and periodic graph $G$ of Figure 3 with five vertices, two boundary vertices and six edges. Multiple eigenvalues are indicated by repeated symbols, compare with the tables in Section 6. The eigenvalues for the Kirchhoff and Dirichlet metric Laplacian are grouped into members of six (light grey and dark grey) belonging to $K_0 = [0, \pi^2]$ and $K_1 = [\pi^2, 4\pi^2]$ as predicted in Lemma 3.1. For a discussion of the relation of the KD intervals with periodic operators see Example 9.1.

We start with a basic definition:

**Definition 6.1.** We define the (metric) Kirchhoff-Dirichlet intervals $I_k = I_k(X, \partial V)$ of the metric graph $X$ with boundary $\partial V$ as

$$I_k := [\lambda_k, \lambda_k^{\partial V}], \qquad k = 1, 2, \ldots$$

Note that by Lemma 3.1, the interval is non-empty and $I_k \subset K_n$ for $k = n|E|+1, \ldots, (n+1)|E|$, where $K_n := [n^2\pi^2, (n+1)^2\pi^2]$ for $n = 0, 1, \ldots$

The aim of the following eigenvalue counting is to understand the nature of the intervals $I_k$, i.e., whether they reduce to points or are contained in $\mathring{K}_n$. It is therefore unavoidable to give a precise account of the eigenvalues repeated according to multiplicity in the Kirchhoff as well as in the Dirichlet case, distinguishing bipartite and non-bipartite graphs.

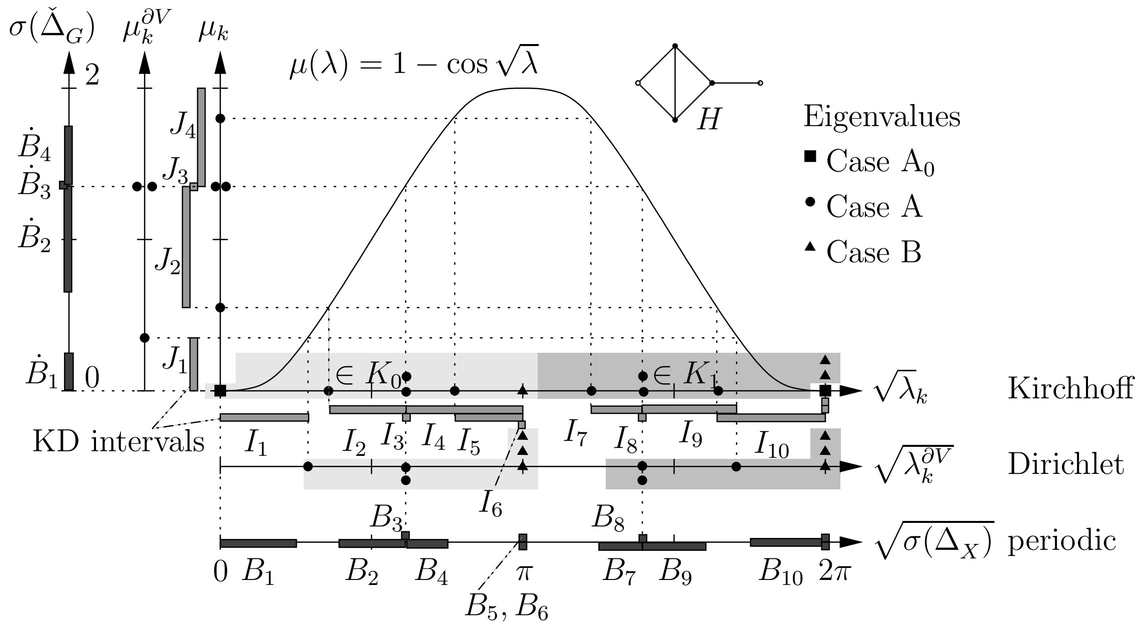


FIGURE 2. The various eigenvalues for the non-bipartite graph with fundamental domain $H$ and periodic graph $G$ of Figure 4 with five vertices, two boundary vertices and six edges. Again, multiple eigenvalues are indicated by repeated symbols; and the eigenvalues for the Kirchhoff and Dirichlet metric Laplacian are grouped into members of six (light grey and dark grey) as in Figure 1 For a discussion of the relation of the KD intervals with periodic operators see Example 9.2.

*Counting the Kirchhoff eigenvalues.* The following result summarises several facts of this and the previous section. In particular it is a consequence of Propositions 4.1, 4.7 and 5.2. We use the abbreviations EF for eigenfunction and EV for eigenvalue. The trivial vertex-based eigenfunction $\varphi_n$ is described in Definition 4.6, the non-trivial vertex-based eigenfunctions are described in Proposition 4.1 (see Definition 4.2) and the topological eigenfunctions are described in Proposition 5.2 (see Definition 4.4).

**Proposition 6.2.** *Let $X$ be a connected compact equilateral metric graph and let $n = 0, 1, 2, \dots$ The EVs $\lambda_k$ of $\Delta_X$ distribute in groups of $|E|$ EVs contained in the intervals $K_n := [n^2\pi^2, (n+1)^2\pi^2]$. We list them in the following tables according to the various possibilities. The brace under the range of the index $k$ denotes the number of such indices.*

*If the graph is bipartite we have:*

***Metric Kirchhoff eigenvalues for bipartite graphs***

| *Case* | *Range of index k* | $\lambda_k$ | *Type of EF* | *EF described in* |
|---|---|---|---|---|
| $A_0$ | $\underbrace{n\lvert E\rvert + 1}_{1}$ | $= n^2\pi^2$ | $\varphi_n$, *trivial vertex-based* | *Proposition 4.7* |
| $A$ | $\underbrace{n\lvert E\rvert + 2, \dots, n\lvert E\rvert + \lvert V\rvert - 1}_{\lvert V\rvert - 2}$ | $\in \mathring{K}_n$ | *vertex-based* | *Proposition 4.1* |
| $B$ | $\underbrace{n\lvert E\rvert + \lvert V\rvert, \dots, (n+1)\lvert E\rvert}_{b_1(X) = \bar{b}_1(X) = \lvert E\rvert - \lvert V\rvert + 1}$ | $= (n+1)^2\pi^2$ | *topological* | *Proposition 5.2* |

*If the graph is not bipartite we have:*

***Metric Kirchhoff eigenvalues for non-bipartite graphs,*** $n$ ***even***

| *Case* | *Range of index k* | $\lambda_k$ | *Type of EF* | *EF described in* |
|---|---|---|---|---|
| $A_0$ | $\underbrace{n\lvert E\rvert+1}_{1}$ | $=n^2\pi^2$ | $\varphi_n$, *trivial vertex-based* | *Proposition 4.7* |
| $A$ | $\underbrace{n\lvert E\rvert+2,\dots,n\lvert E\rvert+\lvert V\rvert}_{\lvert V\rvert-1}$ | $\in \mathring{K}_n$ | *vertex-based* | *Proposition 4.1* |
| $B$ | $\underbrace{n\lvert E\rvert+\lvert V\rvert+1,\dots,(n+1)\lvert E\rvert}_{\bar{b}_1(X)=\lvert E\rvert-\lvert V\rvert}$ | $=(n+1)^2\pi^2$ | *topological* | *Proposition 5.2* |

***Metric Kirchhoff eigenvalues for non-bipartite graphs,*** $n$ ***odd***

| *Case* | *Range of index k* | $\lambda_k$ | *Type of EF* | *EF described in* |
|---|---|---|---|---|
| $A$ | $\underbrace{n\lvert E\rvert+1,\dots,n\lvert E\rvert+\lvert V\rvert-1}_{\lvert V\rvert-1}$ | $\in \mathring{K}_n$ | *non-trivial vertex-based* | *Proposition 4.1* |
| $B$ | $\underbrace{n\lvert E\rvert+\lvert V\rvert,\dots,(n+1)\lvert E\rvert}_{b_1(X)=\lvert E\rvert-\lvert V\rvert+1}$ | $=(n+1)^2\pi^2$ | *topological* | *Proposition 5.2* |

*Remark* 6.3.

(i) For a bipartite graph, the trivial vertex-based eigenfunction $\varphi_n$ corresponds to the constant discrete EF if $n$ is even and to the bipartite eigenfunction if $n$ is odd.

(ii) Note that in the non-bipartite case, there is one eigenvalue of Case A more than in the bipartite case. In the bipartite case, this additional eigenfunction is either a topological one (Case B) if $n$ is even or a trivial vertex-based one (Case A) if $n$ is odd, namely the one corresponding to the bipartite EF.

*Counting the Dirichlet eigenvalues.* The Dirichlet case is simpler and does not distinguish the bipartite and non-bipartite case.

**Proposition 6.4.** *Let $X$ be a connected compact equilateral metric graph with non-trivial boundary $\partial V \neq \emptyset$ and let $n = 0, 1, 2, \dots$ The EVs $\lambda_k^{\partial V}$ of $\Delta^{\partial V} X$ distribute in groups of $\lvert E\rvert$ EVs contained in the intervals $K_n := [n^2\pi^2, (n+1)^2\pi^2]$. We list them in the following table:*

***Metric Dirichlet eigenvalues***

| *Case* | *Range of index k* | $\lambda_k^{\partial V}$ | *Type of EF* |
|---|---|---|---|
| $A$ | $\underbrace{n\lvert E\rvert+1,\dots,n\lvert E\rvert+\lvert V\rvert-\lvert\partial V\rvert}_{\lvert V\rvert-\lvert\partial V\rvert}$ | $\in \mathring{K}_n$ | *non-trivial vertex-based* |
| $B$ | $\underbrace{n\lvert E\rvert+\lvert V\rvert-\lvert\partial V\rvert+1,\dots,(n+1)\lvert E\rvert}_{b_1(X,\partial V)=\bar{b}_1(X,\partial V)=\lvert E\rvert-\lvert V\rvert+\lvert\partial V\rvert}$ | $=(n+1)^2\pi^2$ | *topological* |

*Again, Case A is described in Proposition 4.1 and Case B in Proposition 5.2.*

We can now describe precisely all possible combinations of Kirchhoff-Dirichlet intervals that arise from the previous tables:

**Proposition 6.5.** *Let $X$ be a connected compact equilateral metric graph with non-empty boundary $\partial V \neq \emptyset$ and let $n = 0, 1, 2, \dots$ The metric Kirchhoff-Dirichlet intervals are given in the table below. We call an interval* non-degenerate *if its interior is non-empty. The case-labeling refers to the cases of the Kirchhoff (first letter) and Dirichlet (second letter) eigenvalue.*

*If the graph is bipartite we have:*

***Metric Kirchhoff-Dirichlet intervals for bipartite graphs***

| *Case* | *Range of index* $k$ | $I_k = I_k(X, \partial V)$ | *Type of interval* |
|---|---|---|---|
| $A_0A$ | $\underbrace{n\lvert E\rvert + 1}_{1}$ | $= [n^2\pi^2, \lambda_k^{\partial V}]$ | *non-degenerate* |
| $AA$ | $\underbrace{n\lvert E\rvert + 2, \dots, n\lvert E\rvert + \lvert V\rvert - \lvert\partial V\rvert}_{\lvert V\rvert - \lvert\partial V\rvert - 1}$ | $\subset \mathring{K}_n$ | *degenerate or non-degenerate* |
| $AB$ | $\underbrace{n\lvert E\rvert + \lvert V\rvert - \lvert\partial V\rvert + 1, \dots, n\lvert E\rvert + \lvert V\rvert - 1}_{\lvert\partial V\rvert - 1}$ | $= [\lambda_k, (n+1)^2\pi^2]$ | *non-degenerate* |
| $BB$ | $\underbrace{n\lvert E\rvert + \lvert V\rvert, \dots, (n+1)\lvert E\rvert}_{b_1(X) = \bar{b}_1(X) = \lvert E\rvert - \lvert V\rvert + 1}$ | $= (n+1)^2\pi^2$ | *degenerate* |

*For non-bipartite graphs, we obtain:*

***Metric Kirchhoff-Dirichlet intervals for non-bipartite graphs (*** $n$ ***even)***

| *Case* | *Range of index* $k$ | $I_k = I_k(X, \partial V)$ | *Type of interval* |
|---|---|---|---|
| $A_0A$ | $\underbrace{n\lvert E\rvert + 1}_{1}$ | $= [n^2\pi^2, \lambda_k^{\partial V}]$ | *non-degenerate* |
| $AA$ | $\underbrace{n\lvert E\rvert + 2, \dots, n\lvert E\rvert + \lvert V\rvert - \lvert\partial V\rvert + 1}_{\lvert V\rvert - \lvert\partial V\rvert}$ | $\subset \mathring{K}_n$ | *degenerate or non-degenerate* |
| $AB$ | $\underbrace{n\lvert E\rvert + \lvert V\rvert - \lvert\partial V\rvert + 2, \dots, n\lvert E\rvert + \lvert V\rvert}_{\lvert\partial V\rvert}$ | $= [\lambda_k, (n+1)^2\pi^2]$ | *non-degenerate* |
| $BB$ | $\underbrace{n\lvert E\rvert + \lvert V\rvert + 1, \dots, (n+1)\lvert E\rvert}_{\bar{b}_1(X) = \lvert E\rvert - \lvert V\rvert}$ | $= (n+1)^2\pi^2$ | *degenerate* |

***Metric Kirchhoff-Dirichlet intervals for non-bipartite graphs (*** $n$ ***odd)***

| *Case* | *Range of index* $k$ | $I_k = I_k(X, \partial V)$ | *Type of interval* |
|---|---|---|---|
| $AA$ | $\underbrace{n\lvert E\rvert + 1, \dots, n\lvert E\rvert + \lvert V\rvert - \lvert\partial V\rvert}_{\lvert V\rvert - \lvert\partial V\rvert}$ | $\subset \mathring{K}_n$ | *degenerate or non-degenerate* |
| $AB$ | $\underbrace{n\lvert E\rvert + \lvert V\rvert - \lvert\partial V\rvert + 1, \dots, n\lvert E\rvert + \lvert V\rvert - 1}_{\lvert\partial V\rvert - 1}$ | $= [\lambda_k, (n+1)^2\pi^2]$ | *non-degenerate* |
| $BB$ | $\underbrace{n\lvert E\rvert + \lvert V\rvert, \dots, (n+1)\lvert E\rvert}_{b_1(X) = \bar{b}_1(X) = \lvert E\rvert - \lvert V\rvert + 1}$ | $= (n+1)^2\pi^2$ | *degenerate* |

6.2. **Eigenvalue counting for discrete graphs.** We can now carry over the eigenvalue monotonicity of the Kirchhoff and Dirichlet *metric* Laplacian to the *discrete* one. For this purpose it is is enough to consider only the metric graph eigenvalues in the *first* interval $K_0 = [0, \pi^2]$, since on this interval, the function $\mu(\lambda) = 1 - \cos(\sqrt{\lambda})$ is increasing. Denote by $\mu_k$ ($k = 1, \dots, |V|$) the eigenvalues of the standard discrete Laplacian $\check{\Delta}_G$, and by $\mu_k^{\partial V}$ ($k = 1, \dots, |V| - |\partial V|$) the eigenvalues of the (standard) discrete Laplacian with Dirichlet conditions on $\partial V$ (see Section 2), in both cases counted according to multiplicity.

**Definition 6.6.** We define the (discrete) Kirchhoff-Dirichlet intervals $J_k = J_k(G, \partial V)$ of the metric graph $X$ with boundary $\partial V$ as

$$J_k := [\mu_k, \mu_k^{\partial V}], \qquad k = 1, 2, \dots, |V| - |\partial V|.$$

For higher indices, we set

$$J_k := [\mu_k, 2], \qquad k = |V| - |\partial V| + 1, \dots, |V|.$$

*Remark* 6.7. Note that the names "Kirchhoff" and "Dirichlet" for the standard Laplacian (with Dirichlet conditions on $\partial V$) is justified by Proposition 4.1. Note also, that the operators act in spaces of different dimensions. In particular, the standard Laplacian with $\partial V = \emptyset$ can be written as a $|V| \times |V|$-matrix and has therefore $|V|$ eigenvalues. Similarly, the standard Dirichlet Laplacian has $|V| - |\partial V|$ eigenvalues.

From Proposition 6.5 we immediately obtain:

**Proposition 6.8.** *Let $G$ be a connected finite discrete graph with standard weight ($m(v) = \deg v$ and $m_e = 1$) and non-trivial boundary $\partial V \neq \emptyset$. Then the discrete Kirchhoff-Dirichlet intervals are given in the table below. (Note that the type of the interval is the same as the type for the metric graph.)*

*If the graph is bipartite we have:*

***Discrete Kirchhoff-Dirichlet intervals for bipartite graphs***

| *Case* | *Range of index $k$* | $J_k = J_k(G, \partial V)$ | *Type of interval* |
|---|---|---|---|
| $A_0A$ | $k = 1$ | $= [0, \mu_k^{\partial V}]$ | *non-degenerate* |
| $AA$ | $\underbrace{2, \dots, \lvert V\rvert - \lvert\partial V\rvert}_{\lvert V\rvert - \lvert\partial V\rvert - 1}$ | $\subset (0, 2)$ | *degenerate or non-degenerate* |
| $AB$ | $\underbrace{\lvert V\rvert - \lvert\partial V\rvert + 1, \dots, \lvert V\rvert - 1}_{\lvert\partial V\rvert - 1}$ | $= [\mu_k, 2]$ | *non-degenerate* |
| $BB$ | $k = \lvert V\rvert$ | $= \{2\}$ | *degenerate* |

*For non-bipartite graphs, we obtain:*

***Discrete Kirchhoff-Dirichlet intervals for non-bipartite graphs***

| *Case* | *Range of index $k$* | $J_k = J_k(G, \partial V)$ | *Type of interval* |
|---|---|---|---|
| $A_0A$ | $k = 1$ | $= [0, \mu_k^{\partial V}]$ | *non-degenerate* |
| $AA$ | $\underbrace{2, \dots, \lvert V\rvert - \lvert\partial V\rvert + 1}_{\lvert V\rvert - \lvert\partial V\rvert}$ | $\subset (0, 2)$ | *degenerate or non-degenerate* |
| $AB$ | $\underbrace{\lvert V\rvert - \lvert\partial V\rvert + 2, \dots, \lvert V\rvert}_{\lvert\partial V\rvert}$ | $= [\mu_k, 2]$ | *non-degenerate* |

6.3. **Spectral symmetry for bipartite graphs.** Let us carry over the spectral symmetry for *discrete* bipartite graphs already mentioned in Proposition 2.3 to the metric case. Note that the symmetry function in the discrete case is

$$\theta \colon [0, 2] \longrightarrow [0, 2], \qquad \theta(\mu) = 2 - \mu.$$

In particular, the fixed point of $\theta$, i.e., $\mu = 1$, is always an eigenvalue of $\check{\Delta}_G^{\partial V}$ if $|V| - |\partial V|$ is odd.

We recall the definition $K_n := [n^2\pi^2, (n+1)^2\pi^2]$.

**Proposition 6.9.** *Suppose that $X$ is a bipartite equilateral compact metric graph with boundary $\partial V$ ($\partial V$ may be empty) and let $\lambda \in \mathring{K}_n$. Then*

$$\lambda \in \sigma(\Delta_X^{\partial V}) \qquad \text{iff} \qquad \tau_n(\lambda) \in \sigma(\Delta_X^{\partial V})$$

*and the multiplicity is preserved. Here,*

$$\tau_n \colon K_n \longrightarrow K_n, \qquad \tau_n(\lambda) := \big((2n+1)\pi - \sqrt{\lambda}\big)^2.$$

*If $|V| - |\partial V|$ is odd, then the fixed point of $\tau_n$, i.e., $\lambda = (n+1/2)^2\pi^2$ is an eigenvalue of $\Delta_X^{\partial V}$.*

*Moreover, for $n \geq 1$ the map $\tau_n$ also interchanges the topological eigenvalues $\lambda_n = n^2\pi^2$ and $\lambda_{n+1} = (n+1)^2\pi^2$. The corresponding eigenfunctions for $\tau_n(\lambda_n) = \lambda_{n+1}$ are obtained by those from $\lambda_n$ by keeping the amplitude of the oscilation and interchanging the frequency.*

*If $X$ is non-compact, the spectral symmetry $\tau_n(\sigma(\Delta_X) \cap K_n) = \sigma(\Delta_X) \cap K_n$ still holds.*

*Proof.* The results for eigenvalues $\lambda \in \mathring{K}_n$ follow immediately from Propositions 2.3 and 4.1. Also, the trivial vertex-based eigenfunctions are interchanged by the symmetry, as in the discrete case. For the topological eigenvalues, note that their structure is given in Proposition 5.2, and that the oriented and unoriented Betti numbers agree, namely $b_1(X, \partial V) = \bar{b}_1(X, \partial V)$. The non-compact case follows by the spectral relation for $\lambda \in \mathring{K}_n$ (see Remark 5.4), and by the closedness of the spectrum for the endpoints of $K_n$. □

## 7. Equivariant Laplacians and coverings

In the sequel, we will analyse metric and discrete Laplacians on covering graphs.

7.1. **Equivariant metric Laplacians.** We start with a *metric* covering graph $X \to X_0$ with covering group $\Gamma$ (in general non-abelian) and compact quotient $X_0$, see also [Sun08, Sec. 6] for related aspects. We call the metric Laplacian $\Delta_X$ on $X$ also $\Gamma$-*periodic*. A *fundamental domain* of a metric graph covering $X \to X_0$ is a closed subset $Y$ of $X$ such that

$$\gamma\mathring{Y} \cap \mathring{Y} = \emptyset, \quad \gamma \neq 1, \qquad \bigcup_{\gamma\in\Gamma} \gamma Y = X.$$

Note that the interior of a fundamental domain $\mathring{Y}$ can always be embedded isometrically in the quotient graph $X_0$. Moreover, we assume that the boundary of $Y$ (as topological subset of $X$) consists only of vertices, which are precisely the *boundary* vertices, i.e.,

$$\partial V := \partial Y = Y \setminus \mathring{Y} \subset V. \tag{7.1}$$

Since we can interpret $\mathring{Y}$ as subset of $X_0$, we define the set of *inner vertices* of the quotient $X_0$ by $\mathring{V}_0 := \mathring{Y} \cap V_0$, depending of course on the fundamental domain.

Associated to a fundamental domain is a metric graph also denoted by the symbol $Y$ with boundary vertices $\partial V = \partial Y$ (*not* embedded in the quotient). We define the Dirichlet and Kirchhoff metric Laplacians on this graph, namely, we consider $\Delta_Y^{\partial V}$ and $\Delta_Y$ defined via their quadratic forms on $\mathsf{H}^1_{\partial V}(Y)$ and $\mathsf{H}^1(Y)$.

Let $\rho$ be a unitary representation of $\Gamma$, i.e., $\rho$ is a homomorphism from $\Gamma$ into the group of unitary operators on some Hilbert space $\mathscr{H}$. In order to analyse the spectrum of the periodic operator, we need the following definition:

**Definition 7.1.** A function $f \colon X \longrightarrow \mathscr{H}$ is called *equivariant* iff

$$f(\gamma \cdot x) = \rho(\gamma) f(x), \qquad \forall\, x \in X, \gamma \in \Gamma.$$

Clearly, a $\rho$-equivariant function, locally in $\mathsf{H}^1$, is determined by its values on $Y$, namely, the equivariance condition Definition 7.1 reduces to a condition for the boundary vertices $x = v \in \partial V$ such that $\gamma \cdot v \in \partial V$. We therefore set

$$\mathsf{H}^1_\rho(X_0, \mathscr{H}) := \big\{\, f \in \mathsf{H}^1(Y) \otimes \mathscr{H} \,\big|\, f(\gamma \cdot v) = \rho(\gamma) f(v) \quad \forall\, v \in \partial V \text{ such that } \gamma \cdot v \in \partial V \,\big\}.$$

We can consider functions in $\mathsf{H}^1_\rho(X_0, \mathscr{H})$ as functions on the quotient metric graph $X_0$, where the continuity condition at the boundary vertices is replaced by the equivariance condition.

Denote by $\Delta^\rho_{X_0}$ the operator associated to the quadratic form

$$\mathfrak{h}^\rho(f) := \sum_{e\in E_0} \int_0^{\ell_e} \|f'(x)\|^2_{\mathscr{H}}\, \mathrm{d}x, \qquad \operatorname{dom} \mathfrak{h}^\rho := \mathsf{H}^1_\rho(X_0, \mathscr{H}), \tag{7.2}$$

i.e., functions in the domain of $\Delta^\rho_{X_0}$ fulfill the usual (now vector-valued) continuity and Kirchhoff conditions (3.4b) and (3.4c) on all *inner* vertices $v \in \mathring{V}_0 = \mathring{Y} \cap V_0$. Similarly, we define the Dirichlet and Kirchhoff $\mathscr{H}$-valued operators $\Delta^{\partial V}_Y \otimes \mathbb{1}$ and $\Delta_Y \otimes \mathbb{1}$ via their quadratic forms defined similarly as in (7.2), but with domains $\mathsf{H}^1_{\partial V}(Y) \otimes \mathscr{H}$ and $\mathsf{H}^1(Y) \otimes \mathscr{H}$, respectively. Note that these operators are *decoupled* in the following sense: Assume that $\mathscr{H}$ is $r$-dimensional and

$$(f_1, \dots, f_r) \cong f \in \mathsf{H}^1(Y) \otimes \mathscr{H} \cong \underbrace{\mathsf{H}^1(Y) \oplus \cdots \oplus \mathsf{H}^1(Y)}_{r\text{-times}},$$

then $\Delta_Y \otimes \mathbb{1}$ is unitarily equivalent to the direct sum of $r$ copies of $\Delta_Y$, and therefore the different components *decouple*. The same statement holds for the Dirichlet Laplacian $\Delta^{\partial V}_Y \otimes \mathbb{1}$.

Our crucial observation, already made in [LP07, LP08] is the following inclusion of quadratic form domains

$$\mathsf{H}^1_{\partial V}(Y) \otimes \mathscr{H} \subset \mathsf{H}^1_\rho(X_0, \mathscr{H}) \subset \mathsf{H}^1(Y) \otimes \mathscr{H},$$

implying first, that if $\mathscr{H}$ is finite-dimensional, then $\Delta^\rho_{X_0}$ has purely discrete spectrum (denoted by $\lambda^\rho_k$, written in ascending order and repeated with respect to multiplicity). Moreover we have the following assertion proven via the min-max characterisation of the eigenvalues as in [LP07, LP08]:

**Proposition 7.2.** *Assume that $\rho$ is a $r$-dimensional representation (i.e., $\dim \mathscr{H} = r$). Then*

$$\lambda^{\partial V}_k \ge \lambda^\rho_j \ge \lambda_k, \qquad j = (k-1)r + 1, \dots, kr.$$

*In other words, the $j$-th $\rho$-equivariant eigenvalue is enclosed in the $k$-th metric Kirchhoff-Dirichlet interval*

$$\lambda^\rho_j \in I_k = I_k(Y, \partial V), \qquad j = (k-1)r + 1, \dots, kr.$$

*Moreover, in the equilateral case and for those indices $k$ of case BB described in Proposition 6.5, the $\rho$-equivariant eigenvalues are* independent *of $\rho$, and given by $\lambda^\rho_j = (n+1)^2\pi^2$. The corresponding eigenfunctions are precisely the topological eigenfunctions of the graph $H$ with boundary $\partial V$ and supported in the* interior *of the fundamental domain.*

7.2. **Equivariant discrete Laplacians.** For simplicity, we assume that our discrete graphs have the standard weights. Let $G = (V, E, \partial) \to G_0 = (V_0, E_0, \partial_0)$ be a covering of discrete graphs with covering group $\Gamma$ and finite quotient graph $G_0 = G/\Gamma$. Let $\rho$ be a unitary representation of $\Gamma$ on the Hilbert space $\mathscr{H}$. Denote by

$$\ell^\rho_2(V_0, \mathscr{H}) := \bigl\{\, F \colon V \longrightarrow \mathscr{H} \bigm| F(\gamma \cdot v) = \rho(\gamma) F(v), \quad v \in V \,\bigr\}$$

the space of $\rho$-equivariant functions. Again, functions in $\ell^\rho_2(V_0, \mathscr{H})$ are determined by their values on the vertices of the quotient $V_0$ (as the notation already indicates). We denote by $\check\Delta^\rho_{G_0}$ the *$\rho$-equivariant* or *$\rho$-twisted Laplacian* defined as the restriction of $\check\Delta_G \otimes \mathbb{1}$ from $\ell_2(V) \otimes \mathscr{H}$ onto $\ell^\rho_2(V_0, \mathscr{H})$.

Let $Y$ be a fundamental domain of the associated metric graph, such that (7.1) holds. Now, $Y$ defines a boundary $\partial V$, which we will also consider as boundary of the discrete graph. Of course, $\partial V$ depends on the choice of fundamental domain. Denote the discrete graph associated to $Y$ by $H$.

As in Section 4, we denote by $\check{\mathscr{N}}^\rho(\eta) := \ker(\check\Delta^\rho_G - \eta)$ and $\mathscr{N}^\rho(\lambda) := \ker(\Delta^\rho_X - \lambda)$ the eigenspaces of the equivariant discrete and metric Laplacian, respectively. Moreover, $\mathscr{N}^\rho_0(\lambda)$ denotes the subspace of $\mathscr{N}^\rho(\lambda)$ of eigenfunctions vanishing at all vertices (see Definition 4.4).

For equilateral metric graphs, we have an analogue of Propositions 4.1 and 4.7. Denote by 1 the trivial representation on $\mathscr{H} = \mathbb{C}$ and by $R_\mathrm{a}$ the set of non-trivial involutive unitary representations on $\mathscr{H} = \mathbb{C}$, i.e., $\rho(\gamma)^{-1} = \rho(\gamma)^* = \rho(\gamma)$ for $\gamma \in \Gamma$ and $\rho \ne 1$. We also call $R_\mathrm{a}$ the set of *antisymmtric* representations of $\Gamma$. Note that $R_\mathrm{a}$ may be empty.

**Proposition 7.3.** *Assume that the metric covering graph $X \to X_0$ is equilateral such that the quotient $X_0$ is compact and connected.*

(i) *Assume that* $\lambda \notin \Sigma^{\mathrm{D}}$, *then* $\Phi_\lambda \colon \check{\mathscr{N}}^\rho(\mu(\lambda)) \longrightarrow \mathscr{N}^\rho(\lambda)$ *with* $F \mapsto f = \Phi_\lambda F$ *as defined in Proposition 4.1 is an isomorphism. In particular,*

$$\lambda \in \sigma(\Delta^\rho_{X_0}) \qquad \textit{iff} \qquad \mu(\lambda) \in \sigma(\check{\Delta}^\rho_{G_0}),$$

*preserving multiplicity.*

(ii) *If* $\lambda_n = n^2\pi^2 \in \Sigma^{\mathrm{D}}$, *we have the following cases:*

(a) *If* $\rho \notin R_{\mathrm{a}} \cup \{1\}$ *is irreducible then* $\mathscr{N}^\rho(\lambda_n) = \mathscr{N}_0^\rho(\lambda_n)$.

(b) *If* $\rho = 1$ *is the trivial representation on* $\mathscr{H} = \mathbb{C}$, *then*

$$\mathscr{N}^\rho(\lambda_n) = \begin{cases} \mathscr{N}_0^\rho(\lambda_n) & n \text{ odd and } G_0 \text{ not bipartite,} \\ \mathscr{N}_0^\rho(\lambda_n) \oplus \mathbb{C}\varphi_n & \text{otherwise.} \end{cases}$$

*Here,* $\varphi_n$ *is associated to the graph* $G_0$ *(see Definition 4.6).*

(c) *If* $\rho \in R_{\mathrm{a}}$ *is antisymmetric, then* $\mathscr{N}^\rho(\lambda_n) = \mathscr{N}_0^\rho(\lambda_n) \oplus \mathbb{C}\varphi_n$ *provided* $n$ *is odd and* $G$ *has a bipartite fundamental domain* $H$ *such that the connecting vertices* $\gamma \cdot v, v \in \partial H$ *are joined by a path of odd length. In all other cases,* $\mathscr{N}^\rho(\lambda_n) = \mathscr{N}_0^\rho(\lambda_n)$. *Here,* $\varphi_n$ *is associated to the the bipartite eigenfunction of* $H$.

Note that if $G_0$ is bipartite then any fundamental domain $H$ is, but not vice versa.

*Proof.* The first statement is analogue to the one of Proposition 4.1 and can be shown similarly as e.g. in [P07a]. The proof of the second statement is similar to the proofs of Lemma 4.3 and Proposition 4.7. We only sketch the ideas here. Let $f \in \mathscr{N}^\rho(\lambda_n)$ be an eigenfunction, interpreted as function on a fundamental domain $Y$. Fix $v \in \partial Y$ and let $\gamma \in \Gamma$ such that $\gamma \cdot v \in \partial Y$. Note that the set $\Gamma_0$ of all such $\gamma$'s generate the group $\Gamma$ (see [Rat94]). Let $p_\gamma$ be a path from $v$ to $\gamma \cdot v$ without passing a vertex twice. Denote by $s(p_\gamma)$ the number of edges of $p_\gamma$. Then $f(\gamma \cdot v) = (-1)^{ns(p_\gamma)} f(v)$. If the fundamental domain is bipartite, then $s(\gamma) := s(p_\gamma)$ is independent of the path joining $v$ and $\gamma \cdot v$. Note that $s(\gamma)$ may still depend on $v \in \partial V$. For $\gamma' \in \Gamma_0 \setminus \{\gamma\}$ we set $s(\gamma') = 0$. Now, $\rho_n(\gamma') := (-1)^{ns(\gamma')}$ extends to a unitary representation of $\Gamma$ on $\mathbb{C}$.

The equivariance condition implies that

$$f(v) \in \bigcap_{\gamma_0 \in \Gamma_0} \ker\bigl(\widetilde{\rho}_n(\gamma_0) - \mathrm{id}_{\mathscr{H}}\bigr)$$

where $\widetilde{\rho}_n(\gamma') := \rho_n(\gamma')\rho(\gamma)$ is a representation on $\mathscr{H}$. Since $\rho$ is irreducible, $\widetilde{\rho}_n$ is also irreducible. Moreover, if $\rho \neq \rho_n$, then $f(v) = 0$ and vanishes therefore on all vertices, since an irreducible representation not in $R \cup \{1\}$ cannot have a common eigenvector. This covers Case (iia). Otherwise, $\rho = \rho_n$ and $\mathscr{H} = \mathbb{C}$, and in particular, $\rho = 1$ if $n$ is even or $\rho \in R_{\mathrm{a}}$ if $n$ is odd. The other cases follow step by step. Note that in Case (iib), $n$ odd, it follows from the bipartiteness of the quotient graph $G_0$, that $s$ is even, and therefore $f(v) = \varphi_n(v)$ is a vertex-based solution. □

*Remark* 7.4. We can equivalently define the Laplacian as $\check{\Delta}^\rho_{G_0} = d^*_\rho d_\rho$ where $d_\rho$ is a "twisted" exterior derivative, defined via

$$d_\rho \colon \ell_2^\rho(V_0, \mathscr{H}) \longrightarrow \ell_2(E) \otimes \mathscr{H}, \qquad (d_\rho F)_e = F(\partial_+ e) - F(\partial_- e).$$

Moreover, one can show that the mapping

$$\widetilde{\Psi}_n \colon \ker d^*_\rho \longrightarrow \mathscr{N}_0^\rho(\lambda_n),$$

given by $\widetilde{\Psi}_n \eta = f_\eta$, and $f_{\eta,e}(x) = \eta_e \sin(n\pi x)$ for $n$ even is an isomorphism, i.e., the topological eigenfunctions of the twisted metric graph are related to the twisted cohomology $H^1_\rho(X_0, \mathscr{H}) := \ker d^*_\rho$. Defining the corresponding twisted homologies $H_1^\rho(X_0, \mathscr{H})$ as in Section 5, we obtain the statement analogue to Proposition 5.2.

Similarly, for $n$ odd we obtain the corresponding statements for the unoriented version $\bar{d}_\rho$ and the related (co-)homologies. We skip the details here, as well as an analysis of the twisted Betti numbers, since we do not need the precise spectral information of $\check{\Delta}^\rho_{X_0}$ for the existence of gaps.

We can now carry over the results of Proposition 7.2 to discrete graphs. Note that $\Delta^{\partial V}_H$ is equivalent to a square matrix of size $|V| - |\partial V|$ where $V = V(H)$. Similarly, $\check{\Delta}_H$ is described by an $|V| \times |V|$-matrix and $\check{\Delta}^\rho_{G_0}$ by an matrix of size $r|V_0|$ where $r = \dim \mathscr{H}$ and $V_0 = V(G_0)$. Moreover, $|V| - |\partial V| \le |V_0| \le |V|$.

**Proposition 7.5.** *Assume that $\rho$ is an $r$-dimensional representation (i.e., $\dim \mathscr{H} = r$). Then*[1]

$$\mu^{\partial V}_k \ge \mu^\rho_j \ge \mu_k, \qquad j = (k-1)r + 1, \dots, kr, \quad k = 1, \dots, |V_0|.$$

*In other words, the $j$-th $\rho$-equivariant eigenvalue is enclosed in the discrete Kirchhoff-Dirichlet interval*

$$\mu^\rho_j \in J_k = J_k(H, \partial V), \qquad j = (k-1)r + 1, \dots, kr, \quad k = 1, \dots, |V_0|.$$

Note that the discrete KD intervals are defined for $k \in \{1, \dots, |V|\}$ (see Definition 6.6), whereas the $\rho$-equivariant eigenvalues are given only for $k \le |V_0|$.

## 8. Residually finite coverings

We consider now *infinite* coverings with compact quotient graph and covering group $\Gamma$.

### 8.1. Abelian groups.

Let us start with Abelian covering groups $\Gamma$, for which we have the powerful tool of Floquet-(Bloch)-decomposition. We state the results only for the metric case, the discrete case can be treated similarly. The direct integral decomposition is of the form

$$\mathsf{L}_2(X) \cong \int^\oplus_{\widehat{\Gamma}} \mathsf{L}_2(Y), \qquad \Delta_X \cong \int^\oplus_{\widehat{\Gamma}} \Delta^\rho_{X_0}.$$

Since $\Gamma$ is Abelian, $\rho$ can be parametrised by $\vartheta \in \mathbb{R}^r$ via $\rho(\gamma) = \mathrm{e}^{\mathrm{i}\vartheta \cdot \gamma}$. We also write $\lambda^\vartheta_k$ for $\lambda^\rho_k$. For details we refer to [Sun08, Sec. 6] or [LP07] and the references therein. Moreover, from the direct integral decomposition and the continuous dependence of $\lambda^\rho_k$ on $\rho$, we deduce for the spectrum of the Kirchhoff Laplacian

$$\sigma(\Delta_X) = \bigcup_{\rho \in \widehat{\Gamma}} \sigma(\Delta^\rho_{X_0}) = \bigcup_{k \in \mathbb{N}} B_k \qquad \text{where} \qquad B_k := \{\, \lambda^\rho_k \mid \rho \in \widehat{\Gamma} \,\} \tag{8.1}$$

is called the *$k$-th band* and $\lambda^\rho_k$ denotes the $k$-th eigenvalue of the equivariant Laplacian $\Delta^\rho_{X_0}$. The next proposition is a direct consequence of Proposition 7.2:

**Proposition 8.1.** *Denote by $I_k = I_k(H, \partial V)$ the metric KD interval of the fundamental domain $H$ with vertices $V = V(H)$ and edges $E = E(H)$. Then we have*

$$\sigma(\Delta_X) = \bigcup_{k \in \mathbb{N}} B_k \subset \bigcup_{k \in \mathbb{N}} I_k.$$

*In particular, the bands $B_k$ with index $k = n|E| + |V| + 1 - \alpha_n, \dots, (n+1)|E|$, ($\alpha_n = 1$ if $G$ is bipartite or $G$ is not bipartite and $n$ odd, $\alpha_n = 0$ otherwise) are reduced to points $\{(n+1)^2\pi^2\}$. Moreover, if $\chi(H) = |V| - |E| \le \alpha_n - 1$ then $(n+1)^2\pi^2$ is an eigenvalue of infinite multiplicity for $\Delta_X$. The corresponding eigenspaces are generated by compactly supported edge-based (topological) eigenfunctions of the fundamental domain $H$ and its translates.*

---

[1] If $k > |V| - |\partial V|$, then there are no Dirichlet eigenvalues left. In this case, the inequality is understood as if we would have set $\mu^{\partial V}_k = 2$. This is consistent with the definition of the discrete KD intervals (see Definition 6.6).

8.2. **Residually finite groups.** The following construction of covering graphs is valid for the discrete and metric case by assuming that the projection respects the corresponding structure, i.e., they are graph morphisms respecting orientation in both cases, and additionally, they preserve the length functions.

Assume that $X_0$ is compact (i.e. finite for *discrete* graphs). Moreover, suppose that $\pi\colon X \longrightarrow X_0$ is a covering with covering group $\Gamma = \Gamma_0$. Corresponding to a normal subgroup $\Gamma_i \lhd \Gamma$ we associate a covering $\pi_i\colon X \longrightarrow X_i$ such that

$$\begin{array}{ccccc} & & X & & \\ & \overset{\pi_i}{\swarrow} \;\Gamma_i & & \Gamma\; \overset{\pi}{\searrow} & \\ X_i & & \xrightarrow[\Gamma/\Gamma_i]{p_i} & & X_0 \end{array} \tag{8.2}$$

is a commutative diagram. The groups under the arrows denote the corresponding covering groups.

**Definition 8.2.** A (countable, infinite) discrete group $\Gamma$ is residually finite if there exists a monotonely decreasing sequence of normal subgroups $\Gamma_i \lhd \Gamma$ such that

$$\Gamma = \Gamma_0 \rhd \Gamma_1 \rhd \cdots \rhd \Gamma_i \rhd \cdots, \quad \bigcap_{i \in \mathbb{N}} \Gamma_i = \{e\} \quad \text{and} \quad \Gamma/\Gamma_i \text{ is finite.} \tag{8.3}$$

Suppose now that $\Gamma$ is residually finite. Then there exists a corresponding sequence of coverings $\pi_i\colon X \longrightarrow X_i$ such that $p_i\colon X_i \longrightarrow X_0$ is a *finite* covering (cf. Diagram (8.2)). Such a sequence of covering maps is also called *tower of coverings*.

For more details on residually finite groups we refer to [LP08] and the references therein. The next proposition is provided by Adachi [Ad95] (see also [LP08, Sec. 5]). We just mention the geometric meaning of this algebraic condition: The covering space $X$ with residually finite group can be "exhausted" by the finite covering spaces $X_i$ as one uses in the next proposition. Its proof can be redone literally as in the manifold case.

**Proposition 8.3.** *Suppose $\Gamma$ is residually finite with the associated sequence of coverings $\pi_i\colon X \longrightarrow X_i$ and $p_i\colon X_i \longrightarrow X_0$ as in* (8.2). *Then*

$$\sigma(\Delta_X) \subseteq \overline{\bigcup_{i \in \mathbb{N}} \sigma(\Delta_{X_i})},$$

*and the Laplacian $\Delta_{X_i}$ w.r.t. the finite covering $p_i\colon X_i \longrightarrow X_0$ has discrete spectrum. Equality holds iff $\Gamma$ is amenable.*

Next we analyse the spectrum of the finite covering $X_i \to X_0$ as in [LP08]. Note that a fundamental domain for $X \to X_0$ can also be viewed as fundamental domain for *each* finite covering $X_i \to X_0$, $i \in \mathbb{N}$.

**Proposition 8.4.** *We have*

$$\sigma(\Delta_{X_i}) = \bigcup_{[\rho] \in \widehat{\Gamma/\Gamma_i}} \sigma(\Delta^\rho_{X_0}),$$

*where $\Delta^\rho_{X_0}$ is the equivariant Laplacian introduced in Definition 7.1 and $\Gamma/\Gamma_i$ is a finite group and $\widehat{G_i}$ its dual, i.e., the set of equivalence classes $[\rho]$ of unitary, irreducible representations $\rho$ of $\Gamma$.*

In particular, we have:

**Theorem 8.5.** *Suppose $X \to X_0$ is a $\Gamma$-covering of (not necessarily equilateral) metric graphs with fundamental domain $Y$, where $\Gamma$ is a residually finite group. Then*

$$\sigma(\Delta_X) \subset \bigcup_{k \in \mathbb{N}} I_k =: I,$$

*where* $I_k := [\lambda_k, \lambda_k^{\partial V}]$ *are the Kirchhoff-Dirichlet intervals associated to the fundamental domain* $Y$ *with boundary vertices* $\partial V$ *(see Definition 6.1). In particular, if* $M$ *is an interval with* $M \cap I = \emptyset$*, then* $M \cap \sigma(\Delta_X) = \emptyset$ *(i.e.,* $M$ *is a spectral gap).*

*Moreover if* $G$ *is bipartite, we have the spectral inclusion*

$$\sigma(\Delta_X) \subset \hat{I} \qquad \textit{where} \qquad \hat{I} := \bigcup_{n=0}^{\infty} \tau_n(I \cap K_n) \cap (I \cap K_n)$$

*and where* $\tau_n$ *is the spectral symmetry defined in Proposition 6.9.*

*Proof.* We have

$$\sigma(\Delta_X) \subseteq \overline{\bigcup_{i \in \mathbb{N}} \sigma(\Delta_{X_i})} = \overline{\bigcup_{i \in \mathbb{N}} \bigcup_{[\rho] \in \widehat{\Gamma}_i} \sigma(\Delta_{X_0}^{\rho})} \subseteq \overline{\bigcup_{k \in \mathbb{N}} I_k} = \bigcup_{k \in \mathbb{N}} I_k,$$

where we used Propositions 7.2, 8.3 and 8.4. The results for $\hat{I}$ follow from the spectral symmetry for $\Delta_X$. ☐

Similarly, in the discrete case, we conclude from Propositions 7.5, 8.3 and 8.4:

**Theorem 8.6.** *Suppose* $G \to G_0$ *is a* $\Gamma$*-covering with fundamental domain* $H$*, where* $\Gamma$ *is a residually finite group, then*

$$\sigma(\check{\Delta}_G) \subset \bigcup_{k=1}^{|V|} J_k =: J,$$

*where* $J_k := [\mu_k, \mu_k^{\partial V}]$ *are the discrete Kirchhoff-Dirichlet intervals associated to the fundamental domain* $H$ *with boundary vertices* $\partial V$ *(see Definition 6.6). In particular, if* $M \cap J = \emptyset$*, then* $M \cap \sigma(\check{\Delta}_G) = \emptyset$ *(i.e.,* $M$ *is a spectral gap).*

*Moreover if* $G$ *is bipartite, we have the spectral inclusion*

$$\sigma(\check{\Delta}_G) \subset \hat{J} \qquad \textit{where} \qquad \hat{J} := \theta(J) \cap J$$

*and where* $\theta(\mu) = 2 - \mu$ *is the spectral symmetry defined in Proposition 2.3.*

We refer to $I$ and $J$ as the *KD spectrum* and to $\hat{I}$ and $\hat{J}$ as the *symmetrised* KD spectrum.

Let us mention separately the case when $\Gamma$ is amenable:

**Theorem 8.7.** *Assume that the covering group of the covering is amenable. Then the number of components of* $\sigma(\Delta_X)$ *resp. of* $\sigma(\check{\Delta}_G)$ *is at least as large as the number of components of* $I$ *resp.* $J$ *or* $\hat{I}$ *resp* $\hat{J}$ *in the bipartite case.*

*Proof.* The assertion follows from the fact that, due to amenability, we have equality in Proposition 8.3. In particular, the spectrum of $\Delta_{X_0}$ is contained in $\Delta_X$. Moreover, the quotient spectrum is just the spectrum of $\Delta_{X_0}^{\rho}$ with the trivial representation $\rho = 1$. Therefore, the $k$-th eigenvalue $\lambda_k(X_0)$ of $\Delta_{X_0}$ is contained in $\sigma(\Delta_X)$ and also in the $k$-th KD interval $I_k$ due to Proposition 7.2. The discrete case follows similarly. ☐

## 9. Examples: covering graphs with spectral gaps

In this section we present several examples for which the KD intervals already guarantee the existence of spectral graphs. In some cases, the symmetrised KD spectrum is even *equal* to the $\mathbb{Z}^r$-periodic spectrum, see the bipartite examples below. For the concrete examples one only needs to calculate the spectra of the matrices associated to the discrete operators (see Eq. (2.8)) on a suitable chosen fundamental domain. For brevity, we skip the corresponding spectral results for metric graphs, since they can be obtained straightforward by the results of the previous sections.

In Examples 9.1–9.4, we consider "small" $\mathbb{Z}$-periodic graphs in order to show how our method works in simple examples in which the periodic spectrum can also be calculated directly. One can see that the KD intervals give "good" estimates of the actual location of the bands only for the first and second band. For a larger number of gaps guaranteed by the KD intervals, one should consider graphs with a smaller ratio $|\partial V|/|V|$. Of course, our method is more interesting for non-abelian (residually finite) groups with more than one generator, see Example 9.5.

Note that the choice of fundamental domain is arbitrary, and that the definition of the KD intervals will (in general) depend on the choice of the fundamental domain. Therefore, it might happen, that a "good" choice of fundamental domain leads to a union of the KD intervals having gaps. We do not precise the meaning of "good" here, but as in the case of manifolds and Schrödinger operators (see e.g. [HP03, LP07, LP08]) the fundamental domain should have "small" boundary in order to decouple from its neighbours. In our context, this means that a fundamental domain $H$ should contain a large number of vertices $V(H)$ and edges $E(H)$ compared to the number of boundary vertices $\partial V$, see Examples 9.3 and 9.4.

We start with a bipartite example already used in Figure 1.

**Example 9.1.** Let $G \to G_0$ be the periodic graph with fundamental domain $H$ as given in Figure 3. The spectrum of the discrete (Dirichlet) Laplacian is

$$\sigma(\check{\Delta}_H) = \{0, 1, 1, 1, 2\} \qquad \text{and} \qquad \sigma(\check{\Delta}_H^{\partial V}) = \Big\{1 - \frac{1}{\sqrt{3}}, 1, 1 + \frac{1}{\sqrt{3}}\Big\},$$

resp., where repeated numbers correspond to multiple eigenvalues, so that the KD intervals are

$$J_1 = \Big[0, 1 - \frac{1}{\sqrt{3}}\Big], \quad J_2 = \{1\}, \quad J_3 = \Big[1, 1 + \frac{1}{\sqrt{3}}\Big], \quad J_4 = [1, 2].$$

The equivariant spectrum for $\rho(\gamma) = \mathrm{e}^{\mathrm{i}\vartheta\gamma}$ is

$$\sigma(\check{\Delta}_{G_0}^{\vartheta}) = \Big\{1 - \sqrt{\frac{2 + \cos\vartheta}{3}}, 1, 1, 1 + \sqrt{\frac{2 + \cos\vartheta}{3}}\Big\}.$$

In particular, the bands $\check{B}_k := \{\, \lambda_k^{\vartheta} \mid \vartheta \in [0, 2\pi] \,\}$ (see Eq. (8.1)) are

$$\check{B}_1 = \Big[0, 1 - \frac{1}{\sqrt{3}}\Big], \quad \check{B}_2 = \check{B}_3 = \{1\}, \quad \check{B}_4 = \Big[1 + \frac{1}{\sqrt{3}}, 2\Big].$$

We see that the first and second band agree with the corresponding KD intervals. In particular, the KD intervals detect the first gap $(1-1/\sqrt{3}, 1)$ *precisely*. Moreover, the second KD interval reduces to a point as well as the second band. But the third KD interval is too rough, and the second gap is not detected. See also Figure 1 for the spectral relation with the corresponding metric graph. Nevertheless, since $G$ is bipartite, we also have the spectral inclusion for the symmetrised KD spectrum $\hat{J}$. Here, we even have the equality $\sigma(\check{\Delta}_G) = \hat{J}$ (see Theorem 8.6), showing that the KD intervals can give the actual spectrum of the covering using the spectral symmetry for bipartite graphs.

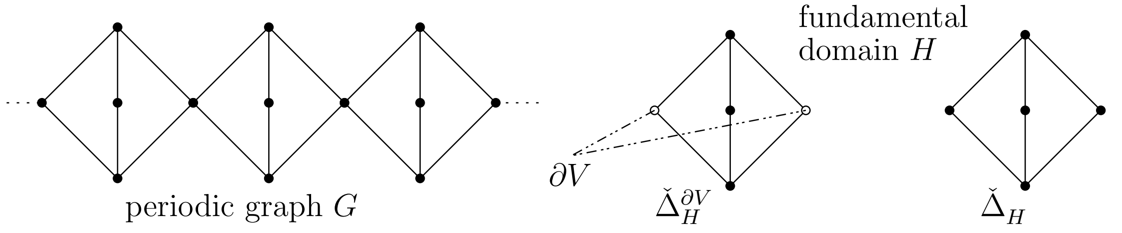


FIGURE 3. A bipartite graph. The related spectral information of this graph is visualised in Figure 1. The full vertices correspond to Kirchhoff conditions and the open vertices correspond to Dirichlet conditions for the associated metric graph.

The next example is a non-bipartite one:

**Example 9.2.** Let $G \to G_0$ be the periodic graph with fundamental domain $H$ as given in Figure 4. The spectrum of the discrete (Dirichlet) Laplacian is

$$\sigma(\check{\Delta}_H) = \Big\{0, \frac{7-\sqrt{13}}{6}, \frac{4}{3}, \frac{4}{3}, \frac{7+\sqrt{13}}{6}\Big\} \qquad \text{and} \qquad \sigma(\check{\Delta}_H^{\partial V}) = \Big\{\frac{1}{3}, \frac{4}{3}, \frac{4}{3}\Big\},$$

resp., where repeated numbers correspond to multiple eigenvalues, so that the KD intervals are

$$J_1 = \Big[0, \frac{1}{3}\Big], \quad J_2 = \Big[\frac{7-\sqrt{13}}{6}, \frac{4}{3}\Big], \quad J_3 = \Big\{\frac{4}{3}\Big\}, \quad J_4 = \Big[\frac{4}{3}, 2\Big].$$

The spectrum of the periodic operator is given by the bands

$$\check{B}_1 = \Big[0, 1 - \frac{\sqrt{5}}{3}\Big], \quad \check{B}_2 = \Big[\frac{2}{3}, \frac{4}{3}\Big], \quad \check{B}_3 = \Big\{\frac{4}{3}\Big\}, \quad \check{B}_4 = \Big[\frac{4}{3}, 1 + \frac{\sqrt{5}}{3}\Big].$$

Here, only the degenerated band $\check{B}_3$ agrees with the KD interval. The corresponding eigenfunction is indicated by the values in Figure 4. Nevertheless, the maximal spectral gap in this example $(1 - \sqrt{5}/3, 2/3) \approx (0.25, 0.66)$ is detected approximately by the KD interval giving the spectral gap $(1/3, (7 - \sqrt{13})/6) \approx (0.33, 0, 57)$. The fourth KD interval gives a too rough upper bound, see also Figure 2 for the spectral relation with the corresponding metric graph.

Note that the (metric) KD intervals do not detect the gap between the nineth and tenth band, the KD intervals even overlap (see Figure 2), whereas the gap between the sixth and seventh band is recognised.

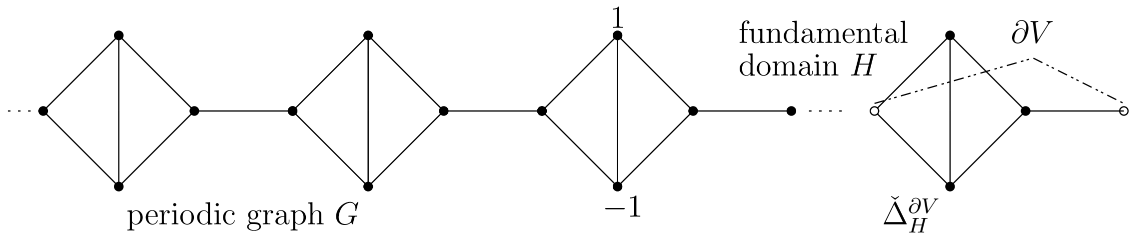


FIGURE 4. A non-bipartite graph. The related spectral information of this graph is visualised in Figure 2. The Laplacian on $H$ without Dirichlet conditions is not plotted here. The two values $\pm 1$ at the vertices indicate the eigenfunction associated to the eigenvalue $\lambda_3 = 4/3$, independent of $\vartheta$; the other vertex values being 0.

The following example (see also [AEL94], where the band-gap ratio of such "onion-like" periodic metric graphs is considered) gives an idea of how to generate gaps by multiple edges:

**Example 9.3.** Let $G \to G_0$ be the periodic graph with fundamental domain $H$ as given in Figure 5 having $r$ repeated edges. The spectrum of the discrete (Dirichlet) Laplacian is

$$\sigma(\check{\Delta}_H) = \Big\{0, 1 - \frac{1}{r+1}, 1 + \frac{1}{r+1}, 2\Big\} \qquad \text{and} \qquad \sigma(\check{\Delta}_H^{\partial V}) = \Big\{\frac{1}{r+1}, 2 - \frac{1}{r+1}\Big\},$$

respectively. The KD intervals are

$$J_1 = \Big[0, \frac{1}{r+1}\Big], \quad J_2 = \Big[1 - \frac{1}{r+1}, 2 - \frac{1}{r+1}\Big] \quad \text{and} \quad J_3 = \Big[1 + \frac{1}{r+1}, 2\Big].$$

Note that as far as $r \geq 2$, we have spectral gaps between the first and second KD interval. Moreover, the KD intervals reduce to the point $\{0\}$ for $k = 1$ and to the interval $[1, 2]$ for $k = 2, 3$ as $r \to \infty$. The spectrum of the periodic operator is given by the bands

$$\check{B}_1 = \Big[0, \frac{1}{r+1}\Big], \quad \check{B}_2 = \Big[1 - \frac{1}{r+1}, 1 + \frac{1}{r+1}\Big], \quad \check{B}_3 = \Big[2 - \frac{1}{r+1}, 2\Big],$$

and only the first KD interval $J_1$ agrees with the first band $\check{B}_1$. Note that in this case, the periodic and antiperiodic equivariant eigenvalues ($\vartheta = 0$ and $\vartheta = \pi$) give already the band edges. For groups with more than one generator, the band edges need not to be on the boundary of the Brillouin zone, see [HKSW07] and appear as KD eigenvalues, but with alternating role ($B_k = [\lambda_k^0, \lambda_k^\pi]$ for $k = 1, 3$ and $B_2 = [\lambda_2^\pi, \lambda_2, 0]$). This phenomena also appears for Schrödinger operators (see [KP07] and the references therein).

Nevertheless, the graph is bipartite, so we can use the spectral symmetry and indeed, we have equality $\sigma(\check{\Delta}_G) = \hat{J}$. Again, the symmetrised KD spectrum gives already the precise spectral information.

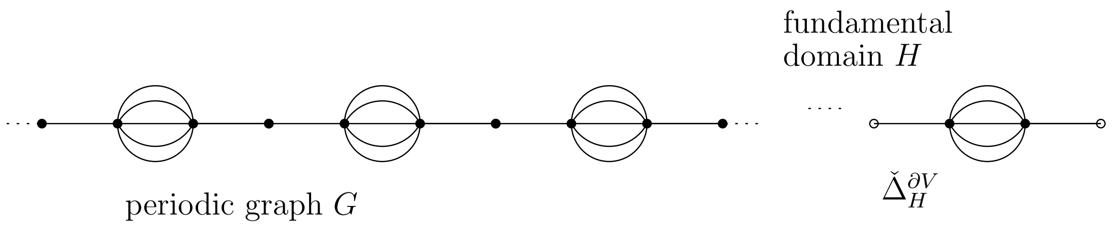


FIGURE 5. Generating gaps by multiple edges. Here, we replaced the middle edge by $r = 5$ edges.

A similar result holds by attaching self-loops to a graph:

**Example 9.4.** Let $G \to G_0$ be the periodic graph with fundamental domain $H$ being a line graph with three vertices and two edges, and $r$ loops attached to the middle vertex. The boundary vertices have degree 1, and the middle vertex has degree $2(r + 1)$. Note that $G$ is not bipartite as long as $r \geq 1$. The spectrum of the discrete (Dirichlet) Laplacian can be calculated as

$$\sigma(\check{\Delta}_H) = \left\{0, 1, 1 + \frac{1}{r+1}\right\} \qquad \text{and} \qquad \sigma(\check{\Delta}_H^{\partial V}) = \left\{\frac{1}{r+1}\right\}.$$

The KD intervals are

$$J_1 = \left[0, \frac{1}{r+1}\right] \qquad \text{and} \qquad J_2 = [1, 2]$$

Note that as far as $r \geq 1$, we have a spectral gap between the two KD intervals. Moreover, the first KD intervals reduce to the point $\{0\}$ as $r \to \infty$. The spectrum of the periodic operator is given by the bands

$$\check{B}_1 = \left[0, \frac{1}{r+1}\right] \qquad \text{and} \qquad \check{B}_2 = \left[1, 1 + \frac{1}{r+1}\right]$$

and again, only the first KD interval $J_1$ agrees with the first band $\check{B}_1$.

We finally present an example with two generators. This example serves aleo as an example for coverings with non-abelian groups.

**Example 9.5.** Let $G \to G_0$ be the $\mathbb{Z}^2$-periodic graph with fundamental domain $H$ as given in Figure 6. One can also construct other coverings associated to a group with two generators by gluing together appropriate copies of the fundamental domain according to the Cayley graph associated with this generator set. The discrete (Dirichlet) Laplacian is

$$\sigma(\check{\Delta}_H) = \left\{0, 1 - \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}, \frac{1}{2}, 1, 1, 1, 1, 1, \frac{3}{2}, 1 + \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}, 2\right\},$$
$$\sigma(\check{\Delta}_H^{\partial V}) = \left\{1 - \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, 1 + \frac{\sqrt{3}}{2}\right\},$$

resp., where repeated numbers correspond to multiple eigenvalues, so that the KD intervals are

$$J_1 = \Bigl[0, 1 - \frac{\sqrt{3}}{2}\Bigr] \approx [0, 0.13], \quad J_2 = J_3 = \Bigl[1 - \frac{1}{\sqrt{2}}, \frac{1}{2}\Bigr] \approx [0.29, 0.5]$$

$$J_4 = \Bigl[\frac{1}{2}, 1\Bigr], \quad J_5 = J_6 = \{1\}, \quad J_7 = J_8 = \Bigl[1, \frac{3}{2}\Bigr] \quad \text{and} \quad J_9 = \Bigl[1, 1 + \frac{\sqrt{3}}{2}\Bigr] \approx [1, 1.87].$$

It is easily seen that there is a spectral gap only between the first and second KD interval. All other intervals overlap. Note that the graph is bipartite, so there there is another gap due to the spectral symmetry.

Here, we can also calculate the spectrum of the $\mathbb{Z}^2$-periodic graph using the Floquet theory (8.1). The periodic ($\vartheta = (0,0)$) and antiperiodic ($\vartheta = (\pi, \pi)$) spectrum is given by

$$\sigma(\check{\Delta}_{G_0}^{(0,0)}) = \Bigl\{0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2\Bigr\},$$

$$\sigma(\check{\Delta}_{G_0}^{(\pi,\pi)}) = \Bigl\{1 - \frac{\sqrt{3}}{2}, 1 - \frac{1}{\sqrt{2}}, 1 - \frac{1}{\sqrt{2}}, 1, 1, 1, 1, 1, 1 + \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}, 1 + \frac{\sqrt{3}}{2}\Bigr\},$$

respectively. Due to the continuous dependence on $\theta$ and the connectedness of $\theta \in [0, 2\pi]^2$, we conclude that the $k$-th band contains the interval $\widetilde{J}_k$ given by the minimum and maximum of the $k$-th periodic and antiperiodic eigenvalues. But for $k = 1, \dots, 6$, the interval $\widetilde{J}_k$ is already the $k$-th KD interval $J_k$, so that $B_k = J_k$ for these $k$. Moreover, $\check{B}_5$ and $\check{B}_6$ are flat bands. Using the spectral symmetry from the bipartiteness, we conclude that the symmetrised KD spectrum $\hat{J}$ already give the spectrum of the $\mathbb{Z}^2$-periodic operator $\check{\Delta}_G$. In particular, the KD intervals give an efficient method to calculate the spectrum of the $\mathbb{Z}^2$-periodic Laplacian with a minimum of calculations needed: We only have to find "good" candidates for $\vartheta$, and do not need the spectrum of $\check{\Delta}_{G_0}^{\vartheta}$ for general $\vartheta$.

Theorem 8.6 assures that $(1 - \sqrt{3}/2, 1 - 1\sqrt{2})$ and $(1 + 1/\sqrt{2}, 1 + \sqrt{3}/2)$ never belongs to the spectrum of *any* covering having $H$ as fundamental domain, in particular for the tree-like graph with covering group $\mathbb{Z} * \mathbb{Z}$, the free group with two generators. Moreover, 1 is always an eigenvalue.

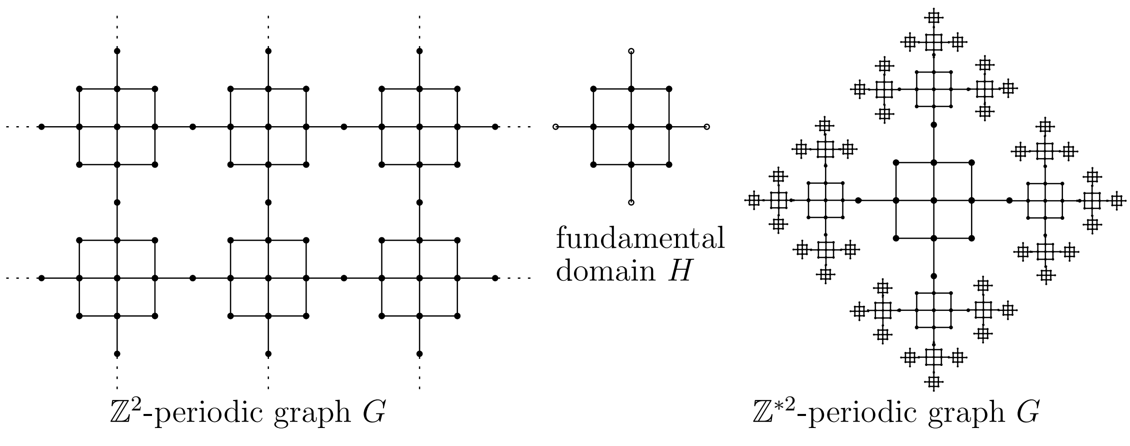


Figure 6. An example with a covering group having two generators. The fundamental domain has 13 vertices and four boundary vertices. On the left, the covering graph with Abelian group is plotted. In this case, the symmetrised KD spectrum $\hat{J}$ already give the spectrum of the covering operator. On the right, we have a $\Gamma$-covering with $\Gamma = \mathbb{Z}^{*2} = \mathbb{Z} * \mathbb{Z}$, the free (non-abelian) group with two generators. Here, we only have the spectral estimate $\sigma(\check{\Delta}_{G^*}) \subset \hat{J}$.

*Remark* 9.6. We could also use (vertex) *Neumann* conditions as lower bound on the equivariant metric eigenvalue instead of the *Kirchhoff* ones. A function $f$ satisfies the (vertex) Neumann condition in a vertex $v \in \partial V$ iff $f_e'(v) = 0$ for each edge $e \in E_v$. Denote by $\mathsf{H}^1_{\partial V,\mathrm{N}}(X)$ the space of functions $f \in \mathsf{H}^1_{\max}(X)$ being continuous in each *inner* vertex, i.e., we do not assume continuity at boundary vertices. Now, we have the additional inclusion $\mathsf{H}^1(X) \subset \mathsf{H}^1_{\partial V,\mathrm{N}}(X) \subset \mathsf{H}^1_{\max}(X)$ in (3.3) and the opposite inequality for the eigenvalues, and a similar statement as in Proposition 7.2 with the Kirchhoff eigenvalue replaced by the vertex Neumann one as lower bound. But a direct calculation of the corresponding eigenvalues (e.g. in Example 9.1) shows, that the corresponding Neumann-Dirichlet intervals do not reveal the spectral gap.

That our Kirchhoff-Dirichlet bracketing is optimal is shown in Examples 9.1 and 9.5, where the (symmetrised) KD spectrum is *exactly* the spectrum of the $\mathbb{Z}^2$-periodic graph (and not only a superset).

## 10. Outlook

We only considered simple examples in which the eigenvalue bracketing guarantees the existence of spectral gaps. It would be interesting to provide quantities estimating the actual number of gaps (at least for Abelian groups $\Gamma = \mathbb{Z}^r$ or amenable groups). As mentioned above, a naive guess would be that the ratio $|\partial V|/|E(H)|$ is related to the number of graphs in the union of the KD intervals (the smaller the ratio is, the more gaps should open up). Moreover, decorations of the graph (like multiple edges or loops, see also [AS00]) should provide examples with open gaps, as the examples in Section 9 indicate. Again, a more systematic treatment would be interesting.

If the groups $\Gamma$ of the covering is amenable and residually finite, we provide a lower bound on the number of spectral gaps. The amenability condition is only needed in order to assure that each KD interval contains at least one spectral point (namely, an eigenvalue of the quotient space). This condition might be weakened, but it is a priori not clear what representation $\rho$ leads to an equivariant eigenvalue inside the KD interval (see Propositions 8.3 and 8.4). In the case of manifolds, we guaranteed the existence of spectrum inside the Neumann-Dirichlet intervals for residually finite groups by the fact that the Dirichlet and Neumann eigenvalues of a suitable chosen fundamental domain were close to each other (cf. [LP08, Thm. 3.3]). An upper bound is given once the covering group has positive Kadison constant (see [Sun92]).

Homology groups have also been used for metric graph Laplacians with *magnetic field*, see [KS03] for details. The type of spectrum for magnetic Laplacians on a metric equilateral square lattice was analysed in [BGP07], and, in particular, for irrational flux, the spectrum has Cantor structure. Magnetic Laplacians may be seen as a generalisation of equivariant Laplacians for Abelian coverings treated in detail in Section 7. It would be interesting to see how the eigenvalue bracketing can be applied to this case in order to make non-trivial statements about the nature of the spectrum of discrete and metric magnetic Laplacians.

Another point we do not address here is the appearance of "degenerated" bands, i.e., eigenvectors localised inside a fundamental domain leading to a spectral band reduced to a point. For metric graphs, this often happens for the exceptional values $\lambda = n^2\pi^2$, but this fact can also happen away from these points, and therefore also for the discrete graph (see Examples 9.2 and 9.5). Moreover, we do not analyse the band-gap ratio which may be estimated by from above by the correspondig ratio for the KD intervals (see [AEL94] and Example 9.3).

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