# CONTINUITY OF THE INTEGRATED DENSITY OF STATES ON RANDOM LENGTH METRIC GRAPHS

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Abstract. We establish several properties of the integrated density of states for random quantum graphs: Under appropriate ergodicity and amenability assumptions, the integrated density of states can be defined using an exhaustion procedure by compact subgraphs. A trace per unit volume formula holds, similarly as in the Euclidean case. Our setting includes periodic graphs. For a model where the edge lengths are random and vary independently in a smooth way we prove a Wegner estimate and related regularity results for the integrated density of states.

These results are illustrated for an example based on the Kagome lattice. In the periodic case we characterise all compactly supported eigenfunctions and calculate the position and size of discontinuities of the integrated density of states.

## 1. Introduction

Quantum graphs are Laplace or Schrödinger operators on metric graphs. As structures intermediate between discrete and continuum objects they have received quite some attention in recent years in mathematics, physics and material sciences, see e.g. the recent proceeding volume [EKK$^+$08] for an overview.

Here, we study periodic and random quantum graphs. Our results concern spectral properties which are related to the integrated density of states (IDS), sometimes called spectral distribution function. As in the case of random Schrödinger operators in Euclidean space, disorder may enter the operator via the potential. Moreover, and this is specific to quantum graphs, randomness may also influence the characteristic geometric ingredients determining the operator, viz.

- the lengths of the edges of the metric graph and
- the vertex conditions at each junction between the edges.

In the present paper we pay special attention to randomness in these geometric data. Our results may be summarised as follows. For quite wide classes of quantum graphs we establish

- the existence, respectively the convergence in the macroscopic limit, of the integrated density of states under suitable ergodicity and amenability conditions (see Theorem 2.6),
- a trace per unit volume formula for the IDS (see equation (2.9)),

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- a Wegner estimate for random edge length models (assuming independence and smoothness for the disorder) (Theorem 2.9). This implies quantitative continuity estimates for the IDS (Corollary 2.10).

These abstract results are illustrated by the thorough discussion of an example concerning a combinatorial and a metric graph based on the Kagome lattice. In this case we calculate positions and sizes of all jumps of the IDS. Our results show the smoothing of the IDS via randomness.

Of course, there are various previous results dealing with related topics. They include results on spectral localisation/delocalisation properties of random quantum graphs with tree structure [ASW06, HP06] and spectral localisation of metric graphs with $\mathbb{Z}^d$ structure [EHS07, Hel07, KP08]. Concerning our specific results we may mention the following: Our construction of the IDS is related to the construction of the IDS for abstract random operators done in [Len99, LPV07]. Our Wegner estimates are closely related to works for Schrödinger operators on metric graphs with random potential [HV07, GV08, GHV08]. Discontinuities of the integrated density of states and compactly supported eigenfunctions have been studied for discrete and quantum graphs in quite a few publications of which we mention [Kuc91, MY02, KLS03, MSY03, KS04, Ves05, Kuc05, KLPS06, GLV07, LMV08, LV08]. Continuity properties of the IDS for models where randomness enters via the geometry or in a non-monotone way have been obtained earlier in [LPV04, LPPV08] and [Klo95, Ves02, HK02] respectively. It goes without saying that our proofs rely on many ideas developed for random Schrödinger operators in Euclidean space. For an overview on this topic see [Ves07].

The article is organised as follows: In the next section, we introduce the random length model and state the main results. In Section 3 we present the Kagome lattice example. In Section 4 we prove Theorem 2.6 concerning the approximability of the IDS. Finally, in Section 5 we prove the Wegner estimate Theorem 2.9.

**Note added:** When we were completing this work, the preprint [KP08b] appeared on arXiv.org, which includes a proof of a Wegner estimate for a model with $\mathbb{Z}^d$-structure related to ours, albeit using different methods.

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## 2. Basic notions, model and results

In the following subsections, we fix basic notions (metric graphs, Laplacians and Schrödinger operators with vertex conditions), introduce the random length model and state our main results. For general treatments and further references on metric graphs, we refer to [EKK$^+$08].

2.1. **Metric graphs.** Since our random model concerns a perturbation of the *metric* structure of a graph, we carefully distinguish between *combinatorial*, *topological* and *metric graphs*. A *combinatorial graph* $G = (V, E, \partial)$ is given by a countable vertex set $V$, a countable set $E$ of edge labels and a map $\partial(e) = \{v_1, v_2\}$ from the edge labels to (unordered) pairs of vertices. If $v_1 = v_2$, we call $e$ a *loop*.

Note that this definition allows multiple edges, but we only consider locally finite combinatorial graphs, i.e., every vertex has only finitely many adjacent edges. A topological graph $X$ is a topological model of a combinatorial graph together with a choice of directions on the edges:

**Definition 2.1.** A *(directed) topological graph* is a CW-complex $X$ containing only (countably many) 0- and 1-cells. The set $V = V(X) \subset X$ of 0-cells is called the *set of vertices*. The 1-cells of $X$ are called *(topological) edges* and are labeled by the elements of $E = E(X)$ (the *(combinatorial) edges*), i.e., for every edge $e \in E$, there is a continuous map $\Phi_e \colon [0,1] \longrightarrow X$ whose image is the corresponding (closed) 1-cell, and $\Phi_e \colon (0,1) \longrightarrow \Phi_e((0,1)) \subset X$ is a homeomorphism. A 1-cell is called a *loop* if $\Phi_e(0) = \Phi_e(1)$. The map $\partial = (\partial_-, \partial_+) \colon E \longrightarrow V \times V$ describes the direction of the edges and is defined by

$$\partial_- e := \Phi_e(0) \in V, \qquad \partial_+ e := \Phi_e(1) \in V.$$

For $v \in V$ we define

$$E_v^{\pm} = E_v^{\pm}(X) := \{\, e \in E \mid \partial_{\pm} e = v \,\}.$$

The set of all adjacent edges is defined as the *disjoint* union[1]

$$E_v = E_v(X) := E_v^+(X) \mathbin{\dot\cup} E_v^-(X).$$

The *degree* of a vertex $v \in V$ in $X$ is defined as

$$\deg v = \deg_X(v) := |E_v| = |E_v^+| + |E_v^-|.$$

A *topological subgraph* $\Lambda$ is a CW-subcomplex of $X$, and therefore $\Lambda$ is itself a topological graph with (possible empty) boundary $\partial\Lambda := \Lambda \cap \overline{\Lambda^{\mathrm{c}}} \subset V(X)$.

Since a topological graph is a topological space, we can introduce the space $\mathsf{C}(X)$ of $\mathbb{C}$-valued continuous functions and the associated notion of measurability. A metric graph is a topological graph where we assign a length to every edge.

**Definition 2.2.** A *(directed) metric* graph $(X, \ell)$ is a topological graph $X$ together with a *length function* $\ell \colon E(X) \longrightarrow (0, \infty)$. The length function induces an identification of the interval $I_e := [0, \ell(e)]$ with the edge $\Phi_e([0,1])$ (up to the end-points of the corresponding 1-cell, which may be identified in $X$ if $e$ is a loop) via the map

$$\Psi_e \colon I_e \longrightarrow X, \qquad \Psi_e(x) = \Phi_e\Big(\frac{x}{\ell(e)}\Big).$$

Note that every topological graph $X$ can be canonically regarded as a metric graph where *all edges have length one*. The corresponding length function $\mathbb{1}_{E(X)}$ is denoted by $\ell_0$. In our random model, we will consider a fixed topological graph $X$ with a random perturbation $\ell_\omega$ of this length function $\ell_0$.

To simplify matters, we canonically identify a metric graph $(X, \ell)$ with the disjoint union $X_\ell$ of the intervals $I_e$ for all $e \in E$ subject to appropriate identifications of the end-points of these intervals (according to the combinatorial structure of the graph), namely

$$X_\ell := \dot{\bigcup_{e \in E}} I_e / {\sim}.$$

[1]The disjoint union is necessary in order to obtain *two* different labels in $E_v(X)$ for a loop.

The coordinate maps $\{\Psi_e\}_e$ can be glued together to a map

$$\Psi_\ell \colon X_\ell \longrightarrow X. \tag{2.1}$$

*Remark* 2.3. A metric graph is canonnically equpped with a metric and a measure. Given the information about the lenght of edges, each path in $X_\ell$ has a well defined lenght. The distance between two arbitrary points $x, y \in X_\ell$ is defined as the infimum of the lenghts of paths joining the two points. The measure on $X_\ell$ is defined in the following way. For each measurable $\Lambda \subset X$ the sets $\Lambda \cap \psi_e(I_e)$ are measurable as well, and are assigned the Lebesgue measure of the preimage $\psi_e^{-1}(\Lambda \cap \psi_e(I_e))$. Consequently, we define the volume of $\Lambda$ by

$$\operatorname{vol}(\Lambda, \ell) := \sum_{e \in E} \psi_e^{-1}(\Lambda \cap \psi_e(I_e)) \tag{2.2}$$

Using the identification (2.1), we define the function space $\mathsf{L}_2(X, \ell)$ as

$$\mathsf{L}_2(X, \ell) := \bigoplus_{e \in E} \mathsf{L}_2(I_e), \qquad f = \{f_e\}_e \quad \text{with } f_e \in \mathsf{L}_2(I_e) \text{ and}$$

$$\|f\|^2_{\mathsf{L}_2(X,\ell)} = \sum_{e \in E} \int_{I_e} |f_e(x)|^2 \, \mathrm{d}x.$$

2.2. **Operators and vertex conditions.** For a given metric graph $(X, \ell)$, we introduce the operator

$$(Df)_e(x) = (D_\ell f)_e(x) = \frac{df_e}{dx}(x),$$

where the derivative is taken in the interval $I_e = [0, \ell(e)]$. Note that both the norm in $\mathsf{L}_2(X, \ell)$ and $D = D_\ell$ depend on the length function. This observation is particularly important in our random length model below, where we perturb the canonical length function $\ell_0 = \mathbb{1}_{E(X)}$ and therefore have (a priori) different spaces on which a function $f$ lives. Our point of view is that $f$ is a function on the fixed underlying topological graph $X$, and that the metric spaces are canonically identified via the maps $\Psi_{\ell_0}^{-1} \circ \Psi_\ell \colon (X, \ell) \longrightarrow (X, \ell_0)$. One easily checks that

$$\|f\|^2_{\mathsf{L}_2(X,\ell)} = \sum_{e \in E} \ell(e) \int_{(0,1)} |f_e(x)|^2 \, \mathrm{d}x, \tag{2.3a}$$

$$(D_\ell f)_e(x) = \frac{1}{\ell(e)} (D_{\ell_0} f)_e\Big(\frac{1}{\ell(e)} x\Big), \tag{2.3b}$$

where $f_e$ and $D_{\ell_0} f$ on the right side are considered as functions on $[0, 1]$ via the identification $\Psi_{\ell_0}^{-1} \circ \Psi_\ell$.

Next we introduce general vertex conditions for Laplacians $\Delta_{(X,\ell)} = -{D_\ell}^2$ and Schrödinger operators $H_{(X,\ell)} = \Delta_{(X,\ell)} + q$ with real-valued potentials $q \in \mathsf{L}_\infty(X)$. The *maximal* or *decoupled* Sobolev space of order $k$ on $(X, \ell)$ is defined by

$$\mathsf{H}^k_{\max}(X, \ell) := \bigoplus_{e \in E} \mathsf{H}^k(I_e)$$

$$\|f\|^2_{\mathsf{H}^k_{\max}(X,\ell)} := \sum_{e \in E} \|f_e\|^2_{\mathsf{H}^k(I_e)}.$$

Note that $D_\ell\colon \mathsf{H}^{k+1}_{\max}(X,\ell) \longrightarrow \mathsf{H}^{k}_{\max}(X,\ell)$ is a bounded operator. We introduce the following two different evaluation maps $\mathsf{H}^1_{\max}(X,\ell) \longrightarrow \bigoplus_{v\in V}\mathbb{C}^{E_v}$:

$$\underline{f}_e(v) := \begin{cases} f_e(0), & \text{if } v = \partial_- e,\\ f_e(\ell(e)), & \text{if } v = \partial_+ e, \end{cases} \quad \text{and} \quad \underset{\longrightarrow}{f}_e(v) := \begin{cases} -f_e(0), & \text{if } v = \partial_- e,\\ f_e(\ell(e)), & \text{if } v = \partial_+ e, \end{cases}$$

and $\underline{f}(v) = \{\underline{f}_e(v)\}_{e\in E_v} \in \mathbb{C}^{E_v}$, $\underset{\longrightarrow}{f}(v) = \{\underset{\longrightarrow}{f}_e(v)\}_{e\in E_v} \in \mathbb{C}^{E_v}$. It follows from standard Sobolev estimates (see e.g. [Kuc04, Lem. 8]) that these evaluation maps are bounded by $\max\{(2/\ell_{\min})^{1/2}, 1\}$, provided the minimal edge length

$$0 < \ell_{\min} := \inf_{e\in E} \ell(e) \tag{2.4}$$

is strictly positive. The second evaluation map is used in connection with the derivative $Df$ of a function $f \in \mathsf{H}^2_{\max}(X,\ell)$. Note that $\underset{\longrightarrow}{Df}$ is independent of the orientation of the edge.

A *single-vertex condition* at $v \in V$ is given by a Lagrangian subspace $L(v)$ of the Hermitian symplectic vector space $(\mathbb{C}^{E_v} \oplus \mathbb{C}^{E_v}, \eta_v)$ with canonical two-form $\eta_v$ defined by

$$\eta_v((x,x'),(y,y')) := \langle x', y\rangle - \langle x, y'\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard unitary inner product in $\mathbb{C}^{E_v}$. The set of all Lagrangian subspaces of $(\mathbb{C}^{E_v} \oplus \mathbb{C}^{E_v}, \eta_v)$ is denoted by $\mathscr{L}_v$ and has a natural manifold structure (see, e.g., [Har00, KS99] for more details on these notions). A Lagrangian subspace $L(v)$ can uniquely be described by the pair $(Q(v), R(v))$ where $Q(v)$ is an orthogonal projection in $\mathbb{C}^{E_v}$ with range $\mathscr{G}(v) := \operatorname{ran} Q(v)$ and $R(v)$ is a symmetric operator on $\mathscr{G}(v)$ such that

$$L(v) := \{\, (x,x') \,|\, (1 - Q(v))x = 0, \quad Q(v)x' = R(v)x \,\} \tag{2.5}$$

(see e.g. [Kuc04]).

A field of single-vertex conditions $L := \{L(v)\}_{v\in V}$ is called a *vertex condition*. We say that $L$ is *bounded*, if

$$C_R := \sup_{v\in V} \|R(v)\| < \infty, \tag{2.6}$$

where the norm is the operator norm on $\mathscr{G}(v)$. For any such bounded vertex condition $L$, a bounded potential $q$ and a metric graph $(X,\ell)$ with $\ell_{\min} > 0$, we obtain a self-adjoint Schrödinger operator $H_{(X,\ell),L} = \Delta_{(X,\ell),L} + q$, by choosing the domain

$$\operatorname{dom} H_{(X,\ell),L} := \{\, f \in \mathsf{H}^2_{\max}(X,\ell) \,|\, (\underline{f}(v), \underset{\longrightarrow}{Df}(v)) \in L(v) \text{ for all } v \in V \,\}.$$

Of particular interest are the following vertex conditions with vanishing vertex operator $R(v) = 0$ for all $v \in V$: *Dirichlet* vertex conditions (where $L(v) = \{0\} \oplus \mathbb{C}^{E_v}$ or $\mathscr{G}(v) = \{0\}$), *Kirchhoff* (also known as *free*) vertex conditions (where $(x,x') \in L(v)$ if all components of $x$ are equal and the sum of all components of $x'$ add up to zero, or equivalently $\mathscr{G}(v) = \mathbb{C}(1,\dots,1)$) and *Neumann* vertex conditions (where $L(v) = \mathbb{C}^{E_v} \oplus \{0\}$ or equivalently $\mathscr{G}(v) = \mathbb{C}^{E_v}$).

2.3. **Random length model.** The underlying *geometric* structure of a random length model is a random length metric graph. A *random length metric graph* is based on a fixed topological graph $X$ with $V$ and $E$ the sets of vertices and edges of $X$, a probability space $(\Omega, \mathbb{P})$, and a *measurable* map $\ell\colon \Omega \times E \longrightarrow (0, \infty)$, which describes the random dependence of the edge lengths. We also assume that there are $\omega$-independent constants $\ell_{\min}, \ell_{\max} > 0$ such that $\ell_{\min} \le \ell_\omega(e) \le \ell_{\max}$ for all $\omega \in \Omega$ and $e \in E$. We will use the notation $\ell_\omega(e) := \ell(\omega, e)$.

A *random length model* associates to such a geometric structure $(X, \Omega, \mathbb{P}, \ell)$ a random family of Schrödinger operators $H_\omega$, by additionally introducing *measurable* maps $L(v)\colon \Omega \longrightarrow \mathscr{L}_v$ for all $v \in V$, and $q\colon \Omega \times X \longrightarrow \mathbb{R}$, describing the random dependence of the vertex conditions and the potentials of these operators. We will use the notation $L_\omega := \{L_\omega(v)\}_{v \in V}$ and $q_\omega(x) = q(\omega, x)$. We assume that we have constants $C_R, C_{\mathrm{pot}} > 0$ such that

$$\|q_\omega\|_\infty \le C_{\mathrm{pot}} \qquad \text{and} \qquad \|R_\omega(v)\| \le C_R \tag{2.7}$$

for almost all $\omega \in \Omega$ and all $v \in V$, where $R_\omega(v)$ is the vertex operator associated to $L_\omega(v)$. From (2.7) and the lower length bound (2.4) it follows that the Schrödinger operators $H_\omega := \Delta_\omega + q_\omega$ are self-adjoint and bounded from below by some constant $\lambda_0 \in \mathbb{R}$ uniformly in $\omega \in \Omega$ (see Lemma 4.1). We call the tuple $(X, \Omega, \mathbb{P}, \ell, L, q)$ a *random length model* with associated Laplacians and Schrödinger operators $\Delta_\omega$ and $H_\omega$ and underlying random metric graphs $(X, \ell_\omega)$.

2.4. **Approximation of the IDS via exhaustions.** Let us describe the setting, for which our first main result holds.

**Assumption 2.4.** Let $(X, \Omega, \mathbb{P}, \ell, L, q)$ be a random length model with the following properties:

(i) The topological graph $X$ is non-compact and connected with underlying (undirected) combinatorial graph $G = (V, E, \partial)$. There is a subgroup $\Gamma \subset \mathrm{Aut}(G)$, acting freely on $V$ with only finitely many orbits. Then $\Gamma$ acts also canonically on $X$ (but does not necessarily respect the directions) by

$$\gamma \Phi_e(x) = \begin{cases} \Phi_{\gamma e}(x) & \text{if } \partial_\pm(\gamma e) = \gamma(\partial_\pm e), \\ \Phi_{\gamma e}(1 - x) & \text{if } \partial_\pm(\gamma e) = \gamma(\partial_\mp e). \end{cases}$$

This action carries over to $\Gamma$-actions on the metric graphs $(X, \ell_0)$ and $(X, \ell_\omega)$ via the identification (2.1). Note that $\Gamma$ acts even *isometrically* on the equilateral graph $(X, \ell_0)$ with $\ell_0 = \mathbb{1}_E$. We can think of $(X, \ell_0)$ as a *covering* of the compact topological graph $(X/\Gamma, \ell_0)$.

(ii) We also assume that $\Gamma$ acts *ergodically* on $(\Omega, \mathbb{P})$ by measure preserving transformations with the following consistencies between the two $\Gamma$-actions on $X$ and $\Omega$:

**Metric consistency:** We assume that

$$\ell_{\gamma\omega}(e) = \ell_\omega(\gamma e) \tag{2.8a}$$

for all $\gamma \in \Gamma$, $\omega \in \Omega$ and $e \in E$. This implies that for every $\gamma \in \Gamma$, the map

$$\gamma\colon (X, \ell_\omega) \longrightarrow (X, \ell_{\gamma\omega})$$

is an isometry between two (different) metric graphs. Moreover, the induced operators

$$U_{(\omega,\gamma)} \colon \mathsf{L}_2(X, \ell_{\gamma^{-1}\omega}) \longrightarrow \mathsf{L}_2(X, \ell_\omega)$$

are unitary.

**Operator consistency:** The transformation behaviour of $q_\omega$ and $L_\omega$ is such that we have for all $\omega \in \Omega$, $\gamma \in \Gamma$,

$$H_\omega = U_{(\omega.\gamma)} H_{\gamma^{-1}\omega} U^*_{(\omega,\gamma)}. \tag{2.8b}$$

Such a random length model $(X, \Omega, \mathbb{P}, \ell, L, q)$ is called a *random length covering model* with associated operators $H_\omega$ and covering group $\Gamma$.

*Remark* 2.5. The simplest random length covering model is given when the probability space $\Omega$ consists of only one element with probability 1. In this case, we have only one length function $\ell = \ell_\omega$, one vertex condition $L = L_\omega$, and one potential $q = q_\omega$. The corresponding family of operators consists then of a single operator $H = H_\omega$. Moreover, the metric consistency means that $\Gamma$ acts isometrically on $(X, \ell)$, and the operator consistency is nothing but the periodicity of $H$, i.e., the property that $H$ commutes with the induced unitary $\Gamma$-action on $\mathsf{L}_2(X, \ell)$.

Next, we introduce some more notation. Let $\mathscr{F}_0$ be a relatively compact topological fundamental domain of the $\Gamma$-action on $(X, \ell_0)$ such that its closure $\mathscr{F} = \overline{\mathscr{F}_0}$ is a topological subgraph. (An example of such a topological fundamental domain is given in Figure 2(a) below.) There is a canonical spectral distribution function $N(\lambda)$, associated to the family $H_\omega$, given by the trace formula

$$N(\lambda) := \frac{1}{\mathbb{E}(\mathrm{vol}(\mathscr{F}, \ell_\bullet))} \mathbb{E}(\mathrm{tr}_\bullet[\mathbb{1}_{\mathscr{F}} P_\bullet((-\infty, \lambda])]), \tag{2.9}$$

where $\mathbb{E}(\cdot)$ denotes the expectation in $(\Omega, \mathbb{P})$, $\mathrm{tr}_\omega$ is the trace on the Hilbert space $\mathsf{L}_2(X, \ell_\omega)$, and $P_\omega(I)$ denotes the spectral projection associated to $H_\omega$ and the interval $I \subset \mathbb{R}$. Moreover, the volume $\mathrm{vol}(\mathscr{F}, \ell_\bullet)$ is defined in (2.2). The function $N$ is called the *(abstract) integrated density of states* with abbreviation IDS.

In the case of an amenable group $\Gamma$ the abstract IDS can also be obtained via appropriate exhaustions. This is the statement of Theorem 2.6 below. A discrete group $\Gamma$ is called *amenable*, if there exist a sequence $I_n \subset \Gamma$ of finite, non-empty subsets with

$$\lim_{n\to\infty} \frac{|I_n \,\Delta\, I_n\gamma|}{|I_n|} = 0, \qquad \text{for all } \gamma \in \Gamma. \tag{2.10}$$

A sequence $I_n$ satisfying (2.10) is called a *Følner sequence.*

For every non-empty finite subset $I \subset \Gamma$, we define $\Lambda(I) := \bigcup_{\gamma \in I} \gamma\mathscr{F}$. A sequence $I_n \subset \Gamma$ of finite subsets is Følner if and only if the associated sequence $\Lambda_n = \Lambda(I_n)$ of topological subgraphs satisfies the *van Hove condition*

$$\lim_{n\to\infty} \frac{|\partial\Lambda(I_n)|}{\mathrm{vol}(\Lambda(I_n), \ell_0)} = 0. \tag{2.11}$$

The proof of this fact is analogous to the proof of [PV02, Lemma 2.4] in the Riemannian manifold case. Note that (2.11) still holds if we replace $\partial\Lambda(I_n)$ by $\partial_r\Lambda(I_n)$ for any $r \geq 1$, where $\partial_r\Lambda$ denotes the *thickened combinatorial boundary*

$\{ v \in V \mid d(v, \partial\Lambda) \leq r \}$ and $d$ denotes the *combinatorial distance* which agrees (on the set of vertices) with the distance function of the unilateral metric graph $(X, \ell_0)$.

A Følner sequence $I_n$ is called *tempered*, if we additionally have

$$\sup_{n\in\mathbb{N}} \frac{|\bigcup_{k\leq n} I_{n+1} I_k^{-1}|}{|I_{n+1}|} < \infty. \tag{2.12}$$

Tempered Følner sequences are needed for an ergodic theorem of Lindenstrauss [Lin01]. This ergodic theorem plays a crucial role in the proof of Theorem 2.6 presented below. However, the additional property (2.12) is not very restrictive since it was also shown in [Lin01] that every Følner sequence $I_n$ has a tempered subsequence $I_{n_j}$.

For any compact topological subgraph $\Lambda$ of $X$, we denote the operator with Dirichlet vertex conditions on the boundary vertices $\partial\Lambda$ and with the original vertex conditions $L_\omega(v)$ on all inner vertices $v \in V(\Lambda) \setminus \partial\Lambda$ by $H_\omega^{\Lambda,\mathrm{D}}$. The label D refers to the Dirichlet conditions on $\partial\Lambda$. For a precise definition of the Dirichlet operator via quadratic forms, we refer to Section 4. The spectral projection corresponding to $H_\omega^{\Lambda,\mathrm{D}}$ is denoted by $P_\omega^{\Lambda,\mathrm{D}}$. It is well-known that compactness of $\Lambda$ implies that the operator $H_\omega^{\Lambda,\mathrm{D}}$ has purely discrete spectrum. The *normalised eigenvalue counting function* associated to the operator $H_\omega^{\Lambda,\mathrm{D}}$ is defined as

$$N_\omega^\Lambda(\lambda) = \frac{1}{\mathrm{vol}(\Lambda, \ell_\omega)} \operatorname{tr}_\omega[P_\omega^{\Lambda,\mathrm{D}}((-\infty, \lambda])].$$

The function $N_\omega^\Lambda$ is the distribution function of a (unique) pure point measure which we denote by $\mu_\omega^\Lambda$.

If $\Lambda = \Lambda(I_n)$ is associated to a Følner sequence $I_n \subset \Gamma$, we use the abbreviations $H_\omega^{n,\mathrm{D}} := H_\omega^{\Lambda(I_n),\mathrm{D}}$ for the Schrödinger operator with Dirichlet conditions on $\partial\Lambda(I_n)$, $N_\omega^n := N_\omega^{\Lambda(I_n)}$ for the normalised eigenvalue counting function and $\mu_\omega^n := \mu_\omega^{\Lambda(I_n)}$ for the corresponding pure point measure on $\Lambda(I_n)$. We can now state our first main result:

**Theorem 2.6.** *Let $(X, \Omega, \mathbb{P}, \ell, L, q)$ be a random length covering model as described in Assumption 2.4 with* amenable *covering group $\Gamma$. Let $N$ be the IDS of the operator family $H_\omega$. Then there exist a subset $\Omega_0 \subset \Omega$ of full $\mathbb{P}$-measure such that we have, for every* tempered *Følner sequence $I_n \subset \Gamma$,*

$$\lim_{n\to\infty} N_\omega^n(\lambda) = N(\lambda)$$

*for all $\omega \in \Omega_0$ and all points $\lambda \in \mathbb{R}$ at which $N$ is continuous.*

The proof is given in Section 4.

*Remark* 2.7. The proof of Theorem 2.6 yields even more. Let $\mu$ denote the measure associated to the distribution function $N$. Then we have

$$\lim_{j\to\infty} \mu_\omega^n(f) = \mu(f) \tag{2.13}$$

for all $\omega \in \Omega_0$ and all functions $f$ of the form $f(x) = g(x)(x+1)^{-1}$ with a function $g$ continuous on $[0, \infty)$ and with limit at infinity. (The behaviour of $g(x)$ for $x < 0$

is of no importance since the spectral measures of all operators under consideration are supported on $\mathbb{R}^+ = [0, \infty)$.)

2.5. **Wegner estimate.** In this subsection, we state a linear Wegner estimate for Laplace operators of a random length model with independently distributed edge lengths and fixed Kirchhoff vertex conditions. This Wegner estimate is linear both in the number of edges and in the length of the considered energy interval. As mentioned in the introduction, a similar result for the case $\mathbb{Z}^d$ was proved recently by different methods in [KP08b]. In contrast to the previous subsection, we do not require periodicity of the graph $X$ associated to a group action. More precisely, we assume the following:

**Assumption 2.8.** Let $(X, \Omega, \mathbb{P}, \ell, L, q)$ be a random length model with the following properties:

(i) We have $q \equiv 0$, i.e., the random family of operators are just the Laplacians ($H_\omega = \Delta_\omega$) and we have no randomness in the vertex condition by fixing $L$ to be Kirchhoff in all vertices. Thus it suffices to look at the tuple $(X, \Omega, \mathbb{P}, \ell)$.

(ii) We have a uniform upper bound $d_{\max} < \infty$ on the vertex degrees $\deg v$, $v \in V(X)$.

(iii) Since the only randomness occurs in the edge lengths satisfying
$$0 < \ell_{\min} \le \ell_\omega(e) \le \ell_{\max} \quad \text{for all } \omega \in \Omega \text{ and } e \in E(X),$$
we think of the probability space $\Omega$ as a Cartesian product $\prod_{e \in E} [\ell_{\min}, \ell_{\max}]$ with projections $\Omega \ni \omega \mapsto \omega_e = \ell_\omega(e) \in [\ell_{\min}, \ell_{\max}]$. The measure $\mathbb{P}$ is assumed to be a product $\bigotimes_{e \in E} \mathbb{P}_e$ of probability measures $\mathbb{P}_e$. Moreover, for every $e \in E$, we assume that $\mathbb{P}_e$ is absolutely continuous with respect to the Lebesgue measure on $[\ell_{\min}, \ell_{\max}]$ with density functions $h_e \in \mathsf{C}^1(\mathbb{R})$ satisfying
$$\|h_e\|_\infty, \|h_e'\|_\infty \le C_h, \tag{2.14}$$
for a constant $C_h > 0$ independent of $e \in E$.

Recall that $\operatorname{tr}_\omega$ is the trace in the Hilbert space $\mathsf{L}_2(\Lambda, \ell_\omega)$. In the next theorem $P_\omega^{\Lambda,\mathrm{D}}$ denotes the spectral projection of the Laplacian $\Delta_\omega^{\Lambda,\mathrm{D}}$ on $(\Lambda, \ell_\omega)$ with Kirchhoff vertex conditions on all interior vertices and Dirichlet boundary conditions on $\partial\Lambda$. Under these assumptions we have:

**Theorem 2.9.** *Let $(X, \Omega, \mathbb{P}, \ell)$ be a random length model satisfying Assumption 2.8. Let $u > 1$ and $J_u = [1/u, u]$. Then there exists a constant $C > 0$ such that*
$$\mathbb{E}(\operatorname{tr} P_\bullet^{\Lambda,\mathrm{D}}(I)) \le C \cdot \boldsymbol{\lambda}(I) \cdot |E(\Lambda)|$$
*for all compact subgraphs $\Lambda \subset X$ and all compact intervals $I \subset J_u$, where $\boldsymbol{\lambda}(I)$ denotes the Lebesgue-measure of $I$, and where $|E(\Lambda)|$ denotes the number of edges in $\Lambda$. The constant $C > 0$ depends only the constants $u$, $d_{\max}$, $\ell_{\min}$, $\ell_{\max}$ and the bound $C_h > 0$ associated to the densities $h_e$ (see* (2.14)*).*

The proof will be given in Section 5. We finish this section with the following corollary. Recall that the periodic situation is a special case of a random length covering model (see Remark 2.5):

**Corollary 2.10.** *Let $(X, \Omega, \mathbb{P}, \ell)$ be a random length covering model, satisfying both Assumptions 2.4 and 2.8, with* amenable *covering group $\Gamma$. Then the IDS $N$ of the Laplacians $\Delta_\omega$ is a continuous function on $\mathbb{R}$ and even Lipschitz continuous on $(0, \infty)$.*

*Proof.* The Lipschitz continuity of $N$ on $(0, \infty)$ follows immediately from Theorems 2.6 and 2.9. It remains to prove continuity of $N$ on $(-\infty, 0]$. Note that our model is a special situation of the general *ergodic groupoid* setting given in [LPV07]. Thus, $N$ is the distribution function of a spectral measure of the direct integral operator $\int_\Omega^\oplus \Delta_\omega \, \mathrm{d}\mathbb{P}(\omega)$. Since $\Delta_\omega \geq 0$ for all $\omega$, $N(\lambda)$ vanishes for all $\lambda < 0$. Moreover, if $N$ would have a jump at $\lambda = 0$, then $\ker \Delta_\omega$ would be non-trivial for almost all $\omega \in \Omega$. But $\Delta_\omega f = 0$ implies

$$0 = \langle f, \Delta_\omega f \rangle = \int_X \left| \frac{df}{dx}(x) \right|^2 dx$$

since $\Delta_\omega$ has Kirchhoff vertex conditions. Thus $f$ is a constant function. Now $X$ is connected as well as non-compact, which implies that $\mathrm{vol}(X, \ell_\omega) = \infty$ by the lower bound $\ell_{\min}$ on the lengths of the edges. Hence constant functions are not in $\mathsf{L}_2$. This gives a contradiction.

□

Our result on Lipschitz continuity of $N$ on $(0, \infty)$ is optimal in the following sense:

*Remark* 2.11. It is well-known that the IDS of the free Laplacian $\Delta_\mathbb{R}$ on $\mathbb{R}$ is proportional to the square root of the energy. Note that this does not change when adding Kirchhoff boundary conditions at arbitrary points. Therefore, every model satisfying Assumptions 2.4 and 2.8 for a metric graph isometric to $\mathbb{R}$ has in fact the above IDS. Therefore, we cannot expect Lipschitz continuity of the IDS at zero for random length models without further assumptions.

## 3. Kagome lattice as an example of a planar graph

In this section, we illustrate the concepts of the previous section for an explicit example. We introduce a particular regular tessellation of the Euclidean plane admitting finitely supported eigenfunctions of the combinatorial Laplacian. We discuss in detail the discontinuities of the IDS of the combinatorial Laplacian and of the Kirchhoff Laplacian of the induced equilateral metric graph. On the other hand, applying Corollary 2.10, we see that the IDS of a random family of Kirchhoff Laplacians for independent distributed edge lengths is continuous. Thus, randomness leads to an improvement of the regularity of the IDS in this example.

We consider the infinite planar topological graph $X \subset \mathbb{C}$ as illustrated in Figure 1. This graph is sometimes called *Kagome lattice.* Every vertex of $X$ has degree four and belongs to a uniquely determined upside triangle. Introducing $w_1 = 1$ and $w_2 = \mathrm{e}^{\pi \mathrm{i}/3}$, we can identify the lower left vertex of a particular upside triangle with the origin in $\mathbb{C}$ and its other two vertices with $w_1, w_2 \in \mathbb{C}$. Consequently, the vertex set of $X$ is given explicitly as the disjoint union of the following three sets:

$$V(X) = (2\mathbb{Z}w_1 + 2\mathbb{Z}w_2) \,\dot\cup\, (w_1 + 2\mathbb{Z}w_1 + 2\mathbb{Z}w_2) \,\dot\cup\, (w_2 + 2\mathbb{Z}w_1 + 2\mathbb{Z}w_2).$$

A pair $v_1, v_2 \in V = V(X)$ of vertices is connected by a straight edge if and only if $|v_2 - v_1| = 1$. We write $v_1 \sim v_2$ for adjacent vertices. The above realisation of the planar graph $X \subset \mathbb{C}$ is an isometric embedding of the metric graph $(X, \ell_0)$.

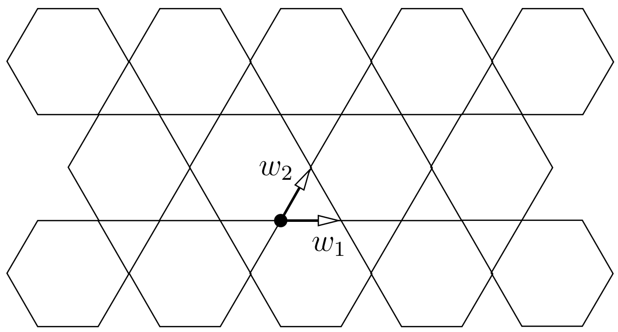


FIGURE 1. Illustration of the planar graph $X$ (Kagome lattice).

The group $\mathbb{Z}^2$ acts on $X$ via the maps $T_\gamma(x) := 2\gamma_1 w_1 + 2\gamma_2 w_2 + x$. A topological fundamental domain $\mathscr{F}_0$ of $X$ is thickened in Figure 2 (a). The set of vertices of the topological subgraph $\mathscr{F} = \overline{\mathscr{F}_0}$ is given by $\{a, b, c, a', b', b'', c''\}$.

Note that we have to distinguish carefully between a topological and a combinatorial fundamental domain. Let $G$ denote the underlying combinatorial graph with set $V$ of vertices and $E$ of combinatorial edges. The maps $T_\gamma$ act also on the set of vertices $V$ and a combinatorial fundamental domain is given by $Q = \{a, b, c\}$. We denote the translates $T_\gamma(Q)$ of $Q$ by $Q_\gamma$.

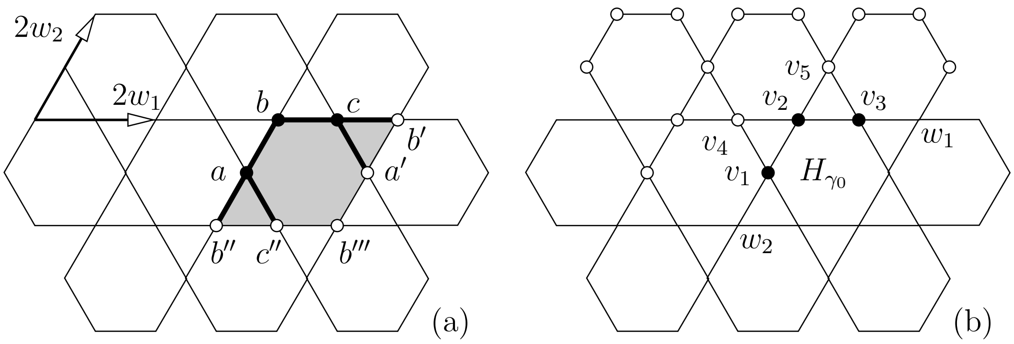


FIGURE 2. (a) The periodic graph with thickened topological fundamental domain $\mathscr{F}_0$ and combinatorial fundamental domain $Q = \{a, b, c\}$ (b) If $\gamma_0$ is vertically extremal for $F$, all white encircled vertices are zeroes of $F$.

3.1. **Spectrum and IDS of the combinatorial Laplacian.** We first observe that $G$ admits finitely supported eigenfunctions of the *combinatorial Laplacian* $\Delta_{\mathrm{comb}}$: Choose an arbitrary hexagon $H \subset X$ with vertices $\{u_0, u_1, \ldots, u_5\}$. Then there exists a centre $w_0 \in \mathbb{C}$ of $H$ such that we have

$$\{u_0, u_1, \ldots, u_5\} = \{\, w_0 + \mathrm{e}^{k\pi \mathrm{i}/3} \mid k = 0, 1, \ldots, 5\,\}.$$

The following function $F_H\colon V \longrightarrow \{0,\pm 1\}$ on the vertices

$$F_H(v) := \begin{cases} 0, & \text{if } v \in V \setminus \{u_0,\dots,u_5\}, \\ (-1)^k, & \text{if } v = w_0 + \mathrm{e}^{k\pi\mathrm{i}/3}, \end{cases} \tag{3.1}$$

satisfies

$$\Delta_{\mathrm{comb}} F_H(v) = \frac{1}{\deg(v)} \sum_{w\sim v} (F_H(v) - F_H(w)) = \frac{3}{2} F_H(v).$$

Thus, the vertices of every hexagon $H \subset X$ are the support of a combinatorial eigenfunction $F_H\colon V \longrightarrow \mathbb{R}$. The functions $F_H$ are the only finitely supported eigenfunctions up to linear combinations:

**Proposition 3.1.** *(a) Let $F\colon V \longrightarrow \mathbb{R}$ be a combinatorial eigenfunction on $X$ with finite support* $\operatorname{supp} F \subset V$*. Then*

$$\Delta_{\mathrm{comb}} F = \frac{3}{2} F$$

*and $F$ is a linear combination of finitely many eigenfunctions $F_H$ of the above type* (3.1)*.*

*(b) Let $H_i$ $(i = 1,\dots,k)$ be a collection of distinct, albeit not necessarily disjoint, hexagons, and $F_i := F_{H_i}$ the associated compactly supported eigenfunctions. Then the set $F_1,\dots,F_k$ is linearly independent.*

*(c) If $g \in \ell^2(V)$ satisfies $\Delta_{comb} g = \mu g$, then $\mu = 3/2$.*

*(d) The space of $\ell^2(V)$-eigenfunctions to the eigenvalue* $3/2$ *is spanned by compactly supported eigenfunctions.*

*Proof.* To prove (a), assume that $F\colon V \longrightarrow \mathbb{R}$ is a finitely supported eigenfunction. Let $Q = \{a,b,c\}$ be a combinatorial fundamental domain of $\mathbb{Z}^2$, as illustrated in Figure 2 (a) and $Q_\gamma := T_\gamma(Q)$. Let $H_\gamma$ be the uniquely defined hexagon containing the three vertices $Q_\gamma$. Moreover, we define

$$A_0 := \{\, \gamma \in \mathbb{Z}^2 \mid \operatorname{supp} F \cap Q_\gamma \neq \emptyset \,\}.$$

Let $\varepsilon_1 = (1,0)$ and $\varepsilon_2 = (0,1)$. We say that $\gamma_0 = (\gamma_{01},\gamma_{02}) \in A_0$ is *vertically extremal* for $F$, if the second coordinate $\gamma_{02}$ is maximal amongst all $\gamma \in A_0$ and if $\gamma_0 - \varepsilon_1 \notin A_0$. This means that $F$ vanishes in the left neighbour of $Q_{\gamma_0}$ and in all vertices vertically above $Q_{\gamma_0}$. Hence, $\gamma_0$ in Figure 2 (b) is vertically extremal if $F$ vanishes in all white encircled vertices and does not vanish in at least one of the black vertices. Obviously, $A_0$ has always vertically extremal elements. Choosing such a $\gamma_0 \in A_0$, we will show below that $F$ is an eigenfunction with eigenvalue $3/2$ and that the following facts hold:

(i) $\gamma_0 + \varepsilon_1$ belongs to $A_0$,

(ii) $\gamma_0 - \varepsilon_2$ or $\gamma_0 - \varepsilon_2 - \varepsilon_1$ belong to $A_0$,

(iii) adding a suitable multiple of $F_{H_{\gamma_0}}$ to $F$, we obtain a new eigenfunction $F_1$ and a set $A_1 := \{\, \gamma \in \mathbb{Z}^2 \mid \operatorname{supp} F_1 \cap Q_\gamma \neq \emptyset \,\}$ satisfying

$$\gamma_0 \notin A_1, \quad A_1 \setminus A_0 \subset \{\gamma_0 - \varepsilon_2, \gamma_0 + \varepsilon_1 - \varepsilon_2\}.$$

To see this, let $\gamma_0 \in A_0$ be vertically extremal and $v_1,\dots,v_5,w_1,w_2$ be chosen as in Figure 2 (b). The eigenvalue equation at the vertices $v_4$ and $v_5$, in which

$F$ vanishes, imply that we have $F(v_1) = -F(v_2) = F(v_3) \neq 0$. Applying the eigenvalue equation again, now at $v_2$, yields that the eigenvalue of $F$ must be $3/2$.

If $\gamma_0 + \varepsilon_1 \notin A_0$, $F$ would vanish in $w_1$ and all its neighbours, except for $v_3$. This would contradict to the eigenvalue equation at $w_1$ and (i) is proven. Similarly, if $\gamma_0 - \varepsilon_2, \gamma_0 - \varepsilon_2 - \varepsilon_1 \notin A_0$, we would obtain a contradiction to the eigenvalue equation at the vertex $w_2$. This proves (ii).

By adding $F(v_1)F_{H_{\gamma_0}}$ to $F$, we obtain a new eigenfunction $F_1$ (again to the eigenvalue $3/2$) which vanishes at all vertices of $Q_{\gamma_0} = \{v_1, v_2, v_3\}$. Thus we have $\gamma_0 \notin A_1$. But $F$ and $F_1$ differ only in the vertices $Q_{\gamma_0}$, $Q_{\gamma_0+\varepsilon_1}$, $Q_{\gamma_0-\varepsilon_2}$ and $Q_{\gamma_0+\varepsilon_1-\varepsilon_2}$, establishing property (iii).

The above procedure can be iteratively (from left to right) applied to the hexagons in the top row of $A_0$: Step (iii) can be applied to the function $F_1$ and a vertically extremal element of $A_1$. After a finite number $n$ of steps the top row of hexagons in $A_0$ is no longer in the support of the function $F_n$. (Note that property (i) implies that when removing the penultimate hexagon form the right, one has simultaneously removed the rightermost one, too.) Again, this procedure can be iterated removing successively rows of hexagons. This time property (ii) guarantees that the procedure stops after a finite number $N$ of steps with $F_N \equiv 0$. We have proven statement (a).

Now we turn to the proof of (b). Since the graph is connected there exists a vertex $v$ in $A := \cup_{i=1}^{k} H_i$ which is adjacent to some vertex outside $A$. Then $v$ is contained in precisely one hexagon $H_{i_0}$. (In the full graph each vertex is in two hexagons.) Thus the condition

$$\sum_{i=1}^{k} \alpha_i F_i = 0 \quad \alpha_i \in \mathbb{C} \tag{3.2}$$

evaluated at the vertex $v$ implies $\alpha_{i_0} = 0$. This shows that all coefficients $\alpha_i$ in (3.2) corresponding to hexagons $H_i$ lying at the boundary of $A$ vanish. This leads to an equation analogous to (3.2) where the indices in the sum run over a strict subset of $\{1, \ldots, k\}$. Now one iterates the pocedure and shows that actually all coefficients $\alpha_1, \ldots, \alpha_k$ in (3.2) are zero. We have shown linear independence of $F_1, \ldots, F_k$.

To prove (c) we recall that the IDS $\Delta_{\text{comb}}$ is a spectral measure (see e.g. [LPV07, Prop. 5.2]). Thus the IDS jumps at the value $\mu$. This in turn implies by [Ves05, Prop. 5.2] that there is a compactly supported $\tilde{g}$ satisfying the eigenvalue equation. Now (a) implies $\mu = 3/2$.

Statement (d) follows from [LV08, Thm. 2.2], cf. also the proof of Proposition 3.3. □

We are primarily interested in $\ell_2$-eigenfunctions of $\Delta_{\text{comb}}$, since their eigenvalues coincide with the *discontinuities* of the corresponding IDS. For combinatorial covering graphs with *amenable* covering group $\Gamma$, every $\ell_2$-eigenfunction $F$ implies the existence of a *finitely supported* eigenfunction to the same eigenvalue which is implied, e.g., by [Ves05, Prop. 5.2] or [LV08, Thm. 2.2]. (Related, but different results have been obtained before in [MY02]. If the group is even abelian, as is the case for the Kagome lattice, the analogous result was proven even earlier in[Kuc91].)

It should be mentioned here that the situation is very different in the smooth category of Riemannian manifolds. There, compactly supported eigenfunctions cannot occur due to the *unique continuation principle.* In the discrete setting of graphs, non-existence of finitely supported combinatorial eigenfunctions is — at present — only be proved for particular examples or in the case of planar graphs of non-positive combinatorial curvature; see [KLPS06] for more details. Hence, Proposition 3.1 tells us that $X$ does not admit combinatorial $\ell_2$-eigenfunctions associated to eigenvalues $\mu \neq 3/2$.

Next, let us discuss spectral informations which can be obtained with the help of *Floquet theory.* Using a general result of Kuchment (see [Kuc91] or [Kuc05, Thm. 8]) for periodic finite difference operators (applying Floquet theory to such operators) we conclude that the *compactly supported* eigenfunctions of $\Delta_{\mathrm{comb}}$ associated to the eigenvalue $3/2$ are already dense in the whole eigenspace $\ker(\Delta_{\mathrm{comb}} - 3/2)$. As for the whole spectrum, we derive the following result:

**Proposition 3.2.** *Denote by $\sigma_{\mathrm{ac}}(\Delta_{\mathrm{comb}})$ and $\sigma_{\mathrm{p}}(\Delta_{\mathrm{comp}})$ the absolutely continuous and point spectrum of $\Delta_{\mathrm{comb}}$ on our $\mathbb{Z}^2$-periodic graph $X$. Then we have*

$$\sigma_{\mathrm{ac}}(\Delta_{\mathrm{comb}}) = \Bigl[0, \frac{3}{2}\Bigr] \quad \textit{and} \quad \sigma_{\mathrm{p}}(\Delta_{\mathrm{comb}}) = \Bigl\{\frac{3}{2}\Bigr\}.$$

The proof follows from standard Floquet theory (for a similar hexagonal graph model see [KP07]):

*Proof.* Note that we have the unitary equivalence

$$\Delta_{\mathrm{comb}} \cong \int_{\mathbb{T}^2}^{\oplus} \Delta_{\mathrm{comb}}^{\theta} \, \mathrm{d}\theta,$$

where $\Delta_{\mathrm{comb}}^{\theta}$ is the $\theta$-equivariant Laplacian on $Q$, $\theta \in \mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2$. This operator is equivalent to the matrix

$$\Delta_{\mathrm{comb}}^{\theta} \cong \frac{1}{4} \begin{pmatrix} 4 & -1 - \mathrm{e}^{-\mathrm{i}\theta_2} & -\mathrm{e}^{-\mathrm{i}\theta_1} - \mathrm{e}^{-\mathrm{i}\theta_2} \\ -1 - \mathrm{e}^{\mathrm{i}\theta_2} & 4 & -1 - \mathrm{e}^{-\mathrm{i}\theta_1} \\ -\mathrm{e}^{\mathrm{i}\theta_1} - \mathrm{e}^{\mathrm{i}\theta_2} & -1 - \mathrm{e}^{\mathrm{i}\theta_1} & 4 \end{pmatrix}$$

using the basis $F \cong (F(a), F(b), F(c))$ for a function on $Q$ and the fact that $F(T_\gamma v) = \mathrm{e}^{\mathrm{i}\langle\theta,\gamma\rangle} F(v)$ (equivariance). The characteristic polynomial is

$$p(\mu) = \Bigl(\mu - \frac{3}{2}\Bigr)\Bigl(\Bigl(\mu - \frac{3}{4}\Bigr)^2 - \frac{3 + 2\kappa}{16}\Bigr),$$

where $\kappa = \cos\theta_1 + \cos\theta_2 + \cos(\theta_1 - \theta_2)$, and the eigenvalues of $\Delta_{\mathrm{comb}}^{\theta}$ are

$$\mu_1 = \frac{3}{2} \quad \text{and } \mu_\pm = \frac{3}{4} \pm \frac{1}{4}\sqrt{3 + 2\kappa}.$$

In particular, we recover the fact that $\Delta_{\mathrm{comb}}$ has an eigenfunction, since $\mu_1$ is independent of $\theta$, only $\mu_\pm$ depend on $\theta$ via $\kappa = \kappa(\theta)$. Note that we have

$$-\frac{3}{2} = \kappa\Bigl(\frac{2\pi}{3}, \frac{4\pi}{3}\Bigr) \le \kappa(\theta) \le \kappa(0,0) = 3,$$

giving the spectral bands $B_- = [0, 3/4]$ and $B_+ = [3/4, 3/2]$. $\square$

The next result discusses (dis)continuity properties of the IDS associated to the combinatorial Laplacian on $X$:

**Proposition 3.3.** *Let $N_{\mathrm{comb}}$ be the (abstract) IDS of the $\mathbb{Z}^2$-periodic operator $\Delta_{\mathrm{comb}}$, given by*

$$N_{\mathrm{comb}}(\mu) = \frac{1}{|Q|} \operatorname{tr}[\mathbb{1}_Q P_{\mathrm{comb}}((-\infty, \mu])],$$

*where* tr *is the trace on the Hilbert space $\ell_2(V)$ and $P_{\mathrm{comb}}$ denotes the spectral projection of $\Delta_{\mathrm{comb}}$. Then $N_{\mathrm{comb}}$ vanishes on $(-\infty, 0]$, is continuous on $\mathbb{R} \setminus \{3/2\}$ and has a jump of size $1/3$ at $\mu = 3/2$. Moreover, $N_{\mathrm{comb}}$ is strictly monotone increasing on $[0, 3/2]$ and $N_{\mathrm{comb}}(\mu) = 1$ for $\mu \geq 3/2$.*

*Proof.* The following facts are given, e.g., in [MY02, p. 119]:

(i) the points of increase of $N_{\mathrm{comb}}$ coincide with the spectrum $\sigma(\Delta_{\mathrm{comb}})$ and
(ii) $N_{\mathrm{comb}}$ can only have discontinuities at $\sigma_p(\Delta_{\mathrm{comb}})$,

together with Proposition 3.2 yield all statements of the proposition, except for the size of the jump at $\mu = 3/2$.

Now, we choose a Følner sequence $I_n \subset \mathbb{Z}^2$ and define $\Lambda_n = \bigcup_{\gamma \in I_n} Q_\gamma$. Let $\partial\Lambda_n$ denote the set of boundary vertices of the combinatorial graph induced by the vertex set $\Lambda_n$, and

$$\partial_r \Lambda := \{\, v \in V(X) \,|\, d(v, \partial\Lambda_n) \leq r \,\} \tag{3.3}$$

be the thickened (combinatorial) boundary. Let

$$D(\lambda) := N_{\mathrm{comb}}(\mu) - \lim_{\varepsilon \to 0} N_{\mathrm{comb}}(\mu - \varepsilon) = \frac{1}{|\Lambda_n|} \operatorname{tr}\bigl[\mathbb{1}_{\Lambda_n} P_{\mathrm{comb}}(\{\mu\})\bigr]. \tag{3.4}$$

The last equality in (3.4) holds for all $n$ and follows easily from the $\mathbb{Z}^2$-invariance of the operator $\Delta_{\mathrm{comb}}$. It remains to prove that $D(3/2) = 1/3$. Let $\Lambda_n' = \Lambda_n \setminus \partial_1 \Lambda_n$ and

$$D_n(\mu) := \frac{1}{|\Lambda_n|} \dim E_n(\mu),$$

where $E_n(\mu) := \{\, F \in \ker(\Delta_{\mathrm{comb}} - \mu) \,|\, \operatorname{supp} F \subset \Lambda_n' \,\}$. Arguments as in [MSY03] or in [LV08] show that

$$D(\mu) = \lim_{n \to \infty} D_n(\mu). \tag{3.5}$$

For the convenience of the reader, we outline the proof of (3.5) below. Using part (b) of Proposition 3.1 one can show that $\dim E_n(\mu)$ equals up to a boundary term the number of hexagons contained in $\Lambda_n'$. Since every translated combinatorial fundamental domain $Q_\gamma$ uniquely determines a hexagon $H_\gamma$ and $|Q| = 3$, we conclude that $\dim E_n(\mu) \approx \frac{1}{3}|\Lambda_n|$, up to an error proportional to $|\partial_1\Lambda_n|$. The van Hove property (2.11) (which holds also in the combinatorial setting) then implies the desired result $D(3/2) = \lim_{n\to\infty} D_n(3/2) = 1/3$.

Finally, we outline the proof of (3.5): Let $E(\mu) = \ker(\Delta_{\mathrm{comb}} - \mu)$ and $S_n(\mu) = \mathbb{1}_{\Lambda_n} E(\mu)$. Let $b_n \colon S_n(\mu) \longrightarrow \mathbb{R}^{|\partial_1\Lambda_n|}$ be the boundary map, i.e., $b_n(F)$ is the collection of all values of $F$ assumed at the (thickened) boundary vertices $\partial_1\Lambda_n$. Then $\ker b_n = E_n(\mu) \subset S_n(\mu)$, and we have

$$D_n(\mu) \leq D(\mu) \leq \frac{\dim S_n(\mu)}{|\Lambda_n|} = \frac{\dim \ker b_n}{|\Lambda_n|} + \frac{\dim \operatorname{ran}\, b_n}{|\Lambda_n|} \leq D_n(\mu) + \frac{|\partial_1\Lambda_n|}{|\Lambda_n|},$$

which yields (3.5), by taking the limit, as $n \to \infty$. $\square$

3.2. **Spectrum and IDS of the periodic Kirchhoff Laplacian.** There is a well known correspondence between the spectrum $\sigma(\Delta_{\mathrm{comb}})$ on a graph $G$ and the spectrum of the (Kirchhoff) Laplacian $\Delta_0$ on the corresponding (equilateral) metric graph $(X, \ell_0)$ with $\ell_0 = \mathbb{1}_E$ (see e.g. [vB85, Cat97, BGP08, Pos08] and the references therein). Namely, any $\lambda \neq k^2\pi^2$ lies in $\sigma_{\mathrm{p}}(\Delta_0)$ resp. $\sigma_{\mathrm{ac}}(\Delta_0)$ iff $\mu(\lambda) = 1 - \cos\sqrt{\lambda}$ lies in $\sigma_{\mathrm{p}}(\Delta_{\mathrm{comb}})$ resp. $\sigma_{\mathrm{ac}}(\Delta_{\mathrm{comb}})$. Moreover, the eigenspace of the metric Laplacian is isomorphic to the corresponding eigenspace of the combinatorial Laplacian.

Let $F\colon V \longrightarrow \mathbb{C}$ be a finitely supported eigenfunction of $\Delta_{\mathrm{comb}}$ as in the previous section. In particular, the eigenvalue must be $\mu = 3/2$. The above mentioned correspondence shows that, for every $\lambda = (2k+2/3)^2\pi^2$, $k \in \mathbb{Z}$, (i.e. $\mu(\lambda) = 3/2$)), there is a Kirchhoff eigenfunction $f\colon X \longrightarrow \mathbb{R}$ of compact support associated to the eigenvalue $\lambda$, satisfying $f(v) = F(v)$ at all vertices $v \in V$. In addition, if $\lambda = k^2\pi^2$, there are so-called *Dirichlet eigenfunctions* of $\Delta_0$, determined by the topology of the graph (see e.g. [vB85, Kuc05, LP08]), which are also generated by compactly supported eigenfunctions.

Using the results [Cat97, BGP08], we conclude from Proposition 3.2:

**Corollary 3.4.** *Let $\Delta_0$ denote the Kirchhoff Laplacian of the equilateral metric graph $(X, \ell_0)$. Let $\sigma_{\mathrm{p}}$ and $\sigma_{\mathrm{ac}}$ denote the point spectrum and absolutely continuous spectrum and $\sigma_{\mathrm{comp}}$ denote the spectrum given by the compactly supported eigenfunctions. Then we have*

$$\sigma_{\mathrm{comp}}(\Delta_0) = \sigma_{\mathrm{p}}(\Delta_0) = \Big\{ \Big(2k + \frac{2}{3}\Big)^2 \pi^2 \,\Big|\, k \in \mathbb{Z} \Big\} \cup \big\{ k^2\pi^2 \,\big|\, k \in \mathbb{N} \big\}$$

*and*

$$\sigma_{\mathrm{ac}}(\Delta_0) = \Big[0, \Big(\frac{2}{3}\Big)^2\pi^2\Big] \cup \bigcup_{k\in\mathbb{N}} \Big[\Big(2k - \frac{2}{3}\Big)^2\pi^2, \Big(2k + \frac{2}{3}\Big)^2\pi^2\Big]. \tag{3.6}$$

Similarly, as in the discrete setting, we conclude the following (dis)continuity properties of the IDS:

**Proposition 3.5.** *Let $N_0$ be the (abstract) IDS of the $\mathbb{Z}^2$-periodic Kirchhoff Laplacian $\Delta_0$ on the metric graph $(X, \ell_0)$, given by*

$$N_0(\lambda) = \frac{1}{\mathrm{vol}(\mathscr{F}, \ell_0)} \operatorname{tr}[\mathbb{1}_{\mathscr{F}} P_0((-\infty, \lambda])],$$

*where* tr *is the trace on the Hilbert space $\mathsf{L}_2(X, \ell_0)$ and $P_0$ denotes the spectral projection of $\Delta_0$. Then all the discontinuities of $N_0\colon \mathbb{R} \longrightarrow [0, \infty)$ are*

*(i) at $\lambda = (2k + \frac{2}{3})^2\pi^2$, $k \in \mathbb{Z}$, with jumps of size $\frac{1}{6}$,*
*(ii) at $\lambda = k^2\pi^2$, $k \in \mathbb{N}$, with jumps of size $\frac{1}{2}$.*

*Moreover, $N_0$ is strictly monotone increasing on the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(\Delta_0)$ given in* (3.6) *and $N_0$ is constant on the complement of $\sigma(\Delta_0)$.*

*Proof.* Our periodic situation fits into the general setting given in [LPV07], by choosing the trivial probability space $\Omega = \{\omega\}$ with only one element. Proposition 5.2 in [LPV07] states that $N_0$ is the distribution function of a spectral measure for the operator $\Delta_0$. Consequently, discontinuities of $N_0$ can only occur at the $\mathsf{L}_2$-eigenvalues of $\Delta_0$, and the points of increase of $N_0$ coincide with the spectrum $\sigma(\Delta_0)$, which is given in Corollary 3.4. Hence, it only remains to prove the statements about the discontinuities of $N_0$. We know from [Kuc05, Theorem 11] that the compactly supported eigenfunctions densely exhaust every $\mathsf{L}_2$-eigenspace of $\Delta_0$.

Let $I_n \subset \mathbb{Z}^2$ be a Følner sequence. This time, we look at the corresponding topological graphs $\Lambda(I_n)$ and their thickened topological boundaries $\partial_r\Lambda(I_n) = \{ x \in X \mid d(x, \partial\Lambda(I_n)) \leq r \}$, and denote them by $\Lambda_n$ and $\partial_r\Lambda_n$, respectively. We are interested in the jumps

$$D(\lambda) := N_0(\lambda) - \lim_{\varepsilon\to 0} N_0(\lambda - \varepsilon) = \frac{1}{\operatorname{vol}(\Lambda_n, \ell_0)} \operatorname{tr}\big[\mathbb{1}_{\Lambda_n} P_0(\{\lambda\})\big],$$

where the right hand side is, again, independent of the choice of $n$. Let $\Lambda_n'$ be the closure of $\Lambda_n \setminus \partial_1\Lambda_n$ and

$$D_n(\lambda) := \frac{1}{\operatorname{vol}(\Lambda_n, \ell_0)} \dim E_n(\lambda),$$

with $E_n(\lambda) = \{ f \in \ker(\Delta_0 - \lambda) \mid \operatorname{supp} f \subset \Lambda_n' \}$. Arguments analogously to the proof of (3.5) yield

$$D(\lambda) = \lim_{n\to\infty} D_n(\lambda). \tag{3.7}$$

For the proof of (3.7), however, we have to define the boundary map

$$b_n \colon S_n(\lambda) \longrightarrow \bigoplus_{v\in\partial\Lambda_n} (\mathbb{C} \oplus \mathbb{C}^{E_v}) \quad \text{by} \quad (b_n f)_v := (f(v), \underrightarrow{Df}(v)).$$

Let $\lambda = (2k + 2/3)^2\pi^2$, $k \in \mathbb{Z}$. We follow the same arguments as in the proof of Proposition 3.3. Again, $\dim E_n(\lambda)$ is equal to the number of hexagons contained in $\Lambda_n$ up to a boundary term and we have $\operatorname{vol}(\mathscr{F}, \ell_0) = 6$ (see Figure 2 (a)). Therefore, we derive that the corresponding jump is of size 1/6.

Let $\lambda = k^2\pi^2$, $k \in \mathbb{N}$. We know from [vB85] or from [LP08, Lem. 5.1 and Prop. 5.2] that the dimension of $E_n(\lambda)$ is (up to an error proportional to $|\partial\Lambda_n|$) approximately equal to

$$|E(\Lambda_n)| - |V(\Lambda_n)| \approx \frac{1}{2} \operatorname{vol}(\Lambda_n, \ell_0).$$

This implies that $N_0$ has a discontinuity at $\lambda = k^2\pi^2$ of size 1/2. □

*Remark* 3.6. Note that Propositions 3.3 and 3.5 hold also for general covering graphs $X \to X_0$ with amenable covering group $\Gamma$ and compact quotient $X_0 \cong X/\Gamma$, once we have information about the shape of the support of elementary eigenfunctions (i.e., eigenfunctions, which generate the eigenspace by linear combinations and translations). In our Kagome lattice example the elementary eigenfunction is supported on a hexagon. For example, the jump of size 1/3 at the eigenvalue $\mu = 3/2$ in the discrete case is the number $\nu$ of hexagons determined by a combinatorial fundamental domain ($\nu = 1$) divided by the number of vertices in a combinatorial fundamental domain ($|Q| = 3$).

In the metric graph setting, the jump at $\lambda = (2k + 2/3)^2\pi^2$ is of size 1/6 due to the fact that we have six edges in one topological fundamental domain.

For the eigenvalues at $\lambda = k^2\pi^2$ (also called *topological*, see [LP08]) we even have a precise information for any $r$-regular amenable covering graph, namely

$$\dim E_n(\lambda) \approx |E(\Lambda_n)| - |V(\Lambda_n)| \approx \Big(1 - \frac{2}{r}\Big)|E(\Lambda_n)| = \Big(1 - \frac{2}{r}\Big)\operatorname{vol}(\Lambda_n, \ell_0),$$

up to an error proportional to $|\partial\Lambda_n|$, so that the jump of $N_0$ at $\lambda$ is $(1 - 2/r)$.

3.3. **IDS of associated random length models.** Finally, we impose a random length structure $\ell\colon \Omega\times E \longrightarrow [\ell_{\min}, \ell_{\max}]$ on the edges of $(X, \ell_0)$ with independently distributed edge lengths, as described in Assumption 2.8. Then Corollary 2.10 tells us that the associated integrated density of states $N\colon \mathbb{R} \longrightarrow [0, \infty)$ is continuous and even Lipschitz continuous on $(0, \infty)$. Hence, all discontinuities occurring for the IDS of the Kirchhoff Laplacian on the $\mathbb{Z}^2$-periodic graph $(X, \ell_0)$ disappear by introducing this type of randomness.

## 4. Proof of the approximation of the IDS via exhaustions

In this section, we prove Theorem 2.6, namely, that the non-random integrated density of states (2.9) can be approximated by suitably chosen normalised eigenvalue counting functions, for $\mathbb{P}$-almost all random parameters $\omega \in \Omega$.

For the following considerations, we need the quadratic forms associated to the Schrödinger operators. Recall that for each Lagrangian subspace $L_v \subset \mathbb{C}^{E_v} \oplus \mathbb{C}^{E_v}$ describing the vertex condition at $v \in V$ there exists a unique orthogonal projection $Q_v$ on $\mathbb{C}^{E_v}$ with range $\mathscr{G}_v := \operatorname{ran} Q_v$ and a symmetric operator on $\mathscr{G}_v$ such that (2.5) holds.

Let $\Lambda \subset X$ be a topological subgraph. The quadratic form associated to the operator with vertex conditions given by $(\mathscr{G}_v, R_v)$ at inner vertices $V(\Lambda) \setminus \partial\Lambda$ and Dirichlet conditions at $\partial\Lambda$ is defined as

$$\operatorname{dom} \mathfrak{h}^{\Lambda,\mathrm{D}} = \big\{\, f \in \mathsf{H}^1_{\max}(X, \ell) \,\big|\, \underline{f}(v) \in \mathscr{G}_v \ \forall v \in V(\Lambda) \setminus \partial\Lambda,\ \underline{f}(v) = 0 \ \forall v \in \partial\Lambda \,\big\},$$
$$\mathfrak{h}^{\Lambda,\mathrm{D}}(f) = \|Df\|^2_{\mathsf{L}_2(\Lambda,\ell)} + \langle qf, f\rangle_{\mathsf{L}_2(\Lambda,\ell)} + \sum_{v\in V(\Lambda)} \langle R_v\underline{f}(v), \underline{f}(v)\rangle_{\mathscr{G}_v}.$$

In particular, if $\Lambda = X$ is the full graph, then there is no boundary and $\mathfrak{h} = \mathfrak{h}^X$ is the quadratic form associated to the operator $H = H_{(X,\ell),L}$.

If $\ell_{\min} := \inf_e \ell(e) > 0$, $C_{\mathrm{pot}} := \|q\|_\infty < \infty$ and $\sup_v \|R_v\| =: C_R < \infty$, then $\mathfrak{h}^{\Lambda,\mathrm{D}}$ is a closed quadratic form with corresponding self-adjoint operator $H^{\Lambda,\mathrm{D}}$.

**Lemma 4.1.** *For any subgraph $\Lambda$ of $X$, the quadratic form $\mathfrak{h}^{\Lambda,\mathrm{D}}$ is closed. Moreover, the associated self-adjoint operator $H^{\Lambda,\mathrm{D}}$ has domain given by*

$$\operatorname{dom} H^{\Lambda,\mathrm{D}} = \Big\{\, f \in \mathsf{H}^2_{\max}(X, \ell) \,\Big|\, \underline{f}(v) = 0 \ \forall v \in \partial V, \\ \underline{f}(v) \in \mathscr{G}_v,\ Q_v\underrightarrow{Df}(v) = R_v\underline{f}(v) \ \forall v \in V(\Lambda) \setminus \partial\Lambda \,\Big\}.$$

*Moreover, $H^{\Lambda,\mathrm{D}}$ is* uniformly *bounded from below by $-C_0$ where $C_0 \geq 0$ depends only on $\ell_-$, $C_R$ and $C_{\mathrm{pot}}$, but not on $\Lambda$.*

*Proof.* The first assertion follows from [Kuc04, Thm. 17]. The *uniform* lower bound is a consequence of [Kuc04, Cor. 10] where the lower bound is given explicitly. Basically, the statements follow from a standard Sobolev estimate of the type

$$\Big|\sum_v \langle R_v \underline{f}(v), \underline{f}(v)\rangle\Big| \le C_R \sum_{v\in V(\Lambda)} |\underline{f}(v)|^2 \le \eta\|Df\|^2 + C_\eta\|f\|^2$$

for $\eta > 0$, where $C_\eta$ depends only on $\eta$, $C_R$ and $\ell_{\min}$. ☐

The Dirichlet operator will serve as upper bound in the bracketing inequality (4.1) later on. In order to have a lower bound we introduce a Neumann-type operator $H^\Lambda$ via its quadratic form $\mathfrak{h}^\Lambda$. Since the vertex conditions can be negative, we have to use the boundary condition $(\mathbb{C}^{E_v}, -C_R)$ instead of a simple Neumann boundary condition $(\mathbb{C}^{E_v}, 0)$. The quadratic form $\mathfrak{h}^\Lambda$ is defined by

$$\operatorname{dom}\mathfrak{h}^\Lambda = \big\{\, f \in \mathsf{H}^1_{\max}(X,\ell) \,\big|\, \underline{f}(v) \in \mathscr{G}_v \ \forall v \in V(\Lambda)\setminus\partial\Lambda \,\big\},$$

$$\mathfrak{h}^\Lambda(f) = \|Df\|^2_{\mathsf{L}_2(\Lambda,\ell)} + \langle qf, f\rangle_{\mathsf{L}_2(\Lambda,\ell)} + \sum_{v\in V(\Lambda)\setminus\partial\Lambda} \langle R_v\underline{f}(v), \underline{f}(v)\rangle_{\mathscr{G}_v} - C_R \sum_{v\in\partial\Lambda} |\underline{f}(v)|^2_{\mathscr{G}_v}.$$

Note that the boundary condition $\widetilde{R}_v = -C_R$ trivially fulfills the norm bound $\|\widetilde{R}_v\| \le C_R$, and therefore by Lemma 4.1, the form $\mathfrak{h}^\Lambda$ is uniformly bounded from below by the same constant $-C_0$ as $\mathfrak{h}^{\Lambda,\mathrm{D}}$. By adding $C_0$ to the (edge) potential $q$ we may assume that w.l.o.g. $H^X$, $H^{\Lambda,\mathrm{D}}$ and $H^\Lambda$ are all non-negative for all subgraphs $\Lambda$.

We can now show the following bracketing result:

**Lemma 4.2.** *Let $\Lambda$ be a topological subgraph of $X$ and $\Lambda'$ be the closure of the complement $\Lambda^{\mathrm{c}}$. Then*

$$H^{\Lambda,\mathrm{D}} \oplus H^{\Lambda',\mathrm{D}} \ge H \ge H^\Lambda \oplus H^{\Lambda'} \ge 0 \tag{4.1}$$

*in the sense of quadratic forms.*

*Proof.* It is clear from the inclusions $\{0\} \subset \mathscr{G}_v \subset \mathbb{C}^{E_v}$ for all boundary vertices $v \in \partial\Lambda$ that the quadratic form domains fulfil

$$\operatorname{dom}\mathfrak{h}^{\Lambda,\mathrm{D}} \oplus \operatorname{dom}\mathfrak{h}^{\Lambda',\mathrm{D}} \subset \operatorname{dom}\mathfrak{h} \subset \operatorname{dom}\mathfrak{h}^\Lambda \oplus \operatorname{dom}\mathfrak{h}^{\Lambda'}.$$

Moreover, if $f = f_\Lambda \oplus f_{\Lambda'}$ is in the decoupled Dirichlet domain, then

$$\mathfrak{h}^{\Lambda,\mathrm{D}}(f_\Lambda) + \mathfrak{h}^{\Lambda',\mathrm{D}}(f_{\Lambda'}) = \mathfrak{h}(f)$$

since $\underline{f}(v) = 0$ on boundary vertices, if $f \in \operatorname{dom}\mathfrak{h}$, then

$$\mathfrak{h}(f) \ge \mathfrak{h}^\Lambda(f_\Lambda) + \mathfrak{h}^{\Lambda'}(f_{\Lambda'})$$

since $R_v \ge -C_R$. In particular, we have shown the inequality for the quadratic forms. ☐

Next, we provide a useful lemma about the spectral shift function of two operators. For a non-negative operator $H$ with purely discrete spectrum $\{\,\lambda_k(H) \,|\, k \ge 0\,\}$ (repeated according to multiplicity), the eigenvalue counting function is given by

$$n(H,\lambda) := \operatorname{tr} \mathbb{1}_{[0,\lambda)}(H) = \big|\{\, k \ge 0 \,|\, \lambda_k(H) \le \lambda \,\}\big|.$$

The *spectral shift function (SSF)* of two non-negative operators $H_1, H_2$ with purely discrete spectrum is then defined as

$$\xi(H_1, H_2, \lambda) := n(H_2, \lambda) - n(H_1, \lambda).$$

We have the following estimate:

**Lemma 4.3.** *Let $(X, \Omega, \mathbb{P}, \ell)$ be a random length metric graph (as described in Subsection 2.3) and $\Lambda \subset X$ be a compact topological subgraph. Let $L_1, L_2$ be two vertex conditions differing in the vertex set $V_{\mathrm{diff}} \subset V(\Lambda)$ only, and such that the operators $\Delta_{(\Lambda,\ell_\omega),L_i}$ are non-negative. Let $0 \le q$ be a bounded measurable potential and $H_i = \Delta_{(\Lambda,\ell_\omega),L_i} + q$. Then we have*

$$|\xi(H_1, H_2, \lambda)| \le 2 \sum_{v \in V_{\mathrm{diff}}} \deg v. \tag{4.2}$$

*Moreover, if $\rho\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a monotone function with $\rho' \in \mathsf{L}_1(\mathbb{R}_+)$, then*

$$\bigl|\mathrm{tr}[\rho(H_1) - \rho(H_2)]\bigr| \le 2|\rho(\infty) - \rho(0)| \sum_{v \in V_{\mathrm{diff}}} \deg v, \tag{4.3}$$

*where the trace is taken in the Hilbert space $\mathsf{L}_2(\Lambda, \ell_\omega)$.*

*Proof.* Let $\mathscr{D}_0 = \mathrm{dom}\, H_1 \cap \mathrm{dom}\, H_2$. Then $\mathscr{D}_0$ has finite index in $\mathrm{dom}\, H_i$, bounded above by twice the number of all edges adjacent to vertices $v \in V_{\mathrm{diff}}$. This implies $\dim(\mathrm{dom}\, H_i/\mathscr{D}_0) \le 2\sum_{v\in V_{\mathrm{diff}}} \deg v$. Inequality (4.2) follows now from [GLV07, Lemma 9]. The second inequality (4.3) follows readily from Krein's trace identity

$$|\,\mathrm{tr}\, \rho(H_1) - \rho(H_2)| \le \int_0^\infty |\rho'(\lambda)| \cdot |\xi(H_1, H_2, \lambda)|\, \mathrm{d}\lambda. \tag{4.4}$$

□

The following uniform resolvent boundedness holds in every random length covering model:

**Lemma 4.4.** *Let $(X, \Omega, \mathbb{P}, \ell, L, q)$ be a random length covering model with covering group $\Gamma$, as described in Assumption 2.4, and $\lambda > 0$. Then there is a constant $C_\lambda > 0$ such that we have*

$$\mathrm{tr}(H_\omega^\Lambda + \lambda)^{-1} \le C_\lambda \,\mathrm{vol}(\Lambda, \ell_0)$$

*for all compact subgraphs $\Lambda \subset (X, \ell)$ and all $\omega \in \Omega$.*

*Proof.* Let $H_\omega^{\Lambda,0}$ denote the restriction on $\Lambda$ with Dirichlet vertex conditions *at all vertices*. Then $H_\omega^{\Lambda,0} = \bigoplus_{e\in E(\Lambda)} H_\omega^{e,\mathrm{D}}$, where we identify the edge $e$ with the topological subgraph consisting of this edge and its end vertices in $X$. From (4.2) of Lemma 4.3 we conclude that

$$|\,\mathrm{tr}\,(H_\omega^{\Lambda,0} + \lambda)^{-1} - (H_\omega^\Lambda + \lambda)^{-1}| \le \frac{4}{\lambda}\,|E(\Lambda)| = \frac{4}{\lambda}\,\mathrm{vol}(\Lambda, \ell_0).$$

Since $(H_\omega^{e,\mathrm{D}} + \lambda)^{-1}$ is bounded from above by $(\Delta_\omega^{e,\mathrm{D}} + \lambda)^{-1}$, and since the edges are uniformly bounded from above by $\ell_{\max}$, there is a constant $c_\lambda > 0$ such that $\mathrm{tr}\,(H_\omega^{e,\mathrm{D}} + \lambda)^{-1} \le c_\lambda$ for all $e \in E(\Lambda)$ and $\omega \in \Omega$. This implies the desired estimate with constant $C_\lambda = 4\lambda^{-1} + c_\lambda$. □

The proof of Theorem 2.6 will now be given in four lemmata. All of these lemmata are based on a given random length covering model $(X, \Omega, \mathbb{P}, \ell, L, q)$ with an *amenable* covering group $\Gamma$ and a fixed *tempered* Følner sequence $I_n$ with associated compact topological graphs $\Lambda_n := \Lambda(I_n)$.

In the first lemma, we prove the convergence (2.13) for a special family of functions $f_\lambda$ associated to resolvents of the operators. Here, we need to apply an ergodic theorem of Lindenstrauss [Lin01].

In later lemmata we show that the convergence (2.13) carries over to the uniform closure of finite linear combinations of the functions $f_\lambda$, identify this closure with the help of the Stone-Weierstrass Theorem, and finally conclude the desired convergence for characteristic functions $\mathbb{1}_{[0,\lambda]}$ at continuity points $\lambda > 0$ of the IDS.

**Lemma 4.5.** *Let $\lambda > 0$ and $f_\lambda \colon [0,\infty) \longrightarrow \mathbb{R}$, $f_\lambda(x) = \frac{1}{x+\lambda}$. Then there exists a subset $\Omega_0 \subset \Omega$ of full $\mathbb{P}$-measure such that*

$$\lim_{n\to\infty} \frac{1}{\operatorname{vol}(\Lambda_n, \ell_\omega)} \operatorname{tr}[f_\lambda(H^{n,\mathrm{D}}_\omega)] = \frac{1}{\mathbb{E}(\operatorname{vol}(\mathscr{F}, \ell_\bullet))} \mathbb{E}(\operatorname{tr}[\mathbb{1}_{\mathscr{F}} f_\lambda(H_\bullet)])$$

*for all $\omega \in \Omega_0$.*

*Proof.* We first consider a fixed $\omega \in \Omega$ and a fixed $\Lambda = \Lambda(I_n)$ and suppress the parameters $\omega$ and $n$ in the notation. Recall the definitions of $H^{\Lambda,\mathrm{D}}$ and $H^\Lambda$ with quadratic form domains given below. Let $\Lambda'$ denote the closure of the complement $\Lambda^c$ in the metric graph $(X, \ell)$. By Lemma 4.2 we have (4.1) in the sense of quadratic forms. Since taking inverses is operator monotone, this implies

$$(H^{\Lambda,\mathrm{D}} \oplus H^{\Lambda',\mathrm{D}} + \lambda)^{-1} \le (H + \lambda)^{-1} \le (H^\Lambda \oplus H^{\Lambda'} + \lambda)^{-1}$$

for all $\lambda > 0$. In particular, we obtain inequalities for the following restricted quadratic forms: Set $(H+\lambda)^{-1}_\Lambda = p_\Lambda (H+\lambda)^{-1} i_\Lambda$, where $i_\Lambda$ and $p_\Lambda$ denote the canonical inclusions and projections between $\mathsf{L}_2(\Lambda, \ell)$ and $\mathsf{L}_2(X, \ell)$. Then

$$(H^{\Lambda,\mathrm{D}} + \lambda)^{-1} \le (H+\lambda)^{-1}_\Lambda \le (H^\Lambda + \lambda)^{-1}. \tag{4.5}$$

Consequently, $(H+\lambda)^{-1}_\Lambda - (H^{\Lambda,\mathrm{D}} + \lambda)^{-1}$ is non-negative and we have

$$\begin{aligned} 0 &\le \operatorname{tr}_{\mathsf{L}_2(\Lambda,\ell)}\big[(H+\lambda)^{-1}_\Lambda - (H^{\Lambda,\mathrm{D}} + \lambda)^{-1}\big] \\ &\qquad\qquad \le \operatorname{tr}_{\mathsf{L}_2(\Lambda,\ell)}\big[f_\lambda(H^\Lambda) - f_\lambda(H^{\Lambda,\mathrm{D}})\big] \le \frac{2}{\lambda} d_{\max} |\partial\Lambda|, \end{aligned}$$

using Lemma 4.3, where $d_{\max}$ is a finite upper bound on the vertex degree of $X$, which exists due to the $\Gamma$-periodicity of $X$. Using the van Hove property (2.11) and the estimate

$$\ell_{\min} \operatorname{vol}(\Lambda, \ell_0) \le \operatorname{vol}(\Lambda, \ell_\omega) \le \ell_{\max} \operatorname{vol}(\Lambda, \ell_0),$$

we conclude that

$$\lim_{n\to\infty} \frac{1}{\operatorname{vol}(\Lambda_n, \ell_\omega)} \big(\operatorname{tr}[(H_\omega + \lambda)^{-1}_{\Lambda_n}] - \operatorname{tr}[f_\lambda(H^{n,\mathrm{D}}_\omega)]\big) = 0. \tag{4.6}$$

Using additivity of the trace and the operator consistency (2.8b), we obtain

$$\operatorname{tr}_{\mathsf{L}_2(\Lambda_n,\ell_\omega)} (H_\omega + \lambda)^{-1}_{\Lambda_n} = \sum_{\gamma\in I_n} \operatorname{tr}_{\mathsf{L}_2(\gamma\mathscr{F},\ell_\omega)} (H_\omega+\lambda)^{-1}_{\gamma\mathscr{F}} = \sum_{\gamma \in I_n^{-1}} g_\lambda(\gamma\omega),$$

where

$$g_\lambda(\omega) = \operatorname{tr}_{\mathsf{L}_2(\mathscr{F},\ell_\omega)}[(H_\omega+\lambda)^{-1}_{\mathscr{F}}] = \operatorname{tr}[\mathbb{1}_{\mathscr{F}} f_\lambda(H_\omega)]. \tag{4.7}$$

Since, by monotonicity (4.5) and Lemma 4.4,

$$0 \le g_\lambda(\omega) \le \operatorname{tr}_{\mathsf{L}_2(\mathscr{F},\ell_\omega)}[(H^{\mathscr{F}}_\omega+\lambda)^{-1}] \le C_\lambda \operatorname{vol}(\mathscr{F},\ell_0),$$

we conclude that $g_\lambda \in \mathsf{L}_1(\Omega)$. Now, we argue as in the proof of Theorem 7 in [LPV04]: Applying Lindenstrauss' ergodic theorem separately to both expressions

$$\frac{1}{|I_n|}\sum_{\gamma\in I_n^{-1}} g_\lambda(\gamma\omega) \qquad \text{and} \qquad \frac{1}{|I_n|}\sum_{\gamma\in I_n^{-1}} \operatorname{vol}(\mathscr{F},\ell_{\gamma\omega}),$$

we conclude that

$$\lim_{n\to\infty}\frac{1}{\operatorname{vol}(\Lambda_n,\ell_\omega)} \operatorname{tr}[(H_\omega+\lambda)^{-1}_{\Lambda_n}] = \frac{1}{\mathbb{E}(\operatorname{vol}(\mathscr{F},\ell_\bullet))}\mathbb{E}(\operatorname{tr}[\mathbb{1}_{\mathscr{F}} f_\lambda(H_\bullet)]) \tag{4.8}$$

for almost all $\omega\in\Omega$. The lemma follows now immediately from (4.6) and (4.8). □

Let us denote by $\mathscr{L}$ the set of functions $\{\, x \mapsto f_\lambda(x) = (x+\lambda)^{-1} \mid \lambda > 0 \,\}$ and by $\mathscr{A}$ the $\|\cdot\|_\infty$-closure of the linear span of $\mathscr{L}$ and the constant function $\mathbb{1}\colon [0,\infty) \longrightarrow \mathbb{R}$, $\mathbb{1}(x) = 1$. Note that, by monotonicity (4.5) and Lemma 4.4, both expressions $\mu^n_\omega(f_1)$ and $\mu(f_1) = (\mathbb{E}(\operatorname{vol}(\mathscr{F},\ell_\bullet)))^{-1}\mathbb{E}(g_1)$ (with $g_1$ defined in (4.7)) are bounded by a constant $K>0$, independent of $\omega$ and $n$. Let $\Omega_0 \subset \Omega$ be the set of full $\mathbb{P}$-measure from Lemma 4.5.

**Lemma 4.6.** *Let $\omega\in\Omega_0$. Set $\nu^n = f_1\cdot\mu^n_\omega$ (for $n\in\mathbb{N}$) and $\nu = f_1\cdot\mu$. Then we have, for all $g\in\mathscr{A}$,*

$$\lim_{n\to\infty}\nu^n(g) = \nu(g).$$

*Proof.* By Lemma 4.5 we know that the statement holds for the function $g = \mathbb{1}$. We note that $f_\lambda\cdot f_1 = \frac{1}{\lambda-1}(f_\lambda - f_1)$ for $\lambda\ne 1$. Thus, by linearity and Lemma 4.5, the convergence holds also for all functions $g = f_\lambda$ with $\lambda>0$, $\lambda\neq 1$. To deal with the case $\lambda=1$ note that $f_{1+\varepsilon}$ converges to $f_1$ uniformly, as $\varepsilon\to 0$. Thus

$$|\nu^n(f_1) - \nu^n(f_{1+\varepsilon})| \le \|f_1 - f_{1+\varepsilon}\|_\infty\, \nu^n(\mathbb{1}) \le K\varepsilon.$$

An analogous statement holds for $\nu^n$ replaced by $\nu$. Thus

$$|\nu(f_1) - \nu^n(f_1)| \le 2K\varepsilon + \bigl|\nu(f_{1+\varepsilon}) - \nu^n(f_{1+\varepsilon})\bigr| \to 2K\varepsilon, \tag{4.9}$$

as $n\to\infty$. Since $\varepsilon>0$ was arbitrary, we conclude that $\lim_{n\to\infty}\nu^n(f_1) = \nu(f_1)$. By linearity, the convergence statement of the Lemma holds for all functions $g$ in the linear span of $\mathscr{L}\cup\{\mathbb{1}\}$. To show that is holds for all functions in the closure $\mathscr{A}$, as well, one uses uniform approximation and an estimate of the same type as in (4.9). □

The next lemma identifies the space $\mathscr{A}$ explicitly:

**Lemma 4.7.** *The function space $\mathscr{A}$ coincides with the set of continuous functions on $[0,\infty)$ which converge at infinity.*

*Proof.* The statement of the lemma is equivalent to $\mathscr{A} = \mathsf{C}([0,\infty])$, where $[0,\infty]$ is the one-point-compactification of $[0,\infty)$. We want to apply the Stone-Weierstrass Theorem. Any $f_\lambda$ with $\lambda > 0$ separates points and $\mathbb{1}$ is nowhere vanishing in $[0,\infty]$. By definition $\mathscr{A}$ is a linear space. To show that it is an algebra we use again the formula $f_{\lambda_1} \cdot f_{\lambda_2} = \frac{1}{\lambda_2 - \lambda_1}(f_{\lambda_1} - f_{\lambda_2})$, which shows that $f_{\lambda_1} \cdot f_{\lambda_2} \in \mathscr{A}$ for $\lambda_1 \neq \lambda_2$. Since $\mathscr{A}$ is closed in the sup-norm, we can use an approximation as in the proof of the Lemma 4.6 to show $f_\lambda^2 \in \mathscr{A}$. A similar argument shows that the product of two limit points $f, g$ of the linear span of $\mathscr{L} \cup \{\mathbb{1}\}$ is in $\mathscr{A}$. □

We have established the convergence $\mu_\omega^n(g) \to \mu(g)$ for all functions of the form $g \cdot f_1$ with $g \in \mathscr{A}$. The following lemma shows that this is sufficient to conclude the almost sure convergence $N_\omega^n(\lambda) \to N(\lambda)$ at continuity points $\lambda$, finishing the proof of Theorem 2.6. One has only to observe that every continuous function of compact support on $\mathbb{R}^+ = [0,\infty)$ can be written as $g \cdot f_1$, with an element $g \in \mathscr{A}$.

**Lemma 4.8.** *For $n \in \mathbb{N}$, let $\rho^n, \rho$ be locally finite measures on $\mathbb{R}^+$. Then*

$$\lim_{n\to\infty} \rho^n(g) = \rho(g)$$

*for all continuous functions $g$ of compact support implies that*

$$\lim_{n\to\infty} \rho^n([0,\lambda]) = \rho([0,\lambda])$$

*for all $\lambda > 0$ which are not atoms of $\rho$.*

*Proof.* The proof is standard. First note that locally finiteness of $\rho$ implies

$$\lim_{\varepsilon\to 0} \rho([\lambda-\varepsilon, \lambda+\varepsilon]) = \rho(\{\lambda\}) = 0.$$

Now choose monotone functions $g_\varepsilon^-, g_\varepsilon^+ \in \mathsf{C}_\mathrm{c}(\mathbb{R}^+)$ satisfying

$$\mathbb{1}_{[0,\lambda-\varepsilon]} \leq g_\varepsilon^- \leq \mathbb{1}_{[0,\lambda]} \leq g_\varepsilon^+ \leq \mathbb{1}_{[0,\lambda+\varepsilon]}.$$

Then

$$\begin{aligned}\rho([0,\lambda]) - \rho^n([0,\lambda]) &\leq \rho(g_\varepsilon^+) - \rho(g_\varepsilon^-) + \rho(g_\varepsilon^-) - \rho^n(g_\varepsilon^-) \\ &\leq \rho([\lambda-\varepsilon, \lambda+\varepsilon]) + \rho(g_\varepsilon^-) - \rho^n(g_\varepsilon^-).\end{aligned}$$

For any $\delta > 0$ one can choose $\varepsilon > 0$ such that $\rho([\lambda-\varepsilon, \lambda+\varepsilon]) < \delta$. Since $\delta > 0$ was arbitrary, we have shown $\rho([0,\lambda]) \leq \liminf_{n\to\infty} \rho^n([0,\lambda])$. The opposite inequality is shown similarly. □

## 5. Proof of the Wegner estimate

This section is devoted to the proof of Theorem 2.9. Let $(X, \Omega, \mathbb{P}, \ell)$ be a random length model satisfying Assumption 2.8. We first introduce a new measurable map $\alpha\colon \Omega \times E \longrightarrow [\omega_-, \omega_+]$ with $\omega_- = \ln \ell_{\min}$, $\omega_+ = \ln \ell_{\max}$, defined by $\alpha_\omega(e) := \alpha(\omega, e) = \ln \ell_\omega(e)$. The random variables $\alpha(\cdot, e)$, $e \in E$, are independently distributed with density functions $g_e(x) = \mathrm{e}^x h_e(\mathrm{e}^x)$, and we have

$$\|g_e'\|_\infty \leq \ell_{\max} \|h_e\|_\infty + \ell_{\max}^2 \|h_e'\|_\infty \leq (\ell_{\max} + \ell_{\max}^2) C_h =: D_h < \infty. \tag{5.1}$$

Thus, we can re-identify $\Omega$ with the Cartesian product $\prod_{e\in E} [\omega_-, \omega_+]$, and the maps $\alpha(\cdot, e)$ are simply projections to the component with index $e$. The measure $\mathbb{P}$ is now given as the product $\bigotimes_{e\in E} \widetilde{\mathbb{P}}_e$ of marginal measures $\widetilde{\mathbb{P}}_e$ with density functions $g_e \in \mathsf{C}^1(\mathbb{R})$ satisfying the above estimate (5.1). The advantage of the new "rescaled" identification $\Omega = \prod_{e\in E}[\omega_-, \omega_+]$ is the following property of the eigenvalues of the Laplacian on any compact subgraph $(\Lambda, \ell_\omega)$:

$$\lambda_i(\Delta^{\Lambda,\mathrm{D}}_{\omega+s\mathbb{1}}) = \mathrm{e}^{-2s}\lambda_i(\Delta^{\Lambda,\mathrm{D}}_\omega). \tag{5.2}$$

Here, the eigenvalues $\lambda_i$ are counted with multiplicity and $\omega + s\mathbb{1}$ denotes the element $\{\omega_e + s\}_{e\in E(X)} \in \Omega$. Property (5.2) is an immediate consequence of (2.3b), $\ell_{\omega+s\mathbb{1}}(e) = \mathrm{e}^{\alpha_\omega(e)+s} = \mathrm{e}^s\ell_\omega(e)$, and the fact that a rescaling of all lengths by a fixed multiplicative constant does not change the domain of the Kirchhoff Laplacian with Dirichlet boundary conditions on $\partial\Lambda$. Property (5.2) is of crucial importance for the proof of the Wegner estimate.

Henceforth, we use this new interpretation of $\Omega$ and rename $\widetilde{\mathbb{P}}_e$ by $\mathbb{P}_e$, for simplicity.

Let $\Lambda \subset X$ be a compact topological subgraph, $\lambda \in \mathbb{R}$ and $\varepsilon > 0$. We write the interval $I$ as $[\lambda - \epsilon, \lambda + \epsilon]$ and start with a smooth function $\rho\colon \mathbb{R} \longrightarrow [-1, 0]$ satisfying $\rho \equiv -1$ on $(-\infty, -\varepsilon]$, $0 \le \rho' \le 1/\varepsilon$, $\rho \equiv 0$ on $[\varepsilon, \infty)$. Moreover, we set $\rho_\lambda(x) = \rho(x - \lambda)$. Then we have

$$\mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(x) \le \rho_\lambda(x + 2\varepsilon) - \rho_\lambda(x - 2\varepsilon) = \int_{-2\varepsilon}^{2\varepsilon} \rho'_\lambda(x + t)\,\mathrm{d}t.$$

Using the spectral theorem, we obtain

$$P^{\Lambda,\mathrm{D}}_\omega([\lambda - \varepsilon, \lambda + \varepsilon]) = \mathbb{1}_{[\lambda-\varepsilon,\lambda+\varepsilon]}(\Delta^{\Lambda,\mathrm{D}}_\omega) \le \int_{-2\varepsilon}^{2\varepsilon} \rho'_\lambda(\Delta^{\Lambda,\mathrm{D}}_\omega + t)\,\mathrm{d}t,$$

and, consequently,

$$\operatorname{tr} P^{\Lambda,\mathrm{D}}_\omega([\lambda - \varepsilon, \lambda + \varepsilon]) \le \int_{-2\varepsilon}^{2\varepsilon} \operatorname{tr} \rho'_\lambda(\Delta^{\Lambda,\mathrm{D}}_\omega + t)\,\mathrm{d}t.$$

Denote by $(\Omega(\Lambda), \mathbb{P}_\Lambda)$ the space $\Omega(\Lambda) = \prod_{e\in E(\Lambda)}[\omega_-, \omega_+]$ with probability measure $\mathbb{P}_\Lambda = \bigotimes_{e\in E(\Lambda)} \mathbb{P}_e$, and $\mathbb{E}_\Lambda(\cdot)$ denote the associated expectation. $\mathbb{E}(\cdot)$ means expectation with respect to the full space $(\Omega, \mathbb{P})$. Applying expectation yields

$$\begin{aligned}\mathbb{E}(\operatorname{tr} P^{\Lambda,\mathrm{D}}_\bullet([\lambda - \varepsilon, \lambda + \varepsilon])) &= \mathbb{E}_\Lambda(\operatorname{tr} P^{\Lambda,\mathrm{D}}_\bullet([\lambda - \varepsilon, \lambda + \varepsilon])) \\ &\le \int_{\Omega(\Lambda)} \int_{-2\varepsilon}^{2\varepsilon} \operatorname{tr} \rho'_\lambda(\Delta^{\Lambda,\mathrm{D}}_\omega + t)\,\mathrm{d}t\,\mathrm{d}\mathbb{P}_\Lambda(\omega). \end{aligned}\tag{5.3}$$

Using the chain rule and scaling property (5.2), we obtain

$$\begin{aligned}\sum_{e\in E(\Lambda)} \frac{\partial}{\partial\omega_e}\rho_\lambda(\lambda_i(\Delta^{\Lambda,\mathrm{D}}_\omega) + t) &= \rho'_\lambda(\lambda_i(\Delta^{\lambda,\mathrm{D}}_\omega) + t)\,\frac{d}{ds}\Big|_{s=0}\big(s \mapsto \lambda_i(\Delta^{\Lambda,\mathrm{D}}_{\omega+s\mathbb{1}})\big) \\ &= -2\rho'_\lambda(\lambda_i(\Delta^{\lambda,\mathrm{D}}_\omega) + t)\lambda_i(\Delta^{\Lambda,\mathrm{D}}_\omega) \le 0.\end{aligned}$$

Now, we use that $[\lambda - \varepsilon, \lambda + \varepsilon] \subset J_u = [1/u, u]$. Since $\operatorname{supp} \rho'_\lambda \subset [\lambda - \varepsilon, \lambda + \varepsilon]$, we derive

$$0 \le \operatorname{tr} \rho'_\lambda(\Delta^{\Lambda,\mathrm{D}}_\omega + t) \le -\frac{u}{2}\Big(\sum_{e\in E(\Lambda)} \frac{\partial}{\partial\omega_e} \operatorname{tr} \rho_\lambda(\Delta^{\Lambda,\mathrm{D}}_\omega + t)\Big). \tag{5.4}$$

For $e \in E(\Lambda)$, denote by $\Lambda_e$ the topological subgraph with vertex set $V_e := V(\Lambda)$ and edge set $E_e := E(\Lambda) \setminus \{e\}$. Using the estimate (5.4), we obtain from (5.3)

$$\begin{aligned} &\mathbb{E}(\operatorname{tr} P_\bullet^{\Lambda,\mathrm{D}}([\lambda-\varepsilon,\lambda+\varepsilon])) \\ &\quad\le -\frac{u}{2}\sum_{e\in E(\Lambda)}\int_{\Omega(\Lambda_e)}\int_{-2\varepsilon}^{2\varepsilon}\int_{\omega_-}^{\omega_+}\Big(\frac{\partial}{\partial\omega_e}\operatorname{tr}\rho_\lambda(\Delta^{\Lambda,\mathrm{D}}_{(\omega',x)}+t)\Big)g_e(x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\mathbb{P}_{\Lambda_e}(\omega') \end{aligned} \tag{5.5}$$

with $(\omega',x)\in\Omega(\Lambda_e)\times[\omega_-,\omega_+]=\Omega(\Lambda)$. Next, we want to carry out partial integration with respect to $x$ in (5.5). Before doing so, it is useful to observe, for fixed $c\in[\omega_-,\omega_+]$,

$$\frac{\partial}{\partial\omega_e}\operatorname{tr}\rho_\lambda(\Delta^{\Lambda,\mathrm{D}}_{(\omega',x)}+t)=\frac{\partial}{\partial\omega_e}\Big(\operatorname{tr}\rho_\lambda(\Delta^{\Lambda,\mathrm{D}}_{(\omega',x)}+t)-\operatorname{tr}\rho_\lambda(\Delta^{\Lambda,\mathrm{D}}_{(\omega',c)}+t)\Big). \tag{5.6}$$

Using (5.6) and applying partial integration, we obtain

$$\begin{aligned} &\Big|\int_{\omega_-}^{\omega_+}\Big(\frac{\partial}{\partial\omega_e}\operatorname{tr}\rho_\lambda(\Delta^{\Lambda,\mathrm{D}}_{(\omega',x)}+t)\Big)g_e(x)\,\mathrm{d}x\Big| \\ &\qquad\qquad\le\|g_e'\|_{\mathsf{L}^1}\sup_{c'\in[\omega_-,\omega_+]}\Big|\operatorname{tr}\rho_{\lambda-t}(\Delta^{\Lambda,\mathrm{D}}_{(\omega',c')})-\operatorname{tr}\rho_{\lambda-t}(\Delta^{\Lambda,\mathrm{D}}_{(\omega',c)})\Big|. \end{aligned} \tag{5.7}$$

For notational convenience, we identify the compact topological graph consisting only of the edge $e$ and its end-points with $e$, and we denote by $\Delta_c^{e,\mathrm{D}}$ be the Dirichlet-Laplacian on the metric graph $(e,\ell_c)$ defined by $\ell_c(e)=\exp(c)$. Using (4.3) in Lemma 4.3, we conclude that

$$\Big|\operatorname{tr}\rho_{\lambda-t}(\Delta^{\Lambda,\mathrm{D}}_{(\omega',c)})-\operatorname{tr}\rho_{\lambda-t}(\Delta^{\Lambda_e,\mathrm{D}}_{\omega'}\oplus\Delta_c^{e,\mathrm{D}})\Big|\le 2\,|\rho(\infty)-\rho(t-\lambda)|\,2d_{\max}\le 4d_{\max},$$

for all values $c\in[\omega_-,\omega_+]$. Consequently, $\sup\big|\operatorname{tr}\rho_{\lambda-t}(\Delta^{\Lambda,\mathrm{D}}_{(\omega',c')})-\operatorname{tr}\rho_{\lambda-t}(\Delta^{\Lambda,\mathrm{D}}_{(\omega',c)})\big|$ in (5.7) can be estimated from above by

$$8d_{\max}+\big|\operatorname{tr}\rho(\Delta_{c'}^{e,\mathrm{D}}+t-\lambda)-\operatorname{tr}\rho(\Delta_c^{e,\mathrm{D}}+t-\lambda)\big|.$$

Note that all eigenfunctions of the Dirichlet operator $\Delta_c^{e,\mathrm{D}}$ are explicitly given sine functions. Therefore, since $\lambda\in[1/u+\varepsilon,u-\varepsilon]$ and $t\in[-2\varepsilon,2\varepsilon]$, there is a constant $C_{u,\ell_{\max}}>0$, depending only on $u,\ell_{\max}$, such that

$$\big|\operatorname{tr}\rho(\Delta_c^{e,\mathrm{D}}+t-\lambda)\big|\le C_{u,\ell_{\max}},$$

for all $\exp(c)\in[\ell_{\min},\ell_{\max}]$. This implies

$$\Big|\int_{\omega_-}^{\omega_+}\left(\frac{\partial}{\partial\omega_e}\operatorname{tr}\rho_\lambda(\Delta^{\Lambda,\mathrm{D}}_{(\omega',x)}+t)\right)g_e(x)\,dx\Big|\le(8d_{\max}+2C_{u,\ell_{\max}})\,\|g_e'\|_{\mathsf{L}_1([\omega_-,\omega_+])}.$$

Plugging this into inequality (5.5), we finally obtain

$$\mathbb{E}(\operatorname{tr}P_\bullet^{\Lambda,\mathrm{D}}([\lambda-\varepsilon,\lambda+\varepsilon]))\le u\,(4d_{\max}+C_{u,\ell_{\max}})\,D_h\,\ln\frac{\ell_{\max}}{\ell_{\min}}\,4\varepsilon\,|E(\Lambda)|,$$

finishing the proof of Theorem 2.9.

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