Conditional Value-at-Risk (CVaR): Algorithms and Applications

Stan Uryasev

Risk Management and Financial Engineering Lab

University of Florida

e-mail: uryasev@ise.ufl.edu

http://www.ise.ufl.edu/uryasev

OUTLINE OF PRESENTATION

- **Background: percentile and probabilistic functions in optimization**
- **Definition of Conditional Value-at-Risk (CVaR) and basic properties**
- **Optimization and risk management with CVaR functions**
- **Case studies:**
- **Definition of Conditional Drawdown-at-Risk (CDaR)**
- **Conclusion**

PAPERS ON MINIMUM CVAR APPROACH

Presentation is based on the following papers:

[1] Rockafellar R.T. and S. Uryasev (2001): Conditional Value-at-Risk for General Loss Distributions. Research Report 2001-5. ISE Dept., University of Florida, April 2001.

(download: www.ise.ufl.edu/uryasev/cvar2.pdf)

[2] Rockafellar R.T. and S. Uryasev (2000): Optimization of Conditional Value-at-Risk. *The Journal of Risk.* Vol. 2, No. 3, 2000, 21-41 (download: www.ise.ufl.edu/uryasev/cvar.pdf)

Several more papers on applications of Conditional Value-at-Risk and the related risk measure, Conditional Drawdown-at-Risk, can be downloaded from www.ise.ufl.edu/rmfe

ABSTRACT OF PAPER[1]

"Fundamental properties of Conditional Value-at-Risk (CVaR), as a measure of risk with significant advantages over Value-at-Risk, are derived for loss distributions in finance that can involve discreetness. Such distributions are of particular importance in applications because of the prevalence of models based on scenarios and finite sampling. Conditional Value-at-Risk is able to quantify dangers beyond Value-at-Risk, and moreover it is coherent. It provides optimization shortcuts which, through linear programming techniques, make practical many large-scale calculations that could otherwise be out of reach. The numerical efficiency and stability of such calculations, shown in several case studies, are illustrated further with an example of index tracking."

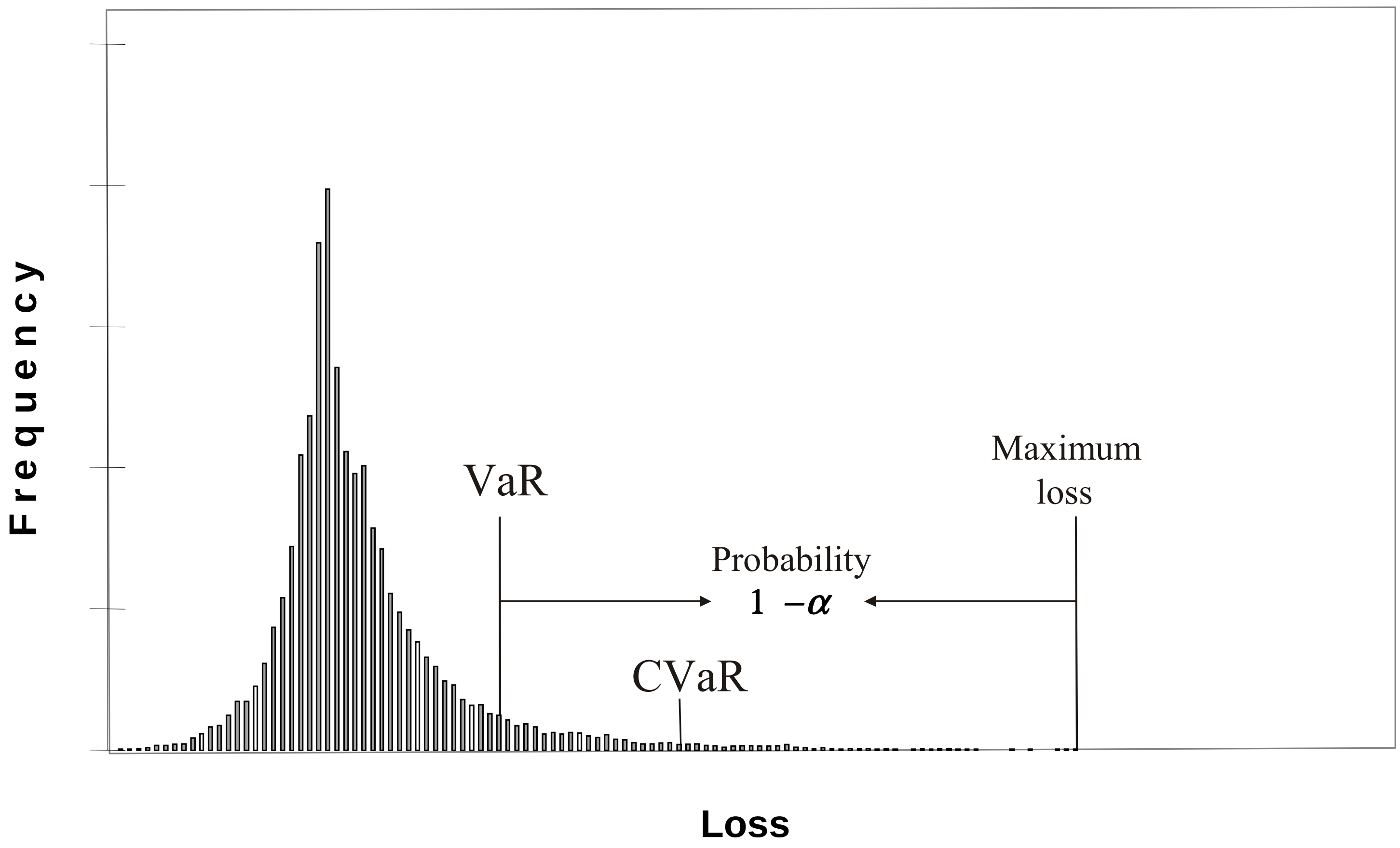
[1]Rockafellar R.T. and S. Uryasev (2001): Conditional Value-at-Risk for General Loss Distributions. Research Report 2001-5. ISE Dept., University of Florida, April 2001. (download: www.ise.ufl.edu/uryasev/cvar2.pdf)

PERCENTILE MEASURES OF LOSS (OR REWARD)

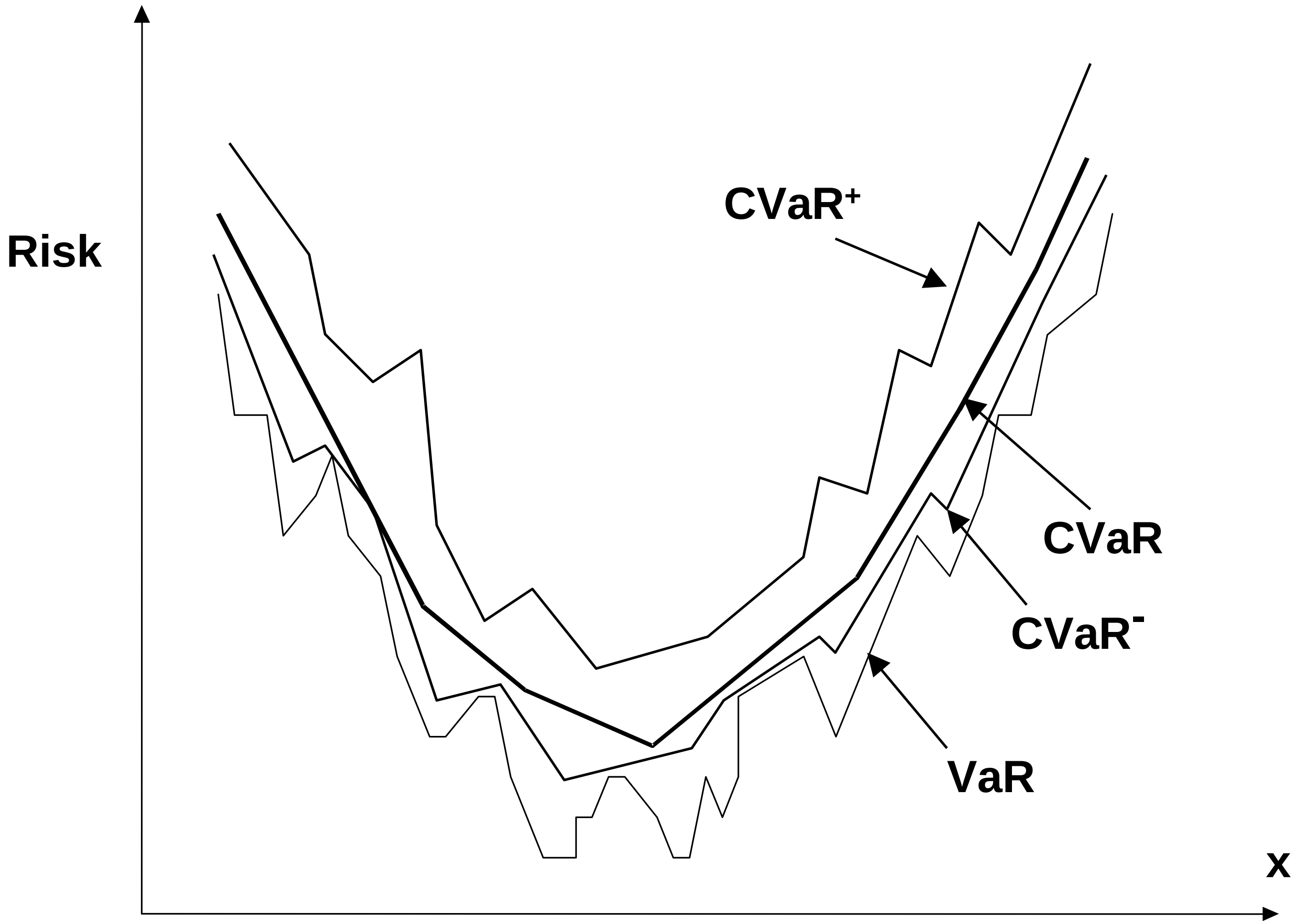
- **Let $f(x,y)$ be a loss functions depending upon a decision vector $x = (x_1 ,\ldots, x_n)$ and a random vector $y = (y_1 ,\ldots, y_m)$**
- **VaR= α – percentile of loss distribution (a smallest value such that probability that losses exceed or equal to this value is greater or equal to α)**
- **$CVaR^+$ ("upper CVaR") = expected losses strictly exceeding VaR (also called Mean Excess Loss and Expected Shortfall)**
- **$CVaR^-$ ("lower CVaR") = expected losses weakly exceeding VaR, i.e., expected losses which are equal to or exceed VaR (also called Tail VaR)**
- **<u>CVaR is a weighted average of VaR and $CVaR^+$</u>**

$$CVaR = \lambda\, VaR + (1- \lambda)\, CVaR^+ , \qquad 0 \le \lambda \le 1$$

VaR, CVaR, $CVaR^+$ and $CVaR^-$



CVaR: NICE CONVEX FUNCTION



CVaR is convex, but VaR, CVaR$^-$,CVaR$^+$ may be non-convex, inequalities are valid: VaR $\leq$CVaR$^-$ $\leq$CVaR $\leq$CVaR$^+$

VaR IS A STANDARD IN FINANCE

- **Value-at-Risk (VaR) is a popular measure of risk: current standard in finance industry**
  various resources can be found at http://www.gloriamundi.org

- **Informally VaR can be defined as a maximum loss in a specified period with some confidence level (e.g., confidence level = 95%, period = 1 week)**

- **Formally, $\alpha-$VaR is the $\alpha-$percentile of the loss distribution:**
  $\alpha-$VaR is a smallest value such that probability that loss exceeds or equals to this value is bigger or equals to α

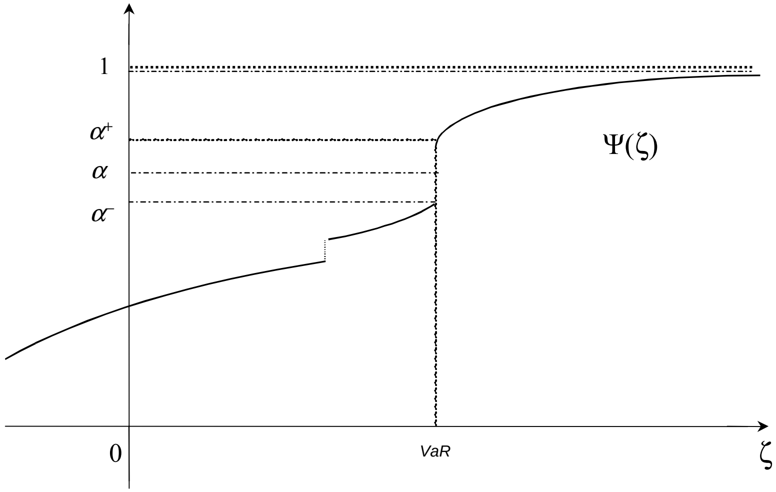
FORMAL DEFINITION OF CVaR

- **Notations:**

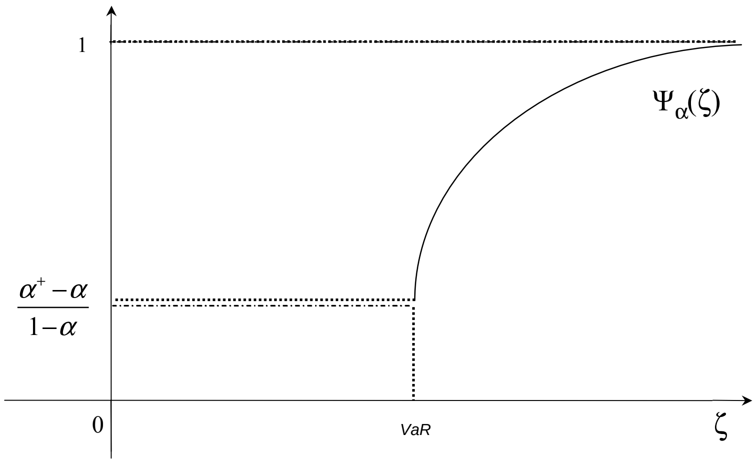
  Ψ = cumulative distribution of losses,

  Ψ_α = α-tail distribution, which equals to zero for losses below VaR, and equals to (Ψ- α)/$(1-\alpha)$ for losses exceeding or equal to VaR

Definition: **CVaR is mean of α-tail distribution Ψ_α**



Cumulative Distribution of Losses, Ψ



α-Tail Distribution, Ψ_α

CVaR: WEIGHTED AVERAGE

- **Notations:**

VaR= α – percentile of loss distribution (a smallest value such that probability that losses exceed or equal to this value is greater or equal to α)

CVaR$^+$ ("upper CVaR") = expected losses strictly exceeding VaR (also called Mean Excess Loss and Expected Shortfall)

Ψ(VaR) = probability that losses do not exceed VaR or equal to VaR

$\lambda = (\Psi(\text{VaR}) - \alpha)/(1-\alpha)\,, \qquad (0 \le \lambda \le 1)$

- **<u>CVaR is weighted average of VaR and CVaR</u>$^+$**

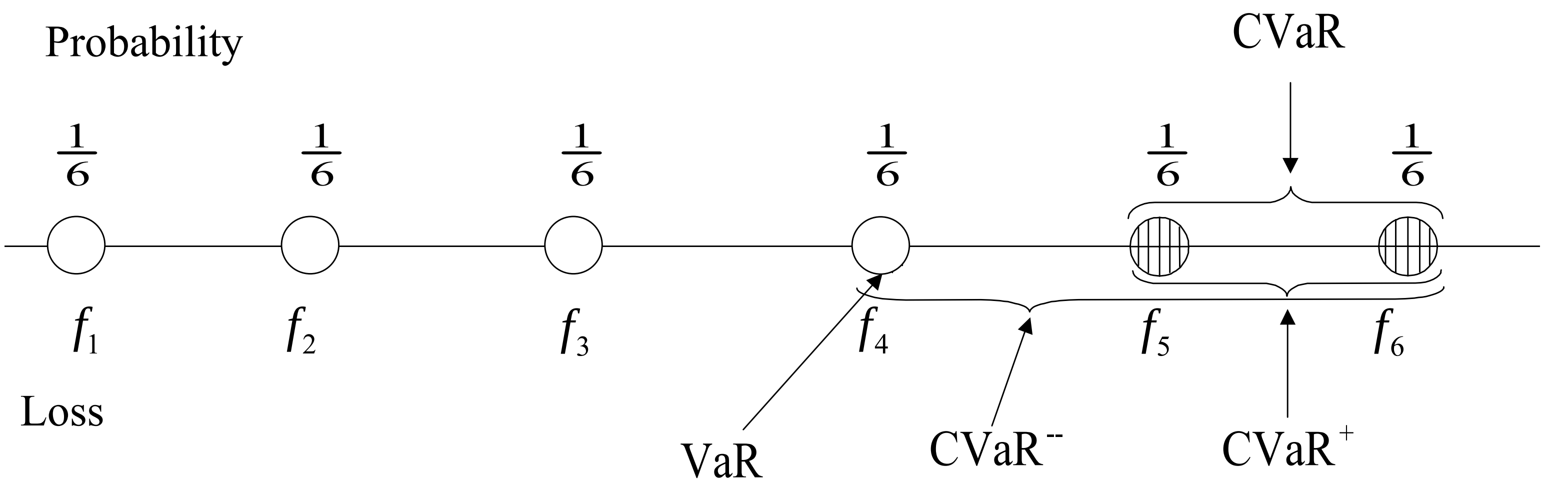
$$\text{CVaR} = \lambda\ \text{VaR} + (1-\lambda)\ \text{CVaR}^+$$

CVaR: DISCRETE DISTRIBUTION, EXAMPLE 1

- **α does not "split" atoms: VaR < CVaR$^-$ < CVaR = CVaR$^+$, $\lambda = (\Psi - \alpha)/(1-\alpha) = 0$**

$$\text{Six scenarios,} \quad p_1 = p_2 = \cdots = p_6 = \tfrac{1}{6}, \ \alpha = \tfrac{2}{3} = \tfrac{4}{6}$$

$$\text{CVaR} = \text{CVaR}^+ = \tfrac{1}{2} f_5 + \tfrac{1}{2} f_6$$

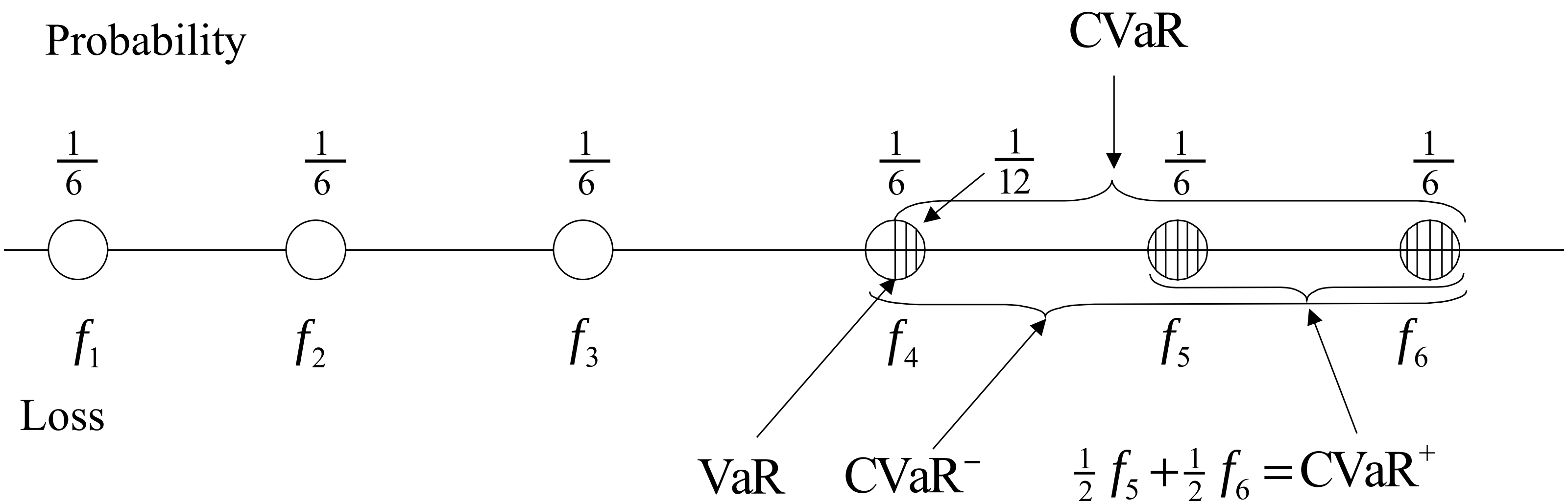


CVaR: DISCRETE DISTRIBUTION, EXAMPLE 2

- **α "splits" the atom: VaR < CVaR$^-$ < CVaR < CVaR$^+$,**
  $\lambda = (\Psi - \alpha)/(1-\alpha) > 0$

$$\text{Six scenarios,}\quad p_1 = p_2 = \cdots = p_6 = \tfrac{1}{6},\ \ \alpha = \tfrac{7}{12}$$

$$\text{CVaR} = \tfrac{1}{5}\text{VaR} + \tfrac{4}{5}\text{CVaR}^+ = \tfrac{1}{5} f_4 + \tfrac{2}{5} f_5 + \tfrac{2}{5} f_6$$

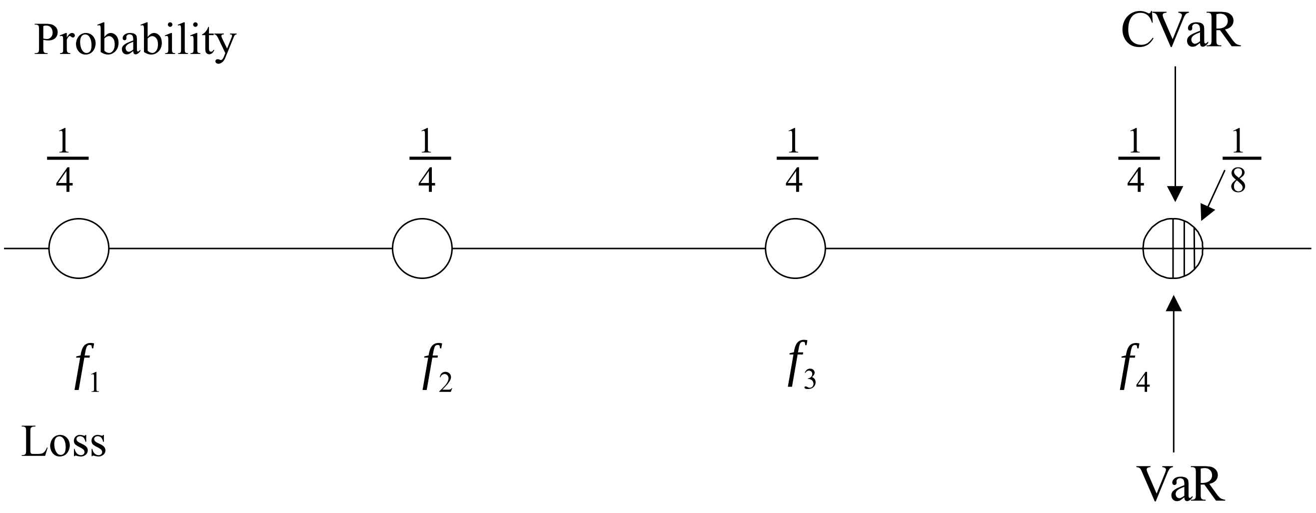


CVaR: DISCRETE DISTRIBUTION, EXAMPLE 3

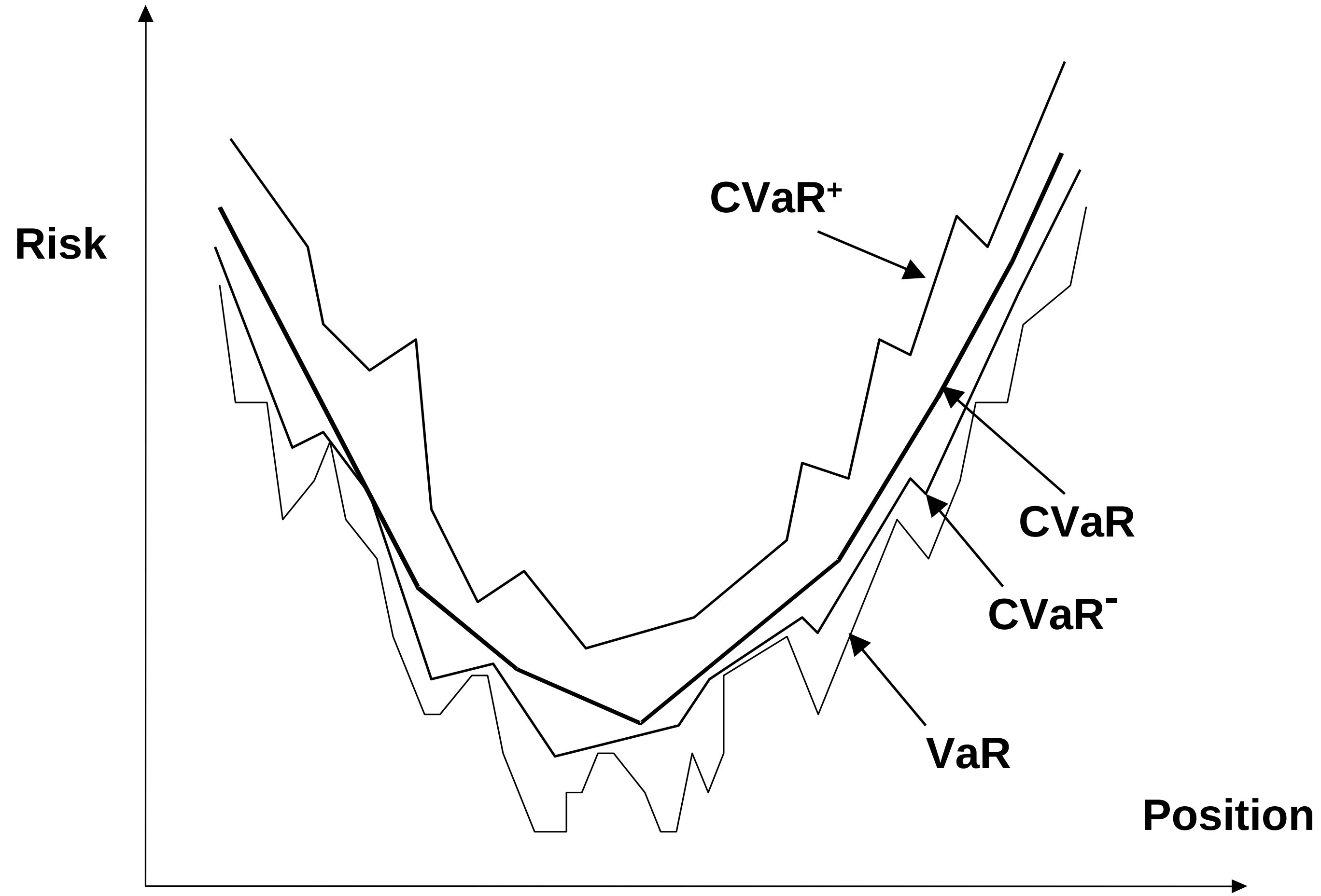
- **α "splits" the last atom: VaR = CVaR$^-$ = CVaR, CVaR$^+$ is not defined, λ = $(\Psi - \alpha)/(1-\alpha) > 0$**

$$\text{Four scenarios,} \quad p_1 = p_2 = p_3 = p_4 = \tfrac{1}{4}, \; \alpha = \tfrac{7}{8}$$

$$\text{CVaR} = \text{VaR} = f_4$$



CVaR: NICE CONVEX FUNCTION



CVaR is convex, but VaR, CVaR$^-$,CVaR$^+$ may be non-convex, inequalities are valid: VaR $\leq$CVaR$^-$ $\leq$CVaR $\leq$CVaR$^+$

CVaR FEATURES[1,2]

- simple convenient representation of risks (one number)
- measures downside risk
- applicable to non-symmetric loss distributions
- CVaR accounts for risks beyond VaR (more conservative than VaR)
- CVaR is convex with respect to portfolio positions
- $VaR \leq CVaR^{-} \leq CVaR \leq CVaR^{+}$
- <u>coherent</u> in the sense of Artzner, Delbaen, Eber and Heath[3]:

  (translation invariant, sub-additive, positively homogeneous, monotonic w.r.t. Stochastic Dominance1)

[1]Rockafellar R.T. and S. Uryasev (2001): Conditional Value-at-Risk for General Loss Distributions. Research Report 2001-5. ISE Dept., University of Florida, April 2001. (Can be downloaded: www.ise.ufl.edu/uryasev/cvar2.pdf)

[2] Pflug, G. Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk, in ``Probabilistic Constrained Optimization: Methodology and Applications'' (S. Uryasev ed.), Kluwer Academic Publishers, 2001.

[3]Artzner, P., Delbaen, F., Eber, J.-M. Heath D. Coherent Measures of Risk, *Mathematical Finance*, 9 (1999), 203--228.

CVaR FEATURES (Cont'd)

- **stable statistical estimates** **(CVaR has integral characteristics compared to VaR which may be significantly impacted by one scenario)**

- **CVaR is continuous with respect to confidence level α, consistent at different confidence levels compared to VaR**
  (VaR, $CVaR^-$, $CVaR^+$ may be discontinuous in α)

- **consistency with mean-variance approach: for normal loss distributions optimal variance and CVaR portfolios coincide**

- **easy to control/optimize for non-normal distributions;**

  linear programming (LP): can be used for optimization of very large problems (over 1,000,000 instruments and scenarios); fast, stable algorithms

- **loss distribution can be shaped using CVaR constraints (many LP constraints with various confidence levels α in different intervals)**

- **can be used in fast online procedures**

CVaR versus EXPECTED SHORTFALL

- **CVaR for <u>continuous</u> distributions usually coincides with <u>conditional expected loss exceeding VaR</u> (also called Mean Excess Loss or Expected Shortfall).**
- **However, for non-continuous (as well as for continuous) distributions CVaR may differ from conditional expected loss exceeding VaR.**
- **Acerbi et al.[1,2] recently redefined Expected Shortfall to be consistent with CVaR definition.**
- **Acerbi et al.[2] proved several nice mathematical results on properties of CVaR, including asymptotic convergence of sample estimates to CVaR.**

[1]**Acerbi, C., Nordio, C., Sirtori, C.** ***Expected Shortfall as a Tool for Financial Risk Management*****, Working Paper, can be downloaded: www.gloriamundi.org/var/wps.html**

[2]**Acerbi, C., and Tasche, D.** ***On the Coherence of Expected Shortfall.*** **Working Paper, can be downloaded: www.gloriamundi.org/var/wps.html**

CVaR OPTIMIZATION

- **Notations:**

  $x = (x_1,...x_n)$ = decision vector (e.g., portfolio weights)

  X = a convex set of feasible decisions

  $y = (y_1,...y_n)$ = random vector

  y^j = scenario of random vector y , ($j=1,...J$)

  $f(x,y)$ = loss functions

- **Example:** ***Two Instrument Portfolio***

  A portfolio consists of two instruments (e.g., options). Let $x=(x_1,x_2)$ be a vector of positions, $m=(m_1,m_2)$ be a vector of initial prices, and $y=(y_1,y_2)$ be a vector of uncertain prices in the next day. The loss function equals the difference between the current value of the portfolio, $(x_1m_1+x_2m_2)$, and an uncertain value of the portfolio at the next day $(x_1y_1+x_2y_2)$, i.e.,

  $$f(x,y) = (x_1m_1+x_2m_2)-(x_1y_1+x_2y_2) = x_1(m_1-y_1)+x_2(m_2-y_2) .$$

  If we do not allow short positions, the feasible set of portfolios is a two-dimensional set of non-negative numbers

  $$X = \{(x_1,x_2),\ x_1 \geq 0,\ x_2 \geq 0\} .$$

  Scenarios $y^j = (y^j_1,y^j_2)$, $j=1,...J$, are sample daily prices (e.g., historical data for J trading days).

CVaR OPTIMIZATION (Cont'd)

- **CVaR minimization**

$$\min_{\{x \in X\}} \text{ CVaR}$$

can be reduced to the following linear programming (LP) problem

$$\min_{\{x \in X,\ \zeta \in R,\ z \in R^J\}} \ \zeta + \nu \sum_{\{j=1,\dots,J\}} z_j$$

subject to

$$z_j \geq f(x, y^j) - \zeta, \quad z_j \geq 0, \quad j = 1,\dots J \qquad (\nu = ((1-\alpha)J)^{-1} = \text{const})$$

- **By solving LP we find an optimal portfolio x* , corresponding VaR, which equals to the lowest optimal ζ^*, and minimal CVaR, which equals to the optimal value of the linear performance function**
- **Constraints, $x \in X$, may account for various trading constraints, including mean return constraint (e.g., expected return should exceed 10%)**
- **Similar to return - variance analysis, we can construct an efficient frontier and find a tangent portfolio**

RISK MANAGEMENT WITH CVaR CONSTRAINTS

- **CVaR constraints in optimization problems can be replaced by a set of linear constraints. E.g., the following <u>CVaR constraint</u>**

$$\text{CVaR} \le C$$

<u>can be replaced by linear constraints</u>

$$\zeta + \nu \sum_{\{j=1,\dots,J\}} z_j \le C$$

$$z_j \ge f(x, y^j) - \zeta\,, \quad z_j \ge 0\,, \quad j = 1,\dots J \qquad (\nu = ((1-\alpha)J)^{-1} = \text{const})$$

- **Loss distribution can be shaped using multiple CVaR constraints at different confidence levels in different times**

- **The reduction of the CVaR risk management problems to LP is a relatively simple fact following from possibility to replace CVaR by some function $F(x,\zeta)$, which is convex and piece-wise linear with respect to x and ζ . A simple explanation of CVaR optimization approach can be found in paper[1] .**

[1]**Uryasev, S. *Conditional Value-at-Risk: Optimization Algorithms and Applications.* Financial Engineering News, No. 14, February, 2000.**

(can be downloaded: www.ise.ufl.edu/uryasev/pubs.html#t).

CVaR OPTIMIZATION: MATHEMATICAL BACKGROUND

Definition

$$F(x,\zeta) = \zeta + \nu \ \Sigma_{j=1,J} \ (f(x,y^j) - \zeta)^+, \qquad \nu = ((1-\alpha)J)^{-1} = \text{const}$$

Theorem 1.

$$CVaR_\alpha(x) = \min_{\zeta\in R} F(x,\zeta) \quad \text{and} \quad \zeta_\alpha(x) \text{ is a smallest minimizer}$$

Remark. **This equality can be used as a definition of CVaR (Pflug).**

Theorem 2.

$$\min_{x\in X} CVaR_\alpha(x) = \min_{\zeta\in R,\, x\in X} F(x,\zeta) \qquad (1)$$

- **Minimizing of $F(x,\zeta)$ simultaneously calculates VaR= $\zeta_\alpha(x)$, optimal decision x, and optimal CVaR**
- **Problem (1) can be reduces to LP using additional variables**

PERCENTILE V.S. PROBABILISTIC CONSTRAINTS

Proposition 1.

Let $f(x,y)$ be a loss functions and $\zeta_\alpha(x)$ be α- percentile (α-VaR) then

$$\zeta_\alpha(x) \le \varepsilon \quad \Leftrightarrow \quad \Pr\{ f(x,y) \le \varepsilon \} \ge \alpha$$

Proof **follows from the definition of α- percentile $\zeta_\alpha(x)$**

$$\zeta_\alpha(x) = \min \{\varepsilon : \Pr\{ f(x,y) \le \varepsilon \} \ge \alpha \}$$

- **Generally, $\zeta_\alpha(x)$ is nonconvex (e.g., discrete distributions), therefore $\zeta_\alpha(x) \le \varepsilon$ as well as $\Pr(f(x,y) \le \varepsilon) \ge \alpha$ may be nonconvex constraints**

- **Probabilistic constraints were considered by Prekopa, Raik, Szantai, Kibzun, Uryasev, Lepp, Mayer, Ermoliev, Kall, Pflug, Gaivoronski, …**

NON-PERCENTILE RISK MEASURES

- **Low partial moment constraint (considered in finance literature from 70-th)**

  $$E\{ ((f(x,y) - \varepsilon)^+)^a \} \le b, \quad a \ge 0, \quad g^+ = \max\{0, g\}$$

  special cases

  $a = 0 \Rightarrow \Pr\{ f(x,y) - \varepsilon \}$

  $a = 1 \Rightarrow E\{ (f(x,y) - \varepsilon)^+ \}$

  $a = 2,\ \varepsilon = E f(x,y) \Rightarrow$ **semi-variance** $E\{ ((f(x,y) - \varepsilon)^+)^2 \}$

- **Regret (King, Dembo) is a variant of low partial moment with**

  $\varepsilon = 0$ **and** $f(x,y)$ **= performance-benchmark**

- **Various variants of low partial moment were successfully applied in stochastic optimization by Ziemba, Mulvey, Zenios, Konno,King, Dembo,Mausser,Rosen,…**

- **Haneveld and Prekopa considered a special case of low partial moment with** $a = 1$, $\varepsilon = 0$: ***integrated chance constraints***

PERCENTILE V.S. LOW PARTIAL MOMENT

- **Low partial moment with *a*>0 does not control percentiles. It is applied when loss can be hedged at additional cost**

total expected value* = *expected cost without high losses
+ *expected cost of high losses*

$$\textit{expected cost of high losses} = p\,E\{ (f(x,y) - \varepsilon)^+ \}$$

- **Percentiles constraints control risks explicitly in percentile terms.**
- **Testury and Uryasev[1] established equivalence between CVaR approach (percentile measure) and low partial moment, *a* = 1 (non-percentile measure) in the following sense:**

a) Suppose that a decision is optimal in an optimization problem with a CVaR constraint, then the same decision is optimal with a low partial moment constraint with some $\varepsilon>0$;

b) Suppose that a decision is optimal in an optimization problem with a low partial moment constraint, then the same decision is optimal with a CVaR constraint at some confidence level α.

[1]Testuri, C.E. and S. Uryasev. On Relation between Expected Regret and Conditional Value-At-Risk. Research Report 2000-9. ISE Dept., University of Florida, August 2000. Submitted to *Decisions in Economics and Finance* journal. (www.ise.ufl.edu/uryasev/Testuri.pdf)

CVaR AND MEAN VARIANCE: NORMAL RETURNS

- $\mathrm{x} = (x_1, \ldots, x_n)$ = positions
- $\mathrm{y} = (y_1, \ldots, y_n)$ = random returns
- $\mathrm{m} = (m_1, \ldots, m_n)$ = mean returns
- loss function

$$f(\mathrm{x}, \mathrm{y}) = -[x_1 y_1 + \cdots + x_n y_n] = -\mathrm{x}^T \mathrm{y}$$

- mean loss and variance

$$\mu(\mathrm{x}) = -\mathrm{x}^T \mathrm{m} \quad \text{and} \quad \sigma(\mathrm{x}) = \mathrm{x}^T \mathrm{V} \mathrm{x}$$

- return constraint

$$\mu(\mathrm{x}) \leq -R \qquad (3)$$

- no-shorts and budget constraints

$$x_j \geq 0 \;\text{ and }\; \textstyle\sum_{j=1}^{n} x_j = 1 \qquad (4)$$

- X = { set of x satisfying (3) and (4) }

CVaR AND MEAN VARIANCE: NORMAL RETURNS

If returns are normally distributed, and return constraint is active, the following portfolio optimization problems have the same solution:

1. Minimize CVaR
   subject to return and other constraints

2. Minimize VaR
   subject to return and other constraints

3. Minimize variance
   subject to return and other constraints

EXAMPLE 1: PORTFOLIO MEAN RETURN AND COVARIANCE

Instrument	Mean Return
S&P	0.0101110
Gov Bond	0.0043532
Small Cap	0.0137058

	S& P	Gov Bond	Small Cap
S&P	0.00324625	0.00022983	0.00420395
Gov Bond	0.00022983	0.00049937	0.00019247
Small Cap	0.00420395	0.00019247	0.00764097

OPTIMAL PORTFOLIO (MIN VARIANCE APPROACH)

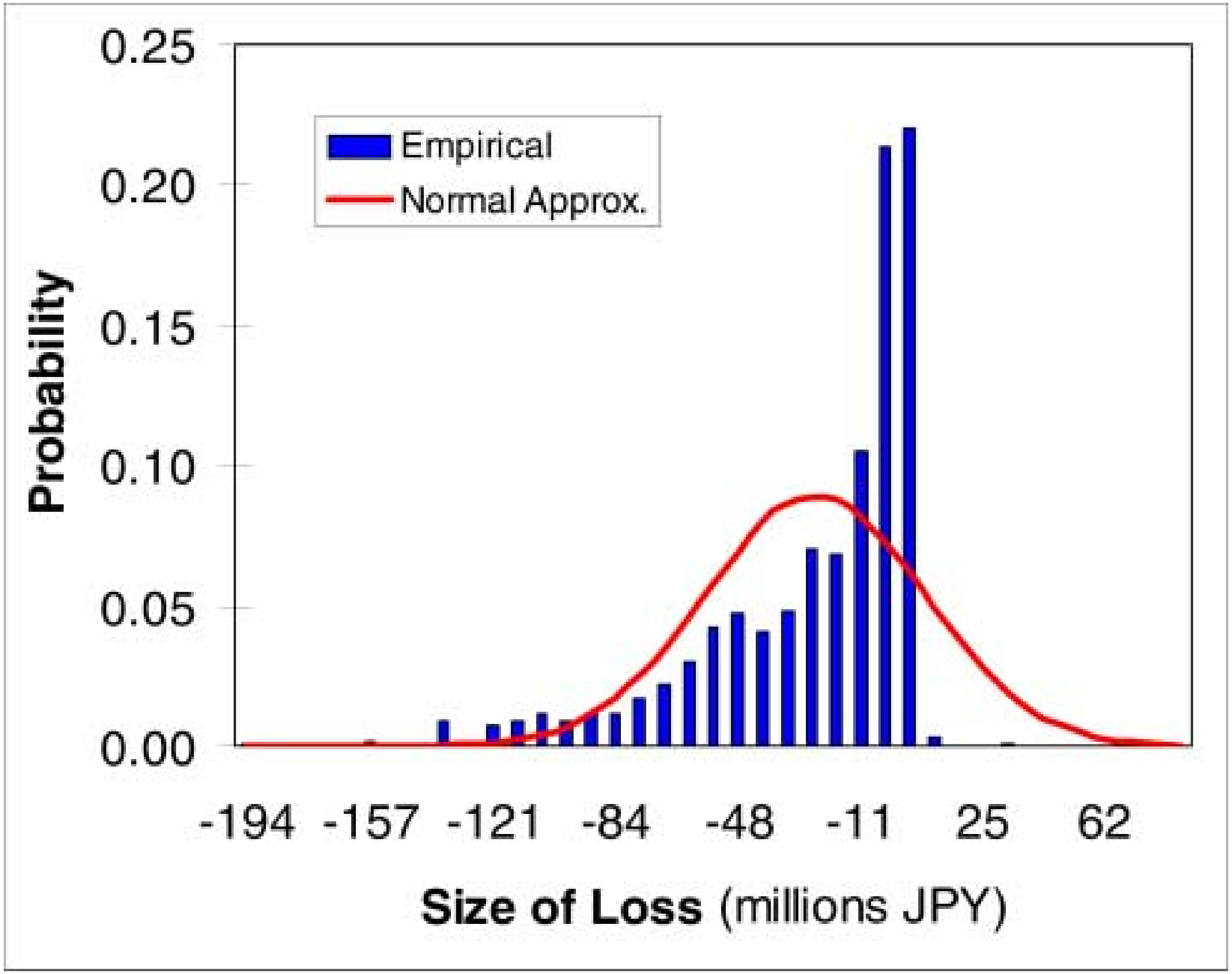
S&P	Gov Bond	Small Cap
0.452013	0.115573	0.432414

	$\alpha = 0.90$	$\alpha = 0.95$	$\alpha = 0.99$
VaR	0.067847	0.090200	0.132128
CVaR	0.096975	0.115908	0.152977

PORTFOLIO, VaR and CVaR (CVaR APPROACH)

α	Smpls #	S&P	Gov Bond	Small Cap	VaR	VaR Dif(%)	CVaR	CVaR Dif(%)	Iter	Time (min)
0.9	1000	0.35250	0.15382	0.49368	0.06795	0.154	0.09962	2.73	1157	0.0
0.9	3000	0.55726	0.07512	0.36762	0.06537	3.645	0.09511	-1.92	636	0.0
0.9	5000	0.42914	0.12436	0.44649	0.06662	1.809	0.09824	1.30	860	0.1
0.9	10000	0.48215	0.10399	0.41386	0.06622	2.398	0.09503	-2.00	2290	0.3
0.9	20000	0.45951	0.11269	0.42780	0.06629	-2.299	0.09602	-0.98	8704	1.5
0.95	1000	0.53717	0.08284	0.37999	0.09224	2.259	0.11516	-0.64	156	0.0
0.95	3000	0.54875	0.07839	0.37286	0.09428	4.524	0.11888	2.56	652	0.0
0.95	5000	0.57986	0.06643	0.35371	0.09175	1.715	0.11659	0.59	388	0.1
0.95	10000	0.47102	0.10827	0.42072	0.08927	-1.03	0.11467	-1.00	1451	0.2
0.95	20000	0.49038	0.10082	0.40879	0.09136	1.284	0.11719	1.11	2643	0.7
0.99	1000	0.41844	0.12848	0.45308	0.13454	1.829	0.14513	-5.12	340	0.0
0.99	3000	0.6196	0.05116	0.32924	0.12791	-3.187	0.14855	-2.89	1058	0.0
0.99	5000	0.63926	0.04360	0.31714	0.13176	-0.278	0.15122	-1.14	909	0.1
0.99	10000	0.45203	0.11556	0.43240	0.12881	-2.51	0.14791	-3.31	680	0.1
0.99	20000	0.45766	0.11340	0.42894	0.13153	-0.451	0.15334	0.24	3083	0.9

EXAMPLE 2: NIKKEI PORTFOLIO



NIKKEI PORTFOLIO

Instrument	Type	Day to Maturity	Strike Price (10^3 JPY)	Position (10^3)	Value (10^3 JPY)
Mitsubishi EC 6mo 860	Call	184	860	11.5	563,340
Mitsubishi Corp	Equity	n/a	n/a	2.0	1,720,00
Mitsubishi Cjul29 800	Call	7	800	-16.0	-967,280
Mitsubishi Csep30 836	Call	70	836	8.0	382,070
Mitsubishi Psep30 800	Put	70	800	40.0	2,418,012
Komatsu Ltd	Equity	n/a	n/a	2.5	2,100,000
Komatsu Cjul29 900	Call	7	900	-28.0	-11,593
Komatsu Cjun2 670	Call	316	670	22.5	5,150,461
Komatsu Cjun2 760	Call	316	760	7.5	1,020,110
Komatsu Paug31 760	Put	40	760	-10.0	-68,919
Komatsu Paug31 830	Put	40	830	10.0	187,167

HEDGING: NIKKEI PORTFOLIO

- **Mausser and Rosen from Algorithmics Inc. developed parametric and simulation VaR techniques for one-dimension hedging**

[1] Mauser, H. and D. Rosen (1991): Beyond VaR: From Measuring Risk to Managing Risk. *ALGO Research Quarterly.* **Vol.1, 2, 5–20.**

- **optimal one-dimension hedge is calculated by changing a position in the portfolio such that VaR is minimal**

- **smoothing techniques were used to cope with nonsmooth multiextremum VaR performance functions**

HEDGING: MINIM CVaR APPROACH

- CVaR techniques for one-dimension and multi-dimension hedging of portfolio

- 1,000 Monte Carlo scenarios of one-day losses generated at Algorithmics Inc.

- loss function and constraints are linear: LP techniques are applicable

- nonsmooth optimization techniques were tested

[1] Uryasev, S. (1991): New Variable-Metric Algorithms for Nondifferential Optimization Problems. *J. of Optim. Theory and Applic.* **Vol. 71, No. 2, 359–388.**

- comparing to LP techniques, nonsmooth optimization may have some advantages for very large number of scenarious

ONE INSTRUMENT HEDGING

Instrument	Best Hedge	VaR	CVaR
Mitsubishi EC 6mo 860	7,337.53	-205,927	1,183,040
Mitsubishi Corp	-926.073	-1,180,000	551,892
Mitsubishi Cjul29 800	-18,978.6	-1,170,000	553,696
Mitsubishi Csep30 836	4381.22	-1,150,000	549,022
Mitsubishi Psep30 800	43,637.1	-1,150,000	542,168
Komatsu Ltd	-196.167	-1,180,000	551,892
Komatsu Cjul29 900	-124,939	-1,200,000	593,078
Komatsu Cjun2 670	19,964.9	-1,220,000	385,698
Komatsu Cjun2 760	4,745.20	-1,200,000	363,556
Komatsu Paug31 760	3,1426.3	-1,120,000	538,662
Komatsu Paug31 830	19,356.3	-1,150,000	536,500

ONE - INSTRUMENT HEDGING

- one-instrument hedging is done with two-dimensional nonsmooth optimization
- MATHEMATICA version of the variable metric code on Pentium II, 450MHz (download from www.ise.ufl.edu/uryasev)
- with $\alpha = 0.95$, initial VaR=657,816 JPY and CVaR=2,022,060 JPY
- one-instrument hedges with Minimum VaR approach (Mausser & Rosen) and Minimum CVaR approach are very close
- e.g., the best one-instrument VaR hedge is Komatsu Cjun2 760, position= 4,800 the best one-instrument CVaR hedge is also Komatsu Cjun2 760, position=4,745 with VaR=-1,200,000 and CVaR=363,556

MULTIPLE INSTRUMENT HEDGING: CVaR APROACH

Instrument	Position in Portfolio	Best Hedge
Komatsu Cjun2 670	22,500	22,500
Komatsu Cjun2 760	7,500	-527
Komatsu Paug31 760	-10,000	-10,000
Komatsu Paug31 830	10,000	-10,000

MULTIPLE - INSTRUMENT HEDGING

- hedging using the last 4 of the 11 instruments

- multiple-instrument hedging gives VaR= -1,400,000 and CVaR=37,335, while the best one-instrument hedge gives VaR= -1,200,000 and CVaR=363,556

- CVaR is a more adequate (conservative) estimate of risk than VaR. VaR shows gain while CVaR shows loss.

- Six correct digits in the performance function and the positions were obtained after 400–800 iterations of the variable metric algorithm. It took about 4–8 minutes with MATHEMATICA code on a Pentium II, 450MHz.

MULTIPLE-HEDGING: MODEL DESCRIPTION

- $\mathrm{z} = (z_1, \ldots, z_{11})$ = initial positions

- K is an index set within $\{1, 2, \ldots, 11\}$ of instruments to be adjusted

- $\mathrm{x} = (x_1, \ldots, x_{11})$ = adjusted positions

- feasible set X is defined by constraints

$$-|z_j| \leq x_j \leq |z_j| \quad \text{for } j \in K,$$
$$x_j = z_j \quad \text{for } j \notin K$$

- random prices one day later
  $\mathrm{y} = (y_1, \ldots, y_{11})$

- mean prices one day later
  $\mathrm{m} = (m_1, \ldots, m_{11})$

- loss function
  $f(x, y) = \mathrm{x}^T\mathrm{m} - \mathrm{x}^T\mathrm{y} = \mathrm{x}^T(\mathrm{m} - \mathrm{y})$

EXAMPLE 3: PORTFOLIO REPLICATION USING CVaR

- **Problem Statement: Replicate an index using $j = 1,\ldots,n$ instruments. Consider impact of CVaR constraints on characteristics of the replicating portfolio.**

- **Daily Data: SP100 index, 30 stocks** (tickers: GD, UIS, NSM, ORCL, CSCO, HET, BS,TXN, HM, INTC, RAL, NT, MER, KM, BHI, CEN, HAL, DK, HWP, LTD, BAC, AVP, AXP, AA, BA, AGC, BAX, AIG, AN, AEP)

- **Notations**

I_t **= price of SP100 index at times** $t = 1,\ldots,T$

p_{jt} **= prices of stocks** $j = 1,\ldots,n$ **at times** $t = 1,\ldots,T$

ν **= amount of money to be on hand at the final time** T

$\theta = \frac{\nu}{I_T}$ **= number of units of the index at the final time** T

x_j **= number of units of *j*-th stock in the replicating portfolio**

- **Definitions (similar to paper[1])**

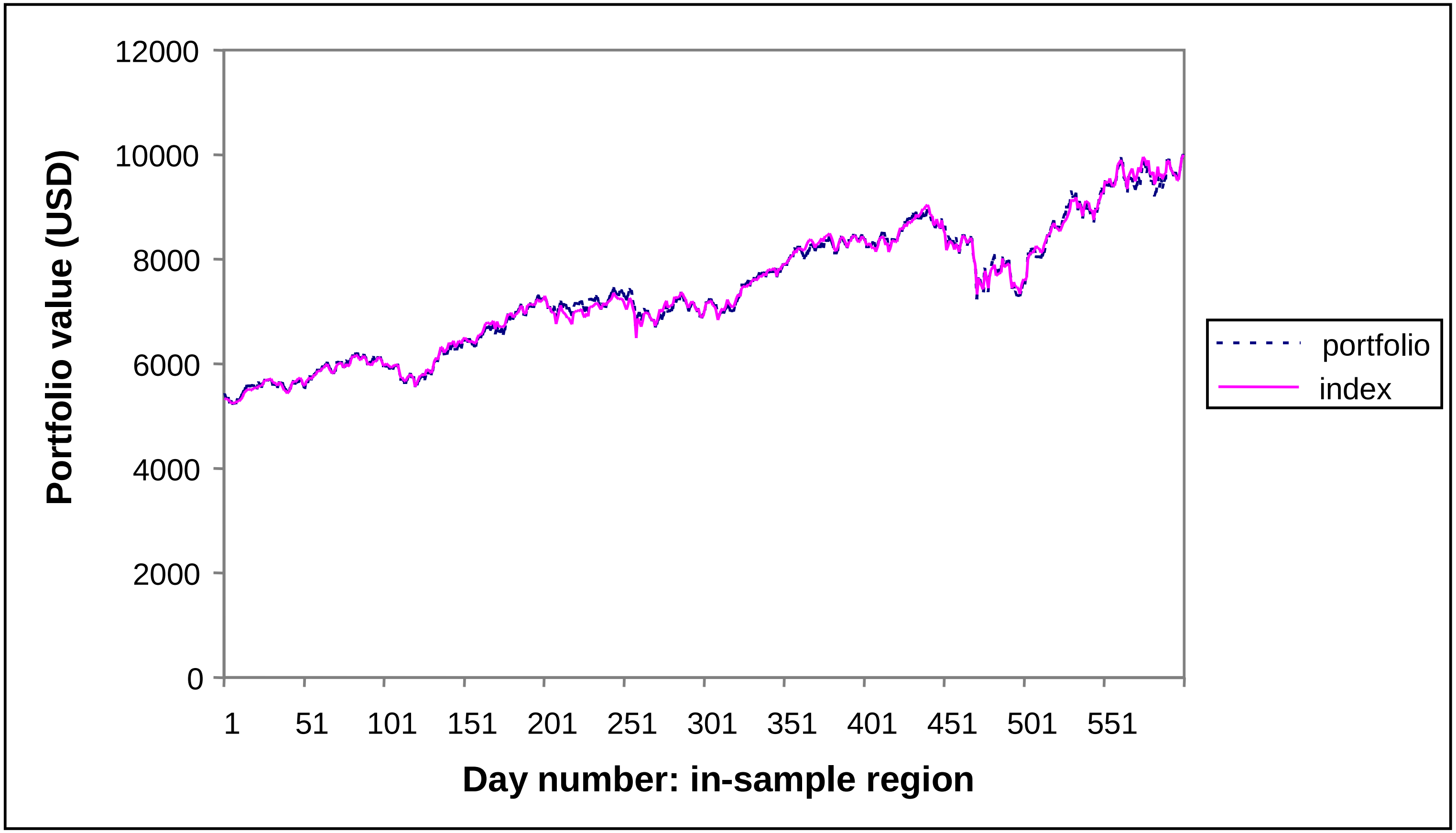
$\sum_{j=1}^{n} p_{jt} x_j$ **= value of the portfolio at time** t

$|(\theta \cdot I_t - \sum_{j=1}^{n} p_{jt} x_j)/(\theta \cdot I_t)|$ **= absolute relative deviation of the portfolio from the target** $\theta \cdot I_t$

$f(x,p_t) = (\theta \cdot I_t - \sum_{j=1}^{n} p_{jt} x_j)/(\theta \cdot I_t)$ **= relative portfolio underperformance compared to target at time** t

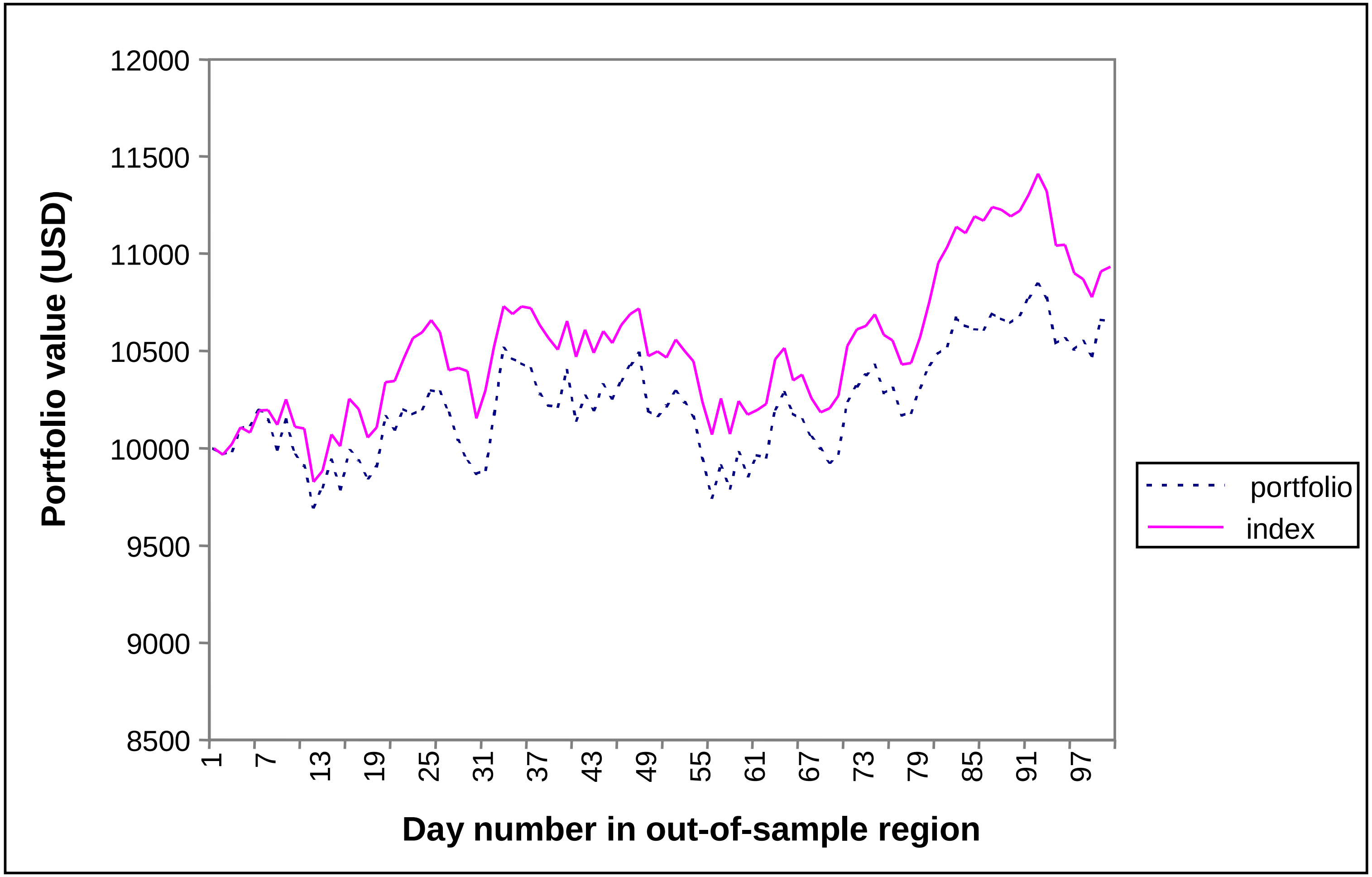
[1]Konno H. and A. Wijayanayake. *Minimal Cost Index Tracking under Nonlinear Transaction Costs and Minimal Transaction Unit Constraints*,Tokyo Institute of Technology, CRAFT Working paper 00-07,(2000).

PORTFOLIO REPLICATION (Cont'd)



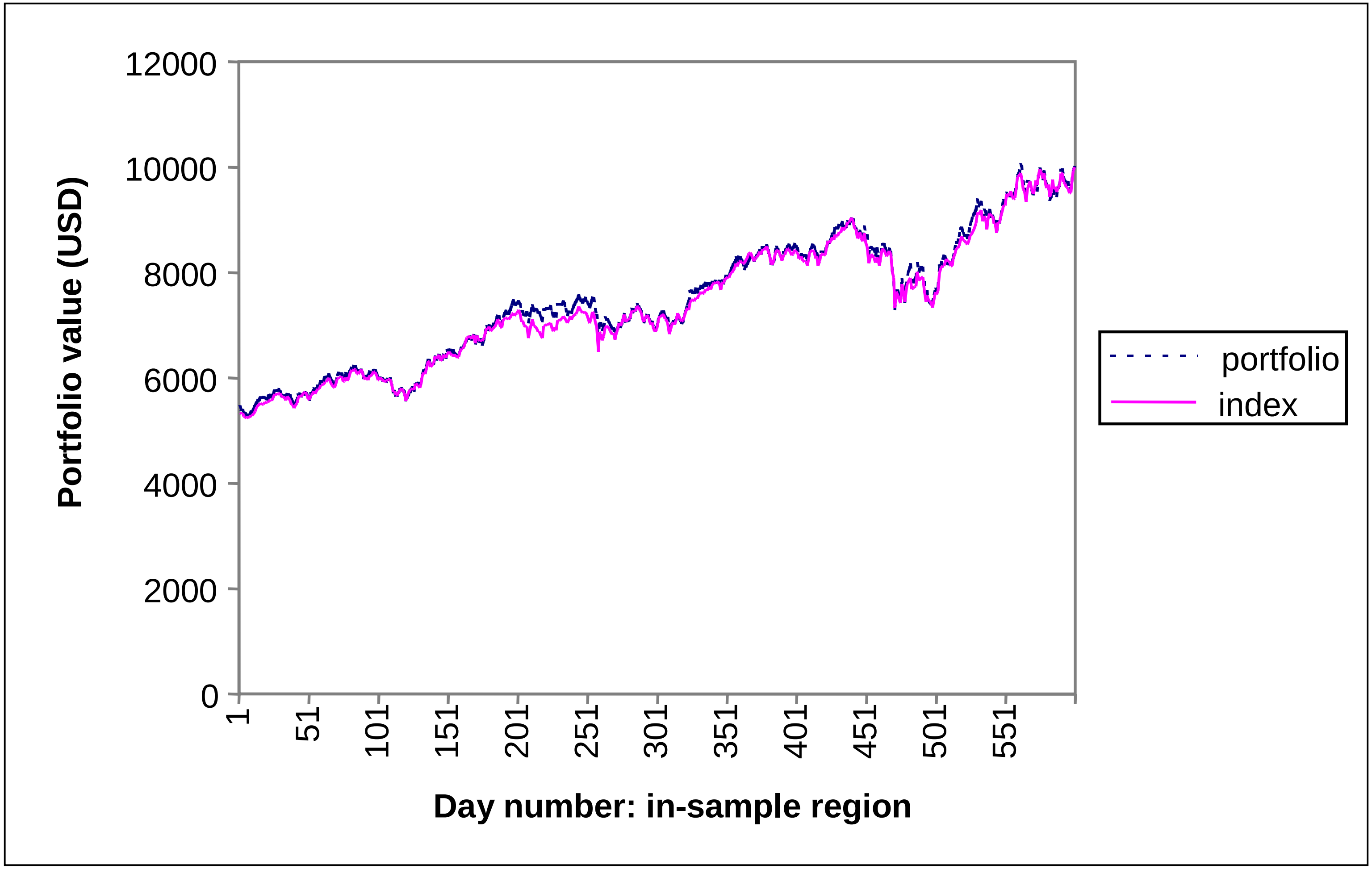
Index and optimal portfolio values in in-sample region, CVaR constraint is inactive (*w* = 0.02)

PORTFOLIO REPLICATION (Cont'd)



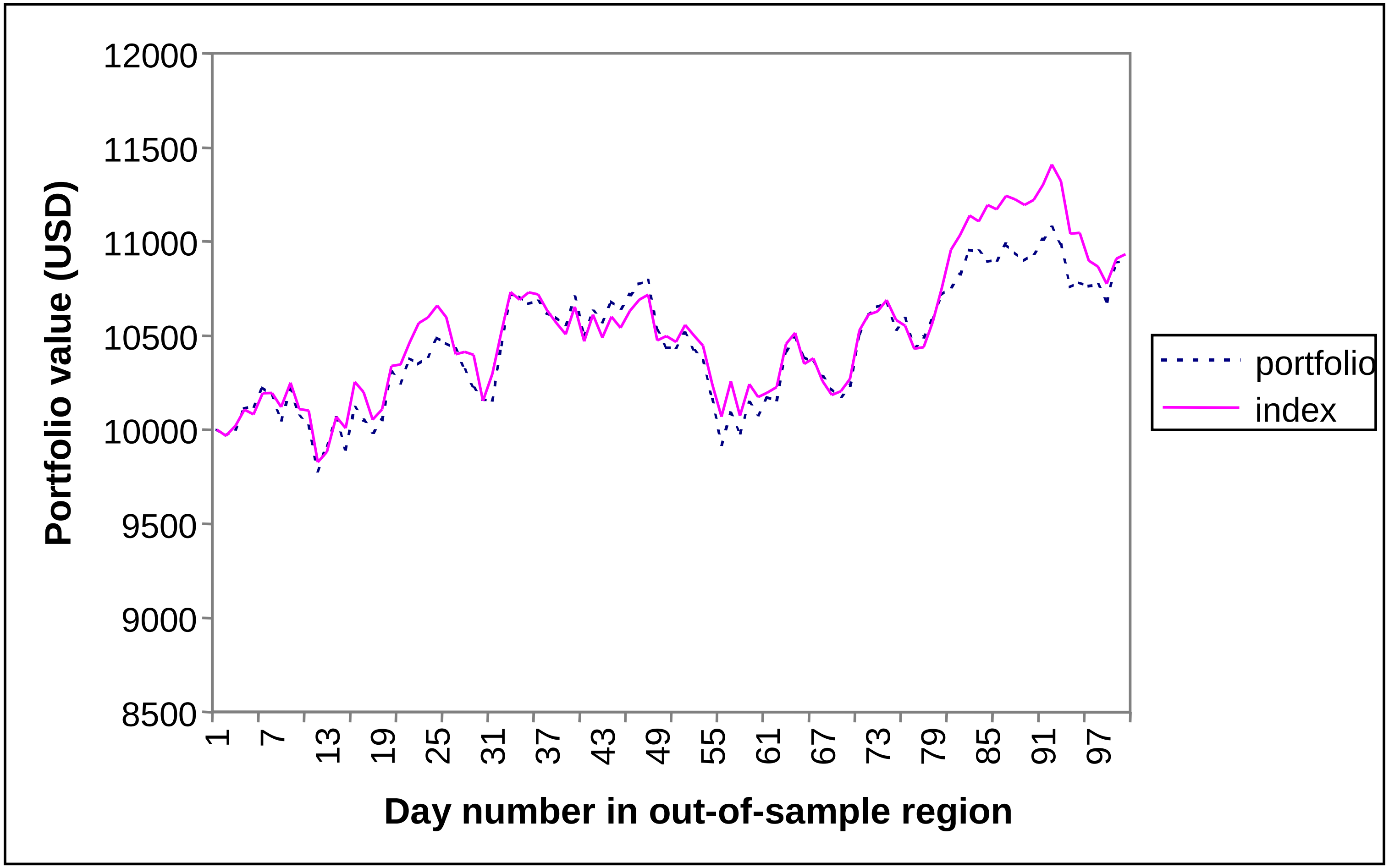
Index and optimal portfolio values in out-of-sample region, CVaR constraint is inactive (*w* = 0.02)

PORTFOLIO REPLICATION (Cont'd)



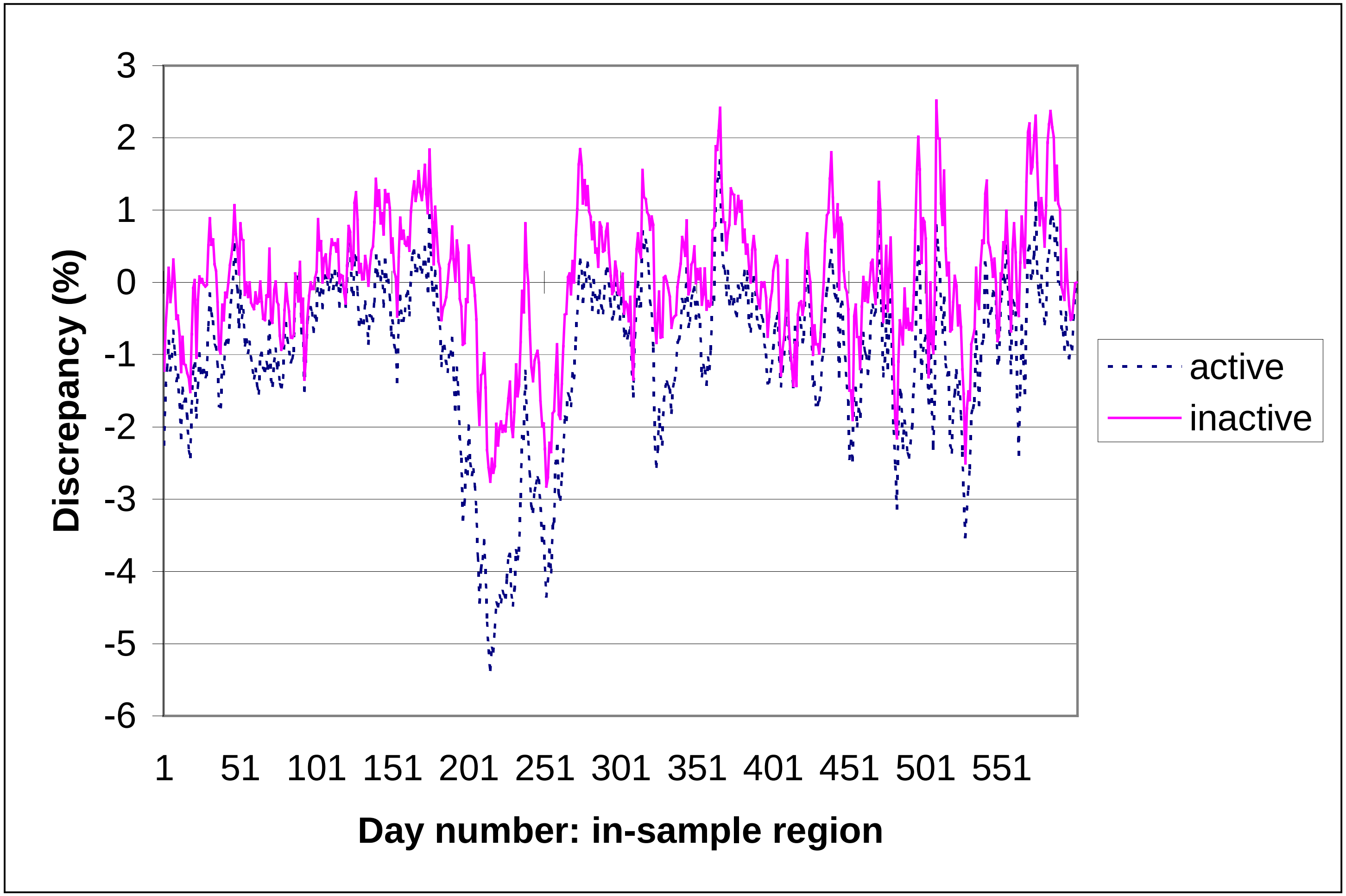
Index and optimal portfolio values in in-sample region, CVaR constraint is active (*w* = 0.005).

PORTFOLIO REPLICATION (Cont'd)



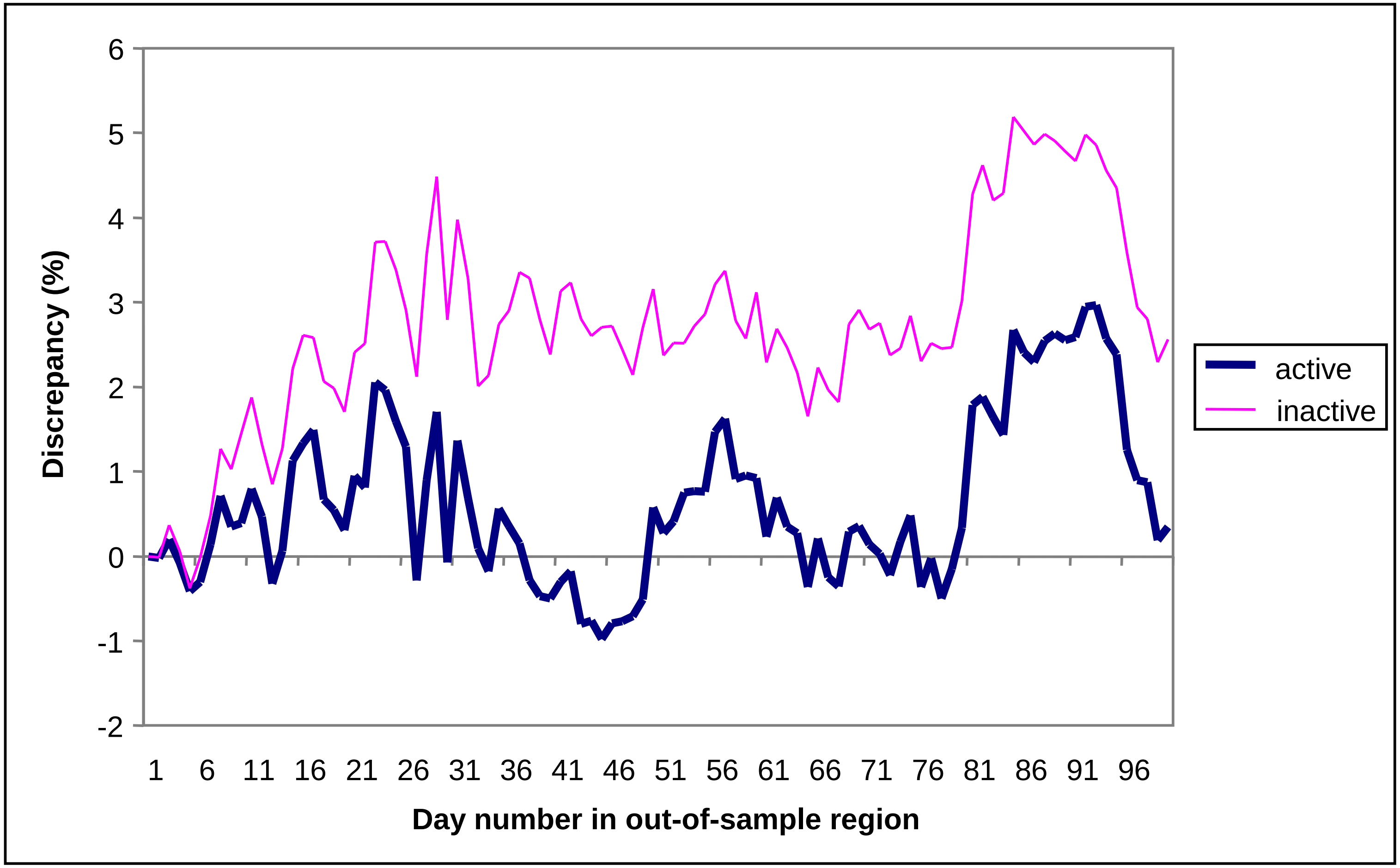
Index and optimal portfolio values in out-of-sample region, CVaR constraint is active (*w* = 0.005).

PORTFOLIO REPLICATION (Cont'd)



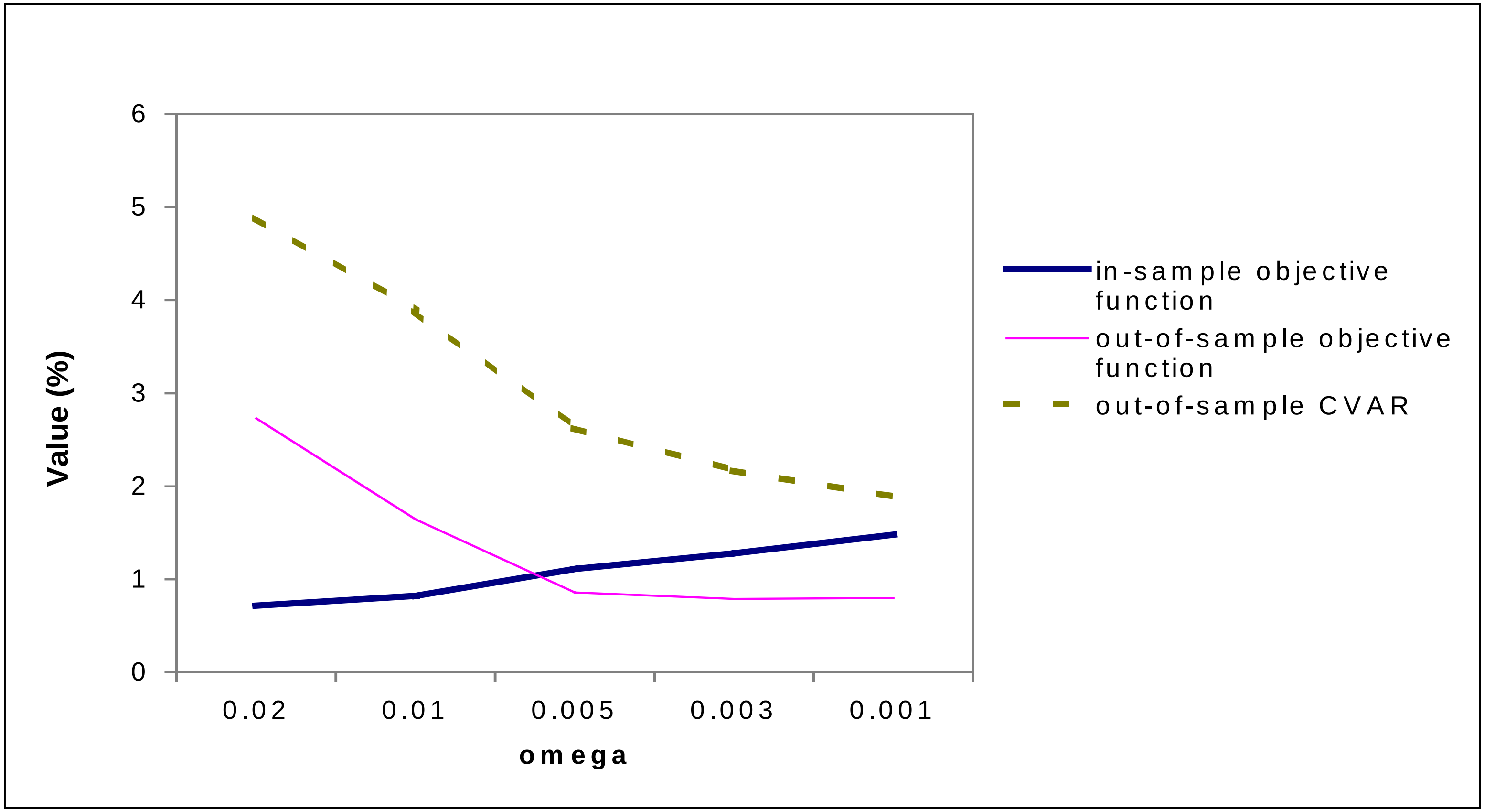
Relative underperformance in in-sample region, CVaR constraint is active (*w* = 0.005) and inactive (*w* = 0.02).

PORTFOLIO REPLICATION (Cont'd)



Relative underperformance in out-of-sample region, CVaR constraint is active (*w* = 0.005) and inactive (*w* = 0.02)

PORTFOLIO REPLICATION (Cont'd)



In-sample objective function (mean absolute relative deviation), out-of-sample objective function, out-of-sample CVaR for various risk levels *w* in CVaR constraint.

PORTFOLIO REPLICATION (Cont'd)

- **Calculation results**

CVaR level w	in-sample (600 days) objective function, in %	out-of-sample (100 days) objective function, in %	out-of-sample CVaR in %
0.02	0.71778	2.73131	4.88654
0.01	0.82502	1.64654	3.88691
0.005	1.11391	0.85858	2.62559
0.003	1.28004	0.78896	2.16996
0.001	1.48124	0.80078	1.88564

- **CVaR constraint reduced underperformance of the portfolio versus the index both in the in-sample region (Column 1 of table) and in the out-of-sample region (Column 4) . For w =0.02, the CVaR constraint is inactive, for $w \leq 0.01$, CVaR constraint is active.**

- **Decreasing of CVaR causes an increase of objective function (mean absolute deviation) in the in-sample region (Column 2).**

- **Decreasing of CVaR causes a decrease of objective function in the out-of-sample region (Column 3). However, this reduction is data specific, it was not observed for some other datasets.**

PORTFOLIO REPLICATION (Cont'd)

***<u>In-sample-calculations</u>*: w=0.005**

- **Calculations were conducted using custom developed software (C++) in combination with CPLEX linear programming solver**

- **For optimal portfolio, CVaR= 0.005. Optimal ζ*= 0.001538627671 gives VaR. Probability of the VaR point is 14/600 (i.e.14 days have the same deviation= 0.001538627671). The losses of 54 scenarios exceed VaR. The probability of exceeding VaR equals 54/600 < 1- α, and**

$$\lambda = (\Psi(\text{VaR}) - \alpha) / (1 - \alpha) = [546/600 - 0.9]/[1 - 0.9] = 0.1$$

- **Since α "splits" VaR probability atom, i.e., Ψ(VaR) - α >0, CVaR is bigger than $CVaR^-$ ("lower CVaR") and smaller than $CVaR^+$ ("upper CVaR", also called expected shortfall)**

  $CVaR^-$ = 0.004592779726 < CVaR = 0.005 < $CVaR^+$=0.005384596925

- **CVaR is the weighted average of VaR and $CVaR^+$**

  CVaR = λ VaR + (1- λ) $CVaR^+$= 0.1 * 0.001538627671 + 0.9 * 0.005384596925= 0.005

- **In several runs, ζ* overestimated VaR because of the nonuniqueness of the optimal solution. VaR equals the smallest optimal ζ*.**

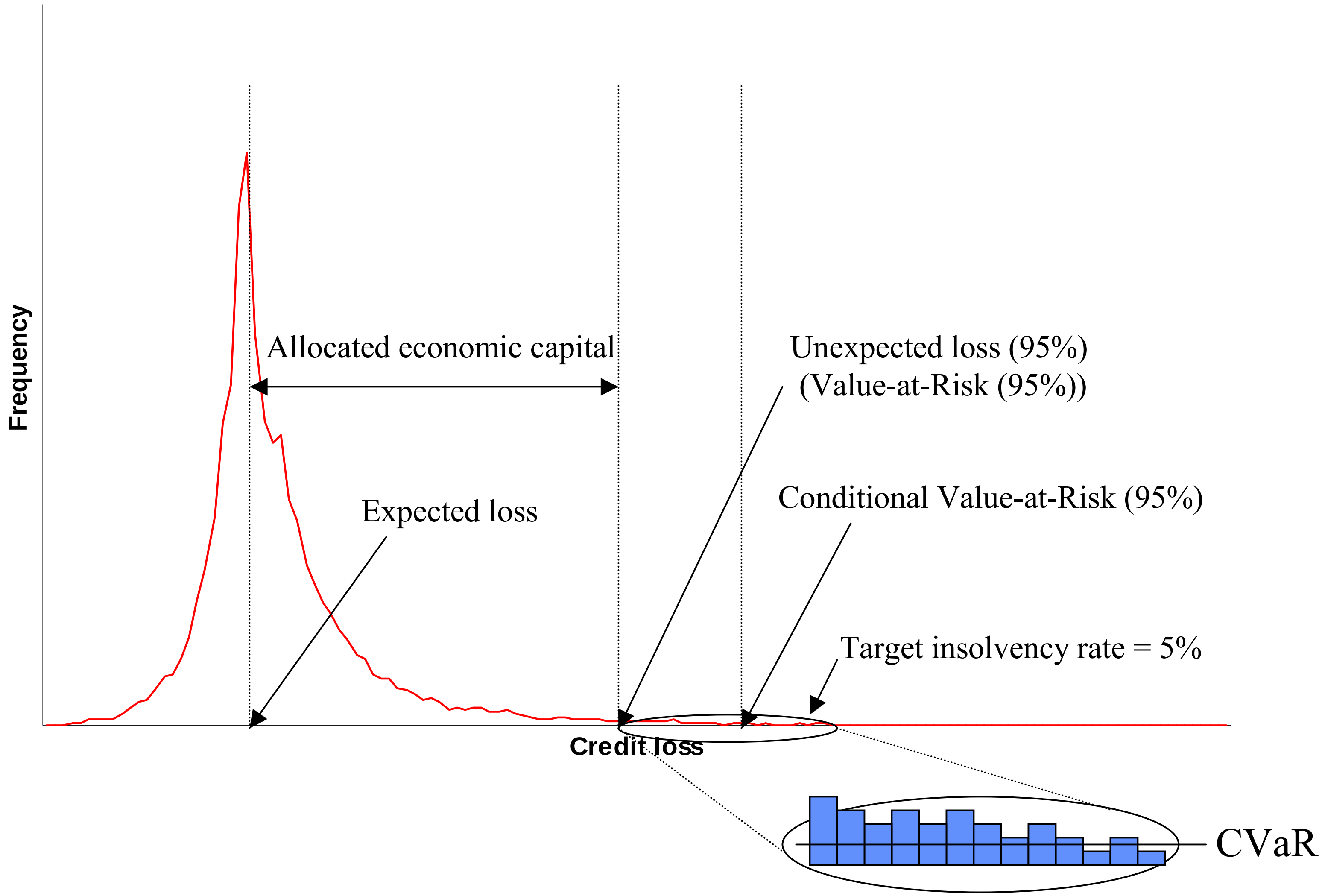
EXAMPLE 4: CREDIT RISK (Related Papers)

- **Andersson, Uryasev, Rosen and Mausser applied the CVaR approach to a credit portfolio of bonds**
  - **Andersson, F., Mausser, Rosen, D. and S. Uryasev (2000), "Credit risk optimization with Conditional Value-at-Risk criterion", *Mathematical Programming,* series B, December)**
- **Uryasev and Rockafellar developed the approach to minimize Conditional Value-at-Risk**
  - **Rockafellar, R.T. and S. Uryasev (2000), "Optimization of Conditional Value-at-Risk", *The Journal of Risk*, Vol. 2 No. 3**
- **Bucay and Rosen applied the CreditMetrics methodology to estimate the credit risk of an international bond portfolio**
  - **Bucay, N. and D. Rosen, (1999)"Credit risk of an international bond portfolio: A case study", *Algo Research Quarterly,* Vol. 2 No. 1, 9-29**
- **Mausser and Rosen applied a similar approach based on the expected regret risk measure**
  - **Mausser, H. and D. Rosen (1999), "Applying scenario optimization to portfolio credit risk", *Algo Research Quarterly*, Vol. 2, No. 2, 19-33**

Basic Definitions

- **Credit risk**
  - **The potential that a bank borrower or counterpart will fail to meet its obligations in accordance with agreed terms**
- **Credit loss**
  - **Losses due to credit events, including both default and credit migration**

Credit Risk Measures

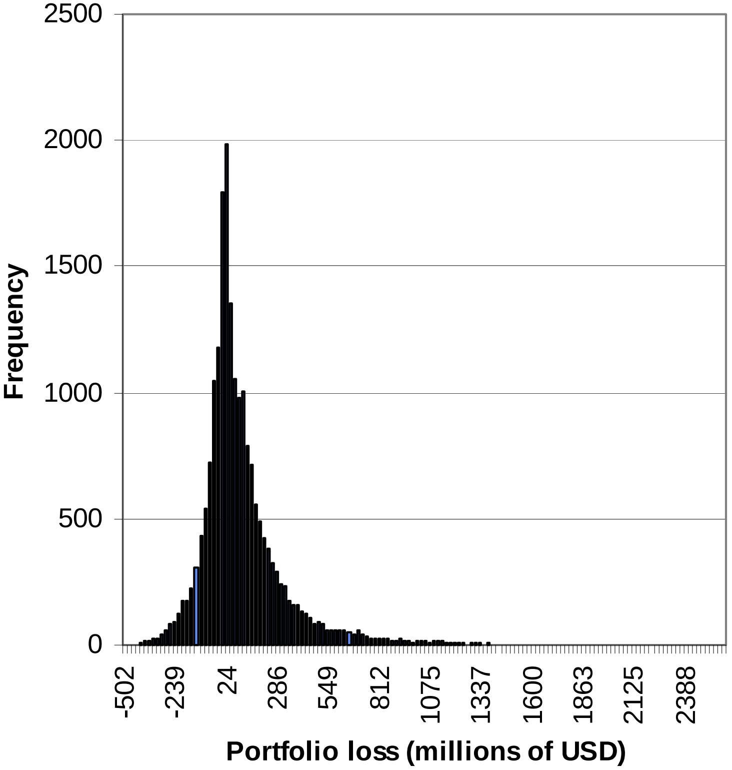


Bond Portfolio

- Compiled to asses to the state-of-the-art portfolio credit risk models
- Consists of 197 bonds, issued by 86 obligors in 29 countries
- Mark-to-market value of the portfolio is 8.8 billions of USD
- Most instruments denominated in USD but 11 instruments are denominated in DEM(4), GBP(1), ITL(1), JPY(1), TRL(1), XEU(2) and ZAR(1)
- Bond maturities range from a few months to 98 years, portfolio duration of approximately five years

Portfolio Loss Distribution

- **Generated by a Monte Carlo simulation based on 20000 scenarios**
- **Skewed with a long fat tail**
- **Expected loss of 95 M USD**
  - **Only credit losses, no interest income**
- **Standard deviation of 232 M USD**
- **VaR (99%) equal 1026 M USD**
- **CVaR (99%) equal 1320 M USD**



Model Parameters

- **Definitions**
  - **1) Obligor weights expressed as multiples of current holdings**
  - **2) Future values without credit migration, i.e. the benchmark scenario**
  - **3) Future scenario dependent values with credit migration**
  - **4) Portfolio loss due to credit migration**

$$\text{(Instrument positions)} \quad \mathbf{x} = (x_1, x_2, \ldots, x_n) \quad (1)$$

$$\text{(Future values without credit migration)} \quad \mathbf{b} = (b_1, b_2, \ldots, b_n) \quad (2)$$

$$\text{(Future values with credit migration)} \quad \mathbf{y} = (y_1, y_2, \ldots, y_n) \quad (3)$$

$$\text{(Portfolio loss)} \quad f(\mathbf{x}, \mathbf{y}) = (\mathbf{b} - \mathbf{y})^T \mathbf{x} \quad (4)$$

OPTIMIZATION PROBLEM

Minimize

$$\alpha + \frac{1}{(1-\beta)J}\sum_{j=1}^{J} z_j$$

Subject to :

(Excess loss) $$z_j \ge \sum_{i=1}^{n}((b_i - y_{j,i})x_i) - \alpha, \quad j = 1,\dots, J,$$

$$z_j \ge 0, \quad j = 1,\dots, J,$$

(Upper/lower bound) $$l_i \le x_i \le u_i, \quad i = 1,\dots, n,$$

(Portfolio value) $$\sum_{i=1}^{n} q_i x_i = \sum_{i=1}^{n} q_i,$$

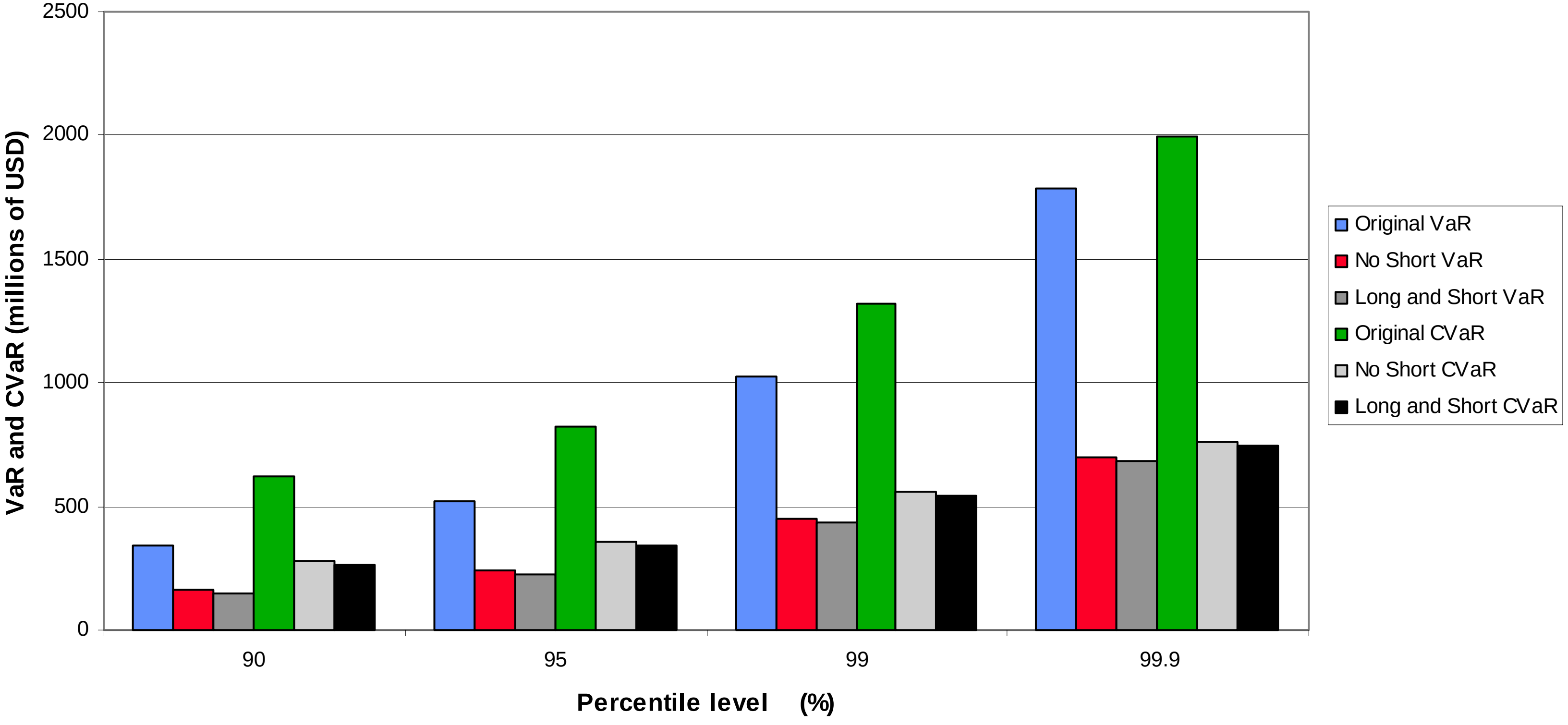
(Expected return) $$\sum_{i=1}^{n} q_i (r_i - R) x_i \ge 0,$$

(Long position) $$x_i q_i \le 0{,}20 \sum_{i=1}^{n} q_i, \quad i = 1,\dots, n$$

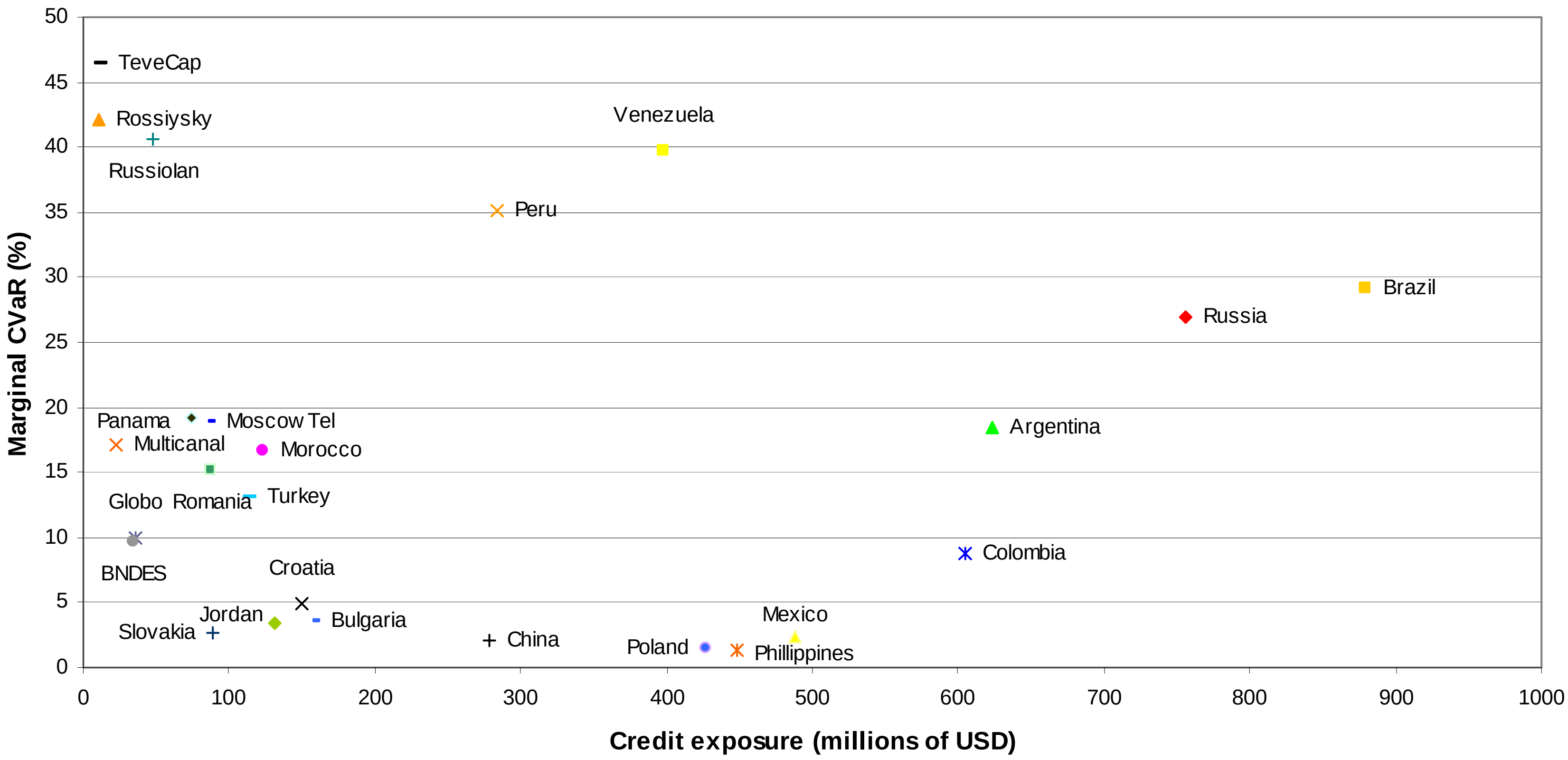
SINGLE - INSTRUMENT OPTIMIZATION

Obligor	Best Hedge	VaR (M USD)	VaR (%)	CVaR (M USD)	CVaR (%)
Brazil	-5.72	612	40	767	42
Russia	-9.55	667	35	863	35
Venezuela	-4.29	683	33	880	33
Argentina	-10.30	751	27	990	25
Peru	-7.35	740	28	980	26
Colombia	-45.07	808	21	1040	21
Morocco	-88.29	792	23	1035	22
Russialan	-21.25	777	24	989	25
MoscowTel	-610.14	727	29	941	29
Romania	-294.23	724	29	937	29
Mexico	-3.75	998	3	1292	2
Philippines	-3.24	1015	1	1309	1

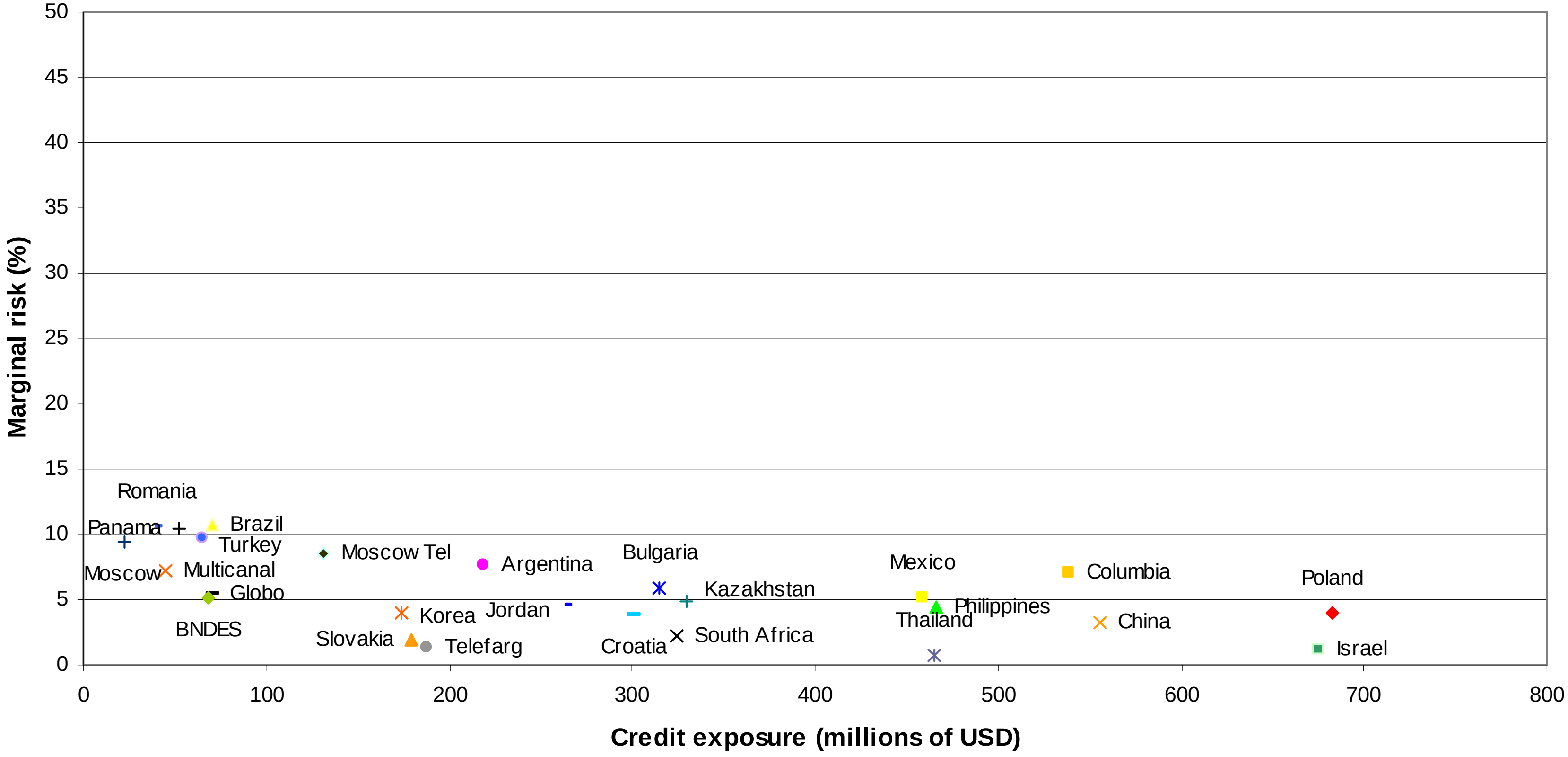
MULTIPLE - INSTRUMENT OPTIMIZATION



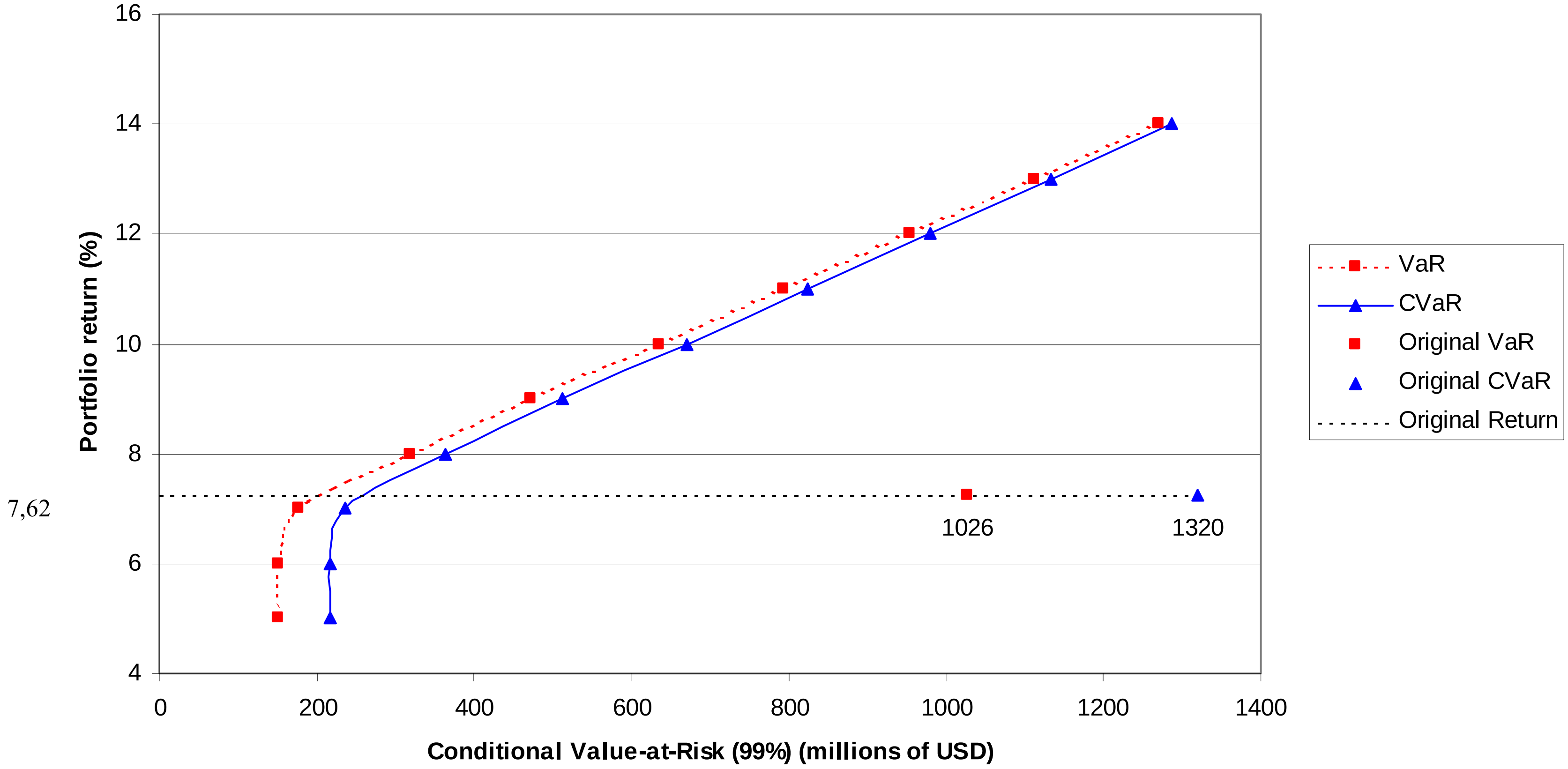
RISK CONTRIBUTION (original portfolio)



RISK CONTRIBUTION (optimized portfolio)



EFFICIENT FRONTIER



CONDITIONAL DRAWDOWN-AT-RISK (CDaR)

- **CDaR[1] is a new risk measure closely related to CVaR**
- **Drawdown is defined as a drop in the portfolio value compared to the previous maximum**
- **CDaR is the average of the worst z% portfolio drawdowns observed in the past (e.q., 5% of worst drawdowns). Similar to CVaR, averaging is done using α-tail distribution.**
- **Notations:**

  $w(x,t)$ = **uncompounded portfolio value**

  t = **time**

  $x = (x_1,...x_n)$ = **portfolio weights**

  $f(x,t) = \max_{\{0 \leq \tau \leq t\}} [w(x,\tau)] - w(x,t)$ = **drawdown**

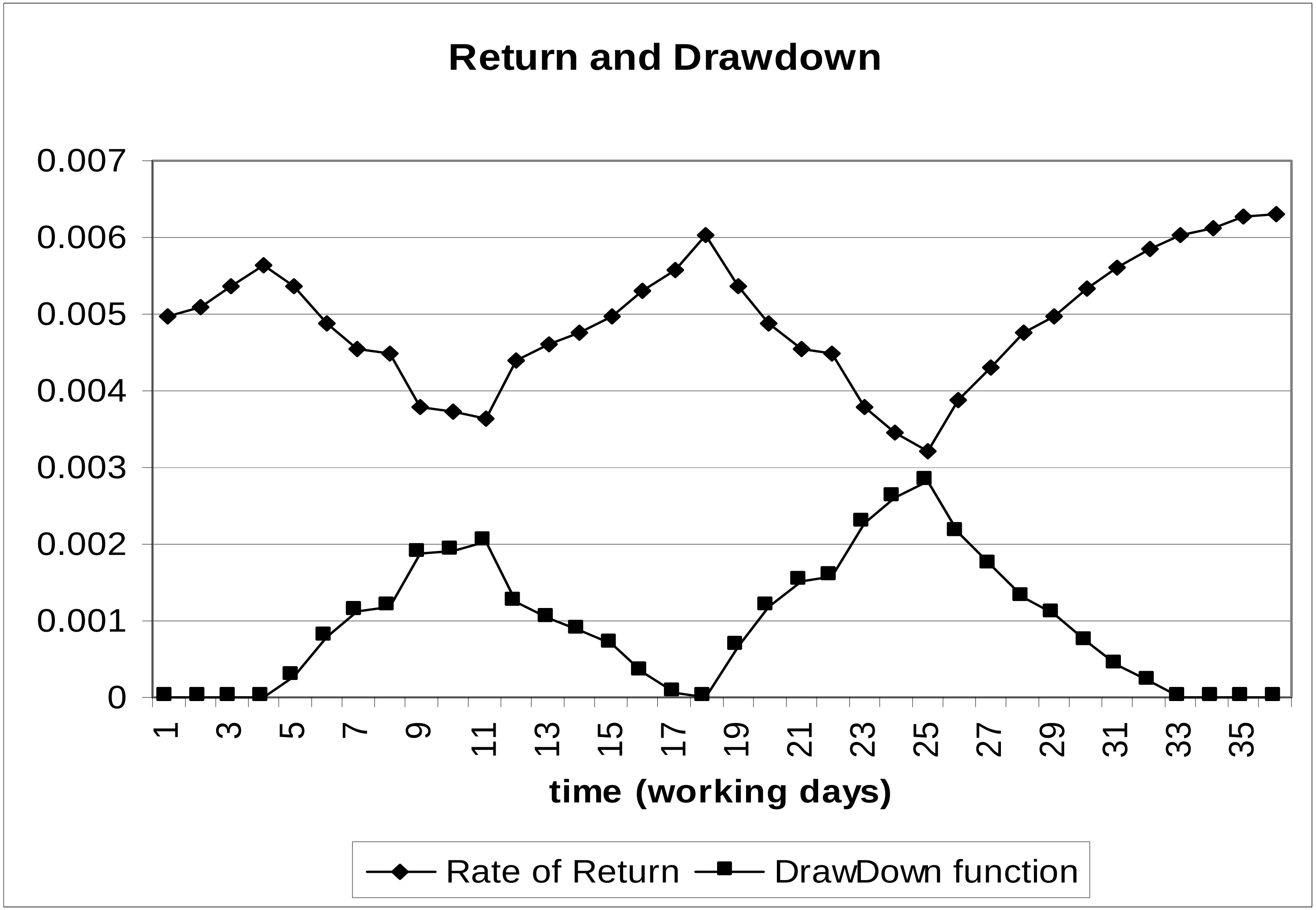
- **<u>Formal definition</u>:**

  CDaR is CVaR with drawdown loss function $f(x,t)$.

- **CDaR can be controlled and optimized using linear programming similar to CVaR**
- **Detail discussion of CDaR is beyond the scope of this presentation**

[1]**Chekhlov, A., Uryasev, S., and M. Zabarankin. *Portfolio Optimization with Drawdown Constraints*. Research Report 2000-5. ISE Dept., University of Florida, April 2000.**

CDaR: EXAMPLE GRAPH



CONCLUSION

- **CVaR is a new risk measure with significant advantages compared to VaR**

  - can quantify risks beyond VaR
  - coherent risk measure
  - consistent for various confidence levels α (smooth w.r.t α)
  - relatively stable statistical estimates (integral characteristics)

- **CVaR is an excellent tool for risk management and portfolio optimization**

  - optimization with linear programming: very large dimensions and stable numerical implementations
  - shaping distributions: multiple risk constraints with different confidence levels at different times
  - fast algorithms which can be used in online applications, such as active portfolio management

- **CVaR methodology is consistent with mean-variance methodology under normality assumption**

  - CVaR minimal portfolio (with return constraint) is also variance minimal for normal loss distributions

CONCLUSION (Cont'd)

- **Various case studies demonstrated high efficiency and stability of of the approach (papers can be downloaded: www.ise.ufl.edu/uryasev)**

  - optimization of a portfolio of stocks
  - hedging of a portfolio of options
  - credit risk management (bond portfolio optimization)
  - asset and liability modeling
  - portfolio replication
  - optimal position closing strategies

- **CVaR has a great potential for further development. It stimulated several areas of applied research, such as Conditional Drawdown-at-Risk and specialized optimization algorithms for risk management**

- **Risk Management and Financial Engineering Lab at UF (www.ise.ufl.edu/rmfe) leads research in CVaR methodology and is interested in applied collaborative projects**

APPENDIX: RELEVANT PUBLICATIONS

[1] Bogentoft, E. Romeijn, H.E. and S. Uryasev (2001): Asset/Liability Management for Pension Funds Using Cvar Constraints. Submitted to The Journal of Risk Finance (download: www.ise.ufl.edu/uryasev/multi_JRB.pdf)

[2] Larsen, N., Mausser H., and S. Uryasev (2001): Algorithms For Optimization Of Value-At-Risk . Research Report 2001-9, ISE Dept., University of Florida, August, 2001 (www.ise.ufl.edu/uryasev/wp_VaR_minimization.pdf)

[3] Rockafellar R.T. and S. Uryasev (2001): Conditional Value-at-Risk for General Loss Distributions. Submitted to The Journal of Banking and Finance (relevant Research Report 2001-5. ISE Dept., University of Florida, April 2001, www.ise.ufl.edu/uryasev/cvar2.pdf)

[4] Uryasev, S. Conditional Value-at-Risk (2000): Optimization Algorithms and Applications. *Financial Engineering News*, No. 14, February, 2000. (www.ise.ufl.edu/uryasev/finnews.pdf)

[5] Rockafellar R.T. and S. Uryasev (2000): Optimization of Conditional Value-at-Risk. *The Journal of Risk.* Vol. 2, No. 3, 2000, 21-41. (www.ise.ufl.edu/uryasev/cvar.pdf).

[6] Andersson, F., Mausser, H., Rosen, D., and S. Uryasev (2000): Credit Risk Optimization With Conditional Value-At-Risk Criterion. Mathematical Programming, Series B, December, 2000. (/www.ise.ufl.edu/uryasev/Credit_risk_optimization.pdf)

APPENDIX: RELEVANT PUBLICATIONS (Cont'd)

[7] Palmquist, J., Uryasev, S., and P. Krokhmal (1999): Portfolio Optimization with Conditional Value-At-Risk Objective and Constraints. Submitted to *The Journal of Risk* (www.ise.ufl.edu/uryasev/pal.pdf)

[8] Chekhlov, A., Uryasev, S., and M. Zabarankin (2000): Portfolio Optimization With Drawdown Constraints. Submitted to *Applied Mathematical Finance* journal.

(www.ise.ufl.edu/uryasev/drd_2000-5.pdf)

[9] Testuri, C.E. and S. Uryasev. On Relation between Expected Regret and Conditional Value-At-Risk. Research Report 2000-9. ISE Dept., University of Florida, August 2000. Submitted to *Decisions in Economics and Finance* journal. (www.ise.ufl.edu/uryasev/Testuri.pdf)

[10] Krawczyk, J.B. and S. Uryasev. Relaxation Algorithms to Find Nash Equilibria with Economic Applications. Environmental Modeling and Assessment, 5, 2000, 63-73. (www.baltzer.nl/emass/articlesfree/2000/5-1/ema505.pdf)

[11] Uryasev, S. Introduction to the Theory of Probabilistic Functions and Percentiles (Value-at-Risk). In "Probabilistic Constrained Optimization: Methodology and Applications," Ed. S. Uryasev, Kluwer Academic Publishers, 2000. (www.ise.ufl.edu/uryasev/intr.pdf).

APPENDIX: BOOKS

[1] Uryasev, S. Ed. "Probabilistic Constrained Optimization: Methodology and Applications," Kluwer Academic Publishers, 2000.

[2] Uryasev, S. and P. Pardalos, Eds. " Stochastic Optimization: Algorithms and Applications," Kluwer Academic Publishers, 2001 (proceedings of the conference on Stochastic Optimization, Gainesville, FL, 2000).