Equilibrium: Stochastic Environment

Roger J-B Wets

University of California, Davis

co: Alejandro Jofre
R.T. Rockafellar, Adib Bagh, Sergio Lucero

Lecture Plan

- stability of equilibrium points
- dynamic & stochastic models
- dealing with nonanticipativity
- disintegration
- equilibrium prices

Review: (Variational Analysis)

Optimization Problem: $\max f_0(x)$ so that $x \in C$

Define: $$f(x) = \begin{cases} f_0(x) & \text{if } x \in C \\ -\infty & \text{otherwise} \end{cases}$$

$$\approx \max f(x), \quad x \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to [-\infty, \infty)$$

Lagrangian: $$L(x, y) = f_0(x) + \langle y, G(x) \rangle \text{ if } x \in C,$$
$$= -\infty \text{ if } x \notin C$$

from: $\max f_0(x)$ so that $G(x) = 0,\ x \in C$

Variational Convergence

- hypo-convergence

$$f^\nu \xrightarrow[h]{} f \Rightarrow \arg\max f^\nu \to \arg\max f$$

continuous conv. ➔ hypo-convergence

- hypo/epi-convergence (Lagrangian fcns)

$$L^\nu(\bullet,\bullet) \xrightarrow[h/e]{} L \Rightarrow \text{saddle-pts } L^\nu \to \text{saddle-pts } L$$

$$\arg\max_x \inf_y L^\nu(x,y) \longrightarrow \arg\max_x \inf_y L(x,y)$$

$$\arg\min_y \sup_x L^\nu(x,y) \longrightarrow \arg\min_y \sup_x L(x,y)$$

Variational Convergence II

- lopsided convergence

$$L^{\nu}(\bullet,\bullet) \xrightarrow[lop]{} L$$

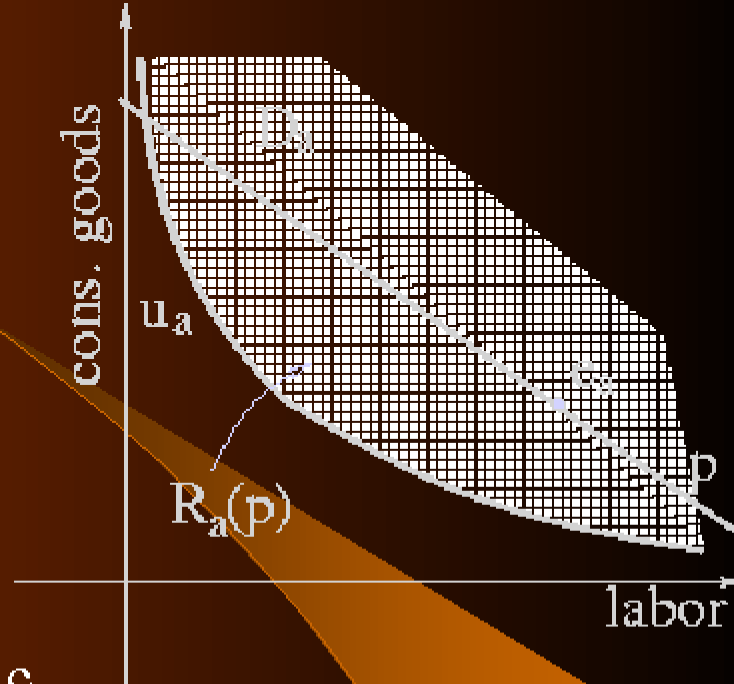
$$\Rightarrow \arg\max_x \inf_y L^{\nu} \to \arg\max_x \inf_y L$$

- definition:

$$\forall\, x^{\nu} \to x, \exists\, y^{\nu} \to y \Rightarrow \limsup L^{\nu}(x^{\nu}, y^{\nu}) \le L(x, y)$$

$$\exists\, x^{\nu} \to x, \forall\, y^{\nu} \to y \Rightarrow \liminf L^{\nu}(x^{\nu}, y^{\nu}) \ge L(x, y)$$

Equilibrium Theory



Agents : $a \in \mathcal{A}$, $|\mathcal{A}|$ finite

$e_a \in R^n$, goods = endowment

$u_a : R^n \to \overline{R}$ utility function, usc

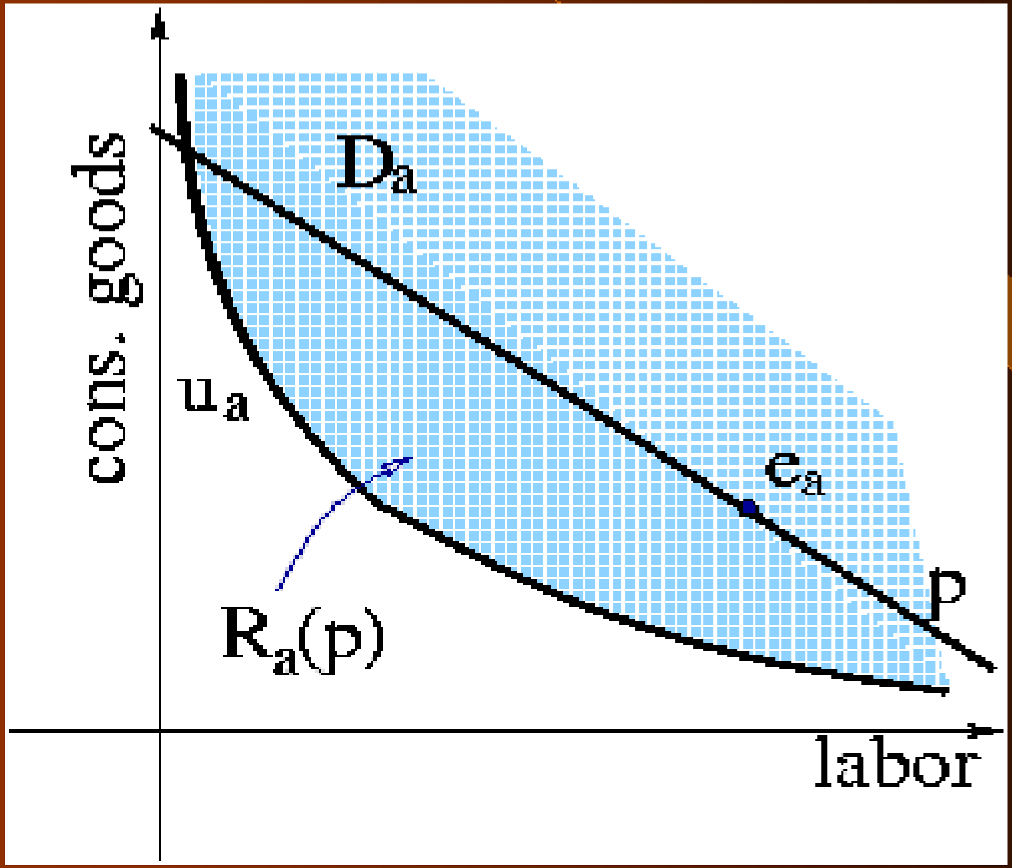
strictly concave, sup-compact

$D_a = \{ c \mid u_a(c) < \infty \}$ survival set

demand function (of agent a)

$$d_a(p) = \underset{c \in R^n}{\text{argmax}} \ \{ u_a(c) \mid \langle p, c \rangle \leq \langle p, e_a \rangle \}$$

Market versus State Regulated



$\exists\ p$ in $\Sigma : s(p) = \Sigma_a e_a - \Sigma_a d_a(p) \geq 0$

some Applications

- Transportation design
  - network layout, new routes
- Financial markets
  - introducing new instruments
- Marketing
  - pricing of new/modified products
- and also … … Economics

The WALRASIAN

excess supply : $s(p) = \sum_{a \in A} e_a - \sum_{a \in A} d_a(p)$

$$W(p,q) = \langle q, s(p) \rangle, \quad W : \Sigma \times \Sigma \to R$$

$$d_a(p) = \underset{c \in R^n}{\operatorname{argmax}} \left\{ u_a(c) \mid \langle p, c \rangle \leq \langle p, e_a \rangle \right\}$$

$p \mapsto d_a(p)$ continuous (by hypo-convergence)

$\Rightarrow \quad p \mapsto s(p)$ is continuous

Ky Fan function

$$W(p,q) = \langle q, s(p) \rangle$$

(a) $\forall \quad q \in \Sigma : \; p \mapsto W(p,q)$ is usc (continuous),

(b) $\forall \quad p \in \Sigma : q \mapsto W(p,q)$ is convex (linear),

(c) $\forall \quad q \in \Sigma : W(q,q) \geq 0,$ (budget constraint)

$$\forall a : \quad \langle q, e_a - d_a(q) \rangle \geq 0$$

$\Rightarrow W$ is a Ky Fan function

Equilibrium price

Ky Fan Inequality: W Ky Fan fcn

& Σ compact, convex

$$\Rightarrow \quad \exists\, \bar{p} \in \arg\max_{p\in\Sigma} \left[\inf_{q\in\Sigma} W(p,q) \right]$$

and $\inf_{q\in\Sigma} W(\bar{p},q) \geq 0$; recall $W(p,q) = \langle p, s(p) \rangle$.

Claim: $\bar{p}$ is an equilibrium price, i.e., $s(\bar{p}) \geq 0$

$$W(\bar{p},q) = \langle q, s(\bar{p}) \rangle \geq 0, \quad \forall q \in \Sigma$$

Stability properties

- Continuous convergence:

$$u_a^\nu \xrightarrow[c]{} u_a,\ p^\nu \to p \Rightarrow d_a^\nu(p^\nu) \to d_a(p)$$

$$\Rightarrow s^\nu(p^\nu) \to s(p) \quad \text{equiv.} \quad s^\nu \xrightarrow[c]{} s$$

- Lopsided convergence of Walrasians

$$W(p,q) = \langle q, s(p) \rangle, \quad W^\nu(p,q) = \langle q, s^\nu(p) \rangle$$

$$W^\nu \xrightarrow[lop]{} W \qquad \text{Ky Fan fcns closed under lopsided}$$

- i.e. $\arg\max_p \inf_q W^\nu \xrightarrow[\text{cluster}]{} \arg\max_p \inf_q W$

$$\exists\, p^\nu \to p \quad \text{(equilibirum points)}$$

Augmented Walrasian

- PL-homotopy methods: Scarf, Eaves, Saigal
- Augmented Walrasian: Bagh, Lucero

$\bar{p}$ max/inf point of W

$\cong$ saddle point $(\bar{p}, \bar{q})$ of $\widetilde{W}_r$

$$\widetilde{W}_r(p,q) = \inf_u \left\{ W(p,u) + r\|u\| - \langle q,u \rangle \right\}$$
$$= \sup_z \left\{ W(p,z) \mid \|z-q\|^{\circ} \le r \right\}$$

with $\|\cdot\|$ a norm and $\|\cdot\|^{\circ}$ its dual norm

Iterations

$$W(p,q) = \langle q, s(p) \rangle \quad \text{on} \quad \Sigma \times \Sigma$$

$$\tilde{W}_r(p,q) = \sup_z \; \{ \; W(p,z) \; : \; \|z-q\|^\circ \le r \; \}$$

$$q^{k+1} = \underset{q\in\Sigma}{\operatorname{argmax}} \; [\; \max_z \langle z, s(p^k) \rangle \; : \; \|z-q\|^\circ \le r_k \;]$$

minimizing a linear form on a ball

$$p^{k+1} = \underset{p\in\Sigma}{\operatorname{argmin}} \; [\; \max_z \langle z, s(p^k) \rangle \; : \; \|z-q^{k+1}\|^\circ \le r_{k+1} \;]$$

reduces to finding the largest element of $s(p^k)$

as $r \uparrow \infty$, $p^k \to \bar{p}$ (max-inf point)

experiments: 10 agents, 150 goods (easy!)

Demand functions

- Cobb-Douglas utility function:

$$u_a(x) = \gamma_a \prod_{l=1}^{n} x_l^{\beta_l^a} \quad \text{with} \quad \sum_{l=1}^{n} \beta_l^a = 1, \ \beta_l^a \geq 0$$

- budget constraint:

$$\sum_l p_l x^l \leq \sum_l p_l e_a^l$$

- demand:

$$d_a^l(p) = (\beta_l^a / p_l)\left(\sum_l p_l e_a^l\right), \quad l = 1, \ldots, n$$

(demand = supply)

A dynamic 2-stage model

Agent's problem:

$$\max_{c^1, x, c^2 \in \mathbb{R}^n} \; u_a^1(c^1) + \langle q, x \rangle + u_a^2(c^2)$$

$$\text{s.t.} \quad \langle p^1, c^1 \rangle \leq \langle p^1, e_a^1 - x \rangle$$

$$\langle p^2, c^2 \rangle \leq \langle p^2, e_a^2 + T_a(x) \rangle$$

Existence of equilibrium: $\cong$ the 1-stage case

$\exists \; p = (p^1, p^2)$ equilibrium prices

A (dynamic) stochastic model

Agent's problem:

$$\max_{c^1, x \in \mathbb{R}^n, c^2_\bullet \in \mathcal{M}} \; u^1_a(c^1) + \langle q, x \rangle + E\{u^2_a(c^2_\xi)\}$$

so that $\langle p^1, c^1 \rangle \le \langle p^1, e^1_a - x \rangle,$

$$\langle p^2_\xi, c^2_\xi \rangle \le \langle p^2_\xi, e^{a,2}_\xi + T^a_\xi(x) \rangle, \; \forall \xi \in \Xi$$

$$\mathcal{M} = \mathcal{M}(\Xi; \mathbb{R}^n)$$

Existence of equilibrium: not like 1-stage case

Here-&-Now vs. Wait-&-See

- Decision → observation → decision

  $$(d_a^1, x_a) \;\Rightarrow\; \xi \;\Rightarrow\; d_a^2(\xi)$$

- Here-&-now problem!
  - not all contingencies available in period 1
  - (d_a^1, x_a) can't depend on ξ!
- Wait-&-see problem
  - implicitly all contingencies available in period 1
  - choose (d_a^1, x_a, d_a^2) after observing ξ.

incomplete ⟲ complete market ?

Fundamental Theorem of Stochastic Optimization

A here-and-now problem can be “reduced” to a wait-and-see problem by introducing the

price of nonanticipativity

Nonanticipativity

Here-&-now

$$\max E\left\{f(\xi, x^1, x^2_\xi)\right\}$$

$$x^1 \in C^1 \subset \mathbb{R}^{n_1}$$

$$x^2_\xi \in C^2(\xi, x^1), \quad \forall \xi$$

Explicit nonanti. constraints

$$\max E\left\{f(\xi, x^1_\xi, x^2_\xi)\right\}$$

$$x^1_\xi \in C^1 \subset \mathbb{R}^{n_1}$$

$$x^1_\xi \in C^2(\xi, x^1_\xi), \quad \forall \xi$$

$$x^1_\xi = E\{x^1_\xi\}, \quad \forall \xi$$

w_ξ perp. to c$^{\text{ste}}$ fcns

$$\Rightarrow \mathrm{E}\left\{\mathrm{w}_\xi\right\} = 0.$$

DISINTEGRATION

$$\max_{x^1_\cdot, x^2_\cdot \in \mathcal{M}} E\left\{ f(\xi; x^1_\xi, x^2_\xi) - \left\langle w_\xi, x^1_\xi \right\rangle \right\}$$

$$x^1_\xi \in C^1,\ x^2_\xi \in C^2(\xi, x^1_\xi),\ \forall \xi \in \Xi$$

solved `separately' for each ξ (in Ξ)

$$(x^{1,*}_\xi, x^{2,*}_\xi) = \arg\max f(\xi, x^1, x^2) - \left\langle w_\xi, x^1 \right\rangle$$

$$x^1 \in C^1 \subset \mathbb{R}^n,\ x^2 \in C^2(\xi, x^1) \subset \mathbb{R}^n$$

$\xi \mapsto x^{1,*}_\xi \equiv \mathrm{c}^{\mathrm{ste}}$, $\xi \mapsto x^{2,*}_\xi$ (collation of sol'ns)

Progressive Hedging

- Step 0. $w^0(\cdot)$ such that $E\left\{w^0(\xi)\right\}=0$
- Step 1. for all ξ:

$$(x_k^1(\xi), x_k^2(\xi)) = \arg\max f(\xi, x^1, x^2) - \left\langle w^k(\xi), x^1 \right\rangle$$

$$x^1 \in C^1 \subset \mathbb{R}^{n_1},\ \ x^2 \in C^2(\xi, x^1) \subset \mathbb{R}^{n_2}$$

- Step 2. $w^{k+1}(\xi) = w^k(\xi) + \rho\left[x_k^1(\xi) - E\left\{x_k^1(\xi)\right\}\right]$
  - and return to Step 1
- Convergence: add prox. term $-\frac{\rho}{2}\left\|x_k^1(\xi) - E\left\{x_k^1(\xi)\right\}\right\|^2$

linear rate in (x_k, w^k)

Agent's problem

with $\quad p_\xi = (p^1, p_\xi^2), \qquad p_\xi^2 = p^2(\xi)$

$$\left(d_a(p_\bullet); x_a\right) = \arg\max_{c^1, x, c_\bullet^2} \left\{u_a^1(c^1) - \langle q, x\rangle + \mathrm{E}\left\{u_a^2(c_\xi^2)\right\}\right\}$$

$$\langle p^1, c^1\rangle \le \langle p^1, e_a^1 - x\rangle,$$

$$\langle p_\xi^2, c_\xi^2\rangle \le \langle p_\xi^2, e_\xi^{a,2} + T_\xi^a(x)\rangle, \quad \forall \xi \in \Xi$$

Disintegration: agent's problem

with $p_\xi = (p^1, p^2_\xi), \qquad p^2_\xi = p^2(\xi)$

$$\left(d^1_a, d^{2,a}_\xi; x_a\right) =$$

$$\arg\max_{c^1, x, c^2} \left\{ u^1_a(c^1) - \left\langle \bar{w}^a_\xi, c^1 \right\rangle - \left\langle q + \tilde{w}^a_\xi, x \right\rangle + u^2_a(c^2) \right\}$$

$$\left\langle p^1, c^1 \right\rangle \le \left\langle p^1, e^1_a - x \right\rangle$$

$$\left\langle p^2_\xi, c^2 \right\rangle \le \left\langle p^2_\xi, e^{a,2}_\xi + T^a_\xi(x) \right\rangle$$

solved for each ξ "separately"

Continuity of "w" multipliers

- $C_1^{\nu} = \left\{(c^1, x) \mid \langle p_\nu^1, c^1 + x \rangle \le \langle p_\nu^1, e_a^1 \rangle \right\}$

$$\xrightarrow[set]{} C_1 = \left\{(c^1, x) \mid \langle p^1, c^1 + x \rangle \le \langle p^1, e_a^1 \rangle \right\}$$

- also $C_2^{\nu}(\xi,\cdot) \xrightarrow[\text{set}]{} C_2(\xi,\cdot)$
- implies Lagrangians hypo/epi-converge

$$L^{\nu}(x,w) \cong E\left\{\tilde{u}(\xi, x_\xi) - \langle w_\xi, x_\xi \rangle \,\middle|\, x_\xi \in \left[C_1^{\nu} \circ C_2^{\nu}(\xi)\right]\right\}$$

- ➔continuity of w with respect to p on Σ

Disintegration: reformulation

with $u^w_{a,1}(\xi; c^1, x) = u^1_a(c^1) - \langle \bar{w}^a_\xi, c^1 \rangle - \langle q + \tilde{w}^a_\xi, x \rangle$

$$\left(d^1_a, d^{2,a}_\xi; x_a\right) = \arg\max_{c^1, x, c^2 \in \mathbb{R}^n} \; u^w_{a,1}(\xi; c^1, x) + u^2_a(c^2)$$

$$\langle p^1, c^1 \rangle \le \langle p^1, e^1_a - x \rangle, \quad \langle p^2_\xi, c^2 \rangle \le \langle p^2_\xi, e^2_a(\xi) + T_a(\xi, x) \rangle$$

$$u^w_{a,1}(\xi; c^1, x) \xrightarrow[c]{} u^{w^*}_{a,1}(\xi; c^1, x) \text{ as } w \to w^*$$

$$\Rightarrow s^w(p^w) \to s^{w^*}(p^{w^*}) \Rightarrow W^w \xrightarrow[lop]{} W^{w^*}$$

➔ Convergence of equilibrium points!

Stochastic Equilibrium?

- Given for each ξ: $\{w_\xi^a = (\bar{w}_\xi^a, \tilde{w}_\xi^a), a \in \mathcal{A}\}$, one can find for each ξ, market prices

$$(p_\xi^1, p_\xi^2)$$

such that for each ξ: $s_1(p_\xi) \geq 0, s_2(p_\xi) \geq 0$

- Given (p^1, p_ξ^2) one can find for each ξ

$$\xi: \{w_\xi^a,\ a \in \mathcal{A}\}$$

such that $[(c^1, x), c_\xi^2]$ are nonanticipative

Believable proof

- Continuity of w_a w.r.to $p = (p^1, p^2(\cdot))$
- Continuity of p w.r.to $\{w_a, a \in A\}$

  & (d_a^1, x_a) are constant w.r.to ξ

$$p \mapsto w(p) : \Sigma^N \to R^{2N}; \quad w \mapsto p(w) : R^{2N} \to \Sigma^N$$

$p \circ w : \Sigma^N \to \Sigma^N$ is continuous

"Stochastic" Walrasian: $\forall\xi$

excess supply: $s(p,w) = \sum_{a\in A} e_a - \sum_{a\in A} d_a(p,w)$

$$W((p,w),q) = \langle q, s(p,w)\rangle,\ W:\tilde{\Sigma}\times\tilde{\Sigma}\to R$$

$$d_a(p,w) = \operatorname*{argmax}_{x\in R^n} \left\{ u_a^{w_a}(c) \mid \langle p,c\rangle \le \langle p,\tilde{e}_a(x)\rangle \right\}$$

$(p,w)\mapsto d_a(p,w)$ continuous

Arrow-Debreu `Dynamics'

Traded Goods:

g_l: good "*l*" traded @ time 0 (now)

g_l^1: good "*l*" traded @ time 1 (tomorrow)

actualized in terms of future contract @ time 0

g_l^2: good "*l*" to be traded at time 2 (later)

also actualized as future contract @ time 0 ….

MARKET ECONOMY: *COMPLETE*

as if all trades take place @ time 0 → ∃ **EQUILIBIUM**

Arrow-Debreu `Stochastics'

Traded Goods:

g_l: good "l" traded @ time 0 (now)

$g_l^1(\xi^1)$: good "l" traded @ time 1 given environment ξ^1

future contract @ time 0, contingent on ξ^1 occurring

$g_l^2(\xi^1,\xi^2)$: good "l" to be traded at time 2

future contract @ time 0, contingent on (ξ^1,ξ^2) occurring

MARKET ECONOMY: *COMPLETE*

as if all trades take place @ time 0 ➔ ∃ **EQUILIBIUM**

PRICES: (….p_l,… …$p_l^1(\xi^1)$,… …$p_l^2(\xi^1, \xi^2)$,…. …)

"static equilibrium"

Dynamic Equilibrium I

- budgetary constraint: period 1
  - c^1 consumption, p^1 market prices
  - x: "invested" goods

$$\langle p^1, c^1 + x \rangle \leq \langle p^1, e_a^1 \rangle$$

- budgetary constraint: period 2
  - c^2 consumption, p^2 market prices
  - $T_a(x)$: "return" on invested goods

$$\langle p^2, c^2 \rangle \leq \langle p^2, e_a^2 + T_a(x) \rangle$$

Dynamic Equilibrium II

- Agent's problem: with $p = (p^1, p^2)$,

$$\left(d_a^1(p), d_a^2(p); x_a(p)\right) \in \arg\max \left\{ u_a^1(c^1) + u_a^2(c^2) + \langle q, x \rangle \mid \text{budget} \right\}$$

$$\langle p^1, c^1 \rangle \le \langle p^1, e_a^1 - x \rangle, \quad \langle p^2, c^2 \rangle \le \langle p^2, e_a^2 + T_a(x) \rangle$$

- Market balance: excess sup., $s(p) = (s^1(p), s^2(p))$

$$s^1(p) = \sum_a [e_a^1 - x_a(p)] - \sum_a d_a^1(p) \ge 0$$

$$s^2(p) = \sum_a [e_a^2 + T_a(x_a(p))] - \sum_a d_a^2(p) \ge 0$$

Stochastic Environment

- budgetary constraint: period 1
  - c^1 consumption, p^1 market prices, e_a^1 endowment
  - x: "invested" goods
- budgetary constraint: period 2
  - c^2 consumption, $p^2(\xi)$ market prices, $e_a^2(\xi)$ endowment
  - $T_a(\xi, x)$: "return" on invested goods

Agent's problem: max E{u}

with $p_\xi = (p^1, p_\xi^2), \qquad p_\xi^2 = p^2(\xi)$

$$d_a^2(\xi, x) = \arg\max_{c^2} \left\{ u_a^2(c^2) \mid \text{budget}(\xi, x, p_\xi^2) \right\}$$

$$\left\langle p_\xi^2, c^2 \right\rangle \le \left\langle p_\xi^2, e_a^2(\xi) + T_a(\xi, x) \right\rangle$$

$$d_a^1(p) = \arg\max_{(c^1, x)} \left\{ u_a^1(c^1) + \left\langle q, x \right\rangle + E\left\{ u_a^2(d_a^2(\xi, x)) \right\} \mid \text{budget}(p^1) \right\}$$

$$\left\langle p^1, c^1 \right\rangle \le \left\langle p^1, e_a^1 - x \right\rangle$$

Agent's problem: version 2

with $p_\xi = (p^1, p_\xi^2), \qquad p_\xi^2 = p^2(\xi)$

$$\left(d_a(p_{(\bullet)}); x_a\right) = \arg\max_{c^1, x, c^2_{(\bullet)}} \left\{u_a^1(c^1) + \langle q, x\rangle + E\left\{u_a^2(c_\xi^2)\right\}\right\}$$

$$\langle p^1, c^1\rangle \le \langle p^1, e_a^1 - x\rangle, \qquad \langle p_\xi^2, c_\xi^2\rangle \le \langle p_\xi^2, e_a^2(\xi) + T_a(\xi, x)\rangle, \quad \forall \xi \in \Xi$$

sol'n: $(\xi, p_\xi) \mapsto (d_a^1(p^1, p_\xi^2), d_a^2(p^1, p_\xi^2); x_a(p^1, p_\xi^2))$

but $(\xi, p_\xi) \mapsto d_a^1(p^1, p_\xi^2)$ & $(\xi, p_\xi) \mapsto x_a(p^1, p_\xi^2)$

are constant functions of ξ

Agent's problem: max E{u}

with $p_\xi = (p^1, p_\xi^2), \qquad p_\xi^2 = p^2(\xi)$

$$d_a^2(\xi, x) = \arg\max_{c^2} \left\{ u_a^2(c^2) \mid \text{budget}(\xi, x, p_\xi^2) \right\}$$

$$\left\langle p_\xi^2, c^2 \right\rangle \le \left\langle p_\xi^2, e_a^2(\xi) + T_a(\xi, x) \right\rangle$$

$$d_a^1(p) = \arg\max_{(c^1, x)} \left\{ u_a^1(c^1) + \langle q, x \rangle + E\left\{ u_a^2(d_a^2(\xi, x)) \right\} \mid \text{budget}(p^1) \right\}$$

$$\left\langle p^1, c^1 \right\rangle \le \left\langle p^1, e_a^1 - x \right\rangle$$

Pure Exchange

Agents: $a \in \mathcal{A}$, $e_a \in R^n$,

$u_a : R^n \to \bar{R}$ usc, strictly concave, sup-compact

$e_a \in D_a = \{ x \mid u_a(x) < \infty \}$ = dom u_a

$p \mapsto d_a(p) = \operatorname*{argmax}_{x \in R^n} \{ u_a(x) \mid \langle p, x \rangle \le \langle p, e_a \rangle \}$ continuous

$$\exists\, p \text{ in } \Sigma : s(p) = \Sigma_a e_a - \Sigma_a d_a(p) \geq 0$$

THM: $p^\nu \to p$ and $u_a^\nu \xrightarrow[c]{} u \Rightarrow s^\nu(p^\nu) \to s(p)$

i.e. s^ν converges continuously to s