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Projective Schemes for Random Operator Equations. I. Weak Compactness of Approximate Solution Measures

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Projective schemes for solving linear random equations are considered. Using an approach to tightness of probability measures on Banach spaces due to de Acosta, we prove a result about the weak compactness of distributions of the approximate solutions. An application is given to the approximate solution of random Fredholm integral equations of the second kind via approximation by random degenerate kernels.

1. Introduction and Preliminaries

In recent years, the study of approximate methods for solving random equations has been an area of active research. Surveys on this subject are presented in [3] (especially, in [21]), [6], and [12]. An important technique for solving random equations is the use of projection schemes. [11] provides a general methodology for adapting projection methods to random operator equations in Hilbert spaces. Note that the general concepts of [12], [13], and [27] for the approximate solution of nonlinear random operator equations also allow applications to projection methods. [18] contains a convergence result for approximate solutions of an equation involving a P-compact random operator via projection schemes.

In this paper we study projection schemes for solving linear random equations, especially random Fredholm integral equations of the second kind. Our main objective is to provide an approach for proving weak

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compactness of approximate "solutions measures," i.e., of probability distributions of approximate random solutions. This approach is motivated by the *flat-concentration property* of probability measures on Banach spaces (introduced by de Acosta [1]), and by Theorem 2.3 of [1]; it is developed in Section 2. Using this approach, in Section 3 we prove a result about the weak compactness of distributions of approximate solutions obtained via projection schemes for linear random operator equations in Banach spaces. This result is applied to a kernel approximation method for random Fredholm equations of the second kind and compared with another result using the approach of [13]. The suggested approach to weak compactness represents an alternative approach to that of [13]. It seems that both approaches provide useful tools for establishing weak compactness and convergence of solution measures associated with random operator equations in Banach spaces.

In the context of stochastic equations driven by martingales and random measures, weak-compactness approaches are well developed. These approaches use criteria for weak compactness of measures on special function spaces and special properties of the stochastic processes that are the solutions of stochastic equations. For a survey we refer to [16, Section 5] and to the remarks in [4]. The situation is rather different in the case of random operator equations in Banach spaces. For this case, only a few papers treating weak compactness of solution measures can be found in the literature (see [4], [13], and [26]). Reference [13] and this paper are concerned with the development of general approaches to this problem, so applications to various random equations and their approximations in concrete Banach spaces can be considered.

Throughout this paper, let X be a real separable Banach space (with norm $\|\cdot\|$), and let $\mathscr{B}(X)$ be the σ-algebra of Borel sets of X. By $\mathscr{P}(X)$ we denote the set of all probability measures on $(X, \mathscr{B}(X))$ equipped with the topology of weak convergence (see e.g. [7]). By X^* we denote the dual of X, and by $\langle \cdot, \cdot \rangle$ the duality relation between X^* and X. $F \subseteq X^*$ will be called *total* if $\langle f, x \rangle = 0$, for all $f \in F$ and some $x \in X$, implies $x = 0$. $(X_n, P_n)_{n \in \mathbb{N}}$ will be called a *projection scheme* for X if $X_n \subset X$ $(n \in \mathbb{N})$ is a sequence of monotonically increasing finite-dimensional subspaces and $P_n: X \to X_n$ $(n \in \mathbb{N})$ are linear continuous projections such that $\lim_{n \to \infty} \|P_n x - x\| = 0$ for all $x \in X$. For any $x \in X$ and any subset $B \subseteq X$, let $d(x, B) = \inf\{\|x - y\| \mid y \in B\}$.

Let $(\Omega, \mathscr{A}, P)$ be a probability space. For any X-valued random variable x [defined on $(\Omega, \mathscr{A}, P)$] we denote by $D(x)$ its probability distribution, i.e., $D(x) \in \mathscr{P}(X)$ defined by $D(x)(B) = P(x^{-1}(B))$, $B \in \mathscr{B}(X)$. A mapping $T: \Omega \times X \to X$ is a *random operator* if for every $x \in X$, $T(\cdot, x): \Omega \to X$ is an X-valued random variable on $(\Omega, \mathscr{A}, P)$. A random operator $T: \Omega \times X \to X$

is said to be *linear*, *continuous*, etc. if for every $\omega \in \Omega$, $T(\omega, \cdot): X \to X$ is linear, continuous, etc. For an introduction to the theory of random operator equations we refer to [2]. As usual, we say that a property depending on $\omega \in \Omega$ holds *almost surely* (a.s.), or *for P-almost all* $\omega \in \Omega$, if there is a set $N \in \mathscr{A}$ with $P(N) = 0$ such that the property holds for all $\omega \in \Omega \setminus N$.

2. Remarks on Weak Compactness and Convergence of Probability Measures on Separable Banach Spaces

In this section, we develop the approach to weak compactness of probability measures which we will use in Section 3. This approach is close in spirit to that of [1] and [22]. Of course, our main tool is Prohorov's theorem [24, 7]. Let X be a separable Banach space, $\mathscr{P}(X)$ be defined as in Section 1, and $(\mu_n)_{n \in \mathbb{N}}$ be a sequence in $\mathscr{P}(X)$. It is known from [24] that $(\mu_n)_{n \in \mathbb{N}}$ is *weakly compact* [i.e., relatively compact with respect to weak convergence in $\mathscr{P}(X)$] if and only if $(\mu_n)_{n \in \mathbb{N}}$ is *tight*, i.e., for all $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset X$ such that

$$\inf_{n \in \mathbb{N}} \mu_n(K_\varepsilon) \geqslant 1 - \varepsilon.$$

In our context (namely, the approximate solution of random operator equations) a straightforward use of tightness seems to be possible only for concrete Banach spaces. The following notions, together with Proposition 2.2, turn out to be useful in this context, because their application does not require a description of compact subsets of the underlying Banach space.

Definition 2.1.

a. $(\mu_n)_{n \in \mathbb{N}}$ will be called *uniformly bounded* iff for all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\inf_{n \in \mathbb{N}} \mu_n(\{x \in X | \|x\| \leqslant C_\varepsilon\}) \geqslant 1 - \varepsilon.$$

b. $(\mu_n)_{n \in \mathbb{N}}$ will be called *flatly concentrated* iff for all $\varepsilon > 0$ and $\delta > 0$, there exists a finite-dimensional subspace L of X such that

$$\inf_{n \in \mathbb{N}} \mu_n(\{x \in X \mid d(x, L) \leqslant \delta\}) \geqslant 1 - \varepsilon$$

(cf. [1]).

The following result, which is essentially Theorem 2.3 of [1], links the introduced notions with tightness.

Proposition 2.2. $(\mu_n)_{n \in \mathbb{N}} \subset \mathscr{P}(X)$ *is tight if and only if* $(\mu_n)_{n \in \mathbb{N}}$ *is uniformly bounded and flatly concentrated.*

PROOF. The necessity is clear by definition. To prove sufficiency we use Theorem 2.3 of [1], and note that the uniform boundedness of $(\mu_n)_{n\in\mathbb{N}}$ implies that for all $f \in X^*$, $(\mu_n \circ f^{-1})_{n\in\mathbb{N}}$ is tight in $\mathscr{P}(\mathbb{R}^1)$. □

REMARK 2.3. In [13], a sequence $(x_n)_{n\in\mathbb{N}}$ of X-valued random variables was called *D-bounded* iff the sequence of their distributions $(D(x_n))_{n\in\mathbb{N}}$ is uniformly bounded (as defined in 2.1). As indicated in [13], the D-boundedness of a sequence of random solutions of random equations may often be verified in applications (see also Theorem 3.3). The next result, which represents a variant of Proposition 2.2, contains a more constructive version of the flat-concentration property, and will be used below.

Proposition 2.4. *Let $(X_n, P_n)_{n\in\mathbb{N}}$ be a projective scheme for X. Then $(\mu_n)_{n\in\mathbb{N}} \subset \mathscr{P}(X)$ is tight if and only if $(\mu_n)_{n\in\mathbb{N}}$ is uniformly bounded and*

for all $\varepsilon > 0$ and $\delta > 0$, there is an $n_0 = n_0(\varepsilon, \delta) \in \mathbb{N}$ such that for every $m \geqslant n_0$

$$\inf_{n\in\mathbb{N}} \mu_n(\{x \in X \mid d(x, X_m) \leqslant \delta\}) \geqslant 1 - \varepsilon \tag{2.1}$$

holds.

PROOF. The "if" part of the statement follows from Proposition 2.2. Now, let $(\mu_n)_{n\in\mathbb{N}}$ be tight; then we will show that (2.1) holds. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary, but fixed. There exists a compact subset $K_\varepsilon \subset X$ such that

$$\inf_{n\in\mathbb{N}} \mu_n(K_\varepsilon) \geqslant 1 - \varepsilon.$$

We note that $(P_n)_{n\in\mathbb{N}}$ converges to the identity uniformly on compact subsets of X. In particular, we note that there is an $n_0 = n_0(\varepsilon, \delta) \in \mathbb{N}$ such that for every $m \geqslant n_0$

$$\sup_{x\in K_\varepsilon} \|x - P_m x\| \leqslant \delta.$$

It follows that for every $m \geqslant n_0$,

$$\begin{aligned}\mu_n(\{x \in X \mid d(x, X_m) \leqslant \delta\}) &\geqslant \mu_n(\{x \in X \mid \|x - P_m x\| \leqslant \delta\}) \\ &\geqslant \mu_n(K_\varepsilon) \geqslant 1 - \varepsilon \qquad \text{for all} \quad n \in \mathbb{N}. \end{aligned}$$ □

REMARK 2.5. Let X be a Banach space with Schauder basis $(e_i)_{i\in\mathbb{N}}$, and let $(f_j)_{j\in\mathbb{N}} \subset X^*$ be the associated coordinate functionals. We define for all

$n \in \mathbb{N}$, $X_n = \operatorname{span}\{e_1, \ldots, e_n\}$,

$$P_n : X \to X_n, \qquad P_n x = \sum_{i=1}^{n} \langle f_i, x \rangle e_i, \quad x \in X,$$

and therefore obtain a projection scheme for X.

For this case, a result analogous to Proposition 2.4 is stated as Theorem 1 in [22].

REMARK 2.6. The following result about the link between tightness and convergence of $(\mu_n)_{n \in \mathbb{N}}$ is well known (e.g., [23, Theorem 13], [8, Theorem 2.2.1]): Let $F \subset X^*$ be a total subset. Then, $(\mu_n)_{n \in \mathbb{N}} \subset \mathscr{P}(X)$ is weakly convergent if and only if $(\mu_n)_{n \in \mathbb{N}}$ is tight and for every $f \in \operatorname{span}(F)$, $(\mu_n \circ f^{-1})_{n \in \mathbb{N}}$ is weakly convergent in $P(\mathbb{R}^1)$.

Note that Theorem 2.6.1 of [8] represents an analogous result about the relation between the convergence in probability of a sequence $(x_n)_{n \in \mathbb{N}}$ of X-valued random variables [on $(\Omega, \mathscr{A}, P)$] and the tightness of $(D(x_n))_{n \in \mathbb{N}}$ (see also Proposition 2.14 in [13]).

3. Weak Compactness of Solution Measures Associated with Projection Schemes for Linear Random Operator Equations

Throughout this section, let $(\Omega, \mathscr{A}, P)$ be a probability space, X be a separable Banach space, and $(X_n, P_n)_{n \in \mathbb{N}}$ be a projection scheme for X. Let $T: \Omega \times X \to X$ and $T_n: \Omega \times X \to X_n$ $(n \in \mathbb{N})$ be linear continuous random operators, and $y: \Omega \to X$ and $y_n: \Omega \to X_n$ be random variables [defined on $(\Omega, \mathscr{A}, P)$].

Let us consider the random operator equation

$$x = T(\omega, x) + y(\omega) \qquad (\omega \in \Omega), \tag{3.1}$$

and its "approximations"

$$x = T_n(\omega, x) + y_n(\omega) \qquad (\omega \in \Omega, \quad n \in \mathbb{N}). \tag{3.2}$$

In this section, we will be concerned with the following

Problem. For all $n \in \mathbb{N}$ let $x_n: \Omega \to X_n$ be a "random solution" of (3.2) for the index n, i.e., x_n is a random variable such that

$$x_n(\omega) = T_n(\omega, x_n(\omega)) + y_n(\omega) \quad \text{a.s.} \tag{3.3}$$

Assume that the sequence $(D(y_n))_{n \in \mathbb{N}} \subset \mathscr{P}(X)$ is tight. Find conditions on T_n $(n \in \mathbb{N})$ that guarantee that the sequence $(D(x_n))_{n \in \mathbb{N}}$ of "solution measures" forms a tight subset of $\mathscr{P}(X)$.

REMARK 3.1.

a. Note that, under the above assumptions, well-known random fixed-point theorems (see e.g. [10, Theorem 8]) yield the existence of a random solution x_n of (3.2) if (3.2) is "deterministically solvable," i.e.,

$$\{x \in X \mid x = T_n(\omega, x) + y_n(\omega)\} \neq \varnothing \quad \text{a.s.} \qquad (n \in \mathbb{N}).$$

b. The above problem was also treated in [4] and [13]. But in [4] the operators are not allowed to be random. The convergence result of [13] states conditions (on T_n) under which the uniform boundedness of $(D(x_n))_{n \in \mathbb{N}}$ is sufficient for its tightness. These conditions are taken from "deterministic operator approximation theory" because the operators are formulated in the following way: $T_n(z_n(\omega), x)$, where T_n is deterministic and z_n is a "stochastic input."

In the following we present an "alternative" approach (to that of [13]) for proving tightness of $(D(x_n))_{n \in \mathbb{N}}$, motivated by Proposition 2.4. The following observation will be used below.

Proposition 3.2. *Let T and T_n $(n \in \mathbb{N})$ be as above, and assume that for P-almost all $\omega \in \Omega$,*

$$I - T(\omega, \cdot) \text{ is injective}, \tag{3.4}$$

$$\lim_{n \to \infty} \|T_n(\omega, \cdot) - T(\omega, \cdot)\| = 0 \tag{3.5}$$

(here $\|\cdot\|$ denotes the usual operator norm). Then

$$\begin{array}{l} \textit{there is a real random variable } \alpha: \Omega \to \mathbb{R}^1 \textit{ such} \\ \textit{that for all } \varepsilon > 0 \textit{ there exist } n_0 = n_0(\varepsilon) \in \mathbb{N} \textit{ and} \\ A_\varepsilon \in \mathscr{A} \textit{ with } P(A_\varepsilon) \geqslant 1 - \varepsilon \textit{ such that we have} \\ \textit{for all } \omega \in A_\varepsilon,\ n \geqslant n_0 \textit{ and } x \in X \\ \|x\| \leqslant \alpha(\omega) \|x - T_n(\omega, x)\|. \end{array} \tag{3.6}$$

PROOF. Let $N \in \mathscr{A}$ with $P(N) = 0$ be such that (3.4) and (3.5) hold for all $\omega \in \Omega \setminus N$. It follows from (3.4) and (3.5) that for every $\omega \in \Omega \setminus N$, $T(\omega, \cdot)$ is completely continuous (see e.g. [15, p. 87]) and

$$I - T(\omega, \cdot): X \to X \text{ is bijective} \qquad [15, \text{p. } 96]. \tag{3.7}$$

Because of [15, Theorem 1], (3.7) yields that the map $(\omega, x) \to [I - T(\omega, \cdot)]^{-1} x$ from $\Omega \times X$ onto X (modified for $\omega \in N$) is a linear continuous random operator.

In particular, the map $\omega \to \|[I - T(\omega, \cdot)]^{-1}\|$ is a real random variable, and we define

$$\alpha(\omega) = \begin{cases} 2\left\|[I - T(\omega, \cdot)]^{-1}\right\|, & \omega \in \Omega \setminus N, \\ 0, & \omega \in N. \end{cases}$$

Let $\varepsilon > 0$ be arbitrary, but fixed. It is well known (e.g., [28, p. 269]) that almost sure convergence in (3.5) implies for any $\delta > 0$

$$\lim_{m \to \infty} P\left(\left\{\omega \mid \sup_{n \geqslant m} \|T_n(\omega, \cdot) - T(\omega, \cdot)\| \left\|[I - T(\omega, \cdot)]^{-1}\right\| \geqslant \delta\right\}\right) = 0. \tag{3.8}$$

Because of (3.8), there is an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for

$$\bar{A}_\varepsilon = \left\{\omega \in \Omega \setminus N \mid \sup_{n \geqslant n_0} \|T_n(\omega, \cdot) - T(\omega, \cdot)\| \left\|[I - T(\omega, \cdot)]^{-1}\right\| \geqslant \tfrac{1}{2}\right\} \tag{3.9}$$

satisfies $P(\bar{A}_\varepsilon) \leqslant \varepsilon$. We define $A_\varepsilon = \Omega \setminus (\bar{A}_\varepsilon \cup N)$ and have $P(A_\varepsilon) \geqslant 1 - \varepsilon$. Because of (3.9), it follows that for all $\omega \in A_\varepsilon$ and $n \geqslant n_0$,

$$\|T_n(\omega, \cdot) - T(\omega, \cdot)\| \left\|[I - T(\omega, \cdot)]^{-1}\right\| < \tfrac{1}{2}. \tag{3.10}$$

Because of (3.7) and (3.10), it follows that from the well-known "perturbation lemma" [19, p. 181] that for all $\omega \in A_\varepsilon$ and $n \geqslant n_0$ the operator $I - T_n(\omega, \cdot)\colon X \to X$ is bijective, and

$$\left\|[I - T_n(\omega, \cdot)]^{-1}\right\| \leqslant 2\left\|[I - T(\omega, \cdot)]^{-1}\right\| = \alpha(\omega). \tag{3.11}$$

Because of (3.11), (3.6) holds. □

The above proposition motivates the condition (3.6), which will play an essential role in the following main result of this section.

Theorem 3.3. *Let T, T_n $(n \in \mathbb{N})$ and y_n $(n \in \mathbb{N})$ be as above, and let* (3.4), (3.5) *be fulfilled for P-almost all $\omega \in \Omega$. For all $n \in \mathbb{N}$ let x_n be a random solution of* (3.2), *i.e., a random variable such that* (3.3) *holds for the index n. Then*

a. *$(D(x_n))_{n \in \mathbb{N}}$ is uniformly bounded if $(D(y_n))_{n \in \mathbb{N}}$ is uniformly bounded,*
b. *$(D(x_n))_{n \in \mathbb{N}}$ is tight if $(D(y_n))_{n \in \mathbb{N}}$ is tight.*

PROOF. a: Let $\varepsilon > 0$ be arbitrary, but fixed. Because of (3.6) (from Proposition 3.2), there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $A_{\varepsilon 1} \in \mathscr{A}$ with $P(A_{\varepsilon 1}) \geqslant 1 - \varepsilon/3$

such that for all $\omega \in A_{\varepsilon 1}$, $n \geqslant n_0$ and $x \in X$,

$$\|x\| \leqslant \alpha(\omega)\|[I - T_n(\omega, \cdot)]x\|. \tag{3.12}$$

There exist an $N \in \mathscr{A}$ with $P(N) = 0$ and an α_ε such that

$$[I - T_n(\omega, \cdot)]x_n(\omega) = y_n(\omega) \qquad \text{for all} \quad \omega \in \Omega \setminus N, \tag{3.13}$$

and

$$P(\{\omega \mid \alpha(\omega) \leqslant \alpha_\varepsilon\}) \geqslant 1 - \frac{\varepsilon}{3}. \tag{3.14}$$

Because of the uniform boundedness of $(D(y_n))_{n \in \mathbb{N}}$, there is a $C_\varepsilon > 0$ such that

$$\inf_{n \in \mathbb{N}} P\left(\left\{\omega \middle| \|y_n(\omega)\| \leqslant C_\varepsilon\right\}\right) \geqslant 1 - \frac{\varepsilon}{3}. \tag{3.15}$$

Let $n \in \mathbb{N}$, $n \geqslant n_0$, be arbitrary, but fixed, and define

$$\begin{aligned} A_{\varepsilon 2} &= \{\omega \mid \alpha(\omega) \leqslant \alpha_\varepsilon\}, \\ A_{\varepsilon 3} &= \left\{\omega \middle| \|y_n(\omega)\| \leqslant C_\varepsilon\right\}, \\ A_{\varepsilon 4} &= \left\{\omega \middle| \|x_n(\omega)\| \leqslant \alpha_\varepsilon C_\varepsilon\right\}. \end{aligned} \tag{3.16}$$

Now, let $\omega \in (\Omega \setminus N) \cap (\bigcap_{i=1}^3 A_{\varepsilon i})$. Because of (3.12), (3.13), and (3.16), it follows that $\|x_n(\omega)\| \leqslant \alpha(\omega)\|y_n(\omega)\| \leqslant \alpha_\varepsilon C_\varepsilon$, so that $\omega \in A_{\varepsilon 4}$. Using (3.14) and (3.15), we have

$$P(\Omega \setminus A_{\varepsilon 4}) \leqslant \sum_{i=1}^{3} P(\Omega \setminus A_{\varepsilon i}) \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since $n \geqslant n_0$ was arbitrary, this means

$$\inf_{n \geqslant n_0} D(x_n)(\{x \in X \mid \|x\| \leqslant \alpha_\varepsilon C_\varepsilon\}) \geqslant 1 - \varepsilon,$$

i.e., $(D(x_n))_{n \geqslant n_0}$ is uniformly bounded. Obviously the finite family $(D(x_n))_{n=1,\ldots,n_0-1}$ is uniformly bounded, and therefore part a is proved.

b: Assume that $(D(y_n))_{n \in \mathbb{N}}$ is tight. From part a we have that $(D(x_n))_{n \in \mathbb{N}}$ is uniformly bounded. To prove tightness of $(D(x_n))_{n \in \mathbb{N}}$, we use Proposition 2.4. Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary, but fixed. We define $n_0 = n_0(\varepsilon) \in \mathbb{N}$, $N \in \mathscr{A}$, $A_{\varepsilon 1}, A_{\varepsilon 2} \in \mathscr{A}$, and α_ε as in part a of the proof, but such that

$$P(A_{\varepsilon i}) \geqslant -\frac{\varepsilon}{6}, \qquad i = 1, 2.$$

By assumption, there is a compact $K_\varepsilon \subset X$ such that

$$\inf_{n \in \mathbb{N}} P(\{\omega \mid y_n(\omega) \in K_\varepsilon\}) \geqslant 1 - \frac{\varepsilon}{6}.$$

Let $\{\bar{y}_i \mid i = 1, \ldots, k\}$ be a finite $\delta/3\alpha_\varepsilon$-net for K_ε, i.e.,

$$K_\varepsilon \subset \bigcup_{i=1}^{k} \left\{ z \in X \mid \|z - \bar{y}_i\| \leqslant \frac{\delta}{3\alpha_\varepsilon} \right\}. \tag{3.17}$$

As in the proof of Proposition 3.2, we conclude that for P-almost all $\omega \in \Omega$, $I - T(\omega, \cdot)$ is bijective. Let $\bar{x}_i : \Omega \to X$ be the a.s. unique random solution of

$$[I - T(\omega, \cdot)]x = \bar{y}_i, \qquad i = 1, \ldots, k,$$

and let $N \in \mathscr{A}$, $P(N) = 0$, be enlarged so that $[I - T(\omega, \cdot)]\bar{x}_i(\omega) = \bar{y}_i$ for all $\omega \in \Omega \setminus N$, $i = 1, \ldots, k$. There exists a constant $C_\varepsilon > 0$ such that it holds for

$$A_{\varepsilon 3} = \left\{ \omega \mid \max_{i=1,\ldots,k} \|\bar{x}_i(\omega)\| \leqslant C_\varepsilon, \|T(\omega, \cdot)\| \leqslant C_\varepsilon \right\}, \tag{3.18}$$

that $P(A_{\varepsilon 3}) \geqslant 1 - \varepsilon/6$.

Let $K > 0$ be chosen such that $\|P_n\| \leqslant K$ for all $n \in \mathbb{N}$. Because of (3.5) and the pointwise convergence of $(P_n)_{n \in \mathbb{N}}$ to the identity I, it follows from [28, p. 269] that

$$\lim_{n \to \infty} P\left(\left\{ \omega \mid \sup_{n \geqslant 1} \|T_n(\omega, \cdot) - T(\omega, \cdot)\| \geqslant \frac{\delta}{3KC_\varepsilon \alpha_\varepsilon} \right\} \right) = 0.$$

$$\lim_{n \to \infty} P\left(\left\{ \omega \mid \sup_{n \geqslant 1} \max_{i=1,\ldots,k} \|\bar{x}_i(\omega) - P_n \bar{x}_i(\omega)\| \geqslant \frac{\delta}{3(1 + C_\varepsilon)\alpha_\varepsilon} \right\} \right) = 0.$$

Thus, there exists an $n_1 = n_1(\varepsilon, \delta) \in \mathbb{N}$, $n_1 \geqslant n_0$, such that for

$$\bar{A}_{\varepsilon 4} = \left\{ \omega \mid \sup_{n \geqslant n_1} \|T_n(\omega, \cdot) - T(\omega, \cdot)\| \geqslant \frac{\delta}{3KC_\varepsilon \alpha_\varepsilon} \right\}$$

$$\bar{A}_{\varepsilon 5} = \left\{ \omega \mid \sup_{n \geqslant n_1} \max_{i=1,\ldots,k} \|\bar{x}_i(\omega) - P_n \bar{x}_i(\omega)\| \geqslant \frac{\delta}{3(1 + C_\varepsilon)\alpha_\varepsilon} \right\} \tag{3.19}$$

we have $P(\bar{A}_{\varepsilon i}) \leqslant \varepsilon/6$, $i = 4, 5$. Let $A_{\varepsilon i} = \Omega \setminus \bar{A}_{\varepsilon i}$, $i = 4, 5$. Now, let $n, m \in \mathbb{N}$,

$n \geqslant m \geqslant n_1$, be arbitrary, but fixed. We define

$$A_{\varepsilon 6} = \{ \omega \mid y_n(\omega) \in K_\varepsilon \}.$$

$$A_{\varepsilon 7} = \{ \omega \mid d(x_n(\omega), X_m) \leqslant \delta \}.$$

Let $\omega \in (\Omega \setminus N) \cap (\bigcap_{i=1}^{6} A_{\varepsilon i})$. By (3.6) and (3.16), it follows that

$$\begin{aligned} d(x_n(\omega), X_m) &\leqslant \alpha_\varepsilon \inf_{x \in X_m} \| [I - T_n(\omega, \cdot)][x_n(\omega) - x] \| \\ &= \alpha_\varepsilon \inf_{x \in X_m} \| y_n(\omega) - [I - T_n(\omega, \cdot)] x \| \\ &\leqslant \alpha_\varepsilon \min_{i=1,\dots,k} \| y_n(\omega) - [I - T_n(\omega, \cdot)] P_m \bar{x}_i(\omega) \|. \quad (3.20) \end{aligned}$$

Let $i \in \{1, \dots, k\}$ be arbitrary. Then we have

$$\begin{aligned} &\| y_n(\omega) - [I - T_n(\omega, \cdot)] P_m \bar{x}_i(\omega) \| \\ &\quad \leqslant \| y_n(\omega) - \bar{y}_i \| + \| [I - T(\omega, \cdot)] \bar{x}_i(\omega) - [I - T_n(\omega, \cdot)] p_m \bar{x}_i(\omega) \| \\ &\quad \leqslant \| y_n(\omega) - \bar{y}_i \| + (1 + \| T(\omega, \cdot) \|) \| \bar{x}_i(\omega) - P_m \bar{x}_i(\omega) \| \\ &\qquad + \| T_n(\omega, \cdot) - T(\omega, \cdot) \| \| P_m \bar{x}_i(\omega) \| \end{aligned}$$

Because of (3.17), (3.18), (3.19), and (3.20), this yields

$$\begin{aligned} d(x_n(\omega), X_m) &\leqslant \alpha_\varepsilon \min_{i=1,\dots,k} \Bigg[\| y_n(\omega) - \bar{y}_i \| \\ &\qquad + (1 + C_\varepsilon) \frac{\delta}{3(1 + C_\varepsilon)\alpha_\varepsilon} + \frac{\delta}{3KC_\varepsilon \alpha_\varepsilon} KC_\varepsilon \Bigg] \\ &\leqslant \alpha_\varepsilon \min_{i=1,\dots,k} \| y_n(\omega) - \bar{y}_i \| + \tfrac{2}{3}\delta \\ &\leqslant \alpha_\varepsilon \frac{\delta}{3\alpha_\varepsilon} + \tfrac{2}{3}\delta = \delta, \end{aligned}$$

i.e., $\omega \in A_{\varepsilon 7}$. Analogous to part a of the proof, we obtain $P(\Omega \setminus A_{\varepsilon 7}) \leqslant \varepsilon$, and thus

$$P(A_{\varepsilon 7}) = D(x_n)(\{ x \in X \mid d(x, X_m) \leqslant \delta \}) \geqslant 1 - \varepsilon.$$

Since $n, m \in \mathbb{N}$, $n \geqslant m \geqslant n_1$, were arbitrary, it holds for all $m \geqslant n_1$ that

$$\inf_{n \geqslant m} D(x_n)(\{ x \in X \mid d(x, X_m) \leqslant \delta \}) \geqslant 1 - \varepsilon.$$

Note that for every $n, m \in \mathbb{N}$ with $n < m$,

$$D(x_n)(\{ x \in X \mid d(x, X_m) \leqslant \delta \}) = 1.$$

This yields for all $m \geqslant n_1$

$$\inf_{n \in \mathbb{N}} D(x_n)(\{x \in X \mid d(x, X_m) \leqslant \delta\}) \geqslant 1 - \varepsilon.$$

Since $\varepsilon > 0$ and $\delta > 0$ were arbitrary, this means that (2.1) is valid for $\mu_n = D(x_n)$ $(n \in \mathbb{N})$. Thus, an application of Proposition 2.4 proves assertion b. □

REMARK 3.4.

a. The condition (3.6) is a generalization of so-called "inverse stability conditions" used in [27, Theorem 2] and [12, Proposition 4.7] (for the case of linear random operators). A generalization of Theorem 3.3 to the case of nonlinear random operators should be possible if "nonlinear versions" of (3.5) and (3.6) are used.
b. Note that the assumptions of Theorem 3.3 are rather strong. This can be seen e.g. from the fact that (3.4), (3.5), and the convergence in probability of $(y_n)_{n \in \mathbb{N}}$ to y imply that $(x_n)_{n \in \mathbb{N}}$ converges in probability to the unique random solution x of (3.1).
c. The approach to tightness of $(D(x_n))_{n \in \mathbb{N}}$ provided by Theorem 3.3 proves to be an alternative to that of [13]. Some of their relations are discussed in Section 4 in the context of an application to random Fredholm integral equations. Section 4 also contains a characterization of weak limits of $(D(x_n))_{n \in \mathbb{N}}$ as distributions of so-called "D-solutions" introduced in [13].

4. Application to Random Fredholm Integral Equations

In Section 3, we developed an approach to weak compactness of a set of measures containing distributions of approximate random solutions of linear random operator equations. Now, we will be concerned with random Fredholm integral equations of the second kind. For an introduction to random Fredholm integral equations we refer to [2, Chapter 4]. Recent surveys on the approximate and numerical solution of such random equations can be found in [5] and [6].

In [9] algorithms for the numerical solution of random Fredholm equations of the second kind by quadrature methods are given. In the following, we will consider another type of approximation procedure based on "kernel approximations." More precisely, we will study the approximation of Fredholm equations with random kernel and random forcing function by a sequence of Fredholm equations with random "degenerate" kernels. Our aim is to show that Theorem 3.3 and the main result of [13] can be applied to this type of approximation scheme.

Let us consider the equation

$$x(t)=\int_0^1 K(\omega,t,s)x(s)\,ds+y(\omega,t),\qquad t\in[0,1],\tag{4.1}$$

under the following assumptions:

$$\begin{aligned}&K:\Omega\times[0,1]\times[0,1]\to\mathbb{R}^1 \text{ is an } \mathscr{A}\times\mathscr{B}([0,1])\times\\ &\mathscr{B}([0,1]\text{-measurable stochastic process such}\\ &\text{that for all } \omega\in\Omega,\ K(\omega,\cdot,\cdot) \text{ is an element of}\\ &Z=L_2((0,1)\times(0,1)),\end{aligned}\tag{4.2}$$

$$\begin{aligned}&y:\Omega\times[0,1]\to\mathbb{R}^1 \text{ is an } \mathscr{A}\times\mathscr{B}([0,1])\text{-mea-}\\ &\text{surable stochastic process such that for all}\\ &\omega\in\Omega,\ y(\omega,\cdot)\in L_2(0,1). \text{ (Here } (\Omega,\mathscr{A},P) \text{ de-}\\ &\text{notes the underlying probability space,}\\ &\mathscr{B}([0,1]) \text{ the Borel-}\sigma\text{-algebra on the interval}\\ &[0,1].)\end{aligned}\tag{4.3}$$

Throughout this section, we define $X=L_2(0,1)$, $T:\Omega\times X\to X$ as follows:

$$[T(\omega,x)](t)=\int_0^1 K(\omega,t,s)x(s)\,ds,\qquad t\in[0,1],\quad \omega\in\Omega,\quad x\in X.$$

From [10, p. 230] it follows that T is a linear, completely continuous random operator, and from [25] that y (K) can be viewed as an X-valued (Z-valued) random variable. Thus, (4.1) fits into the setting of Section 3.

Now, let $(\phi_i)_{i\in\mathbb{N}}$ be an orthonormal basis of X; for each $n\in\mathbb{N}$ let $X_n=\operatorname{span}\{\phi_1,\dots,\phi_n\}$ and P_n be the orthogonal projection from X onto X_n. Clearly, $(X_n,P_n)_{n\in\mathbb{N}}$ is a projection scheme for X.

We assume that the sequences $(K_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of random kernels and random forcing functions, respectively, are given, such that for each $n\in\mathbb{N}$

$$K_n \text{ and } y_n \text{ satisfy (4.2) and (4.3), respectively,}\tag{4.4}$$

and

$$\begin{gathered}K_n(\omega,\cdot,\cdot)\in\operatorname{span}\{\phi_i(\cdot)\phi_j(\cdot),\ i,j=1,\dots,n\},\\ y_n(\omega)\in X_n\qquad\text{for all}\quad \omega\in\Omega.\end{gathered}\tag{4.5}$$

Each kernel K_n is a so-called "random degenerate kernel" [2, p. 150].

Analogously to the above, we define for all $n\in\mathbb{N}$ linear, continuous random operators $T_n:\Omega\times X\to X_n$,

$$[T_n(\omega,x)](t)=\int_0^1 K_n(\omega,t,s)x(s)\,ds,\qquad t\in[0,1],\quad \omega\in\Omega,\quad x\in X,$$

and consider the random Fredholm equations

$$x = T_n(\omega, x) + y_n(\omega) \qquad (\omega \in \Omega, \quad n \in \mathbb{N}). \tag{4.6}$$

The first result of this section is an immediate consequence of Theorem 3.3.

Theorem 4.1. *Let K, K_n $(n \in \mathbb{N})$ and y_n $(n \in \mathbb{N})$ satisfy* (4.2), (4.4), *and* (4.5). *Assume that*

$$I - T(\omega, \cdot) \text{ is injective for } P\text{-almost all } \omega \in \Omega, \tag{4.7}$$

$$\lim_{n \to \infty} \int_0^1 \int_0^1 [K_n(\omega, t, s) - K(\omega, t, s)]^2 \, ds\, dt = 0 \quad a.s., \tag{4.8}$$

$$(D(y_n))_{n \in \mathbb{N}} \text{ is tight}. \tag{4.9}$$

For all $n \in \mathbb{N}$, let x_n be a random solution of (4.6). *Then $(D(x_n))_{n \in \mathbb{N}}$ is tight.*

PROOF. We note that for all $\omega \in \Omega$ we have

$$\|T_n(\omega, \cdot) - T(\omega, \cdot)\|^2 = \int_0^1 \int_0^1 [K_n(\omega, t, s) - K(\omega, t, s)]^2 \, ds\, dt,$$

and the assertion follows from Theorem 3.3b. □

Theorem 4.2. *Let K_n $(n \in \mathbb{N})$ and y_n $(n \in \mathbb{N})$ fulfill* (4.4), (4.5), *and assume that the sequences of distributions $(D(K_n))_{n \in \mathbb{N}}$ and $(D(y_n))_{n \in \mathbb{N}}$ are tight. For all $n \in \mathbb{N}$ let x_n be a random solution of* (4.6) *such that $(D(x_n))_{n \in \mathbb{N}}$ is uniformly bounded. Then $(D(x_n))_{n \in \mathbb{N}}$ is tight.*

PROOF. We use the approach of [13] and define

$$Z = L_2((0,1) \times (0,1)), \qquad \tilde{T}: Z \times X \to X,$$

$$[\tilde{T}(z, x)](t) = \int_0^1 z(t, s) x(s) \, ds, \qquad t \in [0,1], \quad (z, x) \in Z \times X.$$

(4.6) is now equivalent to

$$x = \tilde{T}(K_n(\omega), x) + y_n(\omega) \qquad (\omega \in \Omega, \quad n \in \mathbb{N}). \tag{4.10}$$

Clearly, $\tilde{T}$ is (jointly) continuous and $\tilde{T}(z, \cdot): X \to X$ is compact for all $z \in Z$.

Moreover, it holds for all $z_1, z_2 \in Z$, $x \in X$, that

$$\|\tilde{T}(z_1, x) - \tilde{T}(z_2, x)\| \leqslant \|z_1 - z_2\|_z \|x\|. \tag{4.11}$$

This implies that for all bounded subsets $C \subset X$, $\{\tilde{T}(\cdot, x) | x \in C\}$ is equicontinuous on Z. Thus, (4.10) fits into the setting of [13] and all assumptions of Theorem 2.11 a) of [13] are fulfilled. Now, the assertion follows from that theorem. □

REMARK 4.3a. Theorem 4.2 shows that under weak hypotheses on K_n ($n \in \mathbb{N}$) and y_n ($n \in \mathbb{N}$) the uniform boundedness of $(D(x_n))_{n \in \mathbb{N}}$ implies its tightness. But, for the application of Theorem 4.2, we need conditions that are sufficient for the uniform boundedness of the distributions of some sequence $(x_n)_{n \in \mathbb{N}}$ of random solutions of (4.6). Of course, the best way is the choice of random best approximate solutions (with minimal norm) of (4.6), i.e., $\bar{x}_n(\omega) = (I - T_n)^+(\omega, y_n(\omega))$ ($\omega \in \Omega$, $n \in \mathbb{N}$), where $(I - T_n)^+ : \Omega \times X \to X$ is the random generalized inverse of $I - T_n$ ($n \in \mathbb{N}$) (see e.g., [21, Section II]).

One possibility for proving uniform boundedness of $(D(\bar{x}_n))_{n \in \mathbb{N}}$ seems to be the use of approximation results for generalized inverses (see Theorem 4.1 or other results in [20]). But, these results require strong conditions on T and T_n ($n \in \mathbb{N}$), e.g., (3.5) (in the "stochastic" case). In Section 5 of [20], projection methods for best approximation solutions of linear operator equations are developed. It is shown that appropriate projection schemes should be used. [11] contains a "stochastic" version of that approach. It turns out that in the case of random operators, "stochastic projection schemes" (involving finite-dimensional subspaces that depend on ω) have to be used. Therefore, that approach does not fit into the setting of this paper.

Summarizing these remarks, it appears that although Theorem 4.2 contains a more general result, Theorem 4.1 is of value. Note that the proof of Theorem 4.1 is based on the approach motivated by Proposition 2.4.

REMARK 4.3b. We denote for each $n \in \mathbb{N}$

$$K_n(\omega, t, s) = \sum_{i,j=1}^{n} b_{ij}^{(n)}(\omega)\phi_i(t)\phi_j(s), \qquad t, s \in [0,1],$$

$$y_n(\omega, t) = \sum_{i=1}^{n} c_i^{(n)}(\omega)\phi_i(t), \qquad t \in [0,1],$$

where $b_{ij}^{(n)}, c_i^{(n)}$, $i, j = 1, \ldots, n$, are real random variables (on $(\Omega, \mathscr{A}, P)$).

Then (4.4) and (4.5) are fulfilled, and (4.6) is equivalent to the following random linear algebraic equation:

$$a_i^{(n)}(\omega) = \sum_{j=1}^{n} b_{ij}^{(n)}(\omega) a_j^{(n)}(\omega) + c_i^{(n)}(\omega),$$
$$i = 1, \ldots, n, \quad \omega \in \Omega,$$
$$x_n(\omega) = \sum_{i=1}^{n} a_i^{(n)}(\omega)\phi_i. \tag{4.12}$$

This equation can easily be solved numerically in the case that $b_{ij}^{(n)}, c_i^{(n)}$, $i, j = 1, \ldots, n$, are discrete random variables. Note that e.g. (4.8) is fulfilled if for P-almost all $\omega \in \Omega$,

$$\lim_{n \to \infty} \sum_{i,j=1}^{n} \left[b_{ij}^{(n)}(\omega) - b_{ij}(\omega) \right]^2 = 0,$$

where

$$b_{ij}(\omega) = \int_0^1 \int_0^1 K(\omega, t, s) \phi_i(t) \phi_j(s) \, ds \, dt, \qquad i, j \in \mathbb{N}, \quad \omega \in \Omega.$$

For a further discussion of related questions concerning the approximation approach of this section, we refer to [5]. In particular, in [5] the known results on limit theorems for random linear algebraic equations (with increasing dimensions) are discussed. These results indicate a direct approach [using (4.12)] to weak convergence of the solution measures $(D(x_n))_{n \in \mathbb{N}}$. This approach is also used in [26].

Finally, we will show that every weak limit of probability distributions of random solutions x_n $(n \in \mathbb{N})$ of (4.6) is the distribution of an X-valued random variable [defined on $(\Omega, \mathscr{A}, P)$] that is a so-called "D-solution" of (4.1). The concept of a D-solution was introduced in [13]; for a detailed discussion of this notion we refer to [13, Remarks 2.3 and 2.15].

Although the proof of the following result is essentially the same as that of Theorem 2.11(b) in [13], we will briefly sketch it.

Proposition 4.4. *Let K, K_n $(n \in \mathbb{N})$ and y, y_n $(n \in \mathbb{N})$ satisfy* (4.2), (4.3), (4.4), (4.5), *and assume that the sequence $(D(K_n, y_n))_{n \in \mathbb{N}}$ of joint probability distributions converges weakly to $D(K, y)$ (on $Z \times X$). For all $n \in \mathbb{N}$ let x_n be a random solution of* (4.6), *and assume that $(D(x_n))_{n \in \mathbb{N}}$ is tight. Then there exist a subsequence $(x_{n_k})_k$ of $(x_n)_{n \in \mathbb{N}}$ and an X-valued random variable x (defined on $(\Omega, \mathscr{A}, P)$) such that* (i) *$(D(x_{n_k}))_k$ converges weakly to $D(x)$, and* (ii) *x is a "D-solution" of* (4.1), *i.e., there exist random variables $\bar{K}: \Omega \to Z$ and $\bar{y}: \Omega \to X$ such that $D(K, y) = D(\bar{K}, \bar{y})$ and the distributions of x and w coincide, where*

$$w(\omega) = \int_0^1 \bar{K}(\omega, \cdot, s) x(\omega, s) \, ds + \bar{y}(\omega), \qquad \omega \in \Omega.$$

PROOF. Because of [13, Lemma 2.8], the sequence $(D(K_n, y_n, x_n))_{n \in \mathbb{N}}$ of probability distributions on $Z \times X \times X$ is tight. By Prohorov's theorem and the main result of [14], there exists a subsequence $((K_{n_k}, y_{n_k}, x_{n_k}))_k$ and a $Z \times X \times X$-valued random variable $(\bar{K}, \bar{y}, x)$ (defined on $(\Omega, \mathscr{A}, P)$) such that $(D(K_{n_k}, y_{n_k}, x_{n_k}))_k$ converges weakly to $D(\bar{K}, \bar{y}, x)$. This implies

$$D(K_{n_k}, y_{n_k}) \Rightarrow D(\bar{K}, \bar{y}) = D(K, y) \qquad (\text{see [7]}),$$

and $D(x_{n_k}) \Rightarrow D(x)$ ($\Rightarrow$ denotes weak convergence). We define $\tilde{T}: Z \times X \times X \to X$ as follows:

$$[\tilde{T}(z, u, v)](t) := \int_0^1 z(t, s) v(s)\, ds + u(t), \qquad t \in [0,1],$$

$(z, v, u) \in Z \times X \times X$, and note that $\tilde{T}$ is (jointly) continuous. Because of [7, Theorem 5.1] and (4.6), this implies

$$D(x_{n_k}) = D\big(\tilde{T}(K_{n_k}, y_{n_k}, x_{n_k})\big) \Rightarrow D(\tilde{T}(\bar{K}, \bar{y}, x)) = D(w),$$

and finally, $D(x) = D(w)$. □

REMARK 4.5. Note that a "D-solution" $\tilde{x}$ of (4.1) cannot be interpreted as a stochastic process [on $(\Omega, \mathscr{A}, P)$]. But in Theorem 3.1 of [25] it is shown that there is an $\mathscr{A} \times \mathscr{B}([0,1])$-measurable stochastic process x (defined on some probability space), almost all paths of which belong to $L_2(0,1)$, such that $D(x) = D(\tilde{x})$.

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