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DIFFERENTIAL STABILITY OF TWO-STAGE STOCHASTIC PROGRAMS*

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Abstract. Two-stage stochastic programs with random right-hand side are considered. Optimal values and solution sets are regarded as mappings of the expected recourse functions and their perturbations, respectively. Conditions are identified implying that these mappings are directionally differentiable and semidifferentiable on appropriate functional spaces. Explicit formulas for the derivatives are derived. Special attention is paid to the role of a Lipschitz condition for solution sets as well as of a quadratic growth condition of the objective function.

Key words. two-stage stochastic programs, sensitivity analysis, directional derivatives, semiderivatives, solution sets

AMS subject classifications. 90C15, 90C31

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1. Introduction.

Two-stage stochastic programming is concerned with problems that require a here-and-now decision on the basis of given probabilistic information on the random data without making further observations. The costs to be minimized consist of the direct costs of the here-and-now (or first-stage) decision as well as the costs generated by the need of taking a recourse (or second-stage) decision in response to the random environment. Recourse costs are often formulated by means of expected values with respect to the probability distribution of the involved random data. In this way, two-stage models and their solutions depend on the underlying probability distribution. Since this distribution is often incompletely known in applied models, or it has to be approximated for computational purposes, the stability behavior of stochastic programming models when changing the probability measure is important. This problem is studied in a number of papers. We mention here only the surveys [13], [40] and the papers [1], [12], [18], [26], [27], [34], and [35]. The paper [1] contains general results on continuity properties of optimal values and solutions when perturbing the probability measures with respect to the topology of weak convergence. Quantitative continuity results of solution sets to two-stage stochastic programs with respect to suitable distances of probability measures are obtained in [26] and [27]. Asymptotic properties of statistical estimators of values and solutions to stochastic programs are derived in [18], [34], [35]. They are based on directional differentiability properties of the underlying optimization problems with respect to the parameter that carries the randomness [18], [35] or the probability measure [34]. These directional differentiability results for values [35] and solutions [13], [18], [34] lead to asymptotic results via the so-called *delta-method*. For a description of the delta-method we refer to Chapter 6 in [28], [35], to [36] for an up-to-date presentation, and to [16] for a set-valued variant. These papers illuminate the importance

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of the Hadamard directional differentiability (for single-valued functions) and of the semidifferentiability (for set-valued mappings) in the context of asymptotic statistics.

The present paper aims at contributing to this line of differential stability studies. The results in [18], [34] apply to fairly general stochastic optimization models but impose conditions that are rather restrictive in our context. The present paper deals with special two-stage models and, using structural properties, avoids certain assumptions that complicate or even prevent the applicability of the general results to two-stage stochastic programs. Such assumptions are the (local) uniqueness of solutions and differentiability properties of perturbed problems, which are indispensable in [18], [34]. Before discussing this in more detail, let us introduce the class of two-stage stochastic programs we want to consider:

$$\min\{g(x) + Q_\mu(Ax) : x \in C\}, \tag{1.1}$$

where $g : \mathbb{R}^m \to \mathbb{R}$ is a convex function, $C \subseteq \mathbb{R}^m$ is a nonempty closed convex set, A is an (s, m)-matrix, and Q_μ is the expected recourse function with respect to the (Borel) probability measure μ on $\mathbb{R}^s$;

$$Q_\mu(y) = \int_{\mathbb{R}^s} \tilde{Q}(\omega - y)\mu(\mathrm{d}\omega), \tag{1.2}$$

$$\tilde{Q}(t) = \inf\{\langle q, u\rangle : Wu = t, u \geq 0\}, \quad t \in \mathbb{R}^s. \tag{1.3}$$

Here $q \in \mathbb{R}^{\bar{m}}$ are the recourse costs, W is an $(s, \bar{m})$-matrix and called the recourse matrix, and $\tilde{Q}(\omega - Ax)$ corresponds to the value of the optimal second-stage decision for compensating a possible violation of the (random) constraint $Ax = \omega$. To have the problem (1.1)–(1.3) well defined, we assume

(A1) $\quad \operatorname{pos} W = \{Wu : u \in \mathbb{R}^{\bar{m}}_+\} = \mathbb{R}^s \quad$ (complete recourse),

(A2) $\quad M_D = \{t \in \mathbb{R}^s : W^T t \leq q\} \neq \emptyset \quad$ (dual feasibility),

(A3) $\quad \int_{\mathbb{R}^s} \|\omega\|\mu(\mathrm{d}\omega) < \infty \quad$ (finite first moment).

The assumptions (A1) and (A2) imply that $\tilde{Q}$ is finite, convex, and polyhedral on $\mathbb{R}^s$. Due to (A3), Q_μ is also finite and convex on $\mathbb{R}^s$ (cf. [15], [39]). Observe that, in general, an expected recourse function Q_μ may be nondifferentiable on a certain union of hyperplanes in $\mathbb{R}^s$ and that, indeed, differentiability properties of Q_μ depend on the degree of smoothness induced by the measure μ (cf. [15], [21], [38], [39], and Remark 4.10). Another observation is that the uniqueness of solutions to (1.1) is guaranteed only if the constraint set C picks just one element from the relevant level set of $g(\cdot) + Q_\mu(A\,\cdot)$. As the next example shows, this set may be large since $Q_\mu(A\,\cdot)$ is constant on translates of the null space of the matrix A.

Example 1.1. In (1.1)–(1.3), let $m = 3$, $n = 2$, $g(x) = \frac{1}{4}(x_2 - x_3)$, $C = [0, \frac{1}{2}]^3$, $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$, $q = (1, 1, 1, 1)$, $W = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$, and μ be the uniform distribution on the square $[-\frac{1}{2}, \frac{1}{2}]^2$ in $\mathbb{R}^2$.

Then we have $\tilde{Q}(t) = |t_1| + |t_2|$ and $Q_\mu(y) = y_1^2 + y_2^2 + \frac{1}{2}$ for $y = (y_1, y_2) \in [-\frac{1}{2}, \frac{1}{2}]^2$. The optimization problem (1.1) and its solution set $\psi(Q_\mu)$ take the form

$$\min\left\{\frac{1}{4}(x_2 - x_3) + (x_1 - x_3)^2 + (x_1 - x_2)^2 + \frac{1}{2} : (x_1, x_2, x_3) \in \left[0, \frac{1}{2}\right]^3\right\},$$

$$\psi(Q_\mu) = \left\{ \left(\frac{1}{8} + u,\, u,\, \frac{1}{4} + u\right) \,:\, u \in \left[0,\, \frac{1}{4}\right] \right\} = \left\{ \left(\frac{1}{8},\, 0,\, \frac{1}{4}\right) + \ker A \right\} \cap C,$$

where $\ker A = \{(u,\, u,\, u) \,:\, u \in \mathbb{R}\}$ is the null space of A.

Proposition 2.1 below provides some more insight into the structure of the solution set to (1.1) and elucidates the role of the set-valued mapping $\sigma(y) := \operatorname{argmin}\{g(x) : x \in C, Ax = y\}$ in this respect.

Note that assumption (A1) could be relaxed by introducing the set $\mathcal{K} = \{y \in \mathbb{R}^s : Q_\mu(y) < +\infty\}$. Then (A2) and (A3) imply that $\mathcal{K}$ is a closed convex polyhedron and that Q_μ is convex and continuous on $\mathcal{K}$ (cf. [39]). Now (A1) can be replaced by the condition $\mathcal{K} \supseteq A(C)$ (relatively complete recourse), and much of the work done in this paper carries over to this more general setting by using spaces of functions defined on $\mathcal{K}$ instead of $\mathbb{R}^s$.

Let K_C denote the set of all convex functions on $\mathbb{R}^s$ which forms a convex cone in the space $C^0(\mathbb{R}^s)$ of all continuous functions on $\mathbb{R}^s$. K_C will serve as the set of possible perturbations of the given expected recourse function $Q_\mu \in K_C$. We define

$$\begin{aligned} \varphi(Q) &:= \inf\{g(x) + Q(Ax) : x \in C\}, \\ \psi(Q) &:= \operatorname{argmin}\{g(x) + Q(Ax) : x \in C\} \end{aligned}$$

and regard φ and ψ as mappings from K_C into the extended reals and the set of all closed convex subsets of $\mathbb{R}^m$, respectively.

In this paper we develop a sensitivity analysis for the mappings φ and ψ at some given function Q_μ. The stochastic programming origin of the model (1.1) takes a back seat, and our results are stated in terms of general conditions on Q_μ and its perturbations Q. We identify conditions such that the value function φ has first- and second-order directional derivatives and the solution-set mapping ψ is directionally differentiable at Q_μ into admissible directions. Here, admissibility means that the direction belongs to the radial tangent cone to K_C at Q_μ, i.e.,

$$T^r(K_C; Q_\mu) = \{\lambda(Q - Q_\mu) : Q \in K_C, \lambda > 0\},$$

ensuring that the difference quotients are well defined. For v belonging to $T^r(K_C; Q_\mu)$ the Gateaux directional derivatives of φ and ψ at Q_μ and $(Q_\mu, \bar{x})$, $\bar{x} \in \psi(Q_\mu)$, respectively, are defined as

$$\begin{aligned} \varphi'(Q_\mu; v) &= \lim_{t \to 0+} \frac{1}{t}(\varphi(Q_\mu + tv) - \varphi(Q_\mu)), \\ \varphi''(Q_\mu; v) &= \lim_{t \to 0+} \frac{1}{t^2}(\varphi(Q_\mu + tv) - \varphi(Q_\mu) - t\varphi'(Q_\mu; v)), \\ \psi'(Q_\mu, \bar{x}; v) &= \lim_{t \to 0+} \frac{1}{t}(\psi(Q_\mu + tv) - \bar{x}), \end{aligned}$$

if the limits exist. The third limit is understood in the sense of (Painlevé–Kuratowski) set convergence (e.g. [2]). Recall that the lower and upper set limits of a family $(S_t)_{t>0}$ of subsets of a metric space (X, d) are defined as

$$\begin{aligned} \liminf_{t \to 0+} S_t &= \{x \in X : \lim_{t \to 0+} d(x, S_t) = 0\}, \\ \limsup_{t \to 0+} S_t &= \{x \in X : \liminf_{t \to 0+} d(x, S_t) = 0\}. \end{aligned}$$

Both sets are closed and the lower set limit is contained in the upper limit. If both limits coincide, the family $(S_t)_{t>0}$ is said to converge and its limit set is denoted by $\lim\limits_{t\to 0+} S_t$. For sequences of sets $(S_n)_{n\in\mathbb{N}}$ the definitions of set limits are modified correspondingly.

We also derive conditions implying that the limits defining the directional derivatives exist uniformly with respect to directions v belonging to compact subsets of certain functional spaces. The limits are then called (first- or second-order) Hadamard directional derivatives and semiderivatives for set-valued maps, respectively. The corresponding directional derivatives are defined on tangent cones to the cone of convex functions in certain functional spaces. For more information on concepts of directional differentiability and multifunction differentiability we refer to [4], [33], and to [2], [3], [23], and [25], respectively.

Let us fix some notations used throughout the paper. $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and scalar product, respectively, in some Euclidean space $\mathbb{R}^n$; $B(x,r)$ denotes the open ball around $x\in\mathbb{R}^n$ with radius $r>0$; $d(x,D)$ denotes the distance of $x\in\mathbb{R}^n$ to the set $D\subseteq\mathbb{R}^n$; for a real-valued function f on $\mathbb{R}^n$, ∇f denotes its gradient in $\mathbb{R}^n$ and the (n,n)-matrix $\nabla^2 f$ its Hessian; if f is locally Lipschitzian near $x\in\mathbb{R}^n$, $\partial f(x)$ denotes the Clarke subdifferential of f at x; $f'(x;d)$ denotes the directional derivative of f at x in direction d if it exists; for $x\in C$, $T(C;x)$ denotes the tangent cone to C at x, i.e., $T(C;x)=\liminf_{t\to 0+}\frac{1}{t}(C-x)=\mathrm{cl}\{\lambda(y-x):\, y\in C,\lambda>0\}$, where cl stands for closure; for $x\in C$, $\xi\in T(C;x)$, $T^2(C;x,\xi)$ denotes the second-order tangent set to C at x in direction ξ, i.e., $T^2(C;x,\xi)=\liminf_{t\to 0+}\frac{1}{t^2}(C-x-t\xi)$ (note that $T^2(C;x,\xi)$ is closed and convex; see [10], [6] for further properties).

In our paper, we use the following linear metric spaces of real-valued functions on $\mathbb{R}^s$: The space $C^0(\mathbb{R}^s)$ of continuous functions on $\mathbb{R}^s$ equipped with the distance

$$d_\infty(f,\tilde f)=\sum_{n=1}^{\infty}2^{-n}\frac{\|f-\tilde f\|_{\infty,n}}{1+\|f-\tilde f\|_{\infty,n}},$$

where

$$\|f\|_{\infty,r}=\max_{\|y\|\le r}|f(y)| \text{ for } f,\tilde f\in C^0(\mathbb{R}^s) \text{ and } r>0;$$

the space $C^{0,1}(\mathbb{R}^s)$ of locally Lipschitzian functions on $\mathbb{R}^s$ with the metric

$$d_L(f,\tilde f)=\sum_{n=1}^{\infty}2^{-n}\frac{\|f-\tilde f\|_{\infty,n}+\|f-\tilde f\|_{L,n}}{1+\|f-\tilde f\|_{\infty,n}+\|f-\tilde f\|_{L,n}},$$

where

$$\begin{aligned}\|f\|_{L,r} &= \sup\left\{\frac{|f(y)-f(\tilde y)|}{\|y-\tilde y\|}:\|y\|\le r,\|\tilde y\|\le r, y\ne\tilde y\right\},\\ &= \sup\{\|z\|:z\in\partial f(y),\|y\|\le r\} \text{ for } f,\tilde f\in C^{0,1}(\mathbb{R}^s) \text{ and } r>0;\end{aligned}$$

the space $C^1(\mathbb{R}^s)$ of continuously differentiable functions on $\mathbb{R}^s$ with the metric $d(f,\tilde f)=d_\infty(f,\tilde f)+d_\infty(\nabla f,\nabla\tilde f)$, $f,\tilde f\in C^1(\mathbb{R}^s)$, and the space $C^{1,1}(\mathbb{R}^s)$ of functions in $C^1(\mathbb{R}^s)$ whose gradients are locally Lipschitzian on $\mathbb{R}^s$ equipped with the distance $d(f,\tilde f)=d_\infty(f,\tilde f)+d_\infty(\nabla f,\nabla\tilde f)+d_L(\nabla f,\nabla\tilde f)$, $f,\tilde f\in C^{1,1}(\mathbb{R}^s)$.

The sensitivity analysis of the mappings φ and ψ is carried out by exploiting structural properties of the optimization model (1.1). We obtain novel differentiability properties of solution sets and extend our earlier results on directional differentiability of optimal values in [12] considerably. As one might expect, the basic ingredients of our analysis are a Lipschitz continuity result for solution sets with respect to the distance in $C^{0,1}(\mathbb{R}^s)$ (Theorem 2.3) and a quadratic growth condition near solution sets (Theorem 2.7). Both theorems extend earlier results in [27] to more general situations for the first-stage costs g and constraint set C. All results in the paper apply to the *linear-quadratic case*, i.e., to linear- or convex-quadratic g and polyhedral C. Indeed, all results are formulated as generally as possible and most of them are accompanied by illustrative examples. The second-order analysis of φ in section 3 utilizes some ideas from [31] and [32], but its proof is entirely different and its Gateaux differentiability part is valid for nondifferentiable directions (Theorem 3.4). It is also elaborated that the Hadamard directional differentiability properties require the C^0-topology for the first-order result and the C^1-topology for the second-order one (Theorem 3.8), while the $C^{1,1}$-topology is needed for the semidifferentiability of the solution-set mapping ψ (Theorem 4.9). All results on differentiability properties of ψ in section 4 are new and do not follow from recent sensitivity results (e.g., [5], [8], [7], [17], [32]; see also the survey [8] for further references and Remark 4.4 for a more detailed discussion).

The results of sections 3 and 4 have direct implications to asymptotic properties of values and solution sets of two-stage stochastic programs when applying (smooth) nonparametric estimation procedures to approximate Q_μ. For a discussion of some of the related aspects we refer to the brief exposition in Remark 4.11. Further applications to asymptotics are beyond the scope of this paper and will be done elsewhere.

2. Basic directional properties. The first step in our analysis of directional properties consists in establishing results on the lower Lipschitz continuity of ψ and on the directional uniform quadratic growth of the objective near its solution set. Both results become important for our method of deriving directional differentiability properties for the optimal value function φ and the solution set mapping ψ at some given expected recourse function Q_μ. Their proofs are based on a decomposition of the program

$$\min\{g(x) + Q(Ax) : x \in C\}, \tag{2.1}$$

with Q belonging to K_C, into two auxiliary problems. The first one is a convex program with decisions taken from $A(C)$, and the second represents a parametric convex program which does not depend on Q.

PROPOSITION 2.1. *Let $Q \in K_C$, and let $\psi(Q)$ be nonempty. Then we have*

$$\begin{aligned}
\varphi(Q) &= \inf\{\pi(y) + Q(y) : y \in A(C)\} = \pi(Ax) + Q(Ax), \ \textit{ for any } x \in \psi(Q), \textit{ and} \\
\psi(Q) &= \sigma(Y(Q)), \ \textit{ where} \\
Y(Q) &:= \operatorname{argmin}\{\pi(y) + Q(y) : y \in A(C)\}, \\
\pi(y) &:= \inf\{g(x) : x \in C, Ax = y\}, \ \textit{ and} \\
\sigma(y) &:= \operatorname{argmin}\{g(x) : x \in C, Ax = y\}, \quad y \in A(C).
\end{aligned}$$

Moreover, π is convex on $A(C)$ and $\operatorname{dom}\sigma$ *is nonempty.*

Proof. Let $\bar{x} \in \psi(Q)$. Then we have

$$\varphi(Q) = g(\bar{x}) + Q(A\bar{x}) \geq \pi(A\bar{x}) + Q(A\bar{x}) \geq \inf\{\pi(y) + Q(y) : y \in A(C)\}.$$

For the converse inequality, let $\varepsilon > 0$ and $\bar{y} \in A(C)$ be such that

$$\pi(\bar{y}) + Q(\bar{y}) \leq \inf\{\pi(y) + Q(y) : y \in A(C)\} + \frac{\varepsilon}{2}.$$

Then there exists a $\bar{x} \in C$ such that $A\bar{x} = \bar{y}$ and $g(\bar{x}) \le \pi(\bar{y}) + \frac{\varepsilon}{2}$. Hence

$$\begin{aligned}\varphi(Q) &\le g(\bar{x}) + Q(A\bar{x}) \le \pi(\bar{y}) + Q(\bar{y}) + \frac{\varepsilon}{2}\\ &\le \inf\{\pi(y) + Q(y) : \, y \in A(C)\} + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the first statement has been shown. In particular, $x \in \sigma(Ax)$ and $Ax \in Y(Q)$ for any $x \in \psi(Q)$. Hence, it holds that $\psi(Q) \subseteq \sigma(Y(Q))$. Conversely, let $x \in \sigma(Y(Q))$. Then $x \in \sigma(y)$ for some $y \in Y(Q)$. Thus $Ax = y$ and $g(x) = \pi(y) = \pi(Ax)$, implying

$$\begin{aligned}g(x) + Q(Ax) = \pi(Ax) + Q(Ax) &= \inf\{\pi(y) + Q(y) : \, y \in A(C)\}\\ &= \varphi(Q) \quad \text{and} \quad x \in \psi(Q).\end{aligned}$$

Since the convexity of π is immediate, the proof is complete. ☐

In the following, it will turn out that Lipschitzian properties of the solution set mapping $y \mapsto \sigma(y)$ and a quadratic growth property of g near $\sigma(y)$ are essential. For the linear-quadratic case we are in a comfortable situation in this respect. Namely, we have the following proposition.

PROPOSITION 2.2. *Let g be linear or convex-quadratic, let C be convex polyhedral, and assume* $\operatorname{dom}\sigma$ *to be nonempty. Then σ is a polyhedral multifunction which is Hausdorff Lipschitzian on its domain* $\operatorname{dom}\sigma = A(C)$*, i.e., there exists a constant $L > 0$ such that*

$$d_H(\sigma(y), \sigma(\tilde{y})) \le L\|y - \tilde{y}\| \quad \textit{for all} \quad y, \tilde{y} \in A(C),$$

where d_H denotes the (extended) Hausdorff distance on subsets of $\mathbb{R}^m$.

Moreover, for each $r > 0$ there exists a constant $\eta(r) > 0$ such that

$$g(x) \ge \pi(Ax) + \eta(r) d(x, \sigma(Ax))^2 \quad \textit{for all} \quad x \in C \cap B(0, r).$$

(*Here π and σ are defined as in Proposition* 2.1.)

Proof. The Lipschitz property of σ is shown in [19, Theorem 4.2]. To prove the second statement, let g be of the form $g(x) = \langle Hx, x\rangle + \langle c, x\rangle$, where H is symmetric and positive semidefinite and $c \in \mathbb{R}^m$. For each $y \in A(C)$ we fix some $z(y) \in \sigma(y)$. An elementary characterization of solution sets to convex-quadratic programs with linear constraints yields that

$$\sigma(y) = \{x \in C : \, Ax = y, Hx = Hz(y), \langle c, x\rangle = \langle c, z(y)\rangle\}.$$

Due to the Lipschitz behavior of convex polyhedra (cf. [37]), there exists a constant $L_\sigma > 0$ such that

$$d(x, \sigma(y)) \le L_\sigma(\|Hx - Hz(y)\| + |\langle c, x\rangle - \langle c, z(y)\rangle|)$$

for all $y \in A(C)$ and $x \in C$ with $Ax = y$. Using the decomposition $H = H^{\frac{1}{2}}H^{\frac{1}{2}}$, where $H^{\frac{1}{2}}$ denotes the square root of H, and the representation $\langle c, x\rangle - \langle c, z(y)\rangle = g(x) - \pi(y) - \|H^{\frac{1}{2}}x\|^2 + \|H^{\frac{1}{2}}z(y)\|^2$, one arrives at the estimate

$$d(x, \sigma(y)) \le L_\sigma(\|H^{\frac{1}{2}}\|(1 + \|x\| + \|z(y)\|)\|H^{\frac{1}{2}}(x - z(y))\| + g(x) - \pi(y))$$

for all $y \in A(C)$ and $x \in C$ with $Ax = y$.

Now, let $r > 0$ and let us fix some element $\bar{x} \in C \cap B(0,r)$ and a corresponding $z(A\bar{x}) \in \sigma(A\bar{x})$. For each $y \in A(C)$ we now select $z(y) \in \sigma(y)$ such that $\|z(y) - z(A\bar{x})\| = d(z(A\bar{x}), \sigma(y))$. Since σ is Hausdorff Lipschitzian on $A(C)$, this implies $\|z(y) - z(A\bar{x})\| \leq L\|A\bar{x} - y\|$ for all $y \in A(C)$. Hence, there exists a constant $K(r) > 0$ such that $\|z(Ax)\| \leq K(r)$ for all $x \in C \cap B(0,r)$. Thus our estimate continues to $d(x, \sigma(Ax))^2 \leq \hat{L}(r)(\|H^{\frac{1}{2}}(x - z(Ax))\|^2 + (g(x) - \pi(Ax))^2)$ for all $x \in C \cap B(0,r)$ and some constant $\hat{L}(r) > 0$. Furthermore, the equation

$$g\left(\frac{1}{2}(x + z(y))\right) = \frac{1}{2}g(x) + \frac{1}{2}g(z(y)) - \frac{1}{4}\|H^{\frac{1}{2}}(x - z(y))\|^2$$

implies $\|H^{\frac{1}{2}}(x - z(y))\|^2 \leq 2(g(x) - \pi(y))$ for all $y \in A(C)$, $x \in C$, with $Ax = y$. Therefore, we finally obtain

$$\begin{aligned} d(x, \sigma(Ax))^2 &\leq \hat{L}(r)(2(g(x) - \pi(Ax)) + (g(x) - \pi(Ax))^2) \\ &\leq \hat{L}(r)\max\{2, K(r)\}(g(x) - \pi(Ax)) \end{aligned}$$

for all $x \in C \cap B(0,r)$, where $K(r) := \sup_{x \in C \cap B(0,r)}(g(x) - \pi(Ax))$. □

Due to the above proposition, the main results in this section apply to the linear-quadratic case. Although this case represents the main application of our results, the assumptions of the following theorems are formulated in terms of general conditions on the mapping σ in order to widen the range of applications. The first theorem states (lower) Lipschitz continuity of ψ at Q_μ and supplements Theorem 2.4 in [27].

THEOREM 2.3. *Let $Q_\mu \in K_C$, let $\psi(Q_\mu)$ be nonempty and bounded, and let Q_μ be strongly convex on some open, convex neighborhood of $A\psi(Q_\mu)$. Let $\bar{x} \in \psi(Q_\mu)$ and assume that there exist a constant $L > 0$ and a neighborhood U of $\bar{y}$ with $\{\bar{y}\} = A\psi(Q_\mu)$ such that*

$$d(\bar{x}, \sigma(y)) \leq L\|\bar{y} - y\| \quad \text{for all} \quad y \in A(C) \cap U.$$

Then there exist constants $\hat{L} > 0$, $\delta > 0$, and $r > 0$ such that

$$d(\bar{x}, \psi(Q)) \leq \hat{L}\|Q - Q_\mu\|_{L,r}$$

whenever $Q \in K_C$ and $\|Q - Q_\mu\|_{L,r} < \delta$.

Proof. We may assume that U is open and convex and that Q_μ is strongly convex on U. Let V be an open, convex, bounded subset of $\mathbb{R}^m$ such that $\psi(Q_\mu) \subset V$ and $A(V) \subset U$. It follows from Proposition 2.3 in [27] (where a slightly different terminology is used) that there exists a constant $\delta > 0$ such that $\emptyset \neq \psi(Q) \subset V$ whenever $Q \in K_C$ and

$$\sup\{\|z\| : z \in \partial(Q - Q_\mu)(y),\, y \in \text{cl}\, A(V)\} < \delta.$$

Let $r > 0$ be chosen such that $\text{cl}\, A(V) \subset \bar{B}(0,r)$. Hence, we have $\emptyset \neq \psi(Q) \subset V$ whenever $Q \in K_C$, $\|Q - Q_\mu\|_{L,r} < \delta$. Then Proposition 2.1 yields the relation $\psi(Q) = \sigma(Y(q))$, where $Y(Q) = \text{argmin}\{\pi(y) + Q(y) : y \in A(C)\}$. Since Q_μ is strongly convex on U, there exists a constant $\kappa > 0$ such that

$$\kappa\|y - \bar{y}\|^2 \leq \pi(y) + Q_\mu(y) - (\pi(\bar{y}) + Q_\mu(\bar{y})) \quad \text{for all} \quad y \in U.$$

Let $Q \in K_C$ with $\|Q - Q_\mu\|_{L,r} < \delta$, and let $\tilde{y} \in Y(Q)$. Since y belongs to $A(V) \subset U$, we obtain

$$\begin{aligned} \kappa\|\tilde{y} - \bar{y}\|^2 &\leq \pi(\tilde{y}) + Q_\mu(\tilde{y}) - (\pi(\bar{y}) + Q_\mu(\bar{y})) + \pi(\bar{y}) + Q(\bar{y}) - (\pi(\tilde{y}) + Q(\tilde{y})) \\ &= (Q - Q_\mu)(\bar{y}) - (Q - Q_\mu)(\tilde{y}), \end{aligned}$$

and, hence,

$$\|\tilde{y}-\bar{y}\| \le \frac{1}{\kappa}\,\frac{(Q-Q_\mu)(\bar{y})-(Q-Q_\mu)(\tilde{y})}{\|\bar{y}-\tilde{y}\|} \le \frac{1}{\kappa}\|Q-Q_\mu\|_{L,r}.$$

The proof can now be completed as follows. Let $Q \in K_C$ be such that $\|Q-Q_\mu\|_{L,r} < \delta$. Then

$$\begin{aligned} d(\bar{x},\psi(Q)) = d(\bar{x},\sigma(Y(Q))) &\le \sup_{y\in Y(Q)} d(\bar{x},\sigma(y)) \\ &\le L \sup_{y\in Y(Q)} \|\bar{y}-y\| \le \frac{L}{\kappa}\|Q-Q_\mu\|_{L,r}. \qquad \square \end{aligned}$$

Remark 2.4. The proof shows that a Lipschitz modulus of ψ can be chosen as the quotient of a Lipschitz constant to σ and a strong convexity constant to Q_μ.

From the proof it is immediate that replacing the local Lipschitz condition on σ by stronger conditions like

$$\begin{aligned} &\sup_{x\in\sigma(\bar{y})} d(x,\sigma(y)) \le L\|\bar{y}-y\| \quad \text{or} \\ &d_H(\sigma(\bar{y}),\sigma(y)) \le L\|\bar{y}-y\| \quad \text{for all } y \in A(C)\cap U \end{aligned}$$

leads to corresponding stronger Lipschitz continuity properties of solution sets. Because of Proposition 2.2, all of this applies to the linear-quadratic case. However, it is worth mentioning that the theorem also applies to more general problems such that the corresponding solution sets $\sigma(y)$ enjoy Lipschitzian properties. Conditions ensuring Lipschitz behavior of σ can be derived from stability results for the corresponding parametric generalized equation

$$0 \in \nabla L(x,\lambda;y) + N_{C\times\mathbb{R}^s}(x,\lambda), \tag{2.2}$$

which describes the first-order necessary optimality condition. Here $L(x,\lambda;y) := g(x)+\lambda^T(Ax-y)$ is the Lagrangian function, $\nabla L(x,\lambda;y) = \binom{\nabla g(x)+A^T\lambda}{Ax-y}$, where g is assumed to be continuously differentiable, and $N_{C\times\mathbb{R}^s}$ is the normal cone map of convex analysis. Such stability results are presently available for broad classes of parametric generalized equations (e.g., [17], [22], [24]). A typical recent result in this direction, which applies to our situation for twice continuously differentiable g, is Theorem 5.1 in [22]. It says that the solution set mapping of the parametric generalized equation (2.2) is pseudo-Lipschitzian around $(\bar{x},\bar{\lambda};\bar{y})$ if the adjoint generalized equation

$$0 \in \nabla^2 L(\bar{x},\bar{\lambda};\bar{y})w^* + D^*N_{C\times\mathbb{R}^s}(\bar{x},\bar{\lambda};-\nabla L(\bar{x},\bar{\lambda};\bar{y}))(w^*) \tag{2.3}$$

has only the trivial solution $w^*=0$.

Here $D^*N_{C\times\mathbb{R}^s}(\bar{x},\bar{\lambda};-\nabla L(\bar{x},\bar{\lambda};\bar{y}))$ is the Mordukhovich coderivative [22] of the normal cone multifunction at the point $(\bar{x},\bar{\lambda};-\nabla L(\bar{x},\bar{\lambda};\bar{y}))$ belonging to the graph of $N_{C\times\mathbb{R}^s}$. Translating this into our framework, we obtain that the mapping σ is pseudo-Lipschitzian around $(\bar{x},\bar{y})$ if the following two conditions are satisfied.

(a) There exists an element $\hat{x}$ belonging to the relative interior of C such that $A\hat{x}=\bar{y}$ (Slater condition).

(b) The equations $Aw_1^*=0$ and $0 \in \nabla^2 g(\bar{x})w_1^* + A^Tw_2^* + D^*N_C(\bar{x},\bar{\lambda};-\nabla g(\bar{x})-A^T\bar{\lambda})(w_1^*)$ have only the trivial solution $w_1^*=0$, $w_2^*=0$. (Here $(\bar{x},\bar{\lambda})$ is a solution of (2.2) for $y=\bar{y}$.)

The next examples show that the theorem applies to instances of two-stage stochastic programs with nonunique solutions and with nonpolyhedral convex constraint sets C.

Example 2.5. We revisit Example 1.1 and obtain with the notations of Proposition 2.1 that $A(C) = [-\frac{1}{2}, \frac{1}{2}]^2$, $\pi(y) = \frac{1}{4}(y_1 - y_2)$, $Y(Q_\mu) = \operatorname{argmin}\{\frac{1}{4}(y_1 - y_2) + y_1^2 + y_2^2 + \frac{1}{2} : y \in A(C)\} = \{(-\frac{1}{8}, \frac{1}{8})\}$, and $\sigma(y) = \{(u, u - y_2, u - y_1) : u \in \mathbb{R}\} \cap C$ for $y \in A(C)$. Hence, $Y(Q_\mu)$ is a singleton, but $\psi(Q_\mu) = \sigma(Y(Q_\mu))$ forms a line segment. Moreover, σ is Hausdorff Lipschitzian on $A(C)$ and Theorem 2.3 applies.

Example 2.6. In (1.1)–(1.3) let $m = 2$, $s = 1$, $g(x) \equiv 0$, $A = (1, 0)$, $q = (1, 1)$, $W = (1, -1)$, μ be the uniform distribution on $[-\frac{1}{2}, \frac{1}{2}]$, and $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_2^2 \le x_1\}$. Then we have

$$\tilde{Q}(t) = |t|, Q_\mu(y) = \int_{\mathbb{R}} |\omega - y| \mu(\mathrm{d}\omega) = \begin{cases} y^2 + \frac{1}{4}, & y \in [-\frac{1}{2}, \frac{1}{2}], \\ |y| & \text{otherwise}, \end{cases}$$

$\psi(Q_\mu) = \{(0, 0)\}$, and Q_μ is strongly convex on $(-\frac{1}{2}, \frac{1}{2})$. For $y \in A(C) = \mathbb{R}_+$ we have

$$\sigma(y) = \{x \in C : Ax = y\} = \{(y, x_2) \in \mathbb{R}^2 : x_2^2 \le y\} = \{y\} \times [-\sqrt{y}, \sqrt{y}],$$

and, hence $d((0, 0), \sigma(y)) = y$ for all $y \in \mathbb{R}_+$. Thus Theorem 2.3 applies for $\bar{x} = (0, 0)$.

Example 2.9 shows that Theorem 2.3 gets lost if Q_μ fails to be strongly convex on some neighborhood of $A\psi(Q_\mu)$. Our next result establishes a sufficient condition for the uniform quadratic growth near solution sets.

THEOREM 2.7. *Let $Q_\mu \in K_C$, let $\psi(Q_\mu)$ be nonempty and bounded, and let Q_μ be strongly convex on some open convex neighborhood U of $A\psi(Q_\mu)$. Assume that there exists a constant $L > 0$ such that*

$$d_H(\sigma(y), \sigma(\tilde{y})) \le L\|y - \tilde{y}\| \quad \textit{for all} \quad y, \tilde{y} \in A(C),$$

and for each $r > 0$ there exists a constant $\eta(r) > 0$ such that

$$g(x) \ge \pi(Ax) + \eta(r) d(x, \sigma(Ax))^2 \quad \textit{for all} \quad x \in C \cap B(0, r).$$

Then, for some open, bounded neighborhood V of $\psi(Q_\mu)$ and each $v \in T^r(K_C; Q_\mu)$, there exist constants $c > 0$ and $\delta > 0$ such that the following uniform growth condition holds:

$$g(x) + (Q_\mu + tv)(Ax) \ge \varphi(Q_\mu + tv) + c d(x, \psi(Q_\mu + tv))^2$$

for all $x \in C \cap V$ and $t \in [0, \delta)$.

Proof. Let $v \in T^r(K_C, Q_\mu)$, and let V be an open, bounded subset of $\mathbb{R}^m$ such that $\psi(Q_\mu) \subset V$ and $A(V) \subseteq U$. As in Theorem 2.3 we choose $\delta > 0$ such that $\emptyset \ne \psi(Q_\mu + tv) \subset V$ and, in addition, that $Q_\mu + tv$ is strongly convex on U for all $t \in [0, \delta)$ (with a uniform constant $\kappa > 0$). For each $t \in [0, \delta)$ Proposition 2.1 then yields that $\psi(Q_\mu + tv) = \sigma(y_t)$, where y_t is the unique minimizer of the strongly convex function $\pi + Q_\mu + tv$ on $A(C)$ and, moreover, we have $\kappa\|y - y_t\|^2 \le \pi(y) + (Q_\mu + tv)(y) - \varphi(Q_\mu + tv)$ for all $y \in A(C) \cap U$. Now, we choose $r > 0$ such that

$V \subseteq B(0,r)$ and continue for each $x \in C \cap V$ and $t \in [0,\delta)$ as follows:

$$\begin{aligned}
d(x,\psi(Q_\mu+tv))^2 &= d(x,\sigma(y_t))^2\\
&\le 2(d(x,\sigma(Ax))^2 + d_H(\sigma(Ax),\sigma(y_t))^2)\\
&\le 2\left(\frac{1}{\eta(r)}(g(x)-\pi(Ax)) + L^2\|Ax-y_t\|^2\right)\\
&\le 2\left(\frac{1}{\eta(r)}(g(x)-\pi(Ax))+\frac{L^2}{\kappa}(\pi(Ax)+(Q_\mu+tv)(Ax)-\varphi(Q_\mu+tv))\right)\\
&\le 2\max\left\{\frac{1}{\eta(r)},\frac{L^2}{\kappa}\right\}(g(x)+(Q_\mu+tv)(Ax)-\varphi(Q_\mu+tv)).
\end{aligned}$$

Putting $c^{-1} = 2\max\{\frac{1}{\eta(r)},\frac{L^2}{\kappa}\}$ completes the proof. □

The following examples show that the quadratic growth condition gets lost even for the original problem, i.e., $t = 0$, if either the Lipschitz condition for σ or the strong convexity property for Q_μ are violated.

Example 2.8. Consider again the set-up of Example 2.6. Since it holds that $d_H(\sigma(y),\sigma(0)) = (y^2+y)^{\frac{1}{2}}$ for all $y \in \mathbb{R}_+ = A(C)$, σ is *not* Hausdorff Lipschitzian on $A(C)$. Supposed there exists a neighborhood V of $\psi(Q_\mu) = \{(0,0)\}$ and a constant $\varrho > 0$ such that the growth condition

$$\varrho\, d(x,\psi(Q_\mu))^2 = \varrho\|x\|^2 \le Q_\mu(x_1) - \varphi(Q_\mu) = x_1^2 \quad \text{for all} \quad x \in C \cap V$$

is satisfied. Since the sequence $((\frac{1}{n},\frac{1}{\sqrt{n}}))$ belongs to $C \cap V$ for sufficiently large $n \in \mathbb{N}$, this would imply $\varrho(\frac{1}{n^2}+\frac{1}{n}) \le \frac{1}{n^2}$ for large n, which is a contradiction.

Example 2.9. In (1.1)–(1.3) let $m = s = 1$, $g(x) \equiv 0$, $A = 1$, $C = \mathbb{R}$, $q = (1,1)$, $W = (1,-1)$, and let μ be the probability distribution on $\mathbb{R}$ having the density

$$f_\mu(z) = \begin{cases} |z|, & z \in [-1,1],\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$Q_\mu(y) = \int_{\mathbb{R}} |\omega - y|\mu(\mathrm{d}\omega) = \begin{cases} \frac{1}{3}|y|^3 + \frac{2}{3}, & y \in [-1,1]\\ |y|, & \text{otherwise,}\end{cases}$$

$\psi(Q_\mu) = \{0\}$, and there is *no* neighborhood of $\psi(Q_\mu)$ where Q_μ is strongly convex. It is clear that the quadratic growth condition fails to hold, since the inequality $\varrho x^2 \le Q_\mu(x) - \varphi(Q_\mu) = \frac{1}{3}|x|^3$ cannot be true for some $\varrho > 0$ and all x belonging to some neighborhood of $x = 0$.

With the linear function $v(x) = -x$ $(x \in \mathbb{R})$ we obtain for all $t \in [0,1]$ that $\psi(Q_\mu + tv) = \{\sqrt{t}\}$ (cf. Example 3.7). Hence, the lower Lipschitz property of ψ fails to hold as well. Since the strong convexity and later also the strict convexity of the expected recourse function Q_μ (on certain convex subsets of $\mathbb{R}^s$) form essential conditions in most of our results, we recall a theorem (Theorem 2.2 in [30]) that provides a handy criterion to check these properties for problem (1.1)–(1.3).

PROPOSITION 2.10. *Let $V \subset \mathbb{R}^s$ be open convex, and assume* (A1) *and* (A3). *Consider the following conditions.*

(A2)* int $M_D = \{t \in \mathbb{R}^s : W^T t < q\} \neq \emptyset$.

(A4) μ *is absolutely continuous on* $\mathbb{R}^s$.

(A4)* μ *satisfies* (A4) *and there exist a density f_μ for μ and a constant $\delta > 0$ such that $f_\mu(z) \geq \delta$ whenever $d(z, V) \leq \delta$.*

Then (A2)* *and* (A4) *imply that Q_μ is strictly convex on V if V is a subset of the support of μ, and* (A2)* *and* (A4)* *imply that Q_μ is strongly convex on V.*

In addition, it is shown in [30] that under (A1)–(A4) the condition (A2)* is also necessary for the strict convexity of Q_μ. For extended simple recourse models (i.e., $W = (H, -H)$ with some nonsingular (s, s)-matrix H) (A2)* is equivalent to $q^+ + q^- > 0$ (componentwise), where $q = (q^+, q^-)$ and $q^+, q^- \in \mathbb{R}^s$. This may be used to check strict or strong convexity properties in the Examples 2.6 and 2.9.

3. Directional derivatives of optimal values. In this section, we study first- and second-order directional differentiability properties of the optimal value function φ on its domain K_C. We begin with the first-order analysis and show that φ as a mapping from K_C to the extended reals is Hadamard directionally differentiable at some given expected recourse function $Q_\mu \in K_C$. Here K_C is regarded as a subset of $C^0(\mathbb{R}^s)$. Recall that φ is Hadamard directionally differentiable at Q_μ on K_C iff for all sequences (v_n) converging to some v in $C^0(\mathbb{R}^s)$ and all sequences $t_n \to 0+$ such that the elements $Q_\mu + t_n v_n$ belong to K_C the limit

$$\varphi'(Q_\mu; v) = \lim_{n\to\infty} \frac{1}{t_n}(\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu))$$

exists. Since the condition $Q_\mu + t_n v_n \in K_C$ means that $v_n = \frac{1}{t_n}(Q_n - Q_\mu)$ for some $Q_n \in K_C$, the limit v belongs to the tangent cone $T(K_C; Q_\mu)$ to K_C at Q_μ in $C^0(\mathbb{R}^s)$. In [35], [36] this property is also called Hadamard directional differentiability tangentially to K_C.

PROPOSITION 3.1. *Let $Q_\mu \in K_C$, and assume that $\psi(Q_\mu)$ is nonempty and bounded. Then φ is Hadamard directionally differentiable at Q_μ on K_C, and it holds for all $v \in T(K_C; Q_\mu)$ that*

$$\varphi'(Q_\mu; v) = \min\{v(Ax) : x \in \psi(Q_\mu)\}.$$

If, in addition, Q_μ is strictly convex on some open convex neighborhood of $A\psi(Q_\mu)$, we have

$$\varphi'(Q_\mu; v) = v(\bar{y}), \quad \text{where} \quad \{\bar{y}\} = A\psi(Q_\mu).$$

Proof. Arguing similarly as in the proof of Proposition 2.1 in [26] there exists a neighborhood $\mathcal{N}$ of Q_μ in $C^0(\mathbb{R}^s)$ such that $\psi(Q)$ is nonempty for all $Q \in K_C \cap \mathcal{N}$. Let (t_n) and (v_n) be sequences such that $t_n \to 0+$, $v_n \to v$ in $C^0(\mathbb{R}^s)$, and $Q_\mu + t_n v_n$ belongs to K_C for all $n \in \mathbb{N}$. Then $Q_\mu + t_n v_n \in K_C \cap \mathcal{N}$ for sufficiently large $n \in \mathbb{N}$. Let $x_n \in \psi(Q_\mu + t_n v_n)$ for those $n \in \mathbb{N}$. Since ψ is Berge upper-semicontinuous at Q_μ [26], the sequence (x_n) has an accumulation point $x \in \psi(Q_\mu)$, and we obtain

$$\begin{aligned}\limsup_{n\to\infty} & \frac{1}{t_n}(\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu)) \\ &\geq \limsup_{n\to\infty} \frac{1}{t_n}(g(x_n) + (Q_\mu + t_n v_n)(Ax_n) - g(x_n) - Q_\mu(Ax_n)) \geq v(Ax),\end{aligned}$$

where the last inequality follows from the uniform convergence of (v_n) to v on bounded subsets of $\mathbb{R}^s$. In order to show the reverse inequality for lim inf, let $x \in \psi(Q_\mu)$. Then

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{t_n}(\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu)) \\ \le \liminf_{n\to\infty} \frac{1}{t_n}(g(x) + (Q_\mu + t_n v_n)(Ax) - g(x) - Q_\mu(Ax)) = v(Ax).\end{aligned}$$

This completes the proof of the first part. The second part is an immediate conclusion, since $A\psi(Q_\mu)$ is a singleton whenever Q_μ is strictly convex on some of its open, convex neighborhoods. ∎

The preceding result can also be proved by using the methodology of Theorem 6.4.1 in [28]. There the compactness of the constraint set is assumed, and Gateaux directional differentiability of φ at Q_μ together with its Lipschitz continuity is shown. Here we prefer a direct two-sided argument, which will also be used in the subsequent second-order analysis of φ. Namely, we will first derive an upper bound for the second-order Hadamard directional derivative of φ at some $Q_\mu \in K_C$, where K_C is equipped with the $C^{0,1}$ topology. Second, we identify conditions implying that the upper bound coincides with the Gateaux directional derivative of φ at Q_μ for all directions taken from $T^r(K_C; Q_\mu)$.

LEMMA 3.2. *Let $y \in \mathbb{R}^s$, $Q_\mu \in K_C$, $t_n \to 0+$, (Q_n) be a sequence in K_C such that $v_n := \frac{1}{t_n}(Q_n - Q_\mu) \to v$ in $C^{0,1}(\mathbb{R}^s)$, and let (ξ_n) be a sequence converging to ξ in $\mathbb{R}^s$. Then we have $\limsup_{n\to\infty} \frac{1}{t_n}(v_n(y + t_n\xi_n) - v_n(y)) \le \max_{\zeta\in\partial v(y)}\langle\zeta,\xi\rangle$.*

Proof. Each function v_n is locally Lipschitzian on $\mathbb{R}^s$ and, hence, Lebourg's mean value theorem for Clarke's subdifferential [9] implies the existence of elements $\tilde{y}_n$ belonging to the segments $[y, y + t_n\xi_n]$ such that

$$\frac{1}{t_n}(v_n(y + t_n\xi_n) - v_n(y)) \in \{\langle\zeta,\xi_n\rangle : \zeta \in \partial v_n(\tilde{y}_n)\}.$$

The convergence $v_n \to v$ in $C^{0,1}(\mathbb{R}^s)$ implies that

$$\sup\{\|\zeta\| : \zeta \in \partial(v_n - v)(y), \|y\| \le r\} \underset{n\to\infty}{\longrightarrow} 0$$

holds for any $r > 0$. This yields

$$d_H(\partial v_n(\tilde{y}_n), \partial v(\tilde{y}_n)) \le \sup\{\|\zeta\| : \zeta \in \partial(v_n - v)(\tilde{y}_n)\} \underset{n\to\infty}{\longrightarrow} 0.$$

Here d_H denotes the Hausdorff distance, and the inequality is a consequence of general properties of the subdifferential (cf. Lemma 2.1 in [27]). Hence, there exist elements $\tilde{\zeta}_n$ belonging to $\partial v(\tilde{y}_n)$ such that

$$\frac{1}{t_n}(v_n(y + t_n\xi_n) - v_n(y)) \le \|\xi_n\| d_H(\partial v_n(\tilde{y}_n), \partial v(\tilde{y}_n)) + \langle\tilde{\zeta}_n, \xi_n\rangle$$

and, for some $\tilde{\zeta} \in \partial v(y)$,

$$\limsup_{n\to\infty} \frac{1}{t_n}(v_n(y + t_n\xi_n) - v_n(y)) \le \limsup_{n\to\infty}\langle\tilde{\zeta}_n, \xi_n\rangle = \langle\tilde{\zeta},\xi\rangle \le \max_{\zeta\in\partial v(y)}\langle\zeta,\xi\rangle,$$

where the upper semicontinuity of $\partial v(\cdot)$ is used. This completes the proof. ∎

PROPOSITION 3.3. *Let $Q_\mu \in K_C$, and assume that $\psi(Q_\mu)$ is nonempty and bounded. Let g be twice continuously differentiable, and let Q_μ be strictly convex on* *some open convex neighborhood of* $A\psi(Q_\mu)$ *and twice continuously differentiable at* $\bar{y}$*, where* $\{\bar{y}\} = A\psi(Q_\mu)$*. Let* $\bar{x} \in \psi(Q_\mu)$*,* $t_n \to 0+$*, and* (Q_n) *be a sequence in* K_C *such that* $v_n := \frac{1}{t_n}(Q_n - Q_\mu) \to v$ *in* $C^{0,1}(\mathbb{R}^s)$*. Then*

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{t_n^2}&(\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu) - t_n\varphi'(Q_\mu; v_n))\\ &\le \inf\{\langle \nabla g(\bar{x}), z\rangle + \langle \nabla Q_\mu(\bar{y}), Az\rangle + \frac{1}{2}\langle \nabla^2 g(\bar{x}), \xi, \xi\rangle\\ &+\frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle + \max_{\zeta\in\partial v(\bar{y})}\langle \zeta, A\xi\rangle : \xi \in S(\bar{x}), z \in T^2(C;\bar{x},\xi)\},\end{aligned}$$

where $S(\bar{x}) := \{\xi \in T(C;\bar{x}) : \langle \nabla g(\bar{x}), \xi\rangle + \langle \nabla Q_\mu(\bar{y}), A\xi\rangle = 0\}$*,* $T(C;\bar{x})$ *is the tangent cone to* C *at* $\bar{x}$*, and* $T^2(C;\bar{x},\xi)$ *is the second-order tangent set to* C *at* $\bar{x}$ *in direction* ξ*.*

Proof. Let $\xi \in S(\bar{x})$ and $z \in T^2(C;\bar{x},\xi)$. Then there exists a sequence (z_n) such that $z_n \to z$ and $\bar{x} + t_n\xi + t_n^2 z_n \in C$ for all $n \in \mathbb{N}$. Using Proposition 3.1, this allows for the following estimate:

$$\begin{aligned}\varphi(Q_\mu + t_n v_n) &- \varphi(Q_\mu) - t_n\varphi'(Q_\mu; v_n)\\ &\le g(\bar{x} + t_n\xi + t_n^2 z_n) + Q_\mu(A(\bar{x} + t_n\xi + t_n^2 z_n)) + t_n v_n(A(\bar{x} + t_n\xi + t_n^2 z_n))\\ &\quad -g(\bar{x}) - Q_\mu(A\bar{x}) - t_n v_n(A\bar{x})\\ &= [g(\bar{x} + t_n\xi + t_n^2 z_n) - g(\bar{x}) - t_n\langle \nabla g(\bar{x}), \xi\rangle]\\ &\quad +[Q_\mu(A(\bar{x} + t_n\xi + t_n^2 z_n)) - Q_\mu(A\bar{x}) - t_n\langle \nabla Q_\mu(A\bar{x}), A\xi\rangle]\\ &\quad +t_n[v_n(A(\bar{x} + t_n\xi + t_n^2 z_n)) - v_n(A\bar{x})].\end{aligned}$$

After dividing by t_n^2 and using Lemma 3.2, the limes superior as $n \to \infty$ of the right-hand side can be bounded above by

$$\langle \nabla g(\bar{x}), z\rangle + \frac{1}{2}\langle \nabla^2 g(\bar{x})\xi, \xi\rangle + \langle \nabla Q_\mu(A\bar{x}), Az\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(A\bar{x})A\xi, A\xi\rangle + \max_{\zeta\in\partial v(A\bar{x})}\langle \zeta, A\xi\rangle.$$

Taking the infimum on the right-hand side yields the assertion. ∎

We notice that the upper second-order Hadamard directional derivative $\limsup_{n\to\infty} \frac{1}{t_n^2}(\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu) - t_n\varphi'(Q_\mu; v_n))$ is nonpositive, since φ is concave on K_C and, hence, the inequality $\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu) = \varphi(Q_n) - \varphi(Q_\mu) \le \varphi'(Q_\mu; Q_n - Q_\mu) = t_n\varphi'(Q_\mu; v_n)$ is valid. We also note that the upper bound is nonpositive, since $(0,0)$ belongs to $S(\bar{x}) \times T^2(C;\bar{x},0) = S(\bar{x}) \times T(C;\bar{x})$. Next we consider particular perturbations Q_n of Q_μ, namely, $Q_n := Q_\mu + \lambda t_n(Q - Q_\mu)$ for some $Q \in K_C$, $\lambda > 0$, and sufficiently large $n \in \mathbb{N}$. Then $v_n = \lambda(Q - Q_\mu) \in T^r(K_C; Q_\mu)$. The next result provides conditions implying that the second-order (Gateaux) directional derivative exists and coincides with the upper bound of the previous proposition. To state the result we need the notion of *second-order regularity* (cf. [6]). The constraint set C is called second-order regular at $\bar{x} \in C$ if for any direction $\xi \in T(C;\bar{x})$ and any sequence $x_n \in C$ of the form $x_n = \bar{x} + t_n\xi + t_n^2 r_n$ where, $t_n \to 0+$ and r_n being a sequence in $\mathbb{R}^m$ satisfying $t_n r_n \to 0$, it holds that $\lim_{n\to\infty} d(r_n, T^2(C;\bar{x},\xi)) = 0$. For example, C is second-order regular at $\bar{x} \in C$ if $0 \in T^2(C;\bar{x},\xi)$ for every $\xi \in T(C;\bar{x})$ (cf. [6]). In particular, a polyhedral (convex) set C is second-order regular at any $\bar{x} \in C$.

THEOREM 3.4. *Let* $Q_\mu \in K_C$*, and assume that* $\psi(Q_\mu)$ *is nonempty and bounded. Let* g *be twice continuously differentiable, and let* Q_μ *be strictly convex on some* *open convex neighborhood of* $A\psi(Q_\mu)$ *and twice continuously differentiable at* $\bar{y}$*, where* $\{\bar{y}\} = A\psi(Q_\mu)$*. Let* $\bar{x} \in \psi(Q_\mu)$*,* $v \in T^r(K_C; Q_\mu)$*, and assume that*

(i) $d(\bar{x}, \psi(Q_\mu + tv)) = O(t)$ *for small* $t > 0$*, and*

(ii) C *is second-order regular at* $\bar{x}$*.*

Then the second-order Gateaux directional derivative of φ *at* Q_μ *in direction* v *exists, and it holds that*

$$\begin{aligned}\varphi''(Q_\mu; v) &= \lim_{t\to 0+} \frac{1}{t^2}(\varphi(Q_\mu + t_n v) - \varphi(Q_\mu) - t\varphi'(Q_\mu; v))\\ &= \inf\left\{\frac{1}{2}\langle\nabla^2 g(\bar{x})\xi, \xi\rangle + \frac{1}{2}\langle\nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle + v'(\bar{y}; A\xi) + b(\xi) : \xi \in S(\bar{x})\right\},\end{aligned} \tag{3.1}$$

where $b(\xi) = \inf\{\langle\nabla g(\bar{x}), z\rangle + \langle\nabla Q_\mu(\bar{y}), Az\rangle : z \in T^2(C; \bar{x}, \xi)\}$ *is nonnegative and convex on* $S(\bar{x})$*. Moreover, the infimum in* (3.1) *is attained at some* $\bar{\xi} \in S(\bar{x})$ *having the property that* $\varphi''(Q_\mu; v) = \frac{1}{2}v'(\bar{y}; A\bar{\xi}) + \frac{1}{2}b(\bar{\xi})$*.*

(Here $S(\bar{x})$ and $T^2(C; \bar{x}, \xi)$ are defined as in the previous result, $v'(\bar{y}; \eta)$ is the directional derivative of v at $\bar{y}$ in direction η, and $O(t)$ denotes a real quantity such that $\frac{1}{t}|O(t)|$ is bounded as $t \to 0+$.)

Proof. (i) implies that there exist constants $L > 0$, $\delta > 0$, and elements $x(t) \in \psi(Q_\mu + tv)$ such that $\|x(t) - \bar{x}\| \le Lt$ for all $t \in (0, \delta)$. Now take a sequence (t_n) tending to 0+ in such a way that

$$\begin{aligned}&\liminf_{t\to 0+} \frac{1}{t^2}(\varphi(Q_\mu + tv) - \varphi(Q_\mu) - t\varphi'(Q_\mu; v))\\ &\quad= \lim_{n\to\infty} \frac{1}{t_n^2}(\varphi(Q_\mu + t_n v) - \varphi(Q_\mu) - t_n\varphi'(Q_\mu; v))\end{aligned}$$

and that $\xi_n := \frac{1}{t_n}(x(t_n) - \bar{x}) \underset{n\to\infty}{\longrightarrow} \bar{\xi}$. The latter is possible since $\|\frac{1}{t_n}(x(t_n) - \bar{x})\| \le L$ for $n \in \mathbb{N}$ sufficiently large. Then $\bar{\xi} \in T(C; \bar{x})$ and Proposition 3.1 yields

$$\begin{aligned}v(A\bar{x}) = \varphi'(Q_\mu; v) &= \lim_{n\to\infty} \frac{1}{t_n}(\varphi(Q_\mu + t_n v) - \varphi(Q_\mu))\\ &= \lim_{n\to\infty} \frac{1}{t_n}(g(\bar{x} + t_n\xi_n) + (Q_\mu + t_n v)(A(\bar{x} + t_n\xi_n)) - g(\bar{x}) - Q_\mu(A\bar{x}))\\ &= \langle\nabla g(\bar{x}), \bar{\xi}\rangle + \langle\nabla Q_\mu(A\bar{x}), A\bar{\xi}\rangle + v(A\bar{x}).\end{aligned}$$

This implies $\bar{\xi} \in S(\bar{x})$. We put $r_n = \frac{1}{t_n}(\xi_n - \bar{\xi})$ and $x_n = x(t_n) = \bar{x} + t_n\bar{\xi} + t_n^2 r_n$. By expanding g and Q_μ and using Proposition 3.1, we obtain

$$\begin{aligned}&\varphi(Q_\mu + t_n v) - \varphi(Q_\mu) - t_n\varphi'(Q_\mu; v)\\ &\quad= g(x_n) + Q_\mu(Ax_n) + t_n v(Ax_n) - g(\bar{x}) - Q_\mu(A\bar{x}) - t_n v(A\bar{x})\\ &\quad= \langle\nabla g(\bar{x}), x_n - \bar{x}\rangle + \frac{1}{2}\langle\nabla^2 g(\bar{x})(x_n - \bar{x}), x_n - \bar{x}\rangle\\ &\qquad+ \langle\nabla Q_\mu(A\bar{x}), Ax_n - \bar{x})\rangle + \frac{1}{2}\langle\nabla^2 Q_\mu(A\bar{x})(A(x_n - \bar{x})), A(x_n - \bar{x})\rangle\\ &\qquad+ t_n(v(Ax_n) - v(A\bar{x})) + o(\|x_n - \bar{x}\|^2)\\ &\quad= t_n^2(\langle\nabla g(\bar{x}), r_n\rangle + \frac{1}{2}\langle\nabla^2 g(\bar{x})\bar{\xi}, \bar{\xi}\rangle) + t_n^2(\langle\nabla Q_\mu(A\bar{x}), Ar_n)\rangle\\ &\qquad+ \frac{1}{2}\langle\nabla^2 Q_\mu(A\bar{x})A\bar{\xi}, A\bar{\xi}\rangle) + t_n(v(Ax_n) - v(A\bar{x})) + o(t_n^2).\end{aligned}$$

Here we used that $o(\|x_n - \bar{x}\|^2) = o(t_n^2)$, where $o(t^k)$ denotes a real quantity having the property $\frac{1}{t^k}o(t) \to 0$ as $t \to 0+$ ($k \in \mathbb{N}$).

Since C is second-order regular at $\bar{x}$, there exists a sequence $z_n \in T^2(C; \bar{x}, \bar{\xi})$ such that $\lim_{n\to\infty} \|r_n - z_n\| = 0$, and we get from the previous chain of equalities

$$
\begin{aligned}
&\frac{1}{t_n^2}(\varphi(Q_\mu + t_n v) - \varphi(Q_\mu) - t_n \varphi'(Q_\mu; v)) \\
&\quad = \langle \nabla g(\bar{x}), z_n \rangle + \langle \nabla Q_\mu(\bar{y}), A z_n \rangle + \frac{1}{2}\langle \nabla^2 g(\bar{x})\bar{\xi}, \bar{\xi}\rangle \\
&\qquad + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y}) A\bar{\xi}, A\bar{\xi}\rangle + \frac{1}{t_n}(v(\bar{y} + t_n A\xi_n) - v(\bar{y})) + o(1) \\
&\quad \geq b(\bar{\xi}) + \frac{1}{2}\langle \nabla^2 g(\bar{x})\bar{\xi}, \bar{\xi}\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y}) A\bar{\xi}, A\bar{\xi}\rangle + \frac{1}{t_n}(v(\bar{y} + t_n A\xi_n) - v(\bar{y})) + o(1).
\end{aligned}
$$

Using the fact that v is Hadamard directionally differentiable and Clarke regular [9], i.e., $v'(\bar{y}; \eta) = \max_{\zeta \in \partial v(\bar{y})} \langle \zeta, \eta \rangle$, we obtain

$$
\begin{aligned}
&\liminf_{t \to 0+} \frac{1}{t^2}(\varphi(Q_\mu + tv) - \varphi(Q_\mu) - t\varphi'(Q_\mu; v)) \\
&\quad \geq \frac{1}{2}\langle \nabla^2 g(\bar{x})\bar{\xi}, \bar{\xi}\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y}) A\bar{\xi}, A\bar{\xi}\rangle + v'(\bar{y}; A\bar{\xi}) + b(\bar{\xi}) \\
&\quad \geq \inf\left\{ \frac{1}{2}\langle \nabla^2 g(\bar{x})\xi, \xi\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y}) A\xi, A\xi\rangle + v'(\bar{y}; A\xi) + b(\xi) : \xi \in S(\bar{x}) \right\}.
\end{aligned}
$$

Proposition 3.3 implies that this lower bound for $\liminf_{t\to 0+}$ is also an upper bound for $\limsup_{t\to 0+}$. Hence, the limit $\lim_{t\to 0+} \frac{1}{t^2}(\varphi(Q_\mu + tv) - \varphi(Q_\mu) - t\varphi'(Q_\mu; v))$ exists and is equal to the infimum subject to $\xi \in S(\bar{x})$. Moreover, this infimum is attained at $\bar{\xi} \in S(\bar{x})$.

The nonnegativity of b is due to the fact that the necessary optimality condition for (1.1) at $\bar{x}$ yields

$$\langle \nabla g(\bar{x}), z \rangle + \langle \nabla Q_\mu(\bar{y}), Az \rangle \geq 0 \quad \text{for all} \quad z \in T^2(C; \bar{x}, \xi), \xi \in S(\bar{x}).$$

The convexity of b follows from the property $T^2(C; \bar{x}, \lambda\xi + (1-\lambda)\tilde{\xi}) \supseteq \lambda T^2(C; \bar{x}, \xi) + (1-\lambda)T^2(C; \bar{x}, \tilde{\xi})$ for all $\xi, \tilde{\xi} \in T(C; \bar{x})$, and $\lambda \in [0, 1]$.

For the remainder of the proof we put $a(\xi) := v'(\bar{y}; A\xi)$ and

$$B(\xi) := \frac{1}{2}\langle \nabla^2 g(\bar{x})\xi, \xi\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y}) A\xi, A\xi\rangle \quad \text{for all} \quad \xi \in \mathbb{R}^m.$$

Since $S(\bar{x})$ is a (convex) cone, we have $S(\bar{x}) = \lambda S(\bar{x})$ for any $\lambda > 0$. Moreover, it holds that $T^2(C; \bar{x}, \lambda\bar{\xi}) = \lambda T^2(C; \bar{x}, \bar{\xi})$ and thus that $b(\lambda\bar{\xi}) = \lambda b(\bar{\xi})$ for any $\lambda > 0$. Hence, we obtain

$$
\begin{aligned}
0 \leq f(\lambda) &:= B(\lambda\bar{\xi}) + a(\lambda\bar{\xi}) + b(\lambda\bar{\xi}) - B(\bar{\xi}) - a(\bar{\xi}) - b(\bar{\xi}) \\
&= \lambda^2 B(\bar{\xi}) + (\lambda - 1)(a(\bar{\xi}) + b(\bar{\xi})) - B(\bar{\xi}) \quad \text{for all} \quad \lambda > 0.
\end{aligned}
$$

In the case of $B(\bar{\xi}) > 0$, the quadratic function f vanishes at $\lambda = 1$ with the property $f'(1) = 2B(\bar{\xi}) + a(\bar{\xi}) + b(\bar{\xi}) = 0$, and the final assertion is shown. If $B(\bar{\xi}) = 0$, the fact that $0 \leq f(\lambda) = (\lambda - 1)(a(\bar{\xi}) + b(\bar{\xi}))$ holds for any $\lambda > 0$ implies $a(\bar{\xi}) + b(\bar{\xi}) = 0$. Thus $\varphi''(Q_\mu; v) = 0 = \frac{1}{2}(a(\bar{\xi}) + b(\bar{\xi}))$, and the proof is complete. ∎

The theorem extends our earlier work in [12], where essentially polyhedrality of C is assumed. Compared to [12], the additional term $b(.)$ enters the formula for $\varphi''(Q_\mu; v)$. The convex function $b(.)$ reflects second-order properties of the constraint set C and vanishes if C is polyhedral. Next we state a more handy criterion implying that $\varphi''(Q_\mu; v)$ exists for any direction $v \in T^r(K_C; Q_\mu)$.

COROLLARY 3.5. *Let $Q_\mu \in K_C$, and assume that $\psi(Q_\mu)$ is nonempty and bounded. Let g be twice continuously differentiable, and let Q_μ be strongly convex on some open convex neighborhood of $A\psi(Q_\mu)$ and twice continuously differentiable at $\bar{y}$, where $\{\bar{y}\} = A\psi(Q_\mu)$. Let $\bar{x} \in \psi(Q_\mu)$ and assume that*

(i)$'$ *there exist a constant $L > 0$ and a neighborhood U of $\bar{y}$ such that $d(\bar{x}, \sigma(y)) \le L\|\bar{y} - y\|$ for all $y \in A(C) \cap U$, where $\sigma(y) := \operatorname{argmin}\{g(x) : x \in C, Ax = y\}$, $y \in A(C)$, and*

(ii) *C is second-order regular at $\bar{x}$.*

Then the second-order Gateaux directional derivative of φ at Q_μ exists for any direction $v \in T^r(K_C; Q_\mu)$, and the formula for $\varphi''(Q_\mu; v)$ in Theorem 3.4 holds true. Moreover, conditions (i)$'$ and (ii) are satisfied for any $\bar{x} \in \psi(Q_\mu)$ if C is polyhedral and g is linear or (convex) quadratic.

Proof. Let $v \in T^r(K_C; Q_\mu)$. Theorem 2.3 then says that there exist constants $\hat{L} > 0$, $\delta > 0$, and $r > 0$ such that

$$d(\bar{x}, \psi(Q_\mu + tv)) \le \hat{L}\|v\|_{L,r} t \quad \text{whenever} \quad \|v\|_{L,r} t < \delta.$$

Hence, the strong convexity of Q_μ and condition (i)$'$ imply that condition (i) of the previous theorem is satisfied and that the first part of the assertion is shown. If C is polyhedral and g is linear or (convex) quadratic, (ii) is satisfied and Proposition 2.2 implies (i)$'$ to hold for any $\bar{x} \in \psi(Q_\mu) = \sigma(\bar{y})$. ∎

Let us consider two illustrative examples to provide some insight into the benefit and limits of the previous results.

Example 3.6. We revisit Example 2.6 and know that the general assumptions of Corollary 3.5 and condition (i)$'$ are satisfied for $\bar{x} = (0, 0)$. Furthermore, it holds that $T(C; \bar{x}) = \mathbb{R}_+ \times \mathbb{R}$ and

$$T^2(C; \bar{x}, \xi) = \begin{cases} \mathbb{R}^2, & \xi_1 > 0, \\ \{x_1 \in \mathbb{R} : x_1 \ge \xi_2^2\} \times \mathbb{R}, & \xi_1 = 0, \end{cases} \quad \text{for any} \quad \xi \in T(C; \bar{x}).$$

Moreover, C is second-order regular at $\bar{x}$ (as can be seen from Proposition 4.1 in [6]) and it holds that $b(\xi) = 0$ for all $\xi \in \mathbb{R}^2$. Hence, Corollary 3.5 implies that $\varphi''(Q_\mu; v)$ exists for any $v \in T^r(K_C; Q_\mu)$ and that $\varphi''(Q_\mu; v) = \frac{1}{2}v'(0, \bar{\xi}_1)$, where $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2) \in \operatorname{argmin}\{\xi_1^2 + v'(0, \xi_1) : (\xi_1, \xi_2) \in \mathbb{R}_+ \times \mathbb{R}\}$.

Example 3.7. Here we revisit Example 2.9 and have

$$Q_\mu(y) = \frac{1}{3}|y|^3 + \frac{2}{3} \quad \text{for all} \ \ |y| \le 1, \ \text{and} \ \ \psi(Q_\mu) = \{0\}, \, \varphi(Q_\mu) = \frac{2}{3}.$$

For the function $v(x) = -x$ ($x \in \mathbb{R}$) and $t \in [0, 1)$ we obtain

$$\varphi(Q_\mu + tv) = \inf\{Q_\mu(x) - tx : x \in \mathbb{R}\} = \frac{2}{3}(1 - t^{\frac{3}{2}}),$$
$$\psi(Q_\mu + tv) = \operatorname{argmin}\{Q_\mu(x) - tx : x \in \mathbb{R}\} = \{\sqrt{t}\}.$$

Then $\varphi'(Q_\mu; v) = 0$ and $\frac{1}{t^2}(\varphi(Q_\mu + tv) - \varphi(Q_\mu) - \varphi'(Q_\mu; v)) = -\frac{2}{3}t^{-\frac{1}{2}}$. Hence, φ has no second-order directional derivative at Q_μ in direction v. Note that there is no neighborhood of $\bar{x} = 0$ where Q_μ is strongly convex.

Finally, we aim at showing that φ is even second-order Hadamard directionally differentiable at Q_μ when equipping K_C with a suitable topology. To this end we need a certain counterpart of Lemma 3.2 for the corresponding limes inferior. Since such a bound does not exist for nonsmooth functions, it is a natural idea to consider the space $C^1(\mathbb{R}^s)$, to restrict φ to the subset $K_C \cap C^1$, and to equip $K_C \cap C^1$ with the C^1 topology. Then we are able to show that the assumptions of Corollary 3.5 imply the second-order Hadamard directional differentiability of φ at Q_μ.

THEOREM 3.8. *Let $Q_\mu \in K_C \cap C^1$, and assume that $\psi(Q_\mu)$ is nonempty and bounded. Let g be twice continuously differentiable, and let Q_μ be strongly convex on some open convex neighborhood of $A\psi(Q_\mu)$ and twice continuously differentiable at $\bar{y}$, where $\{\bar{y}\} = A\psi(Q_\mu)$. Let $\bar{x} \in \psi(Q_\mu)$ and assume the conditions* (i)$'$ *and* (ii) *of Corollary* 3.5 *to hold. Then the second order Hadamard directional derivative of φ at Q_μ exists in any direction v belonging to the tangent cone $T(K_C \cap C^1; Q_\mu)$ in $C^1(\mathbb{R}^s)$, i.e., for any such v, and all sequences $t_n \to 0+$ and (Q_n) in K_C such that $v_n := \frac{1}{t_n}(Q_n - Q_\mu) \to v$ in $C^1(\mathbb{R}^s)$ the limit*

$$\varphi''(Q_\mu; v) = \lim_{n\to\infty} \frac{1}{t_n^2}(\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu) - t_n \varphi'(Q_\mu; v_n))$$

exists, and it holds that

$$\varphi''(Q_\mu; v) = \inf\left\{\frac{1}{2}\langle \nabla^2 g(\bar{x})\xi, \xi\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle + \langle \nabla v(\bar{y}), A\xi\rangle + b(\xi) : \xi \in S(\bar{x})\right\}.$$

Proof. Let $v \in T(K_C \cap C^1; Q_\mu)$, $t_n \to 0+$, and (Q_n) be a sequence in K_C such that $v_n = \frac{1}{t_n}(Q_n - Q_\mu) \to v$ in $C^1(\mathbb{R}^s)$. Condition (i)$'$ together with Theorem 2.3 then imply that there exist constants $L > 0$, $r > 0$, $n_0 \in \mathbb{N}$, and elements $x_n \in \psi(Q_\mu + t_n v_n)$ such that

$$\|x_n - \bar{x}\| \le L t_n \|v_n\|_{L,r} \quad \text{for all} \quad n \in \mathbb{N}, n \ge n_0.$$

Since the sequence (v_n) converges in $C^1(\mathbb{R}^s)$, the norms $\|v_n\|_{L,r}$ are uniformly bounded and we have $\|x_n - \bar{x}\| = O(t_n)$. As in the proof of Theorem 3.4 we select a subsequence of (t_n), which is again denoted by (t_n), tending to $0+$ such that $\xi_n := \frac{1}{t_n}(x_n - \bar{x}) \underset{n\to\infty}{\longrightarrow} \bar{\xi} \in S(\bar{x})$. Analogously, we obtain for sufficiently large n:

$$\begin{aligned} &\frac{1}{t_n^2}(\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu) - t_n \varphi'(Q_\mu; v_n)) \\ &\quad \ge b(\bar{\xi}) + \frac{1}{2}\langle \nabla^2 g(\bar{x})\bar{\xi}, \bar{\xi}\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\bar{\xi}, A\bar{\xi}\rangle + \frac{1}{t_n}(v_n(\bar{y} + t_n A\xi_n) - v_n(\bar{y})) + o(1). \end{aligned}$$

Using the mean value theorem for v_n we may continue with some $\bar{y}_n \in [\bar{y}, \bar{y} + t_n A\xi_n]$ as follows:

$$\begin{aligned} &\frac{1}{t_n^2}(\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu) - t_n \varphi'(Q_\mu; v_n)) \\ &\ge \frac{1}{2}\langle \nabla^2 g(\bar{x})\bar{\xi}, \bar{\xi}\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\bar{\xi}, A\bar{\xi}\rangle + \langle \nabla v_n(\bar{y}_n), A\xi_n\rangle + b(\bar{\xi}) + o(1). \end{aligned}$$

Arguing as in the proof of Theorem 3.4 and using $v_n \to v$ in $C^1(\mathbb{R}^s)$, we arrive at the estimate

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{t_n^2}(\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu) - t_n \varphi'(Q_\mu; v_n)) \\ &\ge \frac{1}{2}\langle \nabla^2 g(\bar{x})\bar{\xi}, \bar{\xi}\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\bar{\xi}, A\bar{\xi}\rangle + \langle \nabla v(\bar{y}), A\bar{\xi}\rangle + b(\bar{\xi}) \end{aligned}$$

and, using Proposition 3.3, we arrive at the desired result. ∎

Let us finally note that all minimization problems appearing as bounds or formulas for second-order directional derivatives represent convex programs. Those in the results Theorem 3.4, Corollary 3.5, and Theorem 3.8 have convex cone constraints, which are polyhedral if C is polyhedral. Moreover, the solution sets of the convex minimization problems in Theorem 3.4, Corollary 3.5, and Theorem 3.8 are nonempty. Indeed, we show next that these solution sets represent certain derivatives of the set-valued mapping ψ at the pair $(Q_\mu, \bar{x})$.

4. Differentiability of solution sets. It is well known that second-order differentiability properties of optimal values in perturbed optimization are intrinsic for establishing the differentiability of solutions (see, e.g., [8]). We also pursue this approach and derive conditions implying directional differentiability properties of the solution set mapping by exploiting the results of the previous section. Our first results in this direction concern Gateaux directional differentiability and complement Theorem 3.4 and its corollary.

THEOREM 4.1. *Assume that the general conditions on g, Q_μ, and C of Theorem* 3.4 *are satisfied. Let $\bar{x} \in \psi(Q_\mu)$, $v \in T^r(K_C; Q_\mu)$, and suppose the conditions* (i) *and* (ii) *of Theorem* 3.4 *to be satisfied. In addition, assume that*

(iii) *there exist a neighborhood V of $\psi(Q_\mu)$ and constants $c > 0$, $\delta > 0$ such that the uniform growth condition*

$$g(x) + (Q_\mu + tv)(Ax) \geq \varphi(Q_\mu + tv) + cd(x, \psi(Q_\mu + tv))^2$$

for all $x \in C \cap V$ and $t \in [0, \delta)$ is satisfied.

Then the Gateaux directional derivative of ψ at the pair $(Q_\mu, \bar{x})$ into direction v exists, and it holds that

$$\begin{aligned}\psi'(Q_\mu, \bar{x}; v) &= \lim_{t\to 0+} \frac{1}{t}(\psi(Q_\mu + tv) - \bar{x}) \\ &= \operatorname{argmin}\left\{\frac{1}{2}\langle \nabla^2 g(\bar{x})\xi, \xi\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle + v'(\bar{y}; A\xi) + b(\xi) : \xi \in S(\bar{x})\right\}.\end{aligned}$$

Proof. Let $M(\bar{x}; v)$ denote the solution set in the assertion. First we show that $\limsup_{t\to 0+} \frac{1}{t}(\psi(Q_\mu + tv) - \bar{x}) \subseteq M(\bar{x}; v)$.

Let $\xi \in \limsup_{t\to 0+} \frac{1}{t}(\psi(Q_\mu + tv) - \bar{x})$. Then there exists a sequence (t_n, ξ_n) converging to $(0+, \xi)$ such that $\xi_n \in \frac{1}{t_n}(\psi(Q_\mu + t_n v) - \bar{x})$ and, thus, $\bar{x} + t_n\xi_n \in \psi(Q_\mu + t_n v)$ for all $n \in \mathbb{N}$. Analogously to the proof of Theorem 3.4 we show that ξ belongs to $S(\bar{x})$ and that $\varphi''(Q_\mu; v) = \frac{1}{2}\langle \nabla^2 g(\bar{x})\xi, \xi\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle + v'(\bar{y}; A\xi) + b(\xi)$. Hence $\xi \in M(\bar{x}; v)$.

In the second step we demonstrate that

$$M(\bar{x}; v) \subseteq \liminf_{t\to 0+} \frac{1}{t}(\psi(Q_\mu + tv) - \bar{x}),$$

or, equivalently, that it holds for any $\xi \in M(\bar{x}, v)$ that

$$\lim_{t\to 0} \frac{1}{t}\, d(\bar{x} + t\xi, \psi(Q_\mu + tv)) = 0.$$

Let $\xi \in M(\bar{x}; v)$ and (t_n) be a sequence with $t_n \to 0+$. We have to show that $\lim_{n\to\infty} \frac{1}{t_n}\, d(\bar{x} + t_n\xi, \psi(Q_\mu + t_n v)) = 0$.

Let $\varepsilon > 0$ be given, and let $z \in T^2(C;\bar{x},\xi)$ be such that $\langle \nabla g(\bar{x}), z\rangle + \langle \nabla Q_\mu(\bar{y}), Az\rangle \leq b(\xi)+\varepsilon$. Then there exists a sequence (z_n) converging to z with $x_n = \bar{x}+t_n\xi+t_n^2 z_n \in C$ for all $n \in \mathbb{N}$. Hence, it suffices to show that

$$\lim_{n\to\infty} \frac{1}{t_n} d(\bar{x}+t_n\xi+t_n^2 z_n, \psi(Q_\mu+t_n v)) = 0.$$

Condition (iii) implies the following estimate for all sufficiently large $n \in \mathbb{N}$:

$$\begin{aligned} &cd(\bar{x}+t_n\xi+t_n^2 z_n, \psi(Q_\mu+t_n v))^2 \\ &\qquad \leq g(\bar{x}+t_n\xi+t_n^2 z_n) + (Q_\mu+t_n v)(A(\bar{x}+t_n\xi+t_n^2 z_n)) - \varphi(Q_\mu+t_n v). \end{aligned}$$

By expanding g and Q_μ as in the proof of Theorem 3.4 and using the fact that ξ belongs to $S(\bar{x})$, we may express the right-hand side as

$$\begin{aligned} &t_n^2\langle \nabla g(\bar{x}), z_n\rangle + \frac{1}{2}t_n^2\langle \nabla^2 g(\bar{x})(\xi+t_n z_n), \xi+t_n z_n\rangle \\ &+t_n^2\langle \nabla Q_\mu(\bar{y}), Az_n\rangle + \frac{1}{2}t_n^2\langle \nabla^2 Q_\mu(\bar{y})(A(\xi+t_n z_n)), A(\xi+t_n z_n)\rangle \\ &-(\varphi(Q_\mu+t_n v) - \varphi(Q_\mu) - t_n\varphi'(Q_\mu;v)) \\ &+t_n(v(A(\bar{x}+t_n\xi+t_n^2 z_n)) - v(A\bar{x})) + o(t_n^2\|\xi+t_n z_n\|^2). \end{aligned}$$

After dividing by t_n^2 and taking the $\limsup_{n\to\infty}$, on both sides of the latter inequality, we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \frac{c}{t_n^2} d(\bar{x}+t_n\xi+t_n^2 z_n, \psi(Q_\mu+t_n v))^2 \\ &\leq \langle \nabla g(\bar{x}), z\rangle + \langle \nabla Q_\mu(\bar{y}), Az\rangle + \frac{1}{2}\langle \nabla^2 g(\bar{x})\xi, \xi\rangle \\ &+\frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle - \varphi''(Q_\mu;v) + v'(\bar{y};A\xi) \leq \varepsilon, \end{aligned}$$

where we made use of the choice of z, $\xi \in M(\bar{x};v)$, and Theorem 3.4. This completes the proof. □

Complementing Corollary 3.5, we provide a result on the directional differentiability of ψ at Q_μ into any direction $v \in T^r(K_C;Q_\mu)$.

THEOREM 4.2. *Assume that the general conditions on g, Q_μ, and C of Corollary 3.5 are satisfied. Let $\bar{x} \in \psi(Q_\mu)$, and assume the following.*

(i)″ *There exists a constant $L > 0$ such that*

$$d_H(\sigma(y),\sigma(\tilde{y})) \leq L\|y-\tilde{y}\| \quad \textit{for all } \; y,\tilde{y} \in A(C),$$

and, for each $r > 0$, there exists a constant $\eta(r) > 0$ such that

$$g(x) \geq \pi(Ax) + \eta(r)d(x,\sigma(Ax))^2 \qquad \textit{for all } \; x \in C\cap B(0,r),$$

where $\pi(y) = \inf\{g(x) : x \in C, Ax = y\}$ and

$$\sigma(y) = \operatorname{argmin}\{g(x) : x \in C, Ax = y\}, \; y \in A(C).$$

(ii) *C is second-order regular at $\bar{x}$.*

Then the Gateaux directional derivative $\psi'(Q_\mu, \bar{x}; v)$ *of* ψ *at the pair* $(Q_\mu, \bar{x})$ *exists for any direction* $v \in T^r(K_C; Q_\mu)$ *and satisfies the formula in Theorem* 4.1*. Moreover, conditions* (i)$''$ *and* (ii) *are satisfied if* C *is polyhedral and* g *is linear- or (convex-) quadratic.*

Proof. Let $v \in T^r(K_C; Q_\mu)$. Since Q_μ is strongly convex on some open convex neighborhood of $A\psi(Q_\mu)$, we infer from condition (i)$''$ and Theorem 2.7 that condition (iii) of Theorem 4.1 is satisfied. Moreover, condition (i)$''$ implies (i)$'$, and thus, Corollary 3.5 says that the second-order directional derivative $\varphi''(Q_\mu; v)$ exists. Hence, the first part of the assertion follows from the proof of the previous theorem. Condition (ii) is satisfied if C is polyhedral, and if, in addition, g is convex-quadratic, Proposition 2.2 implies condition (i)$''$ holds. □

We note that Example 3.7 shows that, in general, the directional differentiability property of ψ gets lost at pairs $(Q_\mu, \bar{x})$, $\bar{x} \in \psi(Q_\mu)$, where Q_μ is not strongly convex on some neighborhood of $A\psi(Q_\mu)$. Our next example demonstrates that Theorem 4.2 applies to situations where the solution set and its Gateaux directional derivatives are not singletons.

Example 4.3. We revisit the Examples 1.1 and 2.5 and observe that the assumptions of Theorem 4.2 are satisfied for any $\bar{x} \in \psi(Q_\mu)$. Hence, the Gateaux directional derivative $\psi'(Q_\mu, \bar{x}; v)$ exists at any pair $(Q_\mu, \bar{x})$, $\bar{x} \in \psi(Q_\mu)$ and any direction $v \in T^r(K_C; Q_\mu)$. Since it holds that $\nabla^2 g(\bar{x}) = 0$, $\langle \nabla g(\bar{x}), \xi \rangle + \langle \nabla Q_\mu(A\bar{x}), A\xi \rangle = 0$ for all $\xi \in \mathbb{R}^3$, and $\nabla^2 Q_\mu(A\bar{x}) = 2 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, it takes the form $\psi'(Q_\mu, \bar{x}; v) = \operatorname{argmin}\{\|A\xi\|^2 + v'(A\bar{x}; A\xi) : \xi \in T(C; \bar{x})\}$. Since the function $y \mapsto \|y\|^2 + v'(A\bar{x}; y)$ is strongly convex on $A(T(C; \bar{x}))$, it has a unique minimizer $\bar{y}(v) \in A(T(C; \bar{x}))$. Hence, there exists an element $\bar{\xi}(v) \in T(C; \bar{x})$ such that $A\bar{\xi}(v) = \bar{y}(v)$ and $\psi'(Q_\mu, \bar{x}; v) = (\bar{\xi}(v) + \ker A) \cap T(C; \bar{x})$. In particular, the Gateaux directional derivative $\psi'(Q_\mu, \bar{x}; .)$ is a set-valued mapping of the direction.

Remark 4.4. The approach we followed for deriving Gateaux directional differentiability of solution sets to (1.1) into directions $v \in T^r(K_C; Q_\mu)$ is based on lower and upper estimates for the optimal value function. Compared to the work in [5], [8], and [32], where this approach is developed and reviewed, we assume neither that the data of the perturbed problems $\min\{g(x) + Q(Ax) : x \in C\}$ is differentiable nor that solutions to (1.1) are unique. The (set-valued) Gateaux directional derivatives $\psi'(Q_\mu, \bar{x}; v)$ in the previous results are valid for the case $v = Q - Q_\mu$ with a general $Q \in K_C$. Hence, the results complement earlier work on contaminated distributions (e.g., [13], [14]). They apply to situations where Q is an expected recourse function with respect to a Dirac measure with unit mass placed at ω_*, i.e., $Q(y) = \tilde{Q}(\omega_* - y)$, and, hence, are relevant to study the influence of a specific scenario on changes of solution sets.

Another prominent approach to sensitivity analysis of optimization problems is based on the perturbation analysis of first-order necessary optimality conditions written as generalized equations (e.g., [17], [22], [24]). Applying this technique to study sensitivity of (1.1) requires C^1-properties of perturbed expected recourse functions Q. In the case of (1.1) and $Q \in C^1$, the parametric generalized equation reads

$$0 \in \nabla g(x) + A^T \nabla Q(Ax) + N_C(x),$$

where $N_C(x)$ denotes the normal cone to C at x and Q plays the role of a parameter. Relevant conditions in this context implying Lipschitz and differentiability properties of solutions at some $(Q_\mu, \bar{x})$ are the *strong regularity* of the generalized equation at parameter Q_μ [24], and the *subinvertibility* of the set-valued mapping

$F(x) = \nabla g(x) + A^T\nabla Q_\mu(Ax) + N_C(x)$ [17] together with the single-valuedness of the inverse of the contingent derivative of F at $(\bar{x}, 0)$ (cf., [2]), respectively. To see that both conditions are violated in general, we consider the linear case (i.e., g is linear and C is polyhedral). Then both conditions are equivalent if $Q_\mu \in C^2$ (Theorem 6.1 in [17]). The contingent derivative of F at $(\bar{x}, 0)$ has the form $DF(\bar{x}, 0)(u) = A^T\nabla^2 Q_\mu(A\bar{x})Au + DN_C(\bar{x}, -\nabla g(\bar{x}) - A^T\nabla Q_\mu(A\bar{x}))(u)$ (cf., Section 5.1 in [2]), where the contingent derivative DN_C is again a polyhedral multifunction. Since the first summand remains constant on translates of the null space of the matrix A, single-valuedness of the inverse of $DF(\bar{x}, 0)(u)$ fails to hold in general. This is essentially due to the same structural property, which leads to multiple solutions in Example 1.1 and to set-valued Gateaux directional derivatives in Example 4.3.

Finally, we turn to directional differentiability properties of ψ where the derivatives exist uniformly with respect to directions taken from compact sets of certain functional spaces. For our first result we consider the space $C^1(\mathbb{R}^s)$ and equip the set $K_C \cap C^1$ with the C^1-topology.

PROPOSITION 4.5. *Let $Q_\mu \in K_C \cap C^1$ and assume that the general conditions on g, Q_μ, and C in Proposition* 3.3 *are satisfied. In addition, we suppose condition* (ii) *of Theorem* 3.4 *to be satisfied. Let $\bar{x} \in \psi(Q_\mu)$, $t_n \to 0+$, and let (Q_n) be a sequence in K_C such that $v_n := \frac{1}{t_n}(Q_n - Q_\mu) \to v$ in $C^1(\mathbb{R}^s)$.*

Then the upper set limit of the sequence $(\frac{1}{t_n}(\psi(Q_\mu + t_n v_n) - \bar{x})$ of closed convex subsets in $\mathbb{R}^m$, i.e., $\limsup_{n\to\infty} \frac{1}{t_n}(\psi(Q_\mu + t_n v_n) - \bar{x}))$, is contained in the closed convex set

$$\operatorname{argmin}\left\{\frac{1}{2}\langle\nabla^2 g(\bar{x})\xi, \xi\rangle + \frac{1}{2}\langle\nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle + \langle\nabla v(\bar{y}), A\xi\rangle + b(\xi) : \xi \in S(\bar{x})\right\}.$$

Proof. Let $D_n := \frac{1}{t_n}(\psi(Q_\mu + t_n v_n) - \bar{x})$ for all $n \in \mathbb{N}$, and let $\bar{\xi}$ belong to the upper set limit $\limsup_{n\to\infty} D_n$. Then there exist a subsequence (again denoted by (D_n)) and elements $\xi_n \in D_n$ such that $\xi_n \to \bar{\xi}$. Since $\bar{x} + t_n\xi_n \in \psi(Q_\mu + t_n v_n) \subseteq C$, we have that $\bar{\xi} \in T(C; \bar{x})$, and as in the proof of Theorem 3.4, we deduce that $\bar{\xi} \in S(\bar{x})$. By expanding g and Q_μ as in the proof of Theorem 3.4 we obtain analogously

$$\begin{aligned}
&\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu) - t_n\varphi'(Q_\mu; v_n)\\
&\qquad = g(\bar{x} + t_n\xi_n) + Q_\mu(A(\bar{x} + t_n\xi_n)) + t_n v_n(A(\bar{x} + t_n\xi_n)) - g(\bar{x}) - Q_\mu(A\bar{x})\\
&\qquad\quad - t_n v_n(A\bar{x})\\
&\qquad \geq t_n^2 b(\bar{\xi}) + \frac{1}{2}t_n^2\langle\nabla^2 g(\bar{x})\bar{\xi}, \bar{\xi}\rangle + \frac{1}{2}t_n^2\langle\nabla^2 Q_\mu(A\bar{x})A\bar{\xi}, A\bar{\xi}\rangle\\
&\qquad\quad + t_n(v_n(A(\bar{x} + t_n\xi_n)) - v_n(A\bar{x})) + o(t_n^2).
\end{aligned}$$

After dividing by t_n^2 and taking the $\limsup_{n\to\infty}$ on both sides of the inequality, we obtain, as in the proof of Theorem 3.8,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{t_n^2}(\varphi(Q_\mu + t_n v_n) - \varphi(Q_\mu) - t_n\varphi'(Q_\mu; v_n))\\
&\qquad \geq \frac{1}{2}\langle\nabla^2 g(\bar{x})\bar{\xi}, \bar{\xi}\rangle + \frac{1}{2}\langle\nabla^2 Q_\mu(A\bar{x})A\bar{\xi}, A\bar{\xi}\rangle + \langle\nabla v(A\bar{x}), A\bar{\xi}\rangle + b(\bar{\xi}).
\end{aligned}$$

Hence, we may conclude from Proposition 3.3 that $\bar{\xi}$ belongs to the set

$$\operatorname{argmin}\left\{\frac{1}{2}\langle\nabla^2 \mathrm{g}(\bar{\mathrm{x}})\xi, \xi\rangle + \frac{1}{2}\langle\nabla^2 \mathrm{Q}_\mu(\bar{\mathrm{y}})\mathrm{A}\xi, \mathrm{A}\xi\rangle + \langle\nabla \mathrm{v}(\bar{\mathrm{y}}), \mathrm{A}\xi\rangle + \mathrm{b}(\xi) : \xi \in \mathrm{S}(\bar{\mathrm{x}})\right\},$$

and we are done. ∎

Remark 4.6. The upper limit of the sequence $(\frac{1}{t_n}(\psi(Q_\mu + t_n v_n) - \bar{x})$ in Proposition 4.5 is nonempty if the mapping $d(\bar{x}, \psi(\cdot))$ from K_C into the extended reals has the Lipschitzian property of Theorem 2.3 at Q_μ. Indeed, we may select $x_n \in \psi(Q_\mu + t_n v_n)$ for large $n \in \mathbb{N}$ such that for some constants $\hat{L} > 0$ and $r > 0$, $\|\bar{x} - x_n\| = d(\bar{x}, \psi(Q_\mu + t_n v_n)) \leq \hat{L} t_n \|v_n\|_{L,r}$. Hence, the sequence $(\frac{1}{t_n}(x_n - \bar{x}))$ is bounded and has a convergent subsequence whose limit belongs to $\limsup_{n\to\infty} \frac{1}{t_n}(\psi(Q_\mu + t_n v_n) - \bar{x})$. If the Lipschitz property of $d(\bar{x}, \psi(\cdot))$ is violated, the upper set limit may be empty. This is illustrated by Example 3.7, in which we have $\bar{x} = 0$, $\psi(Q_\mu + t_n v) = \{\sqrt{t_n}\}$, and, thus, $\frac{1}{t_n}(\psi(Q_\mu + t_n v) - \bar{x}) = \{t_n^{-\frac{1}{2}}\}$.

In order to establish the semidifferentiability of ψ at a pair $(Q_\mu, \bar{x})$ belonging to the graph of ψ, it remains to show, according to Proposition 4.5, that the solution set

$$\operatorname{argmin}\left\{\frac{1}{2}\langle \nabla^2 g(\bar{x})\xi, \xi\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle + \langle \nabla v(\bar{y}), A\xi\rangle + b(\xi) : \xi \in S(\bar{x})\right\}$$

is contained in the lower set limit $\liminf_{n\to\infty} \frac{1}{t_n}(\psi(Q_\mu + t_n v_n) - \bar{x})$, where $v_n := \frac{1}{t_n}(Q_n - Q_\mu)$, $Q_n \in K_C$, for all $n \in \mathbb{N}$, and (v_n) converges to v. To this end, a uniform quadratic growth condition of the objective functions $g(\cdot) + (Q_\mu + t_n v_n)(A\,\cdot)$ for large $n \in \mathbb{N}$ is significant. In view of Theorem 2.7, the uniform strong convexity of Q_μ and its approximations Q_n for large $n \in \mathbb{N}$ is decisive for the growth condition. The next example and the following result show that the approximations Q_n do not maintain the strong convexity property of Q_μ in general if the sequence (Q_n) converges to Q_μ in $C^1(\mathbb{R}^s)$, but that the situation is much more advantageous when considering the $C^{1,1}$-topology.

Example 4.7. Let $Q_\mu(y) = y^2$ for all $y \in \mathbb{R}$ and let Q_n be the following differentiable convex function:

$$Q_n(y) := \max\left\{0, -y - \frac{1}{n}\right\}^2 + \max\left\{0, y - \frac{1}{n}\right\}^2 \quad \text{for all } \; y \in \mathbb{R}, n \in \mathbb{N}.$$

Note that $Q_n(y) = 0$ for all $y \in [-\frac{1}{n}, \frac{1}{n}]$, and Q_n is not strongly convex for each $n \in \mathbb{N}$, but (Q_n) converges to Q_μ in $C^1(\mathbb{R}^s)$.

LEMMA 4.8. *Let $Q_\mu \in K_C \cap C^{1,1}(\mathbb{R}^s)$ be strongly convex on some bounded convex set $U \subseteq \mathbb{R}^s$ (with some constant $\kappa > 0$). Then there exists a neighborhood $\mathcal{N}$ of Q_μ in $C^{1,1}(\mathbb{R}^s)$ such that each function Q belonging to $\mathcal{N}$ is strongly convex on U with constant $\frac{\kappa}{2}$.*

Proof. The strong convexity of Q_μ on U (with constant $\kappa > 0$) is equivalent to the condition $\langle \nabla Q_\mu(y) - \nabla Q_\mu(\tilde{y}), y - \tilde{y}\rangle \geq \kappa\|y - \tilde{y}\|^2$ for all $y, \tilde{y} \in U$. Let $r > 0$ be chosen such that $\operatorname{cl} U \subseteq B(0, r)$, and let $\mathcal{N}$ be a neighborhood of Q_μ in $C^{1,1}(\mathbb{R}^s)$ having the property $\|\nabla(Q_\mu - Q)\|_{L,r} \leq \frac{\kappa}{2}$ for all $Q \in \mathcal{N}$. Let $y, \tilde{y} \in U$, with $y \neq \tilde{y}$. Then we obtain for any $Q \in \mathcal{N}$

$$\begin{aligned}
\kappa &\leq \frac{\langle \nabla Q_\mu(y) - \nabla Q_\mu(\tilde{y}), y - \tilde{y}\rangle}{\|y - \tilde{y}\|^2} \\
&= \frac{\langle \nabla Q(y) - \nabla Q(\tilde{y}), y - \tilde{y}\rangle}{\|y - \tilde{y}\|^2} + \frac{\langle \nabla(Q_\mu - Q)(y) - \nabla(Q_\mu - Q)(\tilde{y}), y - \tilde{y}\rangle}{\|y - \tilde{y}\|^2} \\
&\leq \frac{\langle \nabla Q(y) - \nabla Q(\tilde{y}), y - \tilde{y}\rangle}{\|y - \tilde{y}\|^2} + \frac{\|\nabla(Q_\mu - Q)(y) - \nabla(Q_\mu - Q)(\tilde{y})\|}{\|y - \tilde{y}\|^2} \\
&\leq \frac{\langle \nabla Q(y) - \nabla Q(\tilde{y}), y - \tilde{y}\rangle}{\|y - \tilde{y}\|^2} + \|\nabla(Q_\mu - Q)\|_{L,r},
\end{aligned}$$

and, hence,

$$\frac{\kappa}{2}\|y-\tilde{y}\|^2 \leq \langle \nabla Q(y) - \nabla Q(\tilde{y}), y-\tilde{y}\rangle.$$

This means that Q is strongly convex on U with constant $\frac{\kappa}{2}$. ∎

Now we are able to show that the solution set mapping ψ is semidifferentiable on $K_C \cap C^{1,1}$ at some pairs $(Q_\mu, \bar{x})$, $\bar{x} \in \psi(Q_\mu)$, into any direction v from the tangent cone $T(K_C \cap C^{1,1}; Q_\mu)$ to $K_C \cap C^{1,1}(\mathbb{R}^s)$ at Q_μ in $C^{1,1}(\mathbb{R}^s)$. The assumptions are essentially the same as in Theorem 4.2.

THEOREM 4.9. *Let $Q_\mu \in K_C \cap C^{1,1}$, and assume that $\psi(Q_\mu)$ is nonempty and bounded. Let g be twice continuously differentiable, and let Q_μ be strongly convex on some open convex neighborhood U of $A\psi(Q_\mu)$ and twice continuously differentiable at $\bar{y}$, where $\{\bar{y}\} = A\psi(Q_\mu)$. Assume that condition* (i)$''$ *of Theorem* 4.2 *is satisfied.*

Then the solution set mapping ψ from $K_C \cap C^{1,1}$ into $\mathbb{R}^m$ is semidifferentiable at any pair $(Q_\mu, \bar{x})$, $\bar{x} \in \psi(Q_\mu)$, such that C is second-order regular at $\bar{x}$, and into any direction $v \in T(K_C \cap C^{1,1}; Q_\mu)$, i.e., for any such $\bar{x}$ and v, $t_n \to 0+$, and (Q_n) in $K_C \cap C^{1,1}$ with $v_n = \frac{1}{t_n}(Q_n - Q_\mu) \to v$ in $C^{1,1}(\mathbb{R}^s)$ the set limit

$$D\psi(Q_\mu, \bar{x}; v) = \lim_{n\to\infty} \frac{1}{t_n}(\psi(Q_\mu + t_n v_n) - \bar{x})$$

exists. The semiderivative $D\psi(Q_\mu, \bar{x}; v)$ is equal to the set

$$\operatorname{argmin}\left\{\frac{1}{2}\langle \nabla^2 g(\bar{x})\xi, \xi\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle + \langle \nabla v(\bar{y}), A\xi\rangle + b(\xi) : \xi \in S(\bar{x})\right\}.$$

Moreover, ψ is semidifferentiable at any pair $(Q_\mu, \bar{x})$, $\bar{x} \in \psi(Q_\mu)$, into any direction $v \in T(K_C \cap C^{1,1}; Q_\mu)$ if C is polyhedral. Condition (i)$''$ *is satisfied if C is polyhedral and g is linear- or (convex-) quadratic.*

Proof. Let $\bar{x} \in \psi(Q_\mu)$ be such that C is second-order regular at $\bar{x}$, $v \in T(K_C \cap C^{1,1}; Q_\mu)$, and $v_n = \frac{1}{t_n}(Q_n - Q_\mu) \to v$ in $C^{1,1}(\mathbb{R}^s)$, where $t_n \to 0+$ and (Q_n) is a sequence in $K_C \cap C^{1,1}$. We may assume that the neighborhood U is bounded. Since (Q_n) converges to Q_μ in $C^{1,1}(\mathbb{R}^s)$, we obtain from Lemma 4.8 that there exists an $n_0 \in \mathbb{N}$ such that Q_n is strongly convex on U for each $n \geq n_0$ with a uniform constant $\kappa > 0$. Moreover, we choose n_0 sufficiently large such that $\psi(Q_n)$ is nonempty for each $n \geq n_0$. Arguing as in the proof of Theorem 2.7, we obtain a constant $c > 0$ and a neighborhood V of $\psi(Q_n)$ such that the growth condition

$$g(x) + Q_n(Ax) \geq \varphi(Q_n) + cd(x, \psi(Q_n))^2$$

holds for all $x \in C \cap V$ and $n \geq n_0$.

Let $\bar{\xi} \in S(\bar{x})$ be a minimizer of the function $\frac{1}{2}\langle \nabla^2 g(\bar{x})\xi, \xi\rangle + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle + \langle \nabla v(\bar{y}), A\xi\rangle + b(\xi)$ subject to $\xi \in S(\bar{x})$. Because of Proposition 4.5 it remains to show that $\bar{\xi}$ belongs to the lower limit $\liminf_{n\to\infty} \frac{1}{n}(\psi(Q_\mu + t_n v_n) - \bar{x}) = \liminf_{n\to\infty} \frac{1}{t_n}(\psi(Q_n) - \bar{x})$. To this end we argue as in the proof of Theorem 4.1. Let $\varepsilon > 0$ be given, and let $z \in T^2(C; \bar{x}, \bar{\xi})$ be such that $\langle \nabla g(\bar{x}), z\rangle + \langle \nabla Q_\mu(\bar{y}), Az\rangle \leq b(\bar{\xi}) + \varepsilon$. Then there exists a sequence (z_n) converging to z with $x_n = \bar{x} + t_n\xi + t_n^2 z_n \in C$ for all $n \in \mathbb{N}$. Then it suffices to show that

$$\lim_{n\to\infty} \frac{1}{t_n} d(\bar{x} + t_n\bar{\xi} + t_n^2 z_n, \psi(Q_n)) = 0.$$

By using the above growth condition and by expanding the function g and Q_μ, we obtain, similar to the proof of Theorem 4.1, that

$$\begin{aligned}\limsup_{n\to\infty} \frac{c}{t_n^2}\, d(\bar{x}+t_n\bar{\xi}+t_n^2 z_n, \psi(Q_n))^2 \\ \le \langle \nabla g(\bar{x}), z\rangle + \langle \nabla Q_\mu(\bar{y}), Az\rangle + \frac{1}{2}\langle \nabla^2 g(\bar{x})\bar{\xi}, \bar{\xi}\rangle \\ + \frac{1}{2}\langle \nabla^2 Q_\mu(\bar{y})A\bar{\xi}, A\bar{\xi}\rangle - \varphi''(Q_\mu; v) + \langle \nabla v(\bar{y}), A\bar{\xi}\rangle \le \varepsilon\,.\end{aligned}$$

This implies $\bar{\xi} \in \liminf_{n\to\infty} \frac{1}{t_n}(\psi(Q_n)-\bar{x})$ and the semidifferentiability of ψ at $(Q_\mu, \bar{x})$ in direction v is shown. The remaining part of the assertion follows as in the proof of Theorem 4.2. □

For the linear-quadratic case, the essential assumptions in Theorem 4.9 are the strong convexity of Q_μ, and the smoothness properties of Q_μ and its perturbations Q, respectively. While criteria for strong convexity were already discussed in section 2, we now add some comments on $C^{1,1}$ and C^2 properties of expected recourse functions. Later we close by indicating some conclusions of the results of sections 3 and 4 on asymptotic properties of statistical estimators of optimal values and solution sets.

Remark 4.10. Assume (A1)–(A3) and μ to have a density with respect to the Lebesgue measure on $\mathbb{R}^s$. Then the function Q_μ in (1.2) is continuously differentiable on $\mathbb{R}^s$ and its gradient is of the form $\nabla Q_\mu(y) = \sum_{i=1}^{\ell} d_i \mu(y + B_i(\mathbb{R}^s_+))$ for all $y \in \mathbb{R}^s$, where B_i, $i = 1, \ldots, \ell$, are certain basis submatrices of the recourse matrix W such that the simplicial cones $B_i(\mathbb{R}^s_+)$, $i = 1, \ldots, \ell$, are linearity regions of $\tilde{Q}$ and $-d_i$ is the gradient of $\tilde{Q}$ on int $B_i(\mathbb{R}^s_+)$, $i = 1, \ldots, \ell$ (cf., [15], [39]). Denoting by F_μ the distribution function of μ and using the formula

$$\mu(y + B(\mathbb{R}^s_+)) = F_{\mu\circ(-B)}(-B^{-1}y) \quad \text{for all } \ y \in \mathbb{R}^s,$$

for any nonsingular (s, s)-matrix B, $C^{1,1}$ and C^2 properties of Q_μ may thus be formulated in terms of Lipschitz and differentiability properties of the distribution functions $F_{\mu\circ(-B_i)}$ to the linear transforms $\mu \circ (-B_i)$, $i = 1, \ldots, \ell$, of the measure μ.

The distribution function F_μ of a probability measure μ on $\mathbb{R}^s$ is locally Lipschitzian if all one-dimensional marginal distribution functions of μ are locally Lipschitzian (cf. [26], [38]). F_μ is continuously differentiable if μ has a continuous density function and all one-dimensional marginal distribution functions of μ are continuously differentiable (cf. [21], [38]). If μ has a continuous density function, then $\mu \circ B$ has a continuous density for any nonsingular (s, s)-matrix B, too. Hence, we may conclude, for instance, that Q_μ belongs to $C^{1,1}(\mathbb{R}^s)$ (and $C^2(\mathbb{R}^s)$) if μ has a (continuous) density and the above-mentioned conditions on the one-dimensional marginal distribution functions for $F_{\mu\circ B}$ belonging to $C^{0,1}(\mathbb{R}^s)$ (and $C^1(\mathbb{R}^s)$, respectively) are satisfied for any nonsingular (s, s)-matrix B. This criterion is particularly useful for probability distributions μ which have the property that all one-dimensional marginal distributions of μ and all linear transforms $\mu \circ B$ for all nonsingular matrices B belong to the same class of measures. For instance, all multivariate normal and all logarithmic concave probability measures (e.g., [15]) form classes having this property.

Remark 4.11. We consider a sequence (Q_n) of nonparametric estimators of Q_μ and assume that each Q_n is a random variable with values in some linear metric (function) space Z and in K_C. Furthermore, we assume that a central limit result of the form

$$\tau_n^{-1}(Q_n - Q_\mu) \to_d \zeta$$

is satisfied for some sequence of positive numbers (τ_n) decreasing to 0 and for some random variable ζ taking values in a separable subset of Z. Here, we denote by $\rightarrow_d$ the convergence in distribution of Z-valued random variables. Then versions of the delta-method (see, e.g., [36]) together with the second-order Hadamard differentiability of the optimal value φ at Q_μ (Theorem 3.8 and $Z = C^1(\mathbb{R}^s)$) and the semidifferentiability of the solution set ψ at Q_μ (Theorem 4.9 and $Z = C^{1,1}(\mathbb{R}^s)$) lead to central limit formulas for the sequence $(\varphi(Q_n))$ of real random variables and the sequence of random sets $(\psi(Q_n))$, respectively. In particular, we obtain from Theorem 3.8 and a second-order version of the delta-method that

$$\tau_n^{-2}(\varphi(Q_n)-\varphi(Q_\mu)-\varphi'(Q_\mu;Q_n-Q_\mu)) = \tau_n^{-2}(\varphi(Q_n)-g(\bar{x})-Q_n(A\bar{x})) \rightarrow_d \varphi''(Q_\mu;\zeta),$$

where $\bar{x} \in \psi(Q_\mu)$ and $\rightarrow_d$ refer to convergence in distribution of real-valued random variables. Theorem 4.9 and a set-valued version of the delta-method [16], [20] imply

$$\tau_n^{-1}(\psi(Q_n) - \bar{x}) \rightarrow_d D\psi(Q_\mu, \bar{x}; \zeta),$$

where $\bar{x} \in \psi(Q_\mu)$ and $\rightarrow_d$ refer to convergence in distribution of closed-valued measurable multifunctions in $\mathbb{R}^m$ (cf. [29]). The asymptotic distributions in both central limit results are the probability distributions of the optimal value and of the solution set, respectively, of the random convex program that consists in minimizing the (random) objective $\frac{1}{2}\langle\nabla^2 g(\bar{x})\xi,\xi\rangle + \frac{1}{2}\langle\nabla^2 Q_\mu(\bar{y})A\xi, A\xi\rangle + \langle\nabla\zeta(\bar{y}), A\xi\rangle + b(\xi)$ subject to ξ satisfying the (deterministic) constraints $\xi \in T(C;\bar{x})$ and $\langle\nabla g(\bar{x}),\xi\rangle + \langle\nabla Q_\mu(\bar{y}), A\xi\rangle = 0$. Furthermore, in the linear-quadratic case the set-valued central limit result may be complemented by limit theorems for selections forming a Castaing representation of ψ (cf. [11]).

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REFERENCES

[1] Z. Artstein and R. J.-B. Wets, *Stability results for stochastic programs and sensors, allowing for discontinuous objective functions*, SIAM J. Optim., 4 (1994), pp. 537–550.

[2] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[3] A. Auslender and R. Cominetti, *A comparative study of multifunction differentiability with applications in mathematical programming*, Math. Oper. Res., 16 (1991), pp. 240–258.

[4] A. Ben-Tal and J. Zowe, *Directional derivatives in nonsmooth optimization*, J. Optim. Theory Appl., 47 (1985), pp. 483–490.

[5] J. F. Bonnans and R. Cominetti, *Perturbed optimization in Banach spaces* I: *A general theory based on a weak directional constraint qualification*, SIAM J. Control Optim., 34 (1996), pp. 1151–1171.

[6] J. F. Bonnans, R. Cominetti, and A. Shapiro, *Second order optimality conditions based on parabolic second order tangent sets*, SIAM J. Optim., 9 (1999), pp. 466–492.

[7] J. F. Bonnans and A. D. Ioffe, *Quadratic growth and stability in convex programming problems with multiple solutions*, J. Convex Anal., 2 (1995), pp. 41–57.

[8] J. F. Bonnans and A. Shapiro, *Optimization problems with perturbations: A guided tour*, SIAM Rev., 40 (1998), pp. 228–264.

[9] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[10] R. Cominetti, *Metric regularity, tangent sets and second order optimality conditions*, Appl. Math. Optim., 21 (1990), pp. 265–287.

[11] D. Dentcheva, *Differentiable selections and Castaing representations of multifunctions*, J. Math. Anal. Appl., 223 (1998), pp. 371–396.

[12] D. Dentcheva, W. Römisch, and R. Schultz, *Strong convexity and directional derivatives of marginal values in two-stage stochastic programming*, in Stochastic Programming, K. Marti and P. Kall, eds., Lecture Notes in Econom. and Math. Systems 423, Springer-Verlag, Berlin, 1995, pp. 8–21.
[13] J. Dupačová, *Stability and sensitivity analysis for stochastic programming*, Ann. Oper. Res., 27 (1990), pp. 115–142.
[14] J. Dupačová, *Stability in stochastic programming with recourse. Contaminated distributions*, Math. Programming Stud., 27 (1986), pp. 133–144.
[15] P. Kall, *Stochastic Linear Programming*, Springer-Verlag, Berlin, 1976.
[16] A. J. King, *Generalized delta theorems for multivalued mappings and measurable selections*, Math. Oper. Res., 14 (1989), pp. 720–736.
[17] A. J. King and R. T. Rockafellar, *Sensitivity analysis for nonsmooth generalized equations*, Math. Programming, 55 (1992), pp. 193–212.
[18] A. J. King and R. T. Rockafellar, *Asymptotic theory for solutions in statistical estimation and stochastic programming*, Math. Oper. Res., 18 (1993), pp. 148–162.
[19] D. Klatte and G. Thiere, *Error bounds for solutions of linear equations and inequalities*, ZOR—Math. Methods Oper. Res., 41 (1995), pp. 191–214.
[20] P. Lachout, *On multifunction transforms of probability measures*, Ann. Oper. Res., 56 (1995), pp. 241–249.
[21] K. Marti, *Approximationen der Entscheidungsprobleme mit linearer Ergebnisfunktion und positiv homogener, subadditiver Verlustfunktion*, Zeitschrift Wahrscheinlichkeitstheorie und verwandte Gebiete, 31 (1975), pp. 203–233.
[22] B. S. Mordukhovich, *Stability theory for parametric generalized equations and variational inequalities via nonsmooth analysis*, Trans. AMS, 343 (1994), pp. 609–657.
[23] J.-P. Penot, *Differentiability of relations and differential stability of perturbed optimization problems*, SIAM J. Control Optim., 22 (1984), pp. 529–551.
[24] S. M. Robinson, *Strongly regular generalized equations*, Math. Oper. Res., 5 (1980), pp. 43–62.
[25] R. T. Rockafellar, *Proto-differentiability of set-valued mappings and its applications in optimization*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 6 (1989), Suppl., pp. 449–482.
[26] W. Römisch and R. Schultz, *Stability of solutions for stochastic programs with complete recourse*, Math. Oper. Res., 18 (1993), pp. 590–609.
[27] W. Römisch and R. Schultz, *Lipschitz stability for stochastic programs with complete recourse*, SIAM J. Optim., 6 (1996), pp. 531–547.
[28] R. Y. Rubinstein and A. Shapiro, *Discrete Event Systems. Sensitivity Analysis and Stochastic Optimization by the Score Function Method*, Wiley, Chichester, UK, 1993.
[29] G. Salinetti and R. J.-B. Wets, *On the convergence in distribution of measurable multifunctions (random sets), normal integrands, stochastic processes and stochastic infima*, Math. Oper. Res., 11 (1986), pp. 385–419.
[30] R. Schultz, *Strong convexity in stochastic programs with complete recourse*, J. Comput. Appl. Math., 56 (1994), pp. 3–22.
[31] A. Seeger, *Second order directional derivatives in parametric optimization problems*, Math. Oper. Res., 13 (1988), pp. 628–645.
[32] A. Shapiro, *Sensitivity analysis of nonlinear programs and differentiability properties of metric projections*, SIAM J. Control Optim., 26 (1988), pp. 628–645.
[33] A. Shapiro, *On concepts of directional differentiability*, J. Optim. Theory Appl., 66 (1990), pp. 477–487.
[34] A. Shapiro, *On differential stability in stochastic programming*, Math. Programming, 47 (1990), pp. 107–116.
[35] A. Shapiro, *Asymptotic analysis of stochastic programs*, Ann. Oper. Res., 30 (1991), pp. 169–186.
[36] A. W. van der Vaart and J. A. Wellner, *Weak Convergence and Empirical Processes*, Springer Ser. Statist., Springer-Verlag, New York, 1996.
[37] D. Walkup and R. J.-B. Wets, *A Lipschitzian characterization of convex polyhedra*, Proc. AMS, 23 (1969), pp. 167–173.
[38] J. Wang, *Distribution sensitivity analysis for stochastic programs with complete recourse*, Math. Programming, 31 (1985), pp. 286–297.
[39] R. J.-B. Wets, *Stochastic programs with fixed recourse: The equivalent deterministic program*, SIAM Rev., 16 (1974), pp. 309–339.
[40] R. J.-B. Wets, *Stochastic programming*, in Handbooks in Operations Research and Management Science, 1, Optimization, G. L. Nemhauser, A. H. G. Rinnoy Kan, and M. J. Todd, eds., North-Holland, Amsterdam, 1989, pp. 573–629.