QUANTITATIVE STABILITY FOR SCENARIO-BASED STOCHASTIC PROGRAMS

Jitka Dupačová

Department of Probability and Mathematical Statistics, Charles University
Sokolovská 83, 186 75 Prague, Czech Republic
e-mail: dupacova@karlin.mff.cuni.cz

Werner Römisch

Institute of Mathematics, Humboldt University, Unter den Linden 6, D-10099 Berlin, Germany
e-mail: romisch@mathematik.hu-berlin.de

Abstract

General quantitative stability results for stochastic programs are formulated in terms of probability metrics, specified to scenario-based stochastic programs and applied to a bond portfolio management problem.
AMS classification: 90C15, 90C31, 60K30, 90C08
Key words: Probability metrics, stochastic programs, stability wrt. probability measure, random recourse, discrete distributions, application

1 INTRODUCTION

Stability and sensitivity studies for stochastic programs have been motivated by an incomplete information about the probability measure through which the stochastic program is formulated and also by the efforts in designing various discretization and approximation schemes needed in connection with the development and evaluation of algorithms. The solved real life stochastic programs are very complex; in numerical procedures, one uses their specific structure and is interested in robust solutions: Small changes in the input (in our case mainly perturbations of the probability measure) are supposed to cause only small changes of the output (the optimal value, the set of optimal solutions). Evidently, such requirements can be cast under *quantitative stability analysis*, see for instance [1], [3] and references therein, [6], [10], [11] and [15]:

For a general stochastic program with a fixed constraint set

$$\text{minimize} \quad E_P f(\mathbf{x},\omega) \quad \text{on a set} \quad \mathcal{X} \subset R^n \tag{1}$$

where P is a fixed probability measure on $(\Omega, \mathcal{B})$ belonging to a class $\mathcal{P}$, with E_P the corresponding expectation operator, $\mathcal{X} \subset R^n$ a given nonempty closed set and $f : \mathcal{X} \times \Omega \to R^1$ a given function, denote

$$\varphi(P) = \inf_{\mathbf{x}\in\mathcal{X}} E_P f(\mathbf{x},\omega) \tag{2}$$

the optimal value and

$$\psi(P) = \arg\min_{\mathbf{x}\in\mathcal{X}} E_P f(\mathbf{x},\omega)\,\{\mathbf{x} \in \mathcal{X} | f(\mathbf{x}, P) = \varphi(P)\} \tag{3}$$

the solution set. To adapt the general quantitative stability approaches means to select a metric distance d of probability measures which is suitable from the point of view of the structure of the considered stochastic program and/or of the particular type of approximation of probability measure P for to get a Lipschitz (or Hölder) property of the optimal value

$$d(P, P') < \eta \Rightarrow |\varphi(P) - \varphi(P')| < K\eta$$

and possibly also a Lipschitz (or Hölder) property of the Hausdorff distance of the corresponding solution sets with respect to perturbations of P measured by d; naturally, the Lipschitz (or Hölder) constants depend on the chosen metric d.

The first results concerning the optimal value can be found in [12]. Special assumptions are needed for to extend these results to the optimal solutions. A Hölder stability result for solution sets of two-stage stochastic programs with random right-hand side is obtained in [10]. It is formulated in terms of Wasserstein metric on all probability measures having finite first moments (and appearing as metric ζ_1 in Section 2). The result is essentially based on a strong convexity property of the expected recourse function which is now well understood, cf. [14]. Later it has been clarified in [15] and [11] that second order growth conditions for the objective function around the solution set lead to (upper) Lipschitzian stability results for two-stage models. Unfortunately, such growth conditions are only available in special situations (cf. also Section 2). Therefore it is expedient to investigate also quantitative stability for the *sets of ε-optimal solutions*

$$\psi_\varepsilon(P) = \varepsilon - \arg\min_{\mathbf{x}\in\mathcal{X}} E_P f(\mathbf{x},\omega)$$
$$= \{\mathbf{x} \in \mathcal{X} | E_P f(\mathbf{x},\omega) \leq \varphi(P) + \varepsilon\} \quad (4)$$

which hold true under more general circumstances (cf. [2]), an idea suggested in [13].

For the purposes of an algorithmic solution, the prevailing approximation technique is discretization of the initial probability measure: It is replaced by a discrete probability measure concentrated in a finite numbers of atoms, called *scenarios*. To design an approximation which is representative enough and such that the obtained solution enjoys plausible robustness properties is of a great importance. Quantitative stability results for these *scenario-based* programs may help to quantify the desirable robustness properties also in rather complicated instances of stochastic programs with random recourse.

The success and applicability of the quantitative stability results depend essentially on an appropriate choice of the *probability metric* used to measure the perturbations in the model input.

Example. Consider the well known *newsboy problem*:

The newsboy sells newspapers for the cost c each. Before he starts selling, he has to buy the daily supply at the cost b a paper, $c > b > 0$. The demand is random and the unsold newspapers are returned without refund at the end of the day. How many newspapers should he buy?

Assume that the demand is random with a known discrete distribution P concentrated at S points $\omega_1, \ldots, \omega_S$ of a closed interval $[D_1, D_2], D_1 > 0$ with probabilities $p_s > 0, s = 1, \ldots, S, \sum_s p_s = 1$. The problem is

$$\min_{x\geq 0} E_P f(x,\omega) := [(b-c)x + c\sum_s p_s(x-\omega_s)^+]$$

Let an additional scenario $\omega_* \in [D_1, D_2]$ be taken into account; it corresponds to the degenerated probability measure $Q = \delta_{\omega_*}$. The considered perturbed problem is related to a probability measure carried by the initial scenarios $\omega_s, s = 1, \ldots, S$ and by ω_*. Assuming that the proportions between the initial probabilities $p_s, s = 1, \ldots, S$ are kept we can specify this probability measure as $P_\lambda = (1-\lambda)P + \lambda Q$ where $\lambda \in (0,1)$ is the probability of ω_*. Evidently, the difference between the initial and the perturbed objective values $E_P f(x,\omega) - E_{P_\lambda} f(x,\omega) = \lambda(E_P f(x,\omega) - E_Q f(x,\omega))$ can be non-zero only on the interval $[D_1, D_2]$ and at each $x \in [D_1, D_2]$, its *value depends on the probability λ of the additional scenario and on the difference of the two objective functions* $E_P f(x,\omega), E_Q f(x,\omega) = f(x,\omega_*)$. It is easy to bound the differences of the values of the *random* objectives $f(x,\omega) := (b-c)x + c(x-\omega)^+$ for two different realizations:

$$|f(x,\omega) - f(x,\omega')| = c|(x-\omega)^+ - (x-\omega')^+| \leq c|\omega - \omega'| \, \forall x \quad (5)$$

so that the difference of the two considered objective functions

$$|E_P f(x,\omega) - E_Q f(x,\omega)| = c|\sum_s p_s(x-\omega_s)^+ - (x-\omega_*)^+|$$
$$\leq c\sum_s p_s|\omega_s - \omega_*| \quad (6)$$

The difference between the function values depends obviously on the position of the additional scenario with respect to the initial ones. Let us have a look how is this fact reflected by common distances of the one-dimensional probability measures.

Let F, G denote the distribution functions associated with P, Q. The *Kolmogorov* (or *uniform* metric)

$$d_K(P,Q) := \sup_{t\in R} |F(t) - G(t)|$$

equals

$$\max[\sum_{j=1}^{s} p_j, 1 - \sum_{j=1}^{s} p_j] \text{ if } \omega_* \in (\omega_s, \omega_{s+1}) \text{ for some } s$$

and equals 1 otherwise.

Contrary to our expectations and the above results *the Kolmogorov distance does not distinguish the magnitude of the (positive) distance of the additional scenario from the convex hull of the initial ones!* The *least influential additional scenario* $\omega_* \in (\omega_1, \omega_S)$ should minimize the maximal value of $[\sum_{j=1}^s p_j, 1 - \sum_{j=1}^s p_j]$, a condition which is fulfilled for *median* $\tilde{\omega}$ of the distribution P.

An important class of probability metrics in our context, are the *Fortet-Mourier metrics* ζ_p, $p \geq 1$, which are defined in Section 2. Here we use the explicit formulas which are available for ζ_p in the one-dimensional case. With the notation from above, it holds that (cf. Chapter 5 in [7])

$$\zeta_p(P,Q) = \int_{-\infty}^{+\infty} \max\{1, |t|^{p-1}\} |F(t) - G(t)| dt$$

The metric ζ_1 forms the L_1-counterpart of the Kolmogorov metrics and is called (L_1-) Wasserstein or Kantorovich metric. Similarly as for the Kolmogorov metric we have $\zeta_1(P, P_\lambda) = \lambda\zeta_1(P,Q)$ and

$$\zeta_1(P,Q) = \sum_{j=1}^{S} p_j|\omega_j - \omega_*|$$

Notice that the distance between the additional scenario ω_* and all original ones is taken into account and that the least influential additional scenario coincides again with the median $\tilde{\omega}$ of P. For $\omega_* \in [D_1, D_2]$,

$$\sum_j p_j|\omega_j - \tilde{\omega}| \le \zeta_1(P,Q) \le \max\{E_P\omega - D_1, D_2 - E_P\omega\}.$$

The next section summarizes the general quantitative stability results and provides their specification to scenario-based programs. The last section is devoted to an application to a bond portfolio management problem.

2 QUANTITATIVE STABILITY RESULTS

We assume that the constraint set $\mathcal{X}$ is convex and closed, and that the function $f : \mathcal{X} \times \Omega \longrightarrow R^1$ has the properties that $f(\bullet, \omega)$ is convex for each ω and $f(\mathbf{x}, \bullet)$ is measurable for each $\mathbf{x}$. Then the objective function

$$\mathbf{x} \mapsto E_P f(\mathbf{x}, \omega) := \int_\Omega f(\mathbf{x}, \omega) P(d\omega) \tag{7}$$

is convex on R^n for any probability measure P (on $(\Omega, \mathcal{B})$) such that (7) is finite. Later we only consider probability measures having this property.

The structure of the convex program (1) suggests to consider a probability semimetric of the form

$$d_\mathcal{F}(P,Q) := \sup\{|\int_\Omega f(\omega)(P(d\omega) - Q(d\omega))| : f \in \mathcal{F}\} \tag{8}$$

where $\mathcal{F} := \{f(\mathbf{x}, \bullet) : \mathbf{x} \in \mathcal{X}\}$ is the class of all measurable functions from Ω to R^1 that appear as integrands in (7). The probability distance $d_\mathcal{F}(P,Q)$ is finite whenever P and Q belong to the set

$$\mathcal{P}_\mathcal{F}(\Omega) := \{Q : \sup_{f \in \mathcal{F}} |\int_\Omega f(\omega) Q(d\omega)| < \infty\}$$

of probability measures (on $(\Omega, \mathcal{B})$) satisfying a uniform moment condition with respect to $\mathcal{F}$.

Now, (1) is regarded as a convex parametric program with parameter P belonging to the semimetric space $(\mathcal{P}_\mathcal{F}(\Omega), d_\mathcal{F})$. The following stability result is a consequence of a more general perturbation theorem in [8].

Theorem 1. In addition to the general assumptions, let $\psi(P)$ be nonempty and bounded, $P \in \mathcal{P}_\mathcal{F}(\Omega)$ and the function $\mathbf{x} \mapsto E_P f(\mathbf{x}, \omega)$ be locally Lipschitzian on $\mathcal{X}$. Then the solution set mapping ψ is (Berge) upper semicontinuous at P and there exist constants $L > 0$, $\delta > 0$ such that $\psi(Q)$ is nonempty and $|\varphi(P) - \varphi(Q)| \le L d_\mathcal{F}(P,Q)$ whenever $Q \in \mathcal{P}_\mathcal{F}(\Omega)$ and $d_\mathcal{F}(P,Q) < \delta$.

Upper semicontinuity of ψ at P means that for any $\eta > 0$ there exists a $\delta = \delta(\eta) > 0$ such that whenever $Q \in \mathcal{P}_\mathcal{F}(\Omega)$ and $d_\mathcal{F}(P,Q) < \delta$, $\sup_{\mathbf{x} \in \psi(Q)} d(\mathbf{x}, \psi(P)) < \eta$. Of course, it would be desirable to quantify the semicontinuity behavior of ψ, i.e., to derive an explicit representation of the function $\delta(\eta)$ (e.g. of the form $\delta(\eta) = (\eta/K)^k$ with constants $k \ge 1$ and $K > 0$). To obtain such quantitative stability results, it is well known that growth conditions for the objective function $E_P f(\bullet, \omega)$ near $\psi(P)$ play an important role. So far growth conditions have been explored only for stochastic programs with linear recourse and random right-hand sides or certain situations of random technology matrices ([14], [11]). Besides further conditions, the existence of a density to P being positive on certain sets related to $\psi(P)$ is decisive for growth conditions in two-stage models. Since we are interested in models with random recourse and also in purely atomic measures P, these results do not apply. Fortunately, the set of ε-optimal solutions $\psi_\varepsilon(P)$ of (1) enjoys a much better stability behavior when perturbing the probability measure P.

Theorem 2. Adopt the setting of Theorem 1. Then, for any $\varepsilon_0 > 0$, there exists a constant $\hat{L} > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the estimate

$$d_H(\psi_\varepsilon(P), \psi_\varepsilon(Q)) \le (\hat{L}/\varepsilon) d_\mathcal{F}(P,Q)$$

holds whenever $Q \in \mathcal{P}_\mathcal{F}(\Omega)$ and $d_\mathcal{F}(P,Q) < \varepsilon$.
(Here d_H denotes the Hausdorff distance on subsets of R^n.)

The result is taken from [13]. Its proof is based on estimates for ε-optimal solution sets of convex programs (cf. Theorem 7.69 of [9] or [2]) and on further properties of level sets. It is worth mentioning that the Lipschitzian stability result for ψ_ε at P is valid without assuming a growth condition for $E_P f(\bullet, \omega)$ and, hence, applies to many convex stochastic programs.

It is also useful to note that both theorems remain valid true if the class $\mathcal{F}$ of measurable functions from $(\Omega, \mathcal{B})$ to R^1 is replaced by a suitable larger class $\hat{\mathcal{F}} \supseteq \mathcal{F}$ leading to favorable properties of the distances $d_\mathcal{F}$ (e.g. to nice representations or explicit formulas). For two-stage stochastic programs, classes of locally Lipschitzian functions with a prescribed growth of Lipschitz moduli are of particular interest. We assume in the following that Ω is a subset of a Euclidean space and $\mathcal{B}$ is the σ-algebra of Borel sets relative to Ω. We denote by $\mathcal{F}_p$ with $p \ge 1$ the class of real-valued functions f on Ω satisfying the Lipschitzian property

$$|f(\omega) - f(\tilde{\omega})| \le \max\{1, \|\omega\|^{p-1}, \|\tilde{\omega}\|^{p-1}\}\|\omega - \tilde{\omega}\|$$

for all $\omega, \tilde{\omega} \in \Omega$, and by $\mathcal{P}_p(\Omega)$ the class of all probability measures Q on $(\Omega, \mathcal{B})$ having p-th order moments, i.e., $\int_\Omega \|\omega\|^p Q(d\omega) < \infty$. Then the distance $\zeta_p(P,Q) := d_{\mathcal{F}_p}(P,Q)$ is called *Fortet-Mourier metric* and $(\zeta_p, \mathcal{P}_p(\Omega))$ forms a metric space. The metric ζ_p enjoys a well developed duality theory and convergence analysis (cf. [7]). In particular, the following dual representation of ζ_p is valid:

$$\zeta_p(P,Q) = \inf\{\int_{\Omega\times\Omega} \max\{1, \|\omega\|^{p-1}, \|\tilde{\omega}\|^{p-1}\} \|\omega - \tilde{\omega}\| R(d\omega, d\tilde{\omega})\}$$

over all Borel probability measures R on $\Omega \times \Omega$ such that $R(B\times\Omega) - R(\Omega\times B) = P(B) - Q(B)\,\forall\, B \in \mathcal{B}$ (cf. Chapter 5 in [7]). A consequence of this result for $\Omega := R^1$ is the explicit formula for ζ_p that is used in Section 1.

The next result is a conclusion from Theorem 2 in case of discrete probability measures and of integrands $f(\mathbf{x}, \bullet)$ that satisfy a certain Lipschitz property.

Theorem 3. Adopt the setting of Theorem 2 and let P be a discrete probability measure on $(\Omega, \mathcal{B})$ having the form $P = \sum_{i=1}^S p_i \delta_{\omega_i}$. Assume that there exist constants $p \geq 1$ and $L_f > 0$ such that the function $(L_f)^{-1} f(\mathbf{x}, \bullet)$ belongs to $\mathcal{F}_p$ for each $\mathbf{x} \in \mathcal{X}$. Then, for any $\varepsilon_0 > 0$, there exists a constant $L > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the estimate

$$d_H(\psi_\varepsilon(P), \psi_\varepsilon(Q)) \leq \frac{L}{\varepsilon} \inf \left\{ \sum_{i=1}^{S} \sum_{j=1}^{\tilde{S}} \rho_{ij} \|\omega_i - \tilde{\omega}_j\| \max\{1, \|\omega_i\|^{p-1}, \|\tilde{\omega}_j\|^{p-1}\} \right\} \tag{9}$$

subject to

$$\rho_{ij} \in [0,1], \quad \sum_{i=1}^{S} \sum_{j=1}^{\tilde{S}} \rho_{ij} = 1,$$

and

$$\sum_{i=1, \omega_i \in B}^{S} (\sum_{j=1}^{\tilde{S}} \rho_{ij} - p_i) = \sum_{j=1, \tilde{\omega}_j \in B}^{\tilde{S}} (\sum_{i=1}^{S} \rho_{ij} - q_j)\, \forall B \in \mathcal{B}$$

holds whenever Q is a probability measure on $(\Omega, \mathcal{B})$ having the form $Q = \sum_{j=1}^{\tilde{S}} q_j \delta_{\tilde{\omega}_j}$ and the property $\zeta_p(P,Q) < \frac{\varepsilon}{L_f}$.

Proof: Let $\varepsilon_0 > 0$. We choose $\hat{L} > 0$ as in Theorem 2 and select some $\varepsilon \in (0, \varepsilon_0)$. Then Theorem 2 implies that

$$d_H(\psi_\varepsilon(P), \psi_\varepsilon(Q)) \leq \frac{\hat{L} L_f}{\varepsilon} \zeta_p(P,Q)$$

whenever $\zeta_p(P,Q) < \frac{\varepsilon}{L_f}$, where we used that $d_{\mathcal{F}}(P,Q) \leq L_f \zeta_p(P,Q)$. Due to the duality result for ζ_p, we have that

$$\zeta_p(P,Q) \leq \int_{\Omega\times\Omega} \max\{1, \|\omega\|^{p-1}, \|\tilde{\omega}\|^{p-1}\} \|\omega - \tilde{\omega}\| R(d\omega, d\tilde{\omega})$$

holds for any probability measure R on $\Omega \times \Omega$ of the form $R = \sum_{i=1}^S \sum_{j=1}^{\tilde{S}} \rho_{ij} \delta_{\omega_i} \delta_{\tilde{\omega}_j}$ such that for any $B \in \mathcal{B}$

$$R(B \times \Omega) - R(\Omega \times B) = \sum_{i=1}^{S} \sum_{j=1}^{\tilde{S}} \rho_{ij} (\delta_{\omega_i}(B) - \delta_{\tilde{\omega}_j}(B))$$

$$= P(B) - Q(B) = \sum_{i=1}^{S} p_i \delta_{\omega_i}(B) - \sum_{j=1}^{\tilde{S}} q_j \delta_{\tilde{\omega}_j}(B).$$

Taking the infimum subject to all such $\rho_{ij} \in [0,1]$ and putting $L := \hat{L} L_f$ completes the proof.

□

The theorem provides an estimate for the Hausdorff distance of ε-optimal sets to (1) associated with two discrete probability measures in terms of the optimal value of a certain linear program. This estimate can be exploited to develop procedures for deleting scenarios of a given discrete probability measure or for studying the influence of certain scenarios to changes of the problem. To discuss this in more detail, let $P = \sum_{i=1}^S p_i \delta_{\omega_i}$ play the role of a discrete approximation to a certain original probability measure. P might be obtained by a suitable statistical estimation procedure based on a finite (but large) sample. Hence, one might wish to reduce that large number S of scenarios $\omega_1, \ldots, \omega_S$ in order to obtain moderately sized programs in practical applications. Deleting the scenario ω_k of P could be done if the distance $d_H(\psi_\varepsilon(P), \psi_\varepsilon(Q_k))$ is small, where $Q_k = \sum_{j=1, j\neq k}^S q_j \delta_{\omega_j}$ with properly chosen probabilities q_j. Theorem 3 indicates that minimizing the optimal value of the linear program in the right-hand side of the estimate (9) (with $\tilde{S} = S-1$, $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_{S-1}\} = \{\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_S\}$) subject to all weights $q_j \in [0,1]$, $\sum_j q_j = 1$, is such an appropriate choice. A strategy for deleting scenarios could then be based on repeating this argument successively. Finally, we study the influence of an additional scenario $\omega_* \in \Omega$ by looking at the probability measure $P_\lambda = (1-\lambda)P + \lambda Q$, where $Q = \delta_{\omega_*}$ and $\lambda \in (0,1)$, cf. [4]. For small $\lambda > 0$, Theorem 3 provides the estimate

$$d_H(\psi_\varepsilon(P), \psi_\varepsilon(P_\lambda)) \leq \frac{L}{\varepsilon} \zeta_p(P, P_\lambda) = \frac{L\lambda}{\varepsilon} \zeta_p(P,Q)$$

$$\leq \frac{L\lambda}{\varepsilon} \sum_{i=1}^{S} p_i \|\omega_i - \omega_*\| \max\{1, \|\omega_i\|^{p-1}, \|\omega_*\|^{p-1}\}, \tag{10}$$

where (10) contains the explicit solution of the linear program in (9). The least influential additional scenario ω_* then corresponds to the minimizer of the function in (10) subject to $\omega_* \in \Omega$.

3 AN APPLICATION

The main purpose of the considered bond portfolio management problem is to preserve the value of a bond portfolio of a risk averse or risk neutral institutional investor over time. It has been formulated as a multiperiod two-stage scenario-based stochastic program with complete random recourse (e.g., [5]). The main random element is the evolution of the short interest rate over time which is regarded as the only factor that drives the prices of the considered default free government bonds:

Given a sequence of equilibrium future short term interest rates r_t valid for the time interval $(t, t+1], t = 0, \dots, T-1$, the fair price of the j-th bond at time t just after the coupon was paid equals the total cashflow $f_{j\tau}, \tau = t+1, \dots, T$ generated by this bond in subsequent time instances discounted to t:

$$\pi_{jt}(\mathbf{r}) = \sum_{\tau=t+1}^{T} f_{j\tau} \prod_{h=t}^{\tau-1} (1+r_h)^{-1} \tag{11}$$

where T is greater or equal to the time to maturity.

In formulation of the stochastic program one works with a suitable discrete distribution of the T - dimensional vector $\mathbf{r}$ of the short rates $r_t, t = 0, \dots, T-1$, where r_0 (the rate valid in the first period) is known. The possible finitely many realizations of $\mathbf{r}$ are called *scenarios*; we shall index them as $\mathbf{r}^s, s = 1, \dots, S$ and assign them probabilities $p_s > 0, s = 1, \dots, S, \sum_s p_s = 1$. Generation of scenarios is a rather demanding estimation, callibration and sampling procedure. The applied input distribution is thus burdened by various inherent errors and our primal goal is to analyze the influence of these errors on the obtained optimal decisions and on the optimal value of the portfolio.

We denote

$j = 1, \dots, J$ indices of the considered bonds and T_j the dates of their maturities; $T = \max_j T_j$.

$t = 0, \dots, T_0$ the considered discretization of the planning horizon;

$b_j \geq 0$ the initial holdings (in face value) of bond j;

$b_0 \geq 0$ the initial holding in riskless asset;

f_{jt}^s cashflow generated from bond j at time t under scenario s expressed as a fraction of its face value;

ξ_{jt}^s and η_{jt}^s are the selling and purchasing prices of bond j at time t for scenario s obtained from the corresponding fair prices (8) by adding the acrued interest A_{jt}^s and by subtracting or adding scenario independent transaction costs and spread; the initial prices ξ_{j0} and η_{j0} are known, i. e., scenario independent;

x_j/y_j are face values of bond j purchased/sold at the beginning of the planning period, at $t = 0$; x_{jt}^s/y_{jt}^s are the corresponding values for period t under scenario s.

z_{j0} is the face value of bond j held in portfolio after the initial decisions x_j, y_j have been made; z_{jt}^s are the corresponding holdings for period t under scenario s.

The first-stage decision variables x_j, y_j, z_{j0} are nonnegative,

$$y_j + z_{j0} = b_j + x_j \quad \forall j, \tag{12}$$

$$y_0^+ + \sum_j \eta_{j0} x_j = b_0 + \sum_j \xi_{j0} y_j \tag{13}$$

where the auxilliary nonnegative variable y_0^+ denotes the surplus.

Provided that an initial trading strategy determined by feasible scenario independent first-stage decision variables x_j, y_j, y_0^+ (and z_{j0}) for all j has been accepted, the subsequent second-stage scenario dependent decisions have to be made in an optimal way regarding the goal of the model, i. e., to maximize the final wealth subject to constraints on conservation of holdings and rebalancing the portfolio:

$$\text{maximize} \quad V_{T_0}^s := \sum_j \xi_{jT_0}^s z_{jT_0}^s + y_{T_0}^{+s} - \alpha y_{T_0}^{-s} \tag{14}$$

subject to

$$z_{jt}^s + y_{jt}^s = z_{j,t-1}^s + x_{jt}^s \, \forall j, \, 1 \leq t \leq T_0, \tag{15}$$

$$\sum_j \xi_{jt}^s y_{jt}^s + \sum_j f_{jt} z_{j,t-1}^s + (1 - \delta_1 + r_{t-1}^s) y_{t-1}^{+s} + y_t^{-s} =$$

$$\sum_j \eta_{jt}^s x_{jt}^s + (1 + \delta_2 + r_{t-1}^s) y_{t-1}^{-s} + y_t^{+s}, \, 1 \leq t \leq T_0, \tag{16}$$

$$x_{jt}^s \geq 0, y_{jt}^s \geq 0, z_{j,t}^s \geq 0, y_t^{-s} \geq 0, y_t^{+s} \geq 0 \, \forall j, \, 1 \leq t \leq T_0 \tag{17}$$

with $y_0^{-s} = 0, y_0^{+s} = y_0^+, z_{j0}^s = z_{j0} \, \forall j$; the auxilliary variables y_t^{+s}/y_t^{-s} describe the (unlimited) lending /borrowing possibilities for period t under scenario s and with parameters $\delta_1 \geq 0, \delta_2 > 0, \alpha \geq 1$ fixed according to the background of the solved problem.

With $V_{T_0}(\mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+; \mathbf{r}^s)$ the corresponding maximal value of the second-stage scenario subproblem (14)-(17), the full stochastic program can be now written in the form which allows to apply the general results of Section 2: The probability measure $P = \sum_{s=1}^S p_s \delta_{\mathbf{r}^s}$, the vector of the original decision variables $\mathbf{x} \longleftrightarrow [\mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+]$, the set of feasible solutions $\mathcal{X}$ is defined by nonnegativity constraints on all first-stage variables and by constraints (12)–(13), the random objective function $f(\mathbf{x}, \omega) \longleftrightarrow U(V_{T_0}(\mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+; \mathbf{r}))$ with U a concave nondecreasing utility function. (The symbol $\longleftrightarrow$ relates the notation used in Section 2 to that used in the application.) Notice that set of feasible first-stage solutions is nonempty and bounded and that the function $V_{T_0}(\bullet; \mathbf{r})$ is concave in $\mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+$ for any $\mathbf{r} \in R^T$. In this notation, the considered stochastic program $\max_{\mathbf{x}\in\mathcal{X}} E_P f(\mathbf{x}, \omega)$ reads

$$\text{maximize} \quad \sum_{s=1}^{S} p_s U(V_{T_0}(\mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+; \mathbf{r}^s)) \tag{18}$$

subject to nonnegativity constraints on all variables and subject to (12)–(13).

The stochastic program (18) obviously fits into the setting of Section 2. In order to apply the quantitative stability results of Section 2 to study the behavior of (18), we introduce the class $\mathcal{F}$ of relevant integrands as

$$\mathcal{F} := \{U(V_{T_0}(\mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+; \bullet)) : \mathbf{x}, \mathbf{y}, \mathbf{z}_0, y_0^+ \text{ are feasible}\}$$

and the semimetric $d_{\mathcal{F}}$ on the class $\mathcal{P}_{\mathcal{F}}(R^T)$ of probability measures as in Section 2. Moreover, let $\psi_\varepsilon(P)$ be the set of ε-optimal solutions to (18). Then Theorem 2 applies and we obtain the following stability result for (18).

Theorem 4. For any $\varepsilon_0 > 0$, there exists a constant $\hat{L} > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the estimate

$$d_H(\psi_\varepsilon(P), \psi_\varepsilon(Q)) \leq \frac{\hat{L}}{\varepsilon} d_{\mathcal{F}}(P, Q)$$

holds whenever Q is another discrete probability measure on R^T and $d_{\mathcal{F}}(P, Q) < \varepsilon$.

For the proof it remains to note that all discrete probability measures having finite support in R^T belong to $\mathcal{P}_{\mathcal{F}}(R^T)$ and that the assumptions of Theorem 1 are satisfied.

Of course, it would be desirable to identify classes of functions (like the class $\mathcal{F}_p$ in Section 2), that contain $\mathcal{F}$ and allow dual representations for the corresponding metrics. So far this remains an open problem.

4 BIBLIOGRAPHY

[1] Artstein, Z., *Sensitivity with respect to the underlying information in stochastic programs*, JCAM **56** (1994), 127–136.

[2] Attouch, H. and Wets, R. J.-B., *Quantitative stability of variational systems: III. ϵ-approximate solutions*, Math. Progr. **61** (1993), 197–214.

[3] Dupačová, J., *Stability and sensitivity - analysis for stochastic programming*, Annals of Oper. Res. **27** (1990), 115–142.

[4] Dupačová, J., *Scenario-based stochastic programs: Resistance with respect to sample*, Annals of Oper. Res. **64** (1996), 21–38.

[5] Dupačová, J., *Stability properties of a bond portfolio management problem*, paper prepared for APMOD98, March 1998, Cyprus.

[6] Kaňková, V., *On stability in two-stage stochastic nonlinear programming*, in: *Asymptotic Statistics* (P. Mandl and M. Hušková, eds.), Physica Verlag, Heidelberg 1994, 329–440.

[7] Rachev, S. T., *Probability Metrics and the Stability of Stochastic Models*. Wiley, Chichester 1991.

[8] Rachev, S. T. and Römisch, W., *Quantitative stability of stochastic programs: The approach via probability metrics*, manuscript, 1998.

[9] Rockafellar, R. T. and Wets, R. J.-B., *Variational Analysis*. Springer, Berlin 1997.

[10] Römisch, W. and Schultz, R., *Stability analysis for stochastic programs*, Annals of Oper. Res. **30** (1991), 241–266.

[11] Römisch, W. and Schultz, R., *Lipschitz stability for stochastic programs with complete recourse*, SIAM J. Optimization **6** (1996), 531–547.

[12] Römisch, W. and Walkobinger, A., *Obtaining convergence rates for approximations in stochastic programming*, in: *Parametric Optimization and Related Topics*, J. Guddat et al. (eds.), Akademie-Verlag, Berlin 1987, 327–343.

[13] Römisch, W. and Wets, R. J.-B., *Asymptotics of solution sets in stochastic programming*, lecture presented at the 4th World Congress of the Bernoulli Society, Vienna, Aug. 26–31, 1996.

[14] Schultz, R. *Strong convexity in stochastic programs with complete recourse*, JCAM **56** (1994), 3–22.

[15] Shapiro, A., *Quantitative stability in stochastic programming*, Math. Progr. **67** (1994), 99–108.

Acknowledgement. The first author was partly supported by the Grant Agency of the Czech Republic under grants No. 201/96/0230 and 402/96/0631, the second author by the German Research Foundation under grant No. Ro 1006/5-1 with the joint research held up also within the agreement on collaboration between the Charles University, Prague and the Humboldt University, Berlin.