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Stability of multistage stochastic programs incorporating polyhedral risk measures

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We analyse stability aspects of linear multistage stochastic programs with polyhedral risk measures in the objective. In particular, we consider sensitivity of the optimal value with respect to perturbations of the underlying stochastic input process. An existing stability result for multistage stochastic programs with expectation objective is carried forward to the case of polyhedral risk-averse objectives. Beside L_r-distances these results also involve filtration distances of the perturbations of the stochastic process. We discuss additional requirements for the polyhedral risk measures such that the problem dependent filtration distances can be bounded by problem independent ones. Stability and such bounds are the basis for scenario tree approximation techniques used in practical problem solving.

Keywords: multistage stochastic programming; optimization; quantitative stability; filtration distance; polyhedral risk measures; multiperiod risk functionals

AMS Subject Classifications: 90C15; 90C31; 91B30

1. Introduction

Multistage stochastic programs model the situation of a decision maker faced with a finite number of timesteps $t=1,\dots,T$. At each step he/she observes some random outcomes ξ_t and has to make an (optimal) decision x_t based on the exact knowledge of the past ($\xi_1,\dots,\xi_t$ and $x_1,\dots,x_{t-1}$) and on statistical information about the future ($\xi_{t+1},\dots,\xi_T$); cf., e.g. [22]. The random outcomes may affect both, the objective as well as the constraints for the decisions. The presence of statistical information is expressed by assuming $\xi=(\xi_1,\dots,\xi_T)$ to be a (multivariate) stochastic process on some fixed probability space $(\Omega,\mathcal{F},\mathbb{P})$. Note that it is assumed that the stochastic process is a pure input parameter and, hence, does not depend on the decisions.

In the following, it is supposed that $\xi_t\in L_r(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ for $t=1,\dots,T$ with some numbers $r\in[1,\infty]$ and $d\in\mathbb{N}$. We set $\xi^t:=(\xi_1,\dots,\xi_t)$ and we introduce the σ-fields $\mathcal{F}_t:=\sigma(\xi^t)$ for $t=1,\dots,T$. We assume without loss of generality that ξ_1 is deterministic

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and that $\sigma(\xi)=\mathcal{F}$. Thus, we have the following *filtration*: $\mathcal{F}_1=\{\emptyset,\Omega\}\subseteq\mathcal{F}_2\subseteq\cdots\subseteq\mathcal{F}_T=\mathcal{F}$. We will consider linear multistage stochastic programs of the form

$$\min\left\{\mathbb{E}\left[\sum_{t=1}^{T}\langle b_t(\xi_t),x_t\rangle\right]\left|\begin{array}{l} x_t\in L_{r'}(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^{m_t})\ (t=1,\ldots,T),\\ x_t\in X_t \text{ a.s. } (t=1,\ldots,T),\\ \sum_{\tau=0}^{t-1}A_{t,\tau}(\xi_t)x_{t-\tau}=h_t(\xi_t) \text{ a.s. } (t=2,\ldots,T)\end{array}\right.\right\} \tag{1}$$

with some numbers $m_t,n_t\in\mathbb{N}$ and $r'\in[1,\infty]$, polyhedral sets $X_t\subseteq\mathbb{R}^{m_t}$, matrices $A_{t,\tau}\in\mathbb{R}^{n_t\times m_{t-\tau}}$, and vectors $h_t\in\mathbb{R}^{n_t}$ and $b_t\in\mathbb{R}^{m_t}$. We assume that $A_{t,\tau}$, h_t, and b_t depend affinely linearly on ξ_t $(t=1,\ldots,T)$. The matrices $A_{t,0}$ are called the recourse matrices $(t=2,\ldots,T)$. For $\tau>0$ the matrices $A_{t,\tau}$ are called technology matrices. The random vectors x_t contain the decision variables for time $t=1,\ldots,T$. They have to be $\mathcal{F}_t$-measurable, respectively (non-anticipativity). The vectors b_t can be interpreted as cost factors. Note that optimality of the stochastic costs $\langle b_t(\xi_t),x_t\rangle$ is determined in terms of expectation.

For various reasons, it is of interest to analyse stability properties of stochastic programs with respect to perturbations of the underlying stochastic input process $\xi=(\xi_1,\ldots,\xi_T)$. In particular, quantitative stability results have a significant impact on methods for approximating ξ suitably by finite scenario trees. For the special case $T=2$, a lot is known for different types of stochastic programs; see [17,21] for a broad exposition and [10] for applications to scenario approximation. This case is much easier to handle since the information structure is fix: $\mathcal{F}_1=\{\emptyset,\Omega\}$, $\mathcal{F}_2=\mathcal{F}$. For $T>2$, the situation is much more challenging; only few approaches can be found in literature. In [11] a stability result for the optimal values of (1) was stated introducing a so-called *filtration distance*. Scenario tree approximation methods based on this stability result have been presented in [9].

In many applications it is of interest to consider risk functionals alternatively to the expectation functional $\mathbb{E}$ in the objective of (1). Typically, risk measures are inherently non-linear. Since the existing stability results rest to some extent on the linearity of the objective, it seems very difficult to carry them forward to problems with objectives incorporating arbitrary risk measures. However, in [3] the class of polyhedral risk measures was introduced containing ordinary risk measures such as CVaR/AVaR as well as multiperiod risk measures. Polyhedral risk measures are defined as optimal values of certain stochastic programs. As it will be demonstrated in Section 2, these risk measures, though non-linear, behave particularly suitable in the objective of (1). In Section 3, we will prove stability theorems similar to those from [11], which apply to the situation obtained by incorporating a polyhedral risk measure into the objective of (1). These stability results consist of local Lipschitz type estimates involving $L_r(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^s)$ norm distances (where $s=Td$) as well as filtration distances. These filtration distances depend on the solution behaviour of the particular problems. In [9], it has already been discussed how such objects can be estimated by problem independent metrics in the context of scenario tree approximation. However, these estimates are valid only if the sets of ε-optimal solutions (level-sets) are uniformly bounded. Hence, in Section 4, conditions for this level-boundedness will be analysed. It will be seen that none of the risk measures from [3] causes problems with respect to these conditions.

Hence, we conclude in Section 5 that polyhedral risk measures are a meaningful tool for risk aversion in multistage stochastic programming.

2. Polyhedral risk measures

Let $\mathcal{Z}$ denote some space of random variables on the measurable space $(\Omega, \mathcal{F})$ (e.g. $\mathcal{Z} = L_p(\Omega, \mathcal{F}, \mathbb{P})$ with $p \geq 1$ or even $p = 0$) or random processes (e.g. $\mathcal{Z} = \times_{j=1}^{J} L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P})$ with timesteps $1 \leq t_1 < t_2 < \cdots < t_J \leq T$). In the following, risk functionals ρ (risk measures) will be understood as (extended) real-valued mappings on $\mathcal{Z}$, i.e. $\rho : \mathcal{Z} \to \bar{\mathbb{R}}$, $z \mapsto \rho(z)$. Typically, risk functionals are essentially non-linear. The number $\rho(z)$ is intended to represent the chance of ending up with undesired realizations $z(\omega)$ of z or to represent the degree of uncertainty (spread) associated with z. In any case, if there is a choice among different $z \in \mathcal{Z}$, one is interested to find a z such that the value $\rho(z)$ is rather low, i.e. one may want to *minimize* $\rho(z)$ over a subset of $\mathcal{Z}$.

Consider the one-period case, i.e. $\mathcal{Z} = L_p(\Omega, \mathcal{F}, \mathbb{P})$. We will assume in the following that for $z \in \mathcal{Z}$ higher outcomes $z(\omega)$ are preferred to lower ones, e.g. $z = -\sum_{t=1}^{T} \langle b_t(\xi_t), x_t \rangle$. Classical functionals in this context are, e.g. the *variance* [12] ($p \geq 2$ required) or the *Value-at-Risk* at level $\alpha \in (0, 1)$ [6, Chapter 4.4] given by $\mathrm{VaR}_\alpha(z) = -\bar{q}_\alpha(z)$ with $\bar{q}_\alpha(z) = \inf\{a \in \mathbb{R} \colon \mathbb{P}(z \leq a) > \alpha\}$ denoting the upper alpha quantile. Note that both of these functionals are known to have certain drawbacks in particular when being used for optimization. Other well known risk functionals are semideviations [16], expected utility, shortfall risk [6, Chapter 4.6], etc. It may also be desirable to minimize a mixture $\gamma \cdot \rho(z) - (1 - \gamma) \cdot \mathbb{E}[z]$ of a risk measure and the expectation with some number $\gamma \in [0, 1]$ (mean-risk models, cf. [12,16,24]). Important work on axiomatic characterisations of risk measures has been reported in [1] and [6, Chapter 4], but also [16] contains considerations in that direction.

For the case that discrete time random processes $z = (z_{t_1}, \ldots, z_{t_J})$ are to be evaluated, multiperiod risk measures have to be used [2,7,13–15,18]; see also [23, Sections 11–13]. In this case, axiomatic characterisation turned out to be more controversial and fewer instances are suggested in literature, too. If a multiperiod risk measure shall be minimized within a multistage stochastic programming framework such as (1), the risk measure does not necessarily need to take all timesteps $t = 1, \ldots, T$ into account but may be restricted to a subset $t_1, \ldots, t_J$ of timesteps. Hence, for the multiperiod case we will consider $\mathcal{Z} = \times_{j=1}^{J} L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P})$.

For the purpose of being minimized in a (multistage) stochastic program, *polyhedral risk measures* have been introduced in [3] and applied to electricity models in [4,5]. Risk measures from this class are defined as optimal values of certain simple-structured stochastic minimization problems. Consider the multiperiod case with some (fixed) timesteps $1 = t_0 < t_1 < \cdots < t_J = T$ and $\mathcal{Z} = \times_{j=1}^{J} L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P})$. Then a functional ρ is called a (multiperiod) polyhedral risk measure if it has the following form:

$$\rho(z) = \inf \left\{ \mathbb{E}\left[\sum_{j=0}^{J} \langle c_j, y_j \rangle \right] \middle| \begin{array}{l} y_j \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P};\ \mathbb{R}^{k_j})\ (j = 0, \ldots, J), \\ y_j \in Y_j \quad \text{a.s. } (j = 0, \ldots, J), \\ \sum_{\tau=0}^{j} \langle w_{j,\tau}, y_{j-\tau} \rangle = z_{t_j} \quad \text{a.s. } (j = 1, \ldots, J) \end{array} \right\} \tag{2}$$

with some numbers $k_j \in \mathbb{N}$ and vectors $c_j \in \mathbb{R}^{k_j}$ $(j=0,\dots,J)$, $w_{j,\tau} \in \mathbb{R}^{k_{j-\tau}}$, $(j=1,\dots,J,$ $\tau=0,\dots,j)$, a polyhedral set $Y_0 \subseteq \mathbb{R}^{k_0}$, and polyhedral cones $Y_j \subseteq \mathbb{R}^{k_j}$ $(j=1,\dots,J)$. Typically, when using this type of functional in the objective of a multistage stochastic program (cf. (1)), one has $z_{t_j} = -\sum_{t=1}^{t_j} \langle b_t(\xi_t), x_t \rangle$ for $z = (z_{t_1},\dots,z_{t_J}) \in \mathcal{Z}$. Note that the case of minimizing a combination like $\gamma \cdot \rho(z_{t_1},\dots,z_{t_J}) - (1-\gamma) \cdot \mathbb{E}[z_T]$ is fully included in this framework, since such a mixture can be expressed by modifying the vectors c_j in (2) suitably [3, Remark 2.3]. For $\mathcal{Z} = L_p(\Omega, \mathcal{F}, \mathbb{P})$, i.e. for the one-period case, the definition is accordingly ($J=1$ and $t_1=T$).

One reason why polyhedral risk measures are particularly suitable for being minimized is as follows. For a stochastic program of the form (1) with a polyhedral risk measure in the objective

$$\min \left\{ \rho(z_{t_1},\dots,z_{t_J}) \;\middle|\; \begin{array}{l} x_t \in L_{r'}(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{m_t}),\ x_t \in X_t \text{ a.s. } (t=1,\dots,T), \\ \displaystyle\sum_{\tau=0}^{t-1} A_{t,\tau}(\xi_t) x_{t-\tau} = h_t(\xi_t) \text{ a.s. } (t=2,\dots,T) \\ \displaystyle z_t = z_t(\xi^t, x^t) := -\sum_{\tau=1}^{t} \langle b_\tau(\xi_\tau), x_\tau \rangle \ (t=1,\dots,T) \end{array} \right\} \tag{3}$$

there is an obvious equivalence to

$$\min \left\{ \mathbb{E}\left[\sum_{j=0}^{J} \langle c_j, y_j \rangle \right] \;\middle|\; \begin{array}{l} x_t \in L_{r'}(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{m_t}),\ x_t \in X_t \text{ a.s. } (t=1,\dots,T), \\ y_j \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P}; \mathbb{R}^{k_j}),\ y_j \in Y_j \text{ a.s. } (j=0,\dots,J), \\ \displaystyle\sum_{\tau=0}^{t-1} A_{t,\tau}(\xi_t) x_{t-\tau} = h_t(\xi_t) \text{ a.s. } (t=2,\dots,T), \\ \displaystyle\sum_{\tau=0}^{j} \langle w_{j,\tau}, y_{j-\tau} \rangle + \sum_{\tau=1}^{t_j} \langle b_\tau(\xi_\tau), x_\tau \rangle = 0 \text{ a.s. } (j=1,\dots,J) \end{array} \right\} \tag{4}$$

by inserting the definition (2). The equivalence is basically in terms of optimal values rather than in terms of solution sets [3, Proposition 4.1]. The resulting problem (4) is almost of the form (1) (but the matrices $A_{t,\tau}$ then depend on ξ^t rather than ξ_t only). This equivalence is, e.g. useful for algorithmic approaches (see, e.g. [8]).

Example 2.1 For $\mathcal{Z} = L_p(\Omega, \mathcal{F}, \mathbb{P})$, i.e. for the one-period case, the *Conditional* or *Average Value-at-Risk* at level $\alpha \in (0,1)$ (CVaR$_\alpha$ or AVaR$_\alpha$, cf. [19, 24] and [6, Chapter 4.4]) is given by

$$\mathrm{AVaR}_\alpha(z) := \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_{\bar\alpha}(z) \mathrm{d}\bar\alpha = \inf_{y_0 \in \mathbb{R}} \left\{ y_0 + \frac{1}{\alpha} \mathbb{E}[(y_0 + z)^-] \right\} \tag{5}$$

where the second representation on the right is due to [19]. By introducing variables for positive and negative parts of $y_0 + z$, respectively, AVaR$_\alpha$ can be rewritten in the form (2) with $J=1$, $k_0=1$, $k_1=2$, $c_0=1$, $c_1=(0,1/\alpha)$, $w_{1,0}=(1,-1)$, $w_{1,1}=-1$, $Y_0=\mathbb{R}$, and $Y_1=\mathbb{R}_+^2$. Hence, AVaR$_\alpha$ is a polyhedral risk measure. Moreover, AVaR$_\alpha$ is known to be a convex measure of risk in the sense of [6], a coherent risk measure in the sense of [1], and it is consistent with 2nd order stochastic dominance [16].

Example 2.2 Consider the *expected regret* or *expected loss* defined by

$$\mathrm{EL}_\beta(z) = \mathbb{E}[(z-\beta)^-]$$

with some fixed target $\beta \in \mathbb{R}$. This functional, too, can be written in the form (2) with $J=1$, $k_0=1$, $k_1=2$, $c_0=0$, $c_1=(0,\ 1)$, $w_{1,0}=(1,\ -1)$, $w_{1,1}=1$, $Y_0=\{\beta\}$, and $Y_1=\mathbb{R}_+\times\mathbb{R}_+$.

Example 2.3 For the multiperiod case $J>1$, not many instances of risk measures are known. In [3], four different multiperiod polyhedral risk measures, i.e. four possible choices for Y_j, c_j, and $w_{j,\tau}$, have been suggested, cf. Table 1. These instances ρ_1, ρ_2, ρ_3, and ρ_4 of (2) can be understood as multiperiod extensions of the Average Value-at-Risk. As a start, ρ_1 is just an average of AVaRs applied to different time stages, whereas ρ_2 is deduced herefrom by interchanging minimization and summation:

$$\rho_1(z) = \frac{1}{J}\sum_{j=1}^{J} \inf_{y_j\in\mathbb{R}} \left\{ y_j + \frac{1}{\alpha_j}\mathbb{E}\left[\left(z_{t_j}+y_j\right)^-\right]\right\}$$

$$\rho_2(z) = \inf_{y_0\in\mathbb{R}} \left\{ y_0 + \frac{1}{J}\sum_{j=1}^{J}\frac{1}{\alpha_j}\mathbb{E}\left[\left(z_{t_j}+y_0\right)^-\right]\right\}$$

The instances ρ_3 and ρ_4 are such that the information structure of the value process z has a definite impact. In particular, ρ_4 can be understood as the multiperiod extension of AVaR according to [18]. In [3] it is shown that each of these four risk measures is multiperiod coherent in the sense of [2].

Example 2.4 The multiperiod risk measure suggested in [13–15], is based on the concept of the *value of perfect information* (cf., e.g. [22, Chapter 1.2.5]). The risk measure $\mathcal{R}$ is defined as a difference of two functionals assessing the utility of a financial income stream $z=(z_{t_1},\dots,z_{t_J})$ with one functional being derived from the other one by relaxing the information constraints, i.e. by assuming that the actual values of all future incomes are perfectly known from the beginning (clairvoyance). Hence, the difference $\mathcal{R}(z)$ is supposed to measure the financial value of being clairvoyant. The utility functional (including the information constraints) is denoted by ρ_5. It is defined[1] as the optimal value of the following multistage model

$$\rho_5(z) = \inf \left\{ \begin{array}{l} -s_0 y_{0,1} + \mathbb{E}\left[\displaystyle\sum_{j=1}^{J-1}\left(-s_j y_{j,1} + q_j y_{j,3}\right) - d y_{J,2} + q_J y_{J,3}\right]: \\ y_j \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P};\ \mathbb{R}^3)\ (j=0,\dots,J),\ y_{0,2}=y_{0,3}=y_{J,1}=0, \\ y_{j,2}\geq 0\ \text{ a.s.},\ y_{j,3}\geq 0\ \text{ a.s. } (j=1,\dots,J), \\ y_{j,2}-y_{j,3} = y_{j-1,2} + z_{t_j} - y_{j-1,1}\ \text{ a.s. } (j=1,\dots,J) \end{array} \right\} \tag{6}$$

with given constants q_j (shortfall cost factors), s_j (surplus utility factors), and d (discount factor). For economic and mathematical consistency, these constants have to satisfy the relations $d < s_{J-1} < \cdots < s_1 < s_0$ and $s_{j-1} < q_j$ for $j=1,\dots,J$. The functional (6) is of the form (2) with $k_j=3$ $(j\geq 0)$, $Y_0=\mathbb{R}\times\{0\}\times\{0\}$, $Y_j=\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}_+$ $(1\leq j<J)$, $Y_J=\{0\}\times\mathbb{R}_+\times\mathbb{R}_+$, $w_{1,1}=(1,0,0)$, $w_{j,0}=(0,1,-1)$ $(j\geq 1)$, $w_{j,1}=(1,-1,0)$ $(j>1)$, further $w_{j,\tau}=0$ for $\tau>1$, $c_0=(-s_0,0,0)$, $c_j=(-s_j,\ 0,\ q_j)$ $(1\leq j<J)$, and $c_J=(0,-d,q_J)$. Hence, it is a

Table 1. Multiperiod polyhedral risk measures, i.e., instances of (2) from [3]. The notation $y_{j,k}$ refers to k-th component of $y_j \in \mathbb{R}^{k_j}$ for $k = 1, \ldots, k_j$ and $j = 0, \ldots, J$.

No.	Primal representation (2)
ρ_1	$\inf\left\{\frac{1}{J}\sum_{j=1}^{J}\left(y_{0,j}+\frac{1}{\alpha_j}\mathbb{E}[y_{j,2}]\right) \,\middle\vert\, \begin{array}{l} y_0 \in \mathbb{R}^J,\ y_j \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P}; \mathbb{R}^2)\ (j=1,\ldots,J), \\ y_j \in \mathbb{R}_+ \times \mathbb{R}_+ \quad \text{a.s. } (j=1,\ldots,J), \\ y_{j,1} - y_{j,2} = z_{t_j} + y_{0,j} \quad \text{a.s. } (j=1,\ldots,J) \end{array} \right\}$
ρ_2	$\inf\left\{ y_{0,1} + \frac{1}{J}\sum_{j=1}^{J}\frac{1}{\alpha_j}\mathbb{E}[y_{j,2}] \,\middle\vert\, \begin{array}{l} y_0 \in \mathbb{R},\ y_j \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P}; \mathbb{R}^2)\ (j=1,\ldots,J), \\ y_j \in \mathbb{R}_+ \times \mathbb{R}_+ \quad \text{a.s. } (j=1,\ldots,J), \\ y_{j,1} - y_{j,2} = z_{t_j} + y_{0,1} \quad \text{a.s. } (j=1,\ldots,J) \end{array} \right\}$
ρ_3	$\inf\left\{ y_{0,1} + \frac{1}{J}\sum_{j=1}^{J}\frac{1}{\alpha_j}\mathbb{E}[y_{j,2}] \,\middle\vert\, \begin{array}{l} y_0 \in \mathbb{R} \times \{0\},\ y_j \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P}; \mathbb{R}^2)\ (j=1,\ldots,J), \\ y_j \in \mathbb{R}_+ \times \mathbb{R}_+ \quad \text{a.s. } (j=1,\ldots,J), \\ y_{j,1} - y_{j,2} = z_{t_j} + y_{0,1} + y_{j-1,2} \quad \text{a.s. } (j=1,\ldots,J) \end{array} \right\}$
ρ_4	$\inf\left\{\frac{1}{J}\left(y_{0,1}+\sum_{j=1}^{J}\frac{1}{\alpha_j}\mathbb{E}[y_{j,2}]\right) \,\middle\vert\, \begin{array}{l} y_0 \in \mathbb{R},\ y_j \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P}; \mathbb{R}^2)\ (j=1,\ldots,J), \\ y_j \in \mathbb{R} \times \mathbb{R}_+ \quad \text{a.s. } (j=1,\ldots,J-1), \\ y_J \in \mathbb{R}_+ \times \mathbb{R}_+ \text{ a.s. },\ y_{j,1} - y_{j,2} = z_{t_j} + y_{j-1,1} \quad \text{a.s. } (j=1,\ldots,J) \end{array} \right\}$

polyhedral risk measure. It is a coherent risk measure in the sense of [2] if $s_1 = 1$ (cf. [3]). Being clairvoyant with respect to the income process z can be expressed by replacing $y_j \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P}; \mathbb{R}^3)$ in (6) by $y_j \in L_p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^3)$. This relaxation simplifies the utility functional drastically (cf. [14, 15]), such that the overall risk measure, i.e. the difference between ρ_5 and its clairvoyance modification, is given by

$$\mathcal{R}(z_{t_1}, \ldots, z_{t_J}) = \rho_5(z_{t_1}, \ldots, z_{t_J}) + \sum_{j=1}^{J} s_{j-1} \mathbb{E}[z_{t_j}].$$

Observe that $\mathcal{R}$ is always non-negative. Note that it is intended to apply this functional to income processes $z = (z_{t_1}, \ldots, z_{t_J})$ rather than to (accumulated) value processes. Hence, in problem (3), one has to replace

$$z_{t_j} = -\sum_{t=1}^{t_j} \langle b_t(\xi_t), x_t \rangle \quad \text{by } z_{t_j} = -\sum_{t=t_{j-1}+1}^{t_j} \langle b_t(\xi_t), x_t \rangle.$$

Remark 1 In [3], dual representations for (2) have been derived. For these results it is required that the following conditions for Y_j, c_j, and $w_{j,\tau}$ hold:

- *complete recourse*: $\langle w_{j,0},\ Y_j \rangle = \mathbb{R}$ $(j = 1, \ldots, J)$,
- *dual feasibility*: $\bigcap_{j=0}^{J} \mathcal{D}_{\rho,j} \neq \emptyset$ with $\mathcal{D}_{\rho,j} := \{u \in \mathbb{R}^J : c_j + \sum_{\nu=\max\{1,j\}}^{J} u_\nu w_{\nu,\nu-j} \in -Y_j^*\}$

By using the latter notation, the dual representation of (2) reads

$$\rho(z) = \sup \left\{ \inf_{y_0 \in Y_0} \left\langle c_0 + \sum_{\nu=1}^{J} \mathbb{E}[\lambda_\nu] w_{\nu,\nu}, y_0 \right\rangle - \mathbb{E}\left[\sum_{j=1}^{J} \lambda_j z_{t_j} \right] \,\middle|\, \begin{array}{l} \lambda \in \times_{j=1}^{J} L_{p'}(\Omega, \mathcal{F}_{t_j}, \mathbb{P}), \\ \mathbb{E}[\lambda | \mathcal{F}_{t_j}] \in \mathcal{D}_{\rho,j} \text{ a.s.} \\ (j = 1, \ldots, J) \end{array} \right\} \tag{7}$$

with $p' \in [1, \infty]$ such that $1/p + 1/p' = 1$, or

$$\rho(z) = \sup \left\{ -\mathbb{E}\left[\sum_{j=1}^{J} \lambda_j z_{t_j} \right] \,\middle|\, \begin{array}{l} \lambda \in \times_{j=1}^{J} L_{p'}(\Omega, \mathcal{F}_{t_j}, \mathbb{P}), \\ \mathbb{E}[\lambda | \mathcal{F}_{t_j}] \in \mathcal{D}_{\rho,j} \text{ a.s. } (j = 0, \ldots, J) \end{array} \right\} \tag{8}$$

for the case that Y_0 is a cone. Moreover, it has been shown in [3] that, if complete recourse and dual feasibility hold, the polyhedral risk measure ρ is *finite*, *continuous*, and *convex* on $\mathcal{Z}$. Furthermore, a criterion for (multiperiod) coherence (cf. [1,2]) based on the dual representation (8) has been stated.

3. Stability of multistage stochastic programs

Consider a multistage stochastic program of the form (3) with a polyhedral risk measure ρ of the form (2) in the objective. We will study the stability behaviour of its optimal value with respect to perturbations of the stochastic input process $\xi = (\xi_1, \ldots, \xi_T)$. One possible approach for this analysis would be to analyse the equivalent problem (4) which is similar to problem (1). However, it has turned out to be more fruitful to pursue another approach, namely to analyse sensitivity of ρ and then to use these results to analyse problem (3) directly.

For the sensitivity analysis of ρ resp. (2) with regard to perturbations of ξ in (3), observe that ρ does not only depend on $z=(z_{t_1},\dots,z_{t_J})$ but also depends on ξ via the σ-fields $\mathcal{F}_t=\sigma(\xi^t)$. Moreover, perturbations of ξ in (3) may cause variations of x and, hence, variations of z in (2) since $z=z(\xi,x)$ in (3). Therefore, we will use notations like $\rho(z,\xi)$ instead of just $\rho(z)$ in the sequel. Furthermore, we introduce the notation

$$\mathcal{Z}_\Xi := \left\{(z,\xi) : \xi \in L_r(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^s),\ z \in \times_{j=1}^J L_p(\Omega,\sigma(\xi^{t_j}),\mathbb{P})\right\}$$

for pairs of processes such that z is adapted to ξ. For $(z,\xi)\in\mathcal{Z}_\Xi$ we set

$$\mathcal{Y}_\rho(z,\xi) := \left\{ y \in \times_{j=0}^J L_p(\Omega,\sigma(\xi^{t_j}),\mathbb{P};\mathbb{R}^{k_j}) \,\middle|\, \begin{array}{l} y_j \in Y_j \ \text{ a.s. } (j=0,\dots,J), \\ \displaystyle\sum_{\tau=0}^{j} \langle w_{j,\tau}, y_{j-\tau}\rangle = z_{t_j} \ \text{ a.s.} \\ (j=1,\dots,J) \end{array} \right\} \tag{9}$$

for the feasible set of (2). Accordingly, for $y=(y_0,y_1,\dots,y_J)$, we set

$$F_\rho(y) := \mathbb{E}\left[\sum_{j=0}^J \langle c_j, y_j\rangle\right] \tag{10}$$

for the objective. With these notations formula (2) can be written in the following short form: $\rho(z,\xi)=\inf\{F_\rho(y): y\in\mathcal{Y}_\rho(z,\xi)\}$. For a given level $\varepsilon\geq 0$ the sets

$$l_{\rho,\varepsilon}(z,\xi) := \left\{y \in \mathcal{Y}_\rho(z,\xi) : F_\rho(y) \leq \rho(z,\xi)+\varepsilon\right\} \tag{11}$$

are called the *level-sets*. For $\varepsilon>0$ these level-sets are non-empty. For $\varepsilon=0$, $l_{\rho,0}(z,\xi)=: S_\rho(z,\xi)$ is called the *solution set*.

Proposition 3.1 *Consider a multiperiod polyhedral risk measure ρ of the form* (2) *on $\mathcal{Z}_\Xi$ satisfying complete recourse and dual feasibility. Then there exists a constant $K_\rho>0$ such that*

$$|\rho(z,\xi)-\rho(\tilde z,\tilde\xi)| \leq K_\rho\Big(\|z-\tilde z\|_p + D_{f,\rho}\big((z,\xi),(\tilde z,\tilde\xi)\big)\Big) \tag{12}$$

for $(z,\xi),(\tilde z,\tilde\xi)\in\mathcal{Z}_\Xi$. Here, $D_{f,\rho}$ denotes the filtration distance for ρ given by

$$D_{f,\rho}((z,\xi),(\tilde z,\tilde\xi)) := \sup_{\varepsilon>0} D_{f,\rho,\varepsilon}((z,\xi),(\tilde z,\tilde\xi))$$

$$D_{f,\rho,\varepsilon}((z,\xi),(\tilde z,\tilde\xi)) := \inf\left\{\sum_{j=1}^{J-1}\max\left\{\left\|\bar y_j-\mathbb{E}\left[\bar y_j|\sigma(\tilde\xi^{t_j})\right]\right\|_p, \left\|\tilde y_j-\mathbb{E}\left[\tilde y_j|\sigma(\xi^{t_j})\right]\right\|_p\right\} \,\middle|\, \begin{array}{l}\bar y\in l_{\rho,\varepsilon}(z,\xi),\\ \tilde y\in l_{\rho,\varepsilon}(\tilde z,\tilde\xi)\end{array}\right\}.$$

Proof Let $\varepsilon>0$, $(z,\xi),(\tilde z,\tilde\xi)\in\mathcal{Z}_\Xi$, and $\bar y=(\bar y_0,\bar y_1,\dots,\bar y_J)\in l_{\rho,\varepsilon}(z,\xi)$. In the following, an element $\tilde y=(\tilde y_0,\tilde y_1,\dots,\tilde y_J)\in\mathcal{Y}_\rho(\tilde z,\tilde\xi)$ is recursively constructed such that the distance between $\tilde y$ and $\mathbb{E}[\bar y_j|\sigma(\tilde\xi^{t_j})]$ is small in some sense. To this end, consider the set-valued mappings (multifunctions)

$$\begin{aligned} M_j &: \mathbb{R} \rightrightarrows \mathbb{R}^{k_j} \\ &\quad u \mapsto M_j(u) := \{y_j \in Y_j : \langle w_{j,0}, y_j\rangle = u\} \end{aligned}$$

for $j = 1, \ldots, J$. Note that each M_j has polyhedral graph and, hence, is Lipschitz continuous with respect to the Hausdorff distance [20, Example 9.35] with some modulus l_j. Thus,

$$\inf_{y \in M_j(u)} |\hat{y} - y| \leq l_j |\hat{u} - u| \tag{13}$$

for all (non-random) $\hat{u}, u \in \mathbb{R}$ and $\hat{y} \in M_j(\hat{u})$. Here, $|\cdot|$, denotes the Euclidian norm in $\mathbb{R}^{k_j}$. Now, the random element $\tilde{y}$ is constructed as follows: for $j = 0$, we set $\tilde{y}_0 := \bar{y}_0$. For $j > 0$, consider the random elements

$$\bar{u}_j(.) := z_{t_j}(.) - \sum_{\tau=1}^{j} \langle w_{j,\tau}, \bar{y}_{j-\tau}(.) \rangle \qquad \tilde{u}_j(.) := \tilde{z}_{t_j}(.) - \sum_{\tau=1}^{j} \langle w_{j,\tau}, \tilde{y}_{j-\tau}(.) \rangle$$

as well as the following set-valued mappings:

$$\begin{array}{ll} M_j^1 : \Omega \rightrightarrows \mathbb{R}^{k_j} & M_j^2 : \Omega \rightrightarrows \mathbb{R}^{k_j} \\ \omega \mapsto M_j(\tilde{u}_j(\omega)) & \omega \mapsto \arg \min\limits_{y \in M_j^1(\omega)} |\mathbb{E}[\bar{y}_j | \sigma(\tilde{\xi}^{t_j})](\omega) - y| \end{array}$$

Obviously, $M_j^1(\omega)$ is closed, convex, and, due to the complete recourse assumption, non-empty for every $\omega \in \Omega$. $M_j^2(\omega)$ is non-empty for $\omega \in \Omega$ because the distance function $|\mathbb{E}[\bar{y}_j | \sigma(\tilde{\xi}^{t_j})](\omega) - .|$ is coercive. Further, since $\tilde{u}_j$ is measurable with respect to $\sigma(\tilde{\xi}^{t_j}), M_j^1$ and M_j^2 are measurable with respect to $\sigma(\tilde{\xi}^{t_j})$; cf., e.g. [20, Theorem 14.36] and [20, Theorem 14.37]. The latter theorem also guarantees the existence of a $\sigma(\tilde{\xi}^{t_j})$-measurable function $\tilde{y}_j$ with $\tilde{y}_j(\omega) \in M_j^2(\omega)$ for $\omega \in \Omega$. Now, using (13) with $\hat{y} = \mathbb{E}[\bar{y}_j | \sigma(\tilde{\xi}^{t_j})](\omega)$, $\hat{u} = \mathbb{E}[\bar{u}_j | \sigma(\tilde{\xi}^{t_j})](\omega)$, $u = \tilde{u}_j(\omega)$, and $y = \tilde{y}_j(\omega)$ (note that $\tilde{y}_j$ was chosen as a pointwise minimizer) yields the estimate

$$\begin{aligned} \left| \mathbb{E}\left[\bar{y}_j | \sigma\left(\tilde{\xi}^{t_j}\right)\right] - \tilde{y}_j \right| &\leq l_j \left| \mathbb{E}\left[\bar{u}_j | \sigma\left(\tilde{\xi}^{t_j}\right)\right] - \tilde{u}_j \right| \\ &= l_j \left| \mathbb{E}\left[z_{t_j} - \sum_{\tau=1}^{j} \langle w_{j,\tau}, \bar{y}_{j-\tau} \rangle | \sigma(\tilde{\xi}^{t_j}) \right] - \tilde{z}_{t_j} + \sum_{\tau=1}^{j} \langle w_{j,\tau}, \tilde{y}_{j-\tau} \rangle \right| \\ &\leq l_j \left(\left| \mathbb{E}\left[z_{t_j} - \tilde{z}_{t_j} | \sigma\left(\tilde{\xi}^{t_j}\right)\right] \right| + \sum_{\tau=1}^{j} |w_{j,\tau}| \left| \mathbb{E}\left[\bar{y}_{j-\tau} | \sigma\left(\tilde{\xi}^{t_j}\right)\right] - \tilde{y}_{j-\tau} \right| \right) \\ &\leq l_j \left(\mathbb{E}\left[|z_{t_j} - \tilde{z}_{t_j}| | \sigma\left(\tilde{\xi}^{t_j}\right)\right] + \sum_{\tau=1}^{j} |w_{j,\tau}| \left| \mathbb{E}\left[\bar{y}_{j-\tau} | \sigma\left(\tilde{\xi}^{t_{j-\tau}}\right)\right] - \tilde{y}_{j-\tau} \right| \right. \\ &\quad \left. + \sum_{\tau=1}^{j} |w_{j,\tau}| \left| \mathbb{E}\left[\bar{y}_{j-\tau} | \sigma\left(\tilde{\xi}^{t_{j-\tau}}\right)\right] - \mathbb{E}\left[\bar{y}_{j-\tau} | \sigma\left(\tilde{\xi}^{t_j}\right)\right] \right| \right) \end{aligned}$$

pointwise on Ω for $j = 1, \ldots, J$. Note that Jensen's inequality has been used for the first term of the final estimate. Putting these estimates together recursively (recall that $\bar{y}_0 = \tilde{y}_0$) yields

$$\begin{aligned} \left| \mathbb{E}\left[\bar{y}_j | \sigma\left(\tilde{\xi}^{t_j}\right)\right] - \tilde{y}_j \right| &\leq \sum_{i=1}^{j} K_{j,i} \mathbb{E}\left[|z_{t_i} - \tilde{z}_{t_i}| | \sigma\left(\tilde{\xi}^{t_i}\right)\right] \\ &\quad + \sum_{i=1}^{j} \sum_{\tau=1}^{i} C_{j,i,\tau} \left| \mathbb{E}\left[\bar{y}_{i-\tau} | \sigma\left(\tilde{\xi}^{t_{i-\tau}}\right)\right] - \mathbb{E}\left[\bar{y}_{i-\tau} | \sigma\left(\tilde{\xi}^{t_i}\right)\right] \right| \end{aligned}$$

with some positive constants $K_{j,i}$ and $C_{j,i,\tau}$. Hence, since $\bar{y} \in l_{\rho,\varepsilon}(z,\xi)$ and $\tilde{y} \in \mathcal{Y}_\rho(\tilde{z},\tilde{\xi})$, we have

$$\begin{aligned}
\rho(\tilde{z},\tilde{\xi}) - \rho(z,\xi) &\le \mathbb{E}\left[\sum_{j=0}^{J}\langle c_j, \tilde{y}_j\rangle\right] - \mathbb{E}\left[\sum_{j=0}^{J}\langle c_j, \bar{y}_j\rangle\right] + \varepsilon \\
&= \sum_{j=1}^{J}\mathbb{E}\left[\left\langle c_j, \tilde{y}_j - \mathbb{E}\left[\bar{y}_j|\sigma\left(\tilde{\xi}^{t_j}\right)\right]\right\rangle\right] + \varepsilon \\
&\le \sum_{j=1}^{J}|c_j|\mathbb{E}\left[\left|\tilde{y}_j - \mathbb{E}\left[\bar{y}_j|\sigma\left(\tilde{\xi}^{t_j}\right)\right]\right|\right] + \varepsilon \\
&\le \sum_{j=1}^{J}\left(K_j\mathbb{E}\left[|z_{t_j} - \tilde{z}_{t_j}|\right] + \sum_{\tau=1}^{j} C_{j,\tau}\mathbb{E}\left[\left|\mathbb{E}\left[\bar{y}_{j-\tau}|\sigma\left(\tilde{\xi}^{t_{j-\tau}}\right)\right] - \bar{y}_{j-\tau}\right|\right]\right) + \varepsilon \\
&= \sum_{j=1}^{J}K_j\mathbb{E}\left[|z_{t_j} - \tilde{z}_{t_j}|\right] + \sum_{j=1}^{J-1}C_j\mathbb{E}\left[\left|\mathbb{E}\left[\bar{y}_j|\sigma\left(\tilde{\xi}^{t_j}\right)\right] - \bar{y}_j\right|\right] + \varepsilon \\
&\le C\left(\|z-\tilde{z}\|_p + \sum_{j=1}^{J-1}\left\|\mathbb{E}\left[\bar{y}_j|\sigma\left(\tilde{\xi}^{t_j}\right)\right] - \bar{y}_j\right\|_p\right) + \varepsilon
\end{aligned}$$

with some other positive constants K_j, $C_{j,\tau}$, C_j, and C. Observe that the terms in the final line of the previous display do not depend on $\tilde{y}$ which has been constructed dependent on an arbitrary $\bar{y} \in l_{\rho,\varepsilon}(z,\xi)$. Thus, the roles of (z,ξ) and $(\tilde{z},\tilde{\xi})$ can be changed, i.e. for arbitrary $\tilde{y} \in l_{\rho,\varepsilon}(\tilde{z},\tilde{\xi})$ it holds that

$$\rho(z,\xi) - \rho(\tilde{z},\tilde{\xi}) \le \hat{C}\left(\|z-\tilde{z}\|_p + \sum_{j=1}^{J-1}\|\mathbb{E}[\tilde{y}_j|\sigma(\xi^{t_j})] - \tilde{y}_j\|_p\right) + \varepsilon$$

with some positive constant $\hat{C}$. With $K_\rho := \max\{C, \hat{C}\}$ it follows that

$$|\rho(z,\xi) - \rho(\tilde{z},\tilde{\xi})| \le K_\rho\left(\|z-\tilde{z}\|_p + \sum_{j=1}^{J-1}\max\left\{\left\|\bar{y}_j - \mathbb{E}\left[\bar{y}_j|\sigma\left(\tilde{\xi}^{t_j}\right)\right]\right\|_p, \|\tilde{y}_j - \mathbb{E}[\tilde{y}_j|\sigma(\xi^{t_j})]\|_p\right\}\right) + \varepsilon$$

for arbitrary $\bar{y} \in l_{\rho,\varepsilon}(z,\xi)$ and $\tilde{y} \in l_{\rho,\varepsilon}(\tilde{z},\tilde{\xi})$. Hence, we can pass to the infimum arriving at

$$\begin{aligned}
|\rho(z,\xi) - \rho(\tilde{z},\tilde{\xi})| &\le K_\rho\left(\|z-\tilde{z}\|_p + D_{f,\rho,\varepsilon}((z,\xi),(\tilde{z},\tilde{\xi}))\right) + \varepsilon \\
&\le K_\rho\left(\|z-\tilde{z}\|_p + D_{f,\rho}((z,\xi),(\tilde{z},\tilde{\xi}))\right) + \varepsilon
\end{aligned}$$

and because ε was chosen arbitrarily the assertion follows. ∎

Next, we will make use of the latter result for the analysis of the risk-averse stochastic program (3). To this end, we introduce similar notations as for ρ that stress the dependence on ξ:

$$F(\xi, x) := \rho(z(\xi, x))$$

for the objective with

$$z(\xi,x) := (z_{t_1}(\xi,x),\ldots,z_{t_J}(\xi,x)) \quad z_t(\xi,x) = z_t(\xi^t,x^t) := -\sum_{\tau=1}^{t}\langle b_\tau(\xi_\tau),x_\tau\rangle$$

and

$$\mathcal{X}(\xi) := \left\{ x \in \times_{t=1}^{T} L_{r'}(\Omega,\sigma(\xi^t),\mathbb{P};\mathbb{R}^{m_t}) \middle| \begin{array}{l} x_1 \in X_1, \\ x_t \in \mathcal{X}_t(x^{t-1},\xi^t) \text{ a.s. } (t=2,\ldots,T) \end{array} \right\}$$

for the constraint set with

$$\mathcal{X}_t(x^{t-1},\xi_t) := \left\{ x_t \in X_t\colon \sum_{\tau=0}^{t-1} A_{t,\tau}(\xi_t)x_{t-\tau} = h_t(\xi_t) \right\} \subseteq \mathbb{R}^{m_t}$$

for $t=2,\ldots,T$. Then the model (3) can be written in the following short form:

$$\min\{F(\xi,x)\colon x\in\mathcal{X}(\xi)\} \tag{14}$$

and with $v(\xi):=\inf\{F(\xi,x)\colon x\in\mathcal{X}(\xi)\}$ we denote its optimal value. For any $\varepsilon\geq 0$ let

$$l_\varepsilon(F(\xi,\cdot)) := \{x\in\mathcal{X}(\xi)\colon F(\xi,x)\leq v(\xi)+\varepsilon\}$$

denote its ε-*level-set*. For the integrability numbers $r,r',p\geq 1$, we will set r and r' in dependence of the class of problem (3) by the assignment

$$r := \begin{cases} \in [p,\infty) \text{ arbitrarily}, & \text{if only costs or right-hand sides are random} \\ 2p, & \text{if only costs and right-hand sides are random} \\ pT, & \text{if all technology matrices are random} \end{cases}$$
$$r' := \begin{cases} \dfrac{pr}{r-p}, & \text{if only costs are random} \\ r, & \text{if right-hand sides are random but technology matrices aren't} \\ \infty, & \text{if all technology matrices are random} \end{cases} \tag{15}$$

which implies $r\geq p$ and $r'\geq p$. We will consider the following conditions for the optimization model (3):

(A1) $\xi\in L_r(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^s)$

(A2) There exists a $\delta_1>0$ such that for any $\tilde{\xi}\in L_r(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^s)$ with $\|\tilde{\xi}-\xi\|_r\leq\delta_1$, any $t=2,\ldots,T$ and any $x_1\in X_1$, $x_\tau\in L_{r'}(\Omega,\sigma(\tilde{\xi}^\tau),\mathbb{P};\mathbb{R}^{m_\tau})$ with $x_\tau\in\mathcal{X}_\tau(x^{\tau-1},\tilde{\xi}_\tau)$, $\tau=2,\ldots,t-1$, the t-th feasibility set $\mathcal{X}_t(x^{t-1},\tilde{\xi}_t)$ is non-empty (*relatively complete recourse locally around* ξ).

(A3) The optimal values $v(\tilde{\xi})$ of (14) with input $\tilde{\xi}$ are finite for all $\tilde{\xi}$ in a neighbourhood of ξ and the objective function F is *level-bounded locally uniformly at* ξ: for some $\varepsilon_0>0$ there exists a $\delta_2>0$ and a bounded set $B\subseteq L_{r'}(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)$ such that $v(\tilde{\xi})\in\mathbb{R}$ and $l_{\varepsilon_0}(F(\tilde{\xi},\cdot))\subseteq B$ for all $\tilde{\xi}\in L_r(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^s)$ with $\|\tilde{\xi}-\xi\|_r\leq\delta_2$.

(A4) The recourse matrices $A_{t,0}(\xi_t)$ are fixed, i.e. they do not depend on ξ_t $(t=1,\ldots,T)$.

THEOREM 3.2 *For the multistage stochastic program* (3) *respectively* (14), *let* $p \in [1, \infty)$ *and* r *and* r' *be defined by* (15) *and assume that the multiperiod polyhedral risk measure* ρ *on* $\mathcal{Z}_\Xi$ *of the form* (2) *satisfies complete recourse and dual feasibility. Furthermore, let* (A1)–(A4) *be satisfied and* X_1 *be bounded. Then there exist positive constants* K, ε_0 *and* δ *such that the estimate*

$$|v(\xi) - v(\tilde{\xi})| \leq K\Big(\|\xi - \tilde{\xi}\|_r + D_{\mathrm{f}}^{\rho,\mathcal{X}}(\xi, \tilde{\xi})\Big) \tag{16}$$

holds for all random elements $\tilde{\xi} \in L_r(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^s)$ *with* $\|\tilde{\xi} - \xi\|_r \leq \delta$. *Here, the filtration distance* $D_{\mathrm{f}}^{\rho,\mathcal{X}}(\xi, \tilde{\xi})$ *is given by*

$$D_{\mathrm{f}}^{\rho,\mathcal{X}}(\xi, \tilde{\xi}) := \sup_{\varepsilon \in (0, \varepsilon_0]} D_{\mathrm{f},\varepsilon}^{\rho,\mathcal{X}}(\xi, \tilde{\xi})$$

$$D_{\mathrm{f},\varepsilon}^{\rho,\mathcal{X}}(\xi, \tilde{\xi}) := \inf\left\{\begin{array}{l} \displaystyle\sum_{t=2}^{T-1} \max\Big\{\Big\|\mathbb{E}\Big[x_t|\sigma\Big(\tilde{\xi}^t\Big)\Big] - x_t\Big\|_{r'}, \|\mathbb{E}[\tilde{x}_t|\sigma(\xi^t)] - \tilde{x}_t\|_{r'}\Big\} \\ \displaystyle\quad + \sum_{j=1}^{J-1} \max\Big\{\Big\|\mathbb{E}\Big[y_j|\sigma\Big(\tilde{\xi}^{t_j}\Big)\Big] - y_j\Big\|_p, \|\mathbb{E}[\tilde{y}_j|\sigma(\xi^{t_j})] - \tilde{y}_j\|_p\Big\} \end{array}\right\}$$

where the infimum is taken with respect to all $x \in l_\varepsilon(F(\xi, \cdot))$, $\tilde{x} \in l_\varepsilon(F(\tilde{\xi}, \cdot))$, $y \in l_{\rho,\varepsilon}(z(\xi, x), \xi)$, *and* $\tilde{y} \in l_{\rho,\varepsilon}(z(\tilde{\xi}, \tilde{x}), \tilde{\xi})$.

Proof For the sake of clarity and without loss of generality we restrict the following presentation to the case that $A_{t,\tau} = 0$ for $\tau \geq 2$. Since [11, Theorem 2.1] deals with the same problem but with expectation objective, we will use here some formulas from the proof of [11, Theorem 2.1] whose derivation does not depend on the objective.

Let ε_0, δ_1, and δ_2 be selected as in (A2) and (A3) and set $\delta := \min\{\delta_1, \delta_2\} > 0$. Let $\varepsilon \in (0, \varepsilon_0]$. First, recall from the proof of Proposition 3.1 that there exists a positive constant K_ρ such that

$$\begin{aligned} \rho(\tilde{z}, \tilde{\xi}) - \rho(z, \xi) &\leq K_\rho\left(\|z - \tilde{z}\|_p + \sum_{j=1}^{J-1}\Big\|\mathbb{E}\Big[\bar{y}_j|\sigma\Big(\tilde{\xi}^{t_j}\Big)\Big] - \bar{y}_j\Big\|_p\right) + \varepsilon \\ \rho(z, \xi) - \rho(\tilde{z}, \tilde{\xi}) &\leq K_\rho\left(\|z - \tilde{z}\|_p + \sum_{j=1}^{J-1}\|\mathbb{E}[\tilde{y}_j|\sigma(\xi^{t_j})] - \tilde{y}_j\|_p\right) + \varepsilon \end{aligned} \tag{17}$$

holds for all $\bar{y} \in l_{\rho,\varepsilon}(z, \xi)$ and $\tilde{y} \in l_{\rho,\varepsilon}(\tilde{z}, \tilde{\xi})$ and all pairs (z, ξ) and $(\tilde{z}, \tilde{\xi})$ in $\mathcal{Z}_\Xi$. Now, let $\bar{x} \in l_\varepsilon(F(\xi, \cdot))$ and $\tilde{\xi} \in L_r(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^s)$ be such that $\|\tilde{\xi} - \xi\|_r < \delta$. In the following, we construct $\tilde{x} \in \mathcal{X}(\tilde{\xi})$ in the same manner as in the proof of [11, Theorem 2.1] (similarly to $\tilde{y}$ in the proof of Proposition 3.1) such that $\bar{x}_1 = \tilde{x}_1$ and the estimate[2]

$$\begin{aligned} |\mathbb{E}[\bar{x}_t|\sigma(\tilde{\xi}^t)] - \tilde{x}_t| \leq l_t\Bigg(&\sum_{\tau=2}^{t} \max\Big\{1, \Big|\tilde{\xi}^t\Big|^{t-\tau}\Big\}\mathbb{E}\Big[\Big|\xi_\tau - \tilde{\xi}_\tau\Big|\,|\sigma\Big(\tilde{\xi}^\tau\Big)\Big] \\ &+ \sum_{\tau=2}^{t-1} \max\Big\{1, \Big|\tilde{\xi}^t\Big|^{t-\tau}\Big\}\mathbb{E}\Big[\Big|\bar{x}_\tau - \mathbb{E}\Big[\bar{x}_\tau|\sigma\Big(\tilde{\xi}^\tau\Big)\Big]\Big|\,|\sigma\Big(\tilde{\xi}^{\tau+1}\Big)\Big]\Bigg) \end{aligned} \tag{18}$$

holds with some positive constants l_t for $t = 2, \ldots, T$. Note that the first sum on the right-hand side disappears if only costs are random and that both max-terms vanish if the technology matrices are not random.

Now, because $\bar{x} \in l_\varepsilon(F(\xi, .))$ and $\tilde{x} \in \mathcal{X}(\tilde{\xi})$, the optimal values fulfil for any $\bar{y} \in l_{\rho,\varepsilon}(z(\xi, \bar{x}), \xi)$

$$\begin{aligned} v(\tilde{\xi}) - v(\xi) &\le \rho(z(\tilde{\xi}, \tilde{x}), \tilde{\xi}) - \rho(z(\xi, \bar{x}), \xi) + \varepsilon \\ &\le K_\rho\left(\left\|z(\xi, \bar{x}) - z(\tilde{\xi}, \tilde{x})\right\|_p + \sum_{j=1}^{J-1}\left\|\mathbb{E}\left[\bar{y}_j|\sigma\left(\tilde{\xi}^{t_j}\right)\right] - \bar{y}_j\right\|_p\right) + 2\varepsilon \end{aligned} \tag{19}$$

where (17) is used for the second estimate.

Next, we derive an estimate for $\|z(\xi, \bar{x}) - z(\tilde{\xi}, \tilde{x})\|_p$ by making use of (18). With $\hat{x}_t := \mathbb{E}[\bar{x}_t|\sigma(\tilde{\xi}^t)]$ and $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_T)$ we have

$$\|z(\xi, \bar{x}) - z(\tilde{\xi}, \tilde{x})\|_p \le \|z(\xi, \bar{x}) - z(\tilde{\xi}, \bar{x})\|_p + \|z(\tilde{\xi}, \bar{x}) - z(\tilde{\xi}, \hat{x})\|_p + \|z(\tilde{\xi}, \hat{x}) - z(\tilde{\xi}, \tilde{x})\|_p \tag{20}$$

and for the first summand we obtain

$$\begin{aligned} \left\|z(\xi, \bar{x}) - z(\tilde{\xi}, \bar{x})\right\|_p &= \left(\mathbb{E}\left[\sum_{j=1}^{J}|z_{t_j}(\xi, \bar{x}) - z_{t_j}(\tilde{\xi}, \bar{x})|^p\right]\right)^{1/p} \\ &= \left(\sum_{j=1}^{J}\mathbb{E}\left[\left|\sum_{t=1}^{t_j}\left\langle b_t(\xi_t) - b_t\left(\tilde{\xi}_t\right), \bar{x}_t\right\rangle\right|^p\right]\right)^{1/p} \\ &\le \sum_{j=1}^{J}\left(\mathbb{E}\left[\left|\sum_{t=1}^{t_j}\left\langle b_t(\xi_t) - b_t(\tilde{\xi}_t), \bar{x}_t\right\rangle\right|^p\right]\right)^{1/p} \\ &\le \sum_{j=1}^{J}\sum_{t=1}^{t_j}\left(\mathbb{E}\left[\left|\left\langle b_t(\xi_t) - b_t(\tilde{\xi}_t), \bar{x}_t\right\rangle\right|^p\right]\right)^{1/p} \\ &\le J\sum_{t=1}^{T}\left(\mathbb{E}\left[\left|\left\langle b_t(\xi_t) - b_t(\tilde{\xi}_t), \bar{x}_t\right\rangle\right|^p\right]\right)^{1/p} \\ &\le J\sum_{t=1}^{T}\left(\mathbb{E}\left[\left|b_t(\xi_t) - b_t(\tilde{\xi}_t)\right|^p|\bar{x}_t|^p\right]\right)^{1/p} \\ &\le J\sum_{t=1}^{T}\left\|b_t(\xi_t) - b_t(\tilde{\xi}_t)\right\|_r\|\bar{x}_t\|_{r'} \end{aligned}$$

where Minkowski's inequality in $L^p(\Omega, \mathcal{F}, \mathbb{P})$ as well as the Cauchy–Schwarz inequality in $\mathbb{R}^{m_t}$ have been used. For the final estimate, a generalised version of Hölder's inequality has been used which is valid for $1/r + 1/r' = 1/p$ (the case of stochastic cost and deterministic technology matrices) as well as for $p \le r < r' = \infty$ (the case of stochastic technology matrices). For the case that only right-hand sides are random, this estimate is also valid, because then the deterministic[3] cost factors b_t can be moved outside the expectation and Lyapunov's inequality yields the same result. Now, since $\bar{x} \in B$, B is $L_{r'}$-bounded and $b_t(\cdot)$ is affine linear, it holds that that

$$\|z(\xi, \bar{x}) - z(\tilde{\xi}, \bar{x})\|_p \le C_1\|\xi - \tilde{\xi}\|_r$$

with some positive constant C_1 depending on B and b_t $(t=1,\dots,T)$. For the second and the third summand in (20) we conclude analogously:

$$\begin{aligned}\left\|z(\tilde{\xi},\bar{x})-z(\tilde{\xi},\hat{x})\right\|_p &\le J\sum_{t=1}^{T}\left(\mathbb{E}\left[|b_t(\tilde{\xi}_t)|^p\left|\bar{x}_t-\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]\right|^p\right]\right)^{1/p}\\ &\le J\sum_{t=1}^{T}\left\|b_t\left(\tilde{\xi}_t\right)\right\|_r\left\|\bar{x}_t-\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]\right\|_{r'}\\ \|z(\tilde{\xi},\hat{x})-z(\tilde{\xi},\tilde{x})\|_p &\le J\sum_{t=1}^{T}\left(\mathbb{E}\left[\left|b_t(\tilde{\xi}_t)\right|^p\left|\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]-\tilde{x}_t\right|^p\right]\right)^{1/p}\\ &\le J\sum_{t=1}^{T}\left\|b_t(\tilde{\xi}_t)\right\|_r\left\|\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]-\tilde{x}_t\right\|_{r'}\end{aligned}$$

where we have re-substituted $\hat{x}_t=\mathbb{E}[\bar{x}_t|\sigma(\tilde{\xi}^t)]$. Since $\xi\in L_r(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^s)$ and $\|\xi-\tilde{\xi}\|_r\le\delta$, it holds that

$$\begin{aligned}\left\|z(\tilde{\xi},\bar{x})-z(\tilde{\xi},\hat{x})\right\|_p &\le C_2\sum_{t=1}^{T}\left\|\bar{x}_t-\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]\right\|_{r'}\\ \left\|z(\tilde{\xi},\hat{x})-z(\tilde{\xi},\tilde{x})\right\|_p &\le C_2\sum_{t=1}^{T}\left\|\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]-\tilde{x}_t\right\|_{r'}\end{aligned}$$

with some positive constant C_2 depending on ξ,δ, and b_t $(t=1,\dots,T)$. Now, the latter estimate will be continued by inserting (18).

First, we consider the situation that only cost are random and $r'<\infty$. We use Minkowski's and Jensen's inequalities and arrive at

$$\begin{aligned}\left\|z(\tilde{\xi},\hat{x})-z(\tilde{\xi},\tilde{x})\right\|_p &\le C_2\sum_{t=1}^{T}\left(\mathbb{E}\left[\left|\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]-\tilde{x}_t\right|^{r'}\right]\right)^{1/r'}\\ &\le C_2\sum_{t=1}^{T}l_t\left(\mathbb{E}\left[\left|\sum_{\tau=2}^{t-1}\mathbb{E}\left[\left|\bar{x}_\tau-\mathbb{E}\left[\bar{x}_\tau|\sigma\left(\tilde{\xi}^\tau\right)\right]\right|\,|\sigma\left(\tilde{\xi}^{\tau+1}\right)\right]\right|^{r'}\right]\right)^{\frac{1}{r'}}\\ &\le C_3\sum_{t=1}^{T}\left(\mathbb{E}\left[\left|\bar{x}_t-\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]\right|^{r'}\right]\right)^{1/r'}\end{aligned}$$

with some positive constant C_3.

Next, we consider the situation that right-hand sides are random but technology matrices are non-random. Then we have $r=r'<\infty$ and analogously we obtain

$$\begin{aligned}\left\|z\left(\tilde{\xi},\hat{x}\right)-z\left(\tilde{\xi},\tilde{x}\right)\right\|_p &\le C_2\sum_{t=1}^{T}\left(\mathbb{E}\left[\left|\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]-\tilde{x}_t\right|^{r'}\right]\right)^{1/r'}\\ &\le C_2\sum_{t=1}^{T}l_t\left(\mathbb{E}\left[\left|\sum_{\tau=2}^{t}\mathbb{E}\left[\left|\xi_\tau-\tilde{\xi}_\tau\right|\,\left|\sigma\left(\tilde{\xi}^\tau\right)\right.\right]+\sum_{\tau=2}^{t-1}\mathbb{E}\left[\left|\bar{x}_\tau-\mathbb{E}\left[\bar{x}_\tau|\sigma\left(\tilde{\xi}^\tau\right)\right]\right|\,\left|\sigma(\tilde{\xi}^{\tau+1})\right.\right]\right|^{r'}\right]\right)^{1/r'}\\ &\le C_4\left(\left\|\xi-\tilde{\xi}\right\|_r+\sum_{t=1}^{T}\left(\mathbb{E}\left[\left|\bar{x}_t-\mathbb{E}\left[\bar{x}_t|\sigma(\tilde{\xi}^t)\right]\right|^{r'}\right]\right)^{1/r'}\right)\end{aligned}$$

with some constant C_4.

Finally, we consider the case that the technology matrices are random and $r = Tp < r' = \infty$. Then, however, we need to start at the point *before* Hölder's inequality was applied and obtain

$$\begin{aligned}\left\|z\left(\tilde{\xi},\hat{x}\right) - z\left(\tilde{\xi},\tilde{x}\right)\right\|_p &\leq J\sum_{t=1}^{T}\left(\mathbb{E}\left[\left|b_t\left(\tilde{\xi}_t\right)\right|^p\left|\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right] - \tilde{x}_t\right|^p\right]\right)^{1/p} \\ &\leq J\sum_{t=1}^{T} l_t\left(\mathbb{E}\left[\left|b_t\left(\tilde{\xi}_t\right)\right|^p\left|\sum_{\tau=2}^{t}\max\left\{1,\left|\tilde{\xi}^t\right|^{t-\tau}\right\}\mathbb{E}\left[\left|\xi_\tau - \tilde{\xi}_\tau\right|\,|\sigma\left(\tilde{\xi}^\tau\right)\right]\right.\right.\right. \\ &\quad \left.\left.\left. + \sum_{\tau=2}^{t-1}\max\left\{1,\left|\tilde{\xi}^t\right|^{t-\tau}\right\}\mathbb{E}\left[\left|\bar{x}_\tau - \mathbb{E}\left[\bar{x}_\tau|\sigma\left(\tilde{\xi}^\tau\right)\right]\right|\,|\sigma(\tilde{\xi}^{\tau+1})\right]\right|^p\right]\right)^{1/p} \\ &\leq C_5\left(\left\|\xi - \tilde{\xi}\right\|_r + \sum_{t=1}^{T}\left\|\bar{x}_t - \mathbb{E}\left[\bar{x}_t|\sigma(\tilde{\xi}^t)\right]\right\|_\infty\right)\end{aligned}$$

with a constant C_5 depending on $b_t(\cdot)$, $\|\xi\|_r$, and δ^r.

Hence, in all cases we can bound each of the three summands on the right-hand of (20) suitably, i.e. in each case there is a constant C such that

$$\left\|z(\xi,\bar{x}) - z(\tilde{\xi},\tilde{x})\right\|_p \leq C\left(\left\|\xi - \tilde{\xi}\right\|_r + \sum_{t=1}^{T}\left\|\bar{x}_t - \mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]\right\|_{r'}\right)$$

holds for each $\bar{x} \in l_\varepsilon(F(\xi,.))$ (and $\tilde{x}$ constructed appropriately). Hence, we can continue (19) as follows:

$$v(\tilde{\xi}) - v(\xi) \leq \bar{K}\left(\left\|\xi - \tilde{\xi}\right\|_r + \sum_{t=2}^{T-1}\left\|\bar{x}_t - \mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right]\right\|_{r'} + \sum_{j=1}^{J-1}\left\|\bar{y}_j - \mathbb{E}\left[\bar{y}_j|\sigma\left(\tilde{\xi}^{t_j}\right)\right]\right\|_p\right) + 2\varepsilon \quad (21)$$

with some positive constant $\bar{K}$. The estimate is valid for any $\bar{x} \in l_\varepsilon(F(\xi,.))$ and any $\bar{y} \in l_{\rho,\varepsilon}(z(\xi,\bar{x}),\xi)$ and does no longer depend on $\tilde{x}$. Changing the role of ξ and $\tilde{\xi}$ yields another constant $\tilde{K}$ such that

$$v(\xi) - v(\tilde{\xi}) \leq \tilde{K}\left(\left\|\xi - \tilde{\xi}\right\|_r + \sum_{t=2}^{T-1}\left\|\mathbb{E}[\tilde{x}_t|\sigma(\xi^t)] - \tilde{x}_t\right\|_{r'} + \sum_{j=1}^{J-1}\left\|\mathbb{E}[\tilde{y}_j|\sigma(\xi^{t_j})] - \tilde{y}_j\right\|_p\right) + 2\varepsilon \quad (22)$$

for any $\tilde{x} \in l_\varepsilon(F(\tilde{\xi},.))$ and $\tilde{y} \in l_{\rho,\varepsilon}(z(\tilde{\xi},\tilde{x}),\tilde{\xi})$. We note that the second and third summands in (21) and (22) are bounded by

$$\begin{aligned}&\sum_{t=2}^{T-1}\max\left\{\left\|\mathbb{E}\left[\bar{x}_t|\sigma\left(\tilde{\xi}^t\right)\right] - \bar{x}_t\right\|_{r'}, \left\|\mathbb{E}[\tilde{x}_t|\sigma(\xi^t)] - \tilde{x}_t\right\|_{r'}\right\} \\ &\sum_{j=1}^{J-1}\max\left\{\left\|\mathbb{E}\left[\bar{y}_j|\sigma\left(\tilde{\xi}^{t_j}\right)\right] - \bar{y}_j\right\|_p, \left\|\mathbb{E}[\tilde{y}_j|\sigma(\xi^{t_j})] - \tilde{y}_j\right\|_p\right\},\end{aligned}$$

respectively. This leads directly to

$$|v(\xi) - v(\tilde{\xi})| \leq K\left(\|\xi - \tilde{\xi}\|_r + D_{\mathrm{f}}^{\rho,\mathcal{X}}(\xi,\tilde{\xi})\right) + 2\varepsilon$$

with $K := \max\{\bar{K}, \tilde{K}\}$. Finally, it remains to take the infimum of the right-hand side with respect to $\varepsilon > 0$ and the proof is complete. ■

Remark 1 The filtration distance $D_{\mathrm{f}}^{\rho,\mathcal{X}}$ depends on the ε-level-sets, i.e. on the solution behaviour of the problem which is typically unknown in practice. The question arises, whether $D_{\mathrm{f}}^{\rho,\mathcal{X}}$ can be estimated by objects that are better computable. In particular, for making use of Theorem 3.2 for scenario tree approximation of ξ, this question becomes important. For the scenario tree generation procedure described in [9], such an upper bound for $D_{\mathrm{f}}^{\mathbb{E},\mathcal{X}}$ has been used. Analogously, for the situation here, assume that (A3) is satisfied and that the set

$$\bigcup_{\tilde{x}\in l_{\varepsilon}(F(\tilde{\xi},\cdot)),\ \|\tilde{\xi}-\xi\|_r\le\delta} l_{\rho,\varepsilon}\left(z\left(\tilde{\xi},\tilde{x}\right),\tilde{\xi}\right) \tag{23}$$

is bounded in $L_p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^J)$ for some $\varepsilon > 0$. Then, obviously, the following estimate holds

$$\begin{aligned} D_{\mathrm{f}}^{\rho,\mathcal{X}}(\xi,\tilde{\xi}) \le C\Bigg(&\sup_{\|x\|_{r'}\le 1} \sum_{t=2}^{T-1} \left\| \mathbb{E}[x_t|\sigma(\xi^t)] - \mathbb{E}\left[x_t|\sigma\left(\tilde{\xi}^t\right)\right]\right\|_{r'} \\ &+ \sup_{\|y\|_p\le 1} \sum_{j=1}^{J-1} \left\| \mathbb{E}[y_j|\sigma(\xi^{t_j})] - \mathbb{E}\left[y_j|\sigma(\tilde{\xi}^{t_j})\right]\right\|_p \Bigg) \end{aligned}$$

with some constant $C > 0$. The right-hand side here represents a distance measure for the filtrations of ξ and its perturbation $\tilde{\xi}$ and does not depend on the particular problem; cf. [9].

The level-sets $l_\varepsilon(F(\xi,\cdot))$ and $l_\varepsilon(F(\tilde{\xi},\cdot))$ are bounded in $L_{r'}(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)$ due to condition (A3) (e.g. if the sets X_t are bounded for $t = 1,\ldots,T$). However, the corresponding level-sets $l_{\rho,\varepsilon}(z(\xi,x),\xi)$ and $l_{\rho,\varepsilon}(z(\tilde{\xi},\tilde{x}),\tilde{\xi})$ of the polyhedral risk measure may be unbounded in $L_p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^J)$ since the sets $Y_j \subseteq \mathbb{R}^{k_j}$ are assumed to be cones, i.e. unbounded. Hence, (23) can be unbounded in general. By the definition of the elements $z(\tilde{\xi},\tilde{x})$ in $L_p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^J)$, the pairs $(z(\tilde{\xi},\tilde{x}),\tilde{\xi})$ in (23) vary in a bounded subset of $\mathcal{Z}_\Xi$ if (A3) is satisfied. Hence, it remains to clarify the question, under what conditions the level-sets of the polyhedral risk measures are uniformly bounded on bounded subsets of $\mathcal{Z}_\Xi$.

4. Level-sets of polyhedral risk measures

As just motivated in the above remark, it is of interest for the stability analysis to know, whether the sets of ε-optimal solutions are uniformely bounded on bounded subsets $Z \subseteq \mathcal{Z}_\Xi$. However, the following example shows that, for $p > 1$, the level-sets, even for a single element $(z,\xi)\in\mathcal{Z}_\Xi$, are typically unbounded.

Example 4.1 Consider the Average Value-at-Risk at level $\alpha\in(0,1)$ (AVaR_α, cf. Example 2.1) and let $z\in L_p(\Omega,\mathcal{F},\mathbb{P})$ with some $p\in[1,\infty]$. Due to the results in [19] it is known that the solution set of (2) is given by

$$S_{\mathrm{AVaR}_\alpha}(z) = \left\{ \left(y_0, (z+y_0)^+, (z+y_0)^-\right) : y_0 \in [-\bar{q}_\alpha(z), \; -q_\alpha(z)] \right\}$$

with $\bar{q}_\alpha(z) = \inf\{a \in \mathbb{R} : \mathbb{P}(z \le a) > \alpha\}$ and $q_\alpha(z) = \inf\{a \in \mathbb{R} : \mathbb{P}(z \le a) \ge \alpha\}$ denoting the upper and lower quantile of the distribution of z, respectively. Hence, since the interval $[-\bar{q}_\alpha(z), \; -q_\alpha(z)]$ is always compact, the solution set $S_{\mathrm{AVaR}_\alpha}(z)$ is bounded in $L_p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^3)$.

However, things are different for the level-sets $l_{\mathrm{AVaR}_\alpha, \varepsilon}(z)$ for $\varepsilon > 0$. Suppose the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough such that there exist sets $S_n \in \mathcal{F}$ with $\mathbb{P}(S_n) = 1/n$ for $n \in \mathbb{N}$. Consider

$$y^{(n)}(z) := \Big(-\bar{q}_\alpha, (z - \bar{q}_\alpha(z))^+ + \varepsilon n \mathbf{1}_{S_n}, (z - \bar{q}_\alpha(z))^- + \varepsilon n \mathbf{1}_{S_n} \Big)$$

for $n \in \mathbb{N}$. Obviously, $y^{(n)}(z) \in \mathcal{Y}_{\mathrm{AVaR}_\alpha}(z)$, i.e. $y^{(n)}(z)$ is feasible for each n, and $F_{\mathrm{AVaR}_\alpha}(y^{(n)}(z)) = \rho(z) + \varepsilon$, i.e. $y^{(n)}(z) \in l_{\mathrm{AVaR}_\alpha, \varepsilon}(z)$. But even if we assume $z \in L_\infty(\Omega, \mathcal{F}, \mathbb{P})$ it holds that $\|y^{(n)}(z)\|_p \sim n^{1-(1/p)} \to \infty$ for $p \in (1, \infty]$, i.e. the level-set $l_{\mathrm{AVaR}_\alpha, \varepsilon}(z)$ for a single random variable z is unbounded in $L_p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^3)$ for $p > 1$. Thus, for the boundedness of the AVaR_α level-sets, there is only hope for $p = 1$. It will be seen below that $l_{\mathrm{AVaR}_\alpha, \varepsilon}(z)$ is bounded in $L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^3)$ indeed, actually in a uniform manner.

Since the multiperiod polyhedral risk measures (cf. section 2) from [3] boil down to AVaR for $J = 1$, and, hence, their level-sets are unbounded in L_p if $p > 1$, we will assume $p = 1$ from now on (and accordingly $p' = \infty$). In the following, a simple criterion will be derived which guarantees the sort of uniform L_1-boundedness of the level-sets $l_\rho(z, \xi)$ as it is required in Remark 1 in Section 3. This criterion, though appearing to be very specific, applies for most of the polyhedral risk measures ρ introduced so far. Here, the extended real-valued function Φ_ρ, called the *value function* given by

$$\Phi_\rho(y_0, z, \xi) := \inf_{y_1, \ldots, y_J} \{F_\rho(y_0, y_1, \ldots, y_J) : (y_0, y_1, \ldots, y_J) \in \mathcal{Y}_\rho(z, \xi)\}$$

will be used. Observe that $\rho(z, \xi) = \inf_{y_0 \in Y_0} \Phi_\rho(y_0, z, \xi)$. The notation π_j will denote the projection to the j-th component of a vector.

Proposition 4.2 *Let ρ be a functional of the form* (2) *satisfying complete recourse and dual feasibility and assume*

(i) $k_j = 2$, $\langle c_j, Y_j \rangle \subseteq \mathbb{R}_+$ *for* $j = 1, \ldots, J$,
(ii) *the vectors c_j and $w_{j,0}$ are linearly independent for* $j = 1, \ldots, J$,
(iii) $\pi_j(\bigcap_{\nu=j}^J \mathcal{D}_{\rho,\nu})$ *is bounded in* $\mathbb{R}$ *for* $j = 1, \ldots, J$, *and*
(iv) Y_0 *is bounded, or alternatively*
(v) $k_0 = 1$, $c_0 > 0$, *and* $\inf\{\sum_{j=1}^J u_j w_{j,j} : u \in \bigcap_{j=1}^J \mathcal{D}_{\rho,j}\} < -c_0$.

Let $Z \subseteq \mathcal{Z}_\Xi$ such that the projection $\pi_1(Z)$ to the z component is bounded in $L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^J)$. Then the union over all level-sets $\bigcup_{(z,\xi) \in Z} l_{\rho,\varepsilon}(z, \xi)$ is bounded in $L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{\sum k_j})$ for any $\varepsilon > 0$.

Proof First of all, consider the numbers $M_Z := \sup\{\|z\|_1 : (z, \xi) \in Z\}$, $M_D := \sup\{|u|_\infty : u \in \cap_{j=1}^J \mathcal{D}_{\rho,j}\}$, and $M_\rho := \sup\{|\rho(z)| : (z, \xi) \in Z\}$. Observe that $M_Z < \infty$ according to the assumptions about Z and that $M_D < \infty$ due to assumption (iii). First, we show that also $M_\rho < \infty$. To this end, consider the dual representation (7) and note that due to assumption (iii) the feasible set

$$\Lambda_\rho(\xi) := \left\{\lambda \in \times_{j=1}^J L_\infty(\Omega, \sigma(\xi^{t_j}), \mathbb{P}) : \mathbb{E}[\lambda|\xi^{t_j}] \in \mathcal{D}_{\rho,j} \text{ a.s. } (j = 1, \ldots, J)\right\}$$

is bounded in $L_\infty(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^J)$ with a bound M_λ not depending on ξ. Hence, (7) yields the following estimates:

$$\rho(z,\xi) \le \sup_{\lambda\in\Lambda_\rho(\xi)} \inf_{y_0\in Y_0} \left\langle c_0 + \sum_{\nu=1}^J \mathbb{E}[\lambda_\nu] w_{\nu,\nu}, y_0 \right\rangle + M_Z M_\lambda$$
$$\rho(z,\xi) \ge \sup_{\lambda\in\Lambda_\rho(\xi)} \inf_{y_0\in Y_0} \left\langle c_0 + \sum_{\nu=1}^J \mathbb{E}[\lambda_\nu] w_{\nu,\nu}, y_0 \right\rangle - M_Z M_\lambda$$

and since

$$\sup_{\lambda\in\Lambda_\rho(\xi)} \inf_{y_0\in Y_0} \left\langle c_0 + \sum_{\nu=1}^J \mathbb{E}[\lambda_\nu] w_{\nu,\nu}, y_0 \right\rangle = \sup_{u\in\bigcap_{\nu=1}^J \mathcal{D}_{\rho,\nu}} \inf_{y_0\in Y_0} \left\langle c_0 + \sum_{\nu=1}^J u_\nu w_{\nu,\nu}, y_0 \right\rangle$$

it becomes clear that this number, which does not depend on (z, ξ), must be finite (otherwise $\rho(z, \xi)$ would be infinite). Hence, M_ρ is finite indeed.

Now, let $\varepsilon > 0$. We prove boundedness of the level-sets for each component $j = 0, 1, \ldots, J$ successively. For $j = 0$, we show that, if Y_0 is unbounded, the value function $\Phi_\rho(y_0, z, \xi)$ grows to infinity uniformly on Z as $|y_0| \to \infty$. For $y_0 \to +\infty$ this is obvious since $\Phi_\rho(y_0, z, \xi) \ge c_0 y_0$ due to assumption (i) and $c_0 y_0 \to \infty$ due to assumption (iv′). For $y_0 < 0$, we obtain the following estimate by relaxing the non-anticipativity constraints and making use of [20, Theorem 14.60] and LP duality [20, Example 11.43]:

$$\Phi_\rho(y_0, z, \xi) = c_0 y_0 + \inf\left\{ \mathbb{E}\left[\sum_{j=1}^J \langle c_j, y_j\rangle\right] \,\middle|\, \begin{array}{l} y \in \times_{j=1}^J L_1(\Omega, \sigma(\xi^{t_j}), \mathbb{P}; \mathbb{R}^{k_j}), \\ y \in \times_{j=1}^J Y_j \text{ a.s.}, \\ \sum_{\tau=0}^{j-1} \langle w_{j,\tau}, y_{j-\tau}\rangle = z_{t_j} - w_{j,j} y_0 \text{ a.s.} \end{array} \right\}$$

$$\ge c_0 y_0 + \inf\left\{ \mathbb{E}\left[\sum_{j=1}^J \langle c_j, y_j\rangle\right] \,\middle|\, \begin{array}{l} y \in \times_{j=1}^J L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{k_j}), \\ y \in \times_{j=1}^J Y_j \text{ a.s.}, \\ \sum_{\tau=0}^{j-1} \langle w_{j,\tau}, y_{j-\tau}\rangle = z_{t_j} - w_{j,j} y_0 \text{ a.s.} \end{array} \right\}$$

$$= c_0 y_0 + \mathbb{E}\left[\inf\left\{ \sum_{j=1}^J \langle c_j, y_j\rangle \,\middle|\, \begin{array}{l} y \in \times_{j=1}^J Y_j, \\ \sum_{\tau=0}^{j-1} \langle w_{j,\tau}, y_{j-\tau}\rangle = z_{t_j} - w_{j,j} y_0 \end{array} \right\}\right]$$

$$= c_0 y_0 + \mathbb{E}\left[\sup\left\{\sum_{j=1}^{J} u_j(w_{j,j} y_0 - z_{t_j}) \,\middle|\, u = (u_1, \ldots, u_J) \in \bigcap_{j=1}^{J} \mathcal{D}_{\rho,j}\right\}\right]$$

$$\geq c_0 y_0 + y_0 \inf\left\{\sum_{j=1}^{J} u_j w_{j,j} \,\middle|\, u = (u_1, \ldots, u_J) \in \bigcap_{j=1}^{J} \mathcal{D}_{\rho,j}\right\} - M_Z M_D$$

Thus, due to assumption (iv′), $\Phi_\rho(y_0, z, \xi) \to +\infty$ uniformly on Z as $y_0 \to -\infty$. Hence, there is a real number M_0 such that for all $(z, \xi) \in Z$ and for all $\hat{y} = (\hat{y}_0, \ldots, \hat{y}_J) \in l_{\rho,\varepsilon}(z, \xi)$ it holds that $|\hat{y}_0| \leq M_0$. Now, for $j = 1$ it holds due to assumption (i) that

$$\begin{aligned}
\|\langle c_1, \hat{y}_1\rangle\|_1 &= \mathbb{E}[\langle c_1, \hat{y}_1\rangle] \leq \sum_{j=1}^{J} \mathbb{E}[\langle c_j, \hat{y}_j\rangle] = F_\rho(\hat{y}) - \langle c_0, \hat{y}_0\rangle \\
&\leq \rho(z_{t_1}, \ldots, z_{t_J}) + \varepsilon + |c_0| M_0 \;\leq\; M_\rho + \varepsilon + |c_0| M_0 \\
\|\langle w_{1,0}, \hat{y}_1\rangle\|_1 &= \|z_{t_1} - \langle w_{1,1}, \hat{y}_0\rangle\|_1 \\
&\leq \|z_{t_1}\|_1 + |w_{1,1}| M_0 \;\leq\; M_Z + |w_{1,1}| M_0,
\end{aligned}$$

i.e. $(c_1, w_{1,0})' \hat{y}_1$ is bounded in $L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2)$ by a number that does not depend on (z, ξ). The 2×2 matrix $(c_1, w_{1,0})$ is regular due to assumption (ii), hence, $\hat{y}_1$ is L_1-bounded. By induction we conclude analogously for $j > 1$. ■

This proposition applies directly to the exemplary polyhedral risk measures EL_β, AVaR_α, ρ_2, ρ_3, and ρ_4 as far as, say, $\alpha_j = \alpha < 0.5$ for $j = 1, \ldots, J$; cf. Tables 2 and 3. Moreover, uniform level-boundedness of risk measure ρ_1 is guaranteed, too, since its level-sets can be understood as a Cartesian product of level-sets of AVaR_{α_j}.

Example 4.3 Of course, since Proposition 4.2 appears rather technical and for all the examples from Section 2 the level-sets are L_1-bounded, the question arises, whether there exist polyhedral risk measures on L_1 satisfying complete recourse and dual feasibility that have unbounded level-sets. The answer can be given directly: Consider the infimum representation of the Average Value-at-Risk in Example 2.1, i.e. the right-hand side of (5). Set $\alpha = 1$ (though, typically, $\alpha < 1$ is assumed since it is known that $\mathrm{AVaR}_1 = -\mathbb{E}$). The resulting minimization problem still satisfies complete recourse and dual feasibility but neither condition (iv) nor (iv′) of Proposition 4.2 hold. For $z \equiv 0$, formula (5) reveals

Table 2. Feasible sets of the dual representations (7) for the exemplary polyhedral risk measures.

Riskm.	$\mathcal{D}_{\rho,0}$	$\mathcal{D}_{\rho,j}$ $(j = 1, \ldots, J-1)$	$\mathcal{D}_{\rho,J}$
AVaR_α	$\{\ 1\ \}$		$\left[0, \frac{1}{\alpha}\right]$
EL_β	$\mathrm{sign}(\beta)\cdot\mathbb{R}_+$		$[0, 1]$
ρ_1	$\left\{\left(\frac{1}{J}, \ldots, \frac{1}{J}\right)\right\}$	$\left\{u \in \mathbb{R}^J : 0 \leq u_j,\ u_j \leq \frac{1}{J\alpha_j}\right\}$	$\left\{u \in \mathbb{R}^J : u_J \in \left[0, \frac{1}{J\alpha_J}\right]\right\}$
ρ_2	$\left\{u \in \mathbb{R}^J : \sum u_j = 1\right\}$	$\left\{u \in \mathbb{R}^J : 0 \leq u_j,\ u_j \leq \frac{1}{J\alpha_j}\right\}$	$\left\{u \in \mathbb{R}^J : u_J \in \left[0, \frac{1}{J\alpha_J}\right]\right\}$
ρ_3	$\left\{u \in \mathbb{R}^J : \sum u_j = 1\right\{$	$\left\{u \in \mathbb{R}^J : 0 \leq u_j,\ u_j + u_{j+1} \leq \frac{1}{J\alpha_j}\right\}$	$\left\{u \in \mathbb{R}^J : u_J \in \left[0, \frac{1}{J\alpha_J}\right]\right\}$
ρ_4	$\left\{u \in \mathbb{R}^J : u_1 = \frac{1}{J}\right\}$	$\left\{u \in \mathbb{R}^J : u_j = u_{j+1},\ u_j \leq \frac{1}{J\alpha_j}\right\}$	$\left\{u \in \mathbb{R}^J : u_J \in \left[0, \frac{1}{J\alpha_J}\right]\right\}$

Table 3. Details to verify condition (iv′) of Proposition 4.2 for some polyhedral risk measures when α_j is set to $\alpha < 1$ for $j = 1, \ldots, J$.

Riskm.	$\bigcap_{j=1}^{J} D_{\rho,j}$	c_0	$w_{1,1}$	$w_{j,j}$ $(j>1)$	$\inf_{u \in \cap \mathcal{D}_{\rho,j}} \sum u_j w_{j,j}$
AVaR_α	$\left[0, \frac{1}{\alpha}\right]$	1	−1		$-\frac{1}{\alpha}$
ρ_2	$\times_{j=1}^{J}\left[0, \frac{1}{J\alpha}\right]$	1	−1	−1	$-\frac{1}{\alpha}$
ρ_3	$\left\{u \in \mathbb{R}_+^J : u_J \leq \frac{1}{J\alpha}, u_j + u_{j+1} \leq \frac{1}{J\alpha}\right\}$	1	−1	−-1	$-\frac{1}{2\alpha}$
ρ_4	$\left\{u \in \mathbb{R}_+^J : u_1 = \cdots = u_J \leq \frac{1}{J\alpha}\right\}$	$\frac{1}{J}$	−1	0	$-\frac{1}{J\alpha}$

that the y_0 component of the solution set is given by $\pi_0(S_\rho(0, \xi)) = \mathbb{R}_-$, i.e. it is unbounded in $\mathbb{R}$. Hence, $S_\rho(0, \xi)$ and thus $l_{\rho,\varepsilon}(0, \xi)$ are unbounded in $L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^3)$. We conclude that complete recourse and dual feasibility are *not* sufficient conditions for bounded level-sets.

Example 4.4 Regrettably, Proposition 4.2 does not apply to the value of perfect information based risk measure (6) from [14,15] (cf. Example 2.4), because $k_j \neq 2$, i.e. condition (i) is not satisfied. However, it has been observed in [15] that the risk measure decomposes into functionals for each time period:

$$\rho_5(z) = \sum_{j=1}^{J} \mathbb{E}\left[-s_j z_{t_j} + (s_{j-1} - s_j)\mathrm{AVaR}_{\alpha_j}(z_{t_j} | \mathcal{F}_{t_{j-1}})\right]$$

where $\alpha_j = (s_{j-1} - s_j)/(q_j - s_j)$. This decomposition will simplify the analysis of the level-sets drastically. In [15] it has been derived via the dual representation (8) of ρ_5, but it can be deduced directly from (6) by making use of the dynamic constraints $z_{t_j} = y_{j,2} - y_{j,3} - y_{j-1,2} + y_{j-1,1}$:

$$\rho_5(z) + \sum_{j=1}^{J} s_j \mathbb{E}[z_{t_j}]$$

$$= \inf \left\{ \begin{array}{l} -s_0 y_{0,1} + \mathbb{E}\left[\sum_{j=1}^{J}(-s_j y_{j,1} + q_j y_{j,3}) - d y_{j,2}\right] \\ +\mathbb{E}\left[\sum_{j=1}^{J} s_j(y_{j,2} - y_{j,3} - y_{j-1,2} + y_{j-1,1})\right] : \\ y_j \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P};\ \mathbb{R}^3)\ (j = 0, \ldots, J),\ y_{0,2} = y_{0,3} = y_{J,1} = 0, \\ y_{j,2} \geq 0 \ \text{ a.s.},\ y_{j,3} \geq 0 \ \text{ a.s.},\ y_{j,2} - y_{j,3} = y_{j-1,2} + z_{t_j} - y_{j-1,1} \ \text{ a.s.} \\ (j = 1, \ldots, J) \end{array} \right\}$$

$$= \inf \left\{ \begin{array}{l} -y_{0,1}(s_0 - s_1) + \mathbb{E}\left[\sum_{j=1}^{J-1} (s_j - s_{j+1})(y_{j,2} - y_{j,1}) + \sum_{j=1}^{J} (q_j - s_j) y_{j,3} \right] : \\ y_j \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P};\ \mathbb{R}^3)\ (j = 0, \dots, J),\ y_{0,2} = y_{0,3} = y_{J,1} = 0, \\ y_{j,2} \geq 0 \ \text{ a.s.},\ y_{j,3} \geq 0 \ \text{ a.s.},\ y_{j,2} - y_{j,3} = y_{j-1,2} + z_{t_j} - y_{j-1,1} \ \text{ a.s.} \\ (j = 1, \dots, J) \end{array} \right\}$$

where it is set $s_J := d$ for convenience. Substituting $\tilde{y}_{j,1} := y_{j,2} - y_{j,1}$ yields immediately

$$\rho_5(z) = \sum_{j=1}^{J} \big((s_{j-1} - s_j) \rho_{5,j}(z_{t_j}) - s_j \mathbb{E}[z_{t_j}] \big) \ \text{ respectively}$$

$$\mathcal{R}(z) = \sum_{j=1}^{J} (s_{j-1} - s_j) \big(\rho_{5,j}(z_{t_j}) + \mathbb{E}[z_{t_j}] \big)$$

with

$$\rho_{5,j}(z_{t_j}) = \inf \left\{ \mathbb{E}\left[\tilde{y}_{j-1,1} + \frac{q_j - s_j}{s_{j-1} - s_j} y_{j,3} \right] \middle| \begin{array}{l} \tilde{y}_{j-1,1} \in L_p(\Omega, \mathcal{F}_{t_{j-1}}, \mathbb{P}), \\ y_{j,2}, y_{j,3} \in L_p(\Omega, \mathcal{F}_{t_j}, \mathbb{P}), \\ y_{j,2} \geq 0 \text{ a.s.},\ y_{j,3} \geq 0 \text{ a.s.}, \\ y_{j,2} - y_{j,3} = z_{t_j} + \tilde{y}_{j-1,1} \text{ a.s.} \end{array} \right\} \tag{24}$$

for $j = 1, \dots, J$. Interchanging minimization and integration can give the above interpretation $\rho_{5,j}(z_{t_j}) = \mathbb{E}[\mathrm{AVaR}_{\alpha_j}(z_{t_j} | \mathcal{F}_{t_{j-1}})]$ from [15].

Proposition 4.5 *Let $Z \subseteq \mathcal{Z}_\Xi$ such that the projection $\pi_1(Z)$ to the z component is bounded in $L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^J)$. Then, for the risk measure ρ_5 in* (6), *it holds that the union over all ε-level-sets $\bigcup_{(z,\xi) \in Z} l_{\rho_5, \varepsilon}(z, \xi)$ is bounded in $L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{3(J+1)})$ for $\varepsilon > 0$.*

Proof We show that for $\varepsilon > 0$ and for each $j = 1, \dots, J$ the union $\bigcup_{(z,\xi) \in Z} l_{\rho_{5,j}, \varepsilon}(z_{t_j}, \xi)$ of all ε-level-sets of $\rho_{5,j}(z_{t_j}, \xi)$, cf. (24), is bounded in $L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^3)$. To this end, we first note that the number $M_{\rho_{5,j}} := \sup\{ |\rho_{5,j}(z, \xi)| : (z, \xi) \in Z \}$ is finite. This can easily be seen by considering the dual of (24) given by

$$\rho_{5,j}(z, \xi) = \sup \left\{ -\mathbb{E}[\lambda_j z_{t_j}] \middle| \begin{array}{l} \lambda_j \in L_p(\Omega, \sigma(\xi^{t_j}), \mathbb{P}), \\ 0 \leq \lambda_j \leq \dfrac{q_j - s_j}{s_{j-1} - s_j} \ \text{ a.s.},\ \mathbb{E}[\lambda_j | \xi^{t_{j-1}}] = 1 \text{ a.s.} \end{array} \right\}.$$

Now, let $(z^{(n)}, \xi^{(n)}) \in Z$ and $y^{(n)} = (\tilde{y}^{(n)}_{j-1,1}, y^{(n)}_{j,2}, y^{(n)}_{j,3}) \in l_{\rho_{5,j}, \varepsilon}(z^{(n)}_{t_j}, \xi^{(n)})$ for $n \in \mathbb{N}$. Suppose there is a subsequence $(y^{(n_k)})$ such that $\| (\tilde{y}^{(n_k)}_{j-1,1})^- \|_1 \to \infty$. In this case, the following estimate for the objective of (24) would hold:

$$F_{\rho_{5,j}}(y^{(n_k)}) = \mathbb{E}\left[\tilde{y}^{(n_k)}_{j-1,1} + \frac{1}{\alpha_j} y^{(n_k)}_{j,3} \right] \geq \mathbb{E}\left[\mathbf{1}_{\{\tilde{y}^{(n_k)}_{j-1,1} \leq 0\}} \left(\tilde{y}^{(n_k)}_{j-1,1} + \frac{1}{\alpha_j} y^{(n_k)}_{j,3} \right) \right]$$

$$= \mathbb{E}\left[\mathbf{1}_{\{\tilde{y}_{j-1,1}^{(n_k)} \leq 0\}}\left(\tilde{y}_{j-1,1}^{(n_k)} + \frac{1}{\alpha_j}(y_{j,2}^{(n_k)} - \tilde{y}_{j-1,1}^{(n_k)} - z^{(n_k)})\right)\right]$$
$$= \left(\frac{1}{\alpha_j} - 1\right)\left\|(\tilde{y}_{j-1,1}^{(n_k)})^-\right\|_1 + \frac{1}{\alpha_j}\mathbb{E}\left[\mathbf{1}_{\{\tilde{y}_{j-1,1}^{(n_k)} \leq 0\}}\left(y_{j,2}^{(n_k)} - z^{(n_k)}\right)\right] \to \infty.$$

The convergence to infinity holds because $1/\alpha_j > 1$, $y_{j,2}^{(n_k)} \geq 0$, and because the sequence $(z^{(n)})$ is L_1-bounded. However, $F_{\rho_{5,j}}(y^{(n_k)}) \to \infty$ is a contradiction to $y^{(n)} \in l_{\rho_{5,j},\varepsilon}(z^{(n)}, \xi^{(n)})$ since the sequence $(\rho_{5,j}(z^{(n)}, \xi^{(n)}))$ is bounded due to $M_{\rho_{5,j}} < \infty$. Hence, the sequence $((\tilde{y}_{j-1,1}^{(n)})^-)$ is L_1-bounded. Suppose there is a subsequence $(y^{(n_k)})$ such that $\|(\tilde{y}_{j-1,1}^{(n_k)})^+\|_1 \to \infty$. Obviously, this would also imply $F_{\rho_{5,j}}(y^{(n_k)}) \to \infty$ since $y_{j,3}^{(n_k)} \geq 0$ and thus cause a contradiction. Hence, the sequence $(\tilde{y}_{j-1,1}^{(n)})$ is L_1-bounded. Also the existence of subsequences $(y^{(n_k)})$ such that $\|y_{j,2}^{(n_k)}\|_1 \to \infty$ or $\|y_{j,3}^{(n_k)}\|_1 \to \infty$ would cause a contradiction in the same manner. Hence, the overall sequence $(y^{(n)})$ is L_1-bounded. That is, the union over all level-sets of $\rho_{5,\,j}$ is indeed bounded in $L_1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^3)$. Finally, note that this boundedness for $\rho_{5,\,j}$ implies uniform boundedness of the ε-level-sets of (6) for ρ_5, because the substitution $(y_1, y_2) \mapsto (y_2 - y_1, y_2)$ in Example 4.4 is bijective in $\mathbb{R} \times \mathbb{R}_+$. ■

5. Conclusion for stability and scenario tree approximation

In [3], the class of polyhedral risk measures has been suggested. As discussed in Section 2, replacing the expectation in (1) by a (multiperiod) polyhedral risk measure yields problem (3). Problem (4), which is equivalent to (3), has an expectation objective and is of a similar form (but not the same) as (1) with additional stochastic decision variables y_j and additional constraints. The stability theorem from [11], however, does not hold. Here, we provided an equivalent stability theorem (Theorem 3.2) for Problem (3). This result is based on sensitivity analysis for polyhedral risk measures (Proposition 3.1).

Stability according to Theorem 3.2 involves so-called filtration distances which involve the sets of ε-optimal solutions (level-sets) of the underlying problem. In order to make use of Theorem 3.2 in the context of scenario tree approximation, it turns out to be necessary to have these level-sets bounded; cf. Remark 1 in Section 3, see also [9]. However, though in many application the original decision variables x_t can be assumed to be bounded from the outset, the feasible sets of the additional y_j variables arising from the polyhedral risk measures are inherently unbounded. For this reason, criteria for the boundedness of the y_j components of the level-sets are derived in Section 4; in particular, it has been detected that boundedness is guaranteed for all the instances of the class of polyhedral risk measures from [3,15], if the integrability number p of the arguments of the risk measure is set to 1.

As in [11], Theorem 3.2 makes several restrictions for the integrability number r of the stochastic input process ξ. At first glance there seem to be more degrees of freedom for r than in [11] since, theoretically, p may be chosen arbitrarily. But, as mentioned above, in the context of scenario approximation $p = 1$ is the only choice. Then, however, the situation is the same as in [11].

To conclude, by means of the present paper the results from [9,11] apply to Problem (3) where $\mathbb{E}$ in (1) is replaced by a polyhedral risk measure from [3]. In particular, the same scenario approximation techniques can be used as soon as the criteria for the boundedness of the level-sets for the polyhedral risk measure are satisfied.

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Notes

[1] We adopt the notation of [14,15] to the notation of ployhedral risk measures (2). To this end, the original identifies t, a_t, K_t, M_t, L_t and c_t form [14,15] have been replaced by j, $y_{j-1,1}$, $y_{j,2}$, $y_{j,3}$, z_{t_j} and s_{t-1}, respectively.

[2] In the proof of [11, Theorem 2.1] the term $(1+|\bar{x}_{\tau-1}|)$ occurs additionally in the first conditional expectation on the right-hand side of (18) if the technology matrices are not random. However, due to (A3), we have that B is bounded in L_∞ in this case. Hence, since $\bar{x} \in l_\varepsilon(F(\xi,.)) \subseteq B$, $|\bar{x}_{\tau-1}|$ can be estimated by $\|\bar{x}_{\tau-1}\|_\infty$ and we assume the latter norm to be integrated in the constant l_t.

[3] Of course, if b_t are non-random, both sides of the above estimate are zero anyway, but the same argument will be used again below where this is not the case.

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