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CONVERGENCE OF APPROXIMATE SOLUTIONS OF NONLINEAR RANDOM OPERATOR EQUATIONS WITH NON-UNIQUE SOLUTIONS

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ABSTRACT

Let $T(\omega,x) = y(\omega)$ be a nonlinear random operator equation with not necessarily unique solution. For this and similar equations, we prove results about convergence of solutions of suitable approximate problems $T_n(\omega,x) = y_n(\omega)$ to solutions of the original equations. We do this for rather general notions of convergence for random variables. Concepts like consistency, stability, and compactness in sets of measurable functions are introduced and used. For all assumptions that are needed in the general theory, sufficient conditions are given with respect to convergence in probability and almost-sure convergence.
As a specific method for constructing approximate equations we discuss "discretization schemes", where the underlying probability space is discretized.
Some results might be of interest also in different contexts; these include criteria for almost-sure convergence of measurable multifunctions and results about compactness with respect to convergence in probability and almost-sure convergence.

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1.) INTRODUCTION AND PRELIMINARIES

In this paper we want to give a general theory for convergence of solutions of approximations to random operator equations to solutions of the original equation, which is not assumed to be uniquely solvable. For this general theory, we do not fix the notion of convergence we use. Later we illustrate our abstract results for convergence in probability, almost-sure and almost uniform convergence.

This introductory section contains basic material concerning random operator equations, measurable multifunctions, and the abstract notion of convergence we use.

We now fix the terminology and notation for this paper. Throughout the paper, let $(\Omega,\mathcal{A},P)$ be a complete probability space and X,Y be Polish spaces; the metrics in X and Y will, though different, both be denoted by d. By $\mathcal{P}(X)$ we denote the set of all non-empty subsets of X, by 2^X the set of all non-empty closed subsets of X. As usual, $\mathcal{B}(X)$ denotes the Borel-σ-algebra on X, i.e., the σ-algebra generated by the open sets; the product of the σ-algebras $\mathcal{A}$ and $\mathcal{B}(X)$, i.e., the σ-algebra generated by $\{A\times B/A \in \mathcal{A}, B \in \mathcal{B}(X)\}$ will be denoted by $\mathcal{A}\times\mathcal{B}(X)$.

A "(closed-valued) multifunction from Ω into X" is a function from Ω into $\mathcal{P}(X)$ (2^X, respectively); the "graph" of a multifunction C is $Gr\ C: = \{(\omega,x)/\omega \in \Omega,\ x \in C(\omega)\}$. A multifunction C is said to be "weakly measurable" iff for all open $B \subseteq X$, $\{\omega \in \Omega/C(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$; we call C "measurable" iff $Gr\ C \in \mathcal{A}\times \mathcal{B}(X)$. This latter property is usually referred to as "Gr-measurable". A survey about properties of measurable multifunctions is given in [32], where the following simple facts can be found:

Proposition 1.1.: Let C be a multifunction from Ω into X.

a) *If C is measurable, then C is weakly measurable. If C is closed-valued, the converse holds.*

b) *C is weakly measurable if and only if for all $x \in X$, the real-valued function $\omega \to d(x,C(\omega)): = \inf\ \{d(x,z)/z \in C(\omega)\}$ is measurable, where d denotes the metric on X.*

As usual, we say that a property depending on $\omega \in \Omega$ *holds "almost surely (a.s.)" if there is a set* $N \in A$ *with* $P(N) = 0$ *such that the property holds for all* $\omega \in \Omega \setminus N$. *For a multifunction C from* Ω *into X, we will denote the set of all random variables which are a.s. selectors of C by S(C), i.e.*

$$(1.1) \qquad S(C): = \{x: \Omega \to X / x \text{ measurable}, x(\omega) \in C(\omega) \text{ a.s.}\}.$$

For the case that $C(\omega) = X$ *for all* $\omega \in \Omega$, *S(C) equals the set of all X-valued random variables, so that the notation*

$$(1.2) \qquad S(X): = \{x: \Omega \to X / x \text{ measurable}\}$$

is consistent. Results that state when $S(C) \neq \emptyset$ *are called "measurable selection theorems"; see [32] and [18] for a variety of such theorems. Note that in our definition of S(C), the "exceptional sets" where* $x(\omega) \in C(\omega)$ *need not hold may be different for each* $x \in S(C)$. *Since (because of the completeness of* (Ω, A, P)*) we can redefine a measurable function on* Ω *arbitrarily on a set of measure 0 without destroying measurability, for each* $x \in S(C)$ *there exists a measurable* $\bar{x}: \Omega \to X$ *such that* $\bar{x}(\omega) \in C(\omega)$ *for* __all__ $\omega \in \Omega$. *Thus, the exceptional sets where elements of S(C) need not be selectors, which give us more flexibility later, do not cause problems.*

Measurable multifunctions will serve as domains for the operators that appear in our random equations, which we will call "random operators on stochastic domains":

__*Definition 1.2:*__ *Let C be a multifunction from* Ω *into X.*
$T: Gr\, C \to Y$ *will be called "random operator with stochastic domain C" if C is weakly measurable and for all open* $D \subseteq Y$ *and all* $x \in X$,

$$\{\omega \in \Omega / x \in C(\omega),\ T(\omega,x) \in D\} \in A.$$

$T: Gr\, C \to Y$ *will be called "*$A \times B(X)$*-measurable" if for all* $B \in B(Y)$,

$$T^{-1}(B): = \{(\omega,x) \in Gr\, C / T(\omega,x) \in B\} \in A \times B(X).$$

Note that if $T: Gr\, C \to Y$ *is* $A \times B(X)$*-measurable then the multifunction C is measurable and T is a random operator with stochastic domain C in the sense of Definition 1.2. The two concepts defined in Definition 1.2 are equivalent if C is separable (see [8] for*

this concept, which we do not need here) and T is continuous (see below). These statements can be found e.g. in [25].
The following observation, that will frequently be needed below, is obvious: If T: Gr $C \to Y$ is $\mathcal{A}\times\mathcal{B}(X)$-measurable, then the mapping $\omega \to T(\omega,x(\omega))$ is measurable for all measurable $x: \Omega \to X$ with $x(\omega) \in C(\omega)$ for all $\omega \in \Omega$.
An operator T: Gr $C \to Y$ will be called "linear, continuous, differentiable, ..." if for all $\omega \in \Omega$, $T(\omega,.) : C(\omega) \to Y$ has the corresponding property.

A "random operator equation" is an equation of the form

$$(1.3) \quad T(\omega,x) = y(\omega),$$

where T: Gr $C \to Y$ is a random operator with stochastic domain C and $y \in S(Y)$. In (2.14), a formally (but not conceptually) more general equation will also be called "random operator equation"; the terms "wide-sense solution" and "random solution" to be defined now will also be used there.
Any function $x: \Omega \to X$ such that $x(\omega) \in C(\omega)$ a.s. and $T(\omega,x(\omega))=y(\omega)$ a.s. is called a "wide-sense solution" of (1.3); a "random solution" is a wide-sense solution which is an element of $S(C)$. The two exceptional sets appearing in this definition can be combined, so that a measurable function $x: \Omega \to X$ is a random solution of (1.3) iff there exists a set $N \in \mathcal{A}$ with $P(N) = 0$ such that for all $\omega \in \Omega\setminus N$, $x(\omega) \in C(\omega)$ and $T(\omega,x(\omega)) = y(\omega)$. We emphasize this, since throughout the paper we want to be careful in treating exceptional sets.

In recent years, a number of authors have obtained sufficient conditions for the existence of random solutions of random operator equations. E.g., see [21], [22], [1o], [11] and the references quoted there.

As mentioned above, we will study the question of convergence of random solutions of random equations approximating (1.3) to a random solution of (1.3). We want to keep our results as general as possible in order to include different modes of convergence. For this reason, we formulate those concepts and results where *this is possible not for special types of convergence, but for a general "convergence" fulfilling various axioms, which are chosen in such a way that they are fulfilled for the special modes of convergence we have in mind. As a guideline for the concept of convergence we use we take the fundamental paper [3o]. However, the concepts of that paper have to be adapted to our situation, where we have to deal with random variables. As we will see below, the conditions of [3o] are not fulfilled for the important concept of almost-sure convergence of random variables.*

Let $S(X)^{\mathbb{N}}$ *denote the set of all sequences in* $S(X)$ *and* ρ_X *be a multifunction from a subset* $D(\rho_X)$ *of* $S(X)^{\mathbb{N}}$ *to* $S(X)$*. The following properties that* ρ_X *may or may not have will be of interest:*

(1.4) If $(x_n) = (x,x,x,\dots)$ with $x \in S(X)$ is a constant sequence, then $(x_n) \in D(\rho_X)$ and $x \in \rho_X((x_n))$.

(1.5) If $(x_n) \in D(\rho_X)$ and $x \in \rho_X((x_n))$ and (x_{n_k}) is a subsequence of (x_n), then $(x_{n_k}) \in D(\rho_X)$ and $x \in \rho_X((x_{n_k}))$.

(1.6) Let K,M be disjoint infinite subsets of $\mathbb{N}$ with $K \cup M = \mathbb{N}$, $(x_k)_{k \in K} \in D(\rho_X)$, $(\bar{x}_m)_{m \in M} \in D(\rho_X)$, $x \in S(X)$ be such that $x \in \rho_X((x_k)) \cap \rho_X((\bar{x}_m))$; for all $n \in \mathbb{N}$, let

$$\tilde{x}_n := \begin{cases} x_n & \text{if } n \in K \\ \bar{x}_n & \text{if } n \in M \end{cases}$$

Then $(\tilde{x}_n) \in D(\rho_X)$ and $x \in \rho_X((\tilde{x}_n))$.

$$(1.7)\quad \begin{cases} \text{For any sequence } (x_n) \in S(X)^{\mathbb{N}} \text{ and } x \in S(X) \text{ we have:} \\ (x_n) \in D(\rho_X) \text{ and } x \in \rho_X((x_n)) \text{ iff each subsequence} \\ (x_{n_k}) \text{ of } (x_n) \text{ has a subsequence } (x_{n_{k_i}}) \text{ such that} \\ (x_{n_{k_i}}) \in D\,(\rho_X) \text{ and } x \in \rho_X(x_{n_{k_i}}). \end{cases}$$

$$(1.8)\quad \begin{cases} \text{For every sequence } (x_n) \in D(\rho_X),\ x \in \rho_X((x_n)) \text{ and} \\ \bar{x} \in S(X) \text{ we have: } \bar{x} \in \rho_X((x_n)) \text{ if and only if } x = \bar{x} \text{ a.s.;} \\ \text{furthermore, if } (\bar{x}_n) \in S(X)^{\mathbb{N}} \text{ is such that for all } n \in \mathbb{N}, \\ \bar{x}_n = x_n \text{ a.s., then } (\bar{x}_n) \in D(\rho_X) \text{ and } x \in \rho_X((\bar{x}_n)). \end{cases}$$

Definition 1.3: A multifunction ρ_X from $D(\rho_X) \subset S(X)^{\mathbb{N}}$ to $S(X)$ will be called a "convergence (on $S(X)$)" if it has the properties (1.4), (1.5), (1.6), and (1.8). A sequence in $D(\rho_X)$ will be called "ρ_X-convergent", any element of $\rho_X((x_n))$ will be called a "ρ_X-limit" of (x_n).

Of course the symbolism used so far is inconvenient, so that we write

(1.9) "$x = \rho_X - \lim\limits_{n\in\mathbb{N}} x_n$" for $x \in \rho_X((x_n))$,

keeping in mind that the equality sign is (1.9) is only a useful symbol and does not suggest uniqueness of the ρ_X-limit. With this notation, the properties (1.4) - (1.8) can be thought of in the following form:

(1.4') If $(x_n) = (x,x,\ldots)$, then $\rho_X - \lim\limits_{n\in\mathbb{N}} x_n = x$.

$$(1.5')\quad \begin{cases} \text{If } \rho_X - \lim\limits_{n\in\mathbb{N}} x_n = x \text{ and } (x_{n_k}) \text{ is a subsequence of} \\ (x_n), \text{ then } \rho_X - \lim\limits_{k\in\mathbb{N}} x_{n_k} = x. \end{cases}$$

$$(1.6')\quad \begin{cases} \text{With } K,M \text{ as in (1.6), let } x = \rho_X - \lim\limits_{k\in K} x_k \text{ and} \\ x = \rho_X - \lim\limits_{m\in M} \bar{x}_m. \text{ With } (\tilde{x}_n) \text{ defined as in (1.6),} \\ x = \rho_X - \lim\limits_{n\in \mathbb{N}} \tilde{x}_n. \end{cases}$$

$$(1.7')\quad \begin{cases} x = \rho_X - \lim\limits_{n\in \mathbb{N}} x_n \text{ if and only if every subsequence} \\ (x_{n_k}) \text{ of } (x_n) \text{ has a subsequence } (x_{n_{k_i}}) \text{ such that} \\ x = \rho_X - \lim\limits_{i\in \mathbb{N}} x_{n_{k_i}}. \end{cases}$$

$$(1.8')\quad \begin{cases} \text{If } x = \rho_X - \lim\limits_{n\in \mathbb{N}} x_n, \text{ then } \bar{x} = \rho_X - \lim\limits_{n\in \mathbb{N}} x_{\bar{n}} \text{ iff} \\ x = \bar{x} \text{ a.s.; furthermore, if } (\bar{x}_n) \text{ is such that for all} \\ n \in \mathbb{N},\ \bar{x}_n = x_n \text{ a.s., then } x = \rho_X - \lim\limits_{n\in \mathbb{N}} \bar{x}_n. \end{cases}$$

We note that (1.4), (1.5), and (1.7) are analogous to the conditions (LO), (L1), and (L2), respectively, used by Stummel in [3o] to describe the concept of discrete convergence; our condition (1.6) is weaker than (1.7) and is chosen instead of (1.6) so as not to exclude almost-sure convergence (see below). Finally, (1.8) is a natural condition if one has modes of convergence for random variables in mind; however, (1.8) rules out convergence in distribution (see section 5).

<u>Example 1.4:</u> Let $(x_n) \in S(X)^{\mathbb{N}}$, $x \in S(X)$. We recall the definitions of the basic modes of convergence. Let d be the metric on X. (x_n) converges to x

a) "almost surely" ("a.s.- $\lim\limits_{n\in\mathbb{N}} x_n = x$") iff there is an $N \in A$ with $P(N)=0$ such that for all $\omega \in \Omega\backslash N$, $\lim\limits_{n\to\infty} d(x_n(\omega), x(\omega)) = 0$.

b) "almost uniformly" ("a.u. - $\lim_{n\in\mathbb{N}} x_n = x$") iff for every $\varepsilon > 0$ there exists $A_\varepsilon \in A$ such that $P(\Omega\setminus A_\varepsilon) \le \varepsilon$ and $\lim_{n\to\infty} d(x_n(.), x(.)) = 0$ uniformly on A_ε.

c) "in probability" ("P - $\lim_{n\in\mathbb{N}} x_n = x$") iff for every $\varepsilon > 0$ we have $\lim_{n\to\infty} P(\{\omega \in \Omega / d(x_n(\omega), x(\omega)) \ge \varepsilon\}) = 0$.

If X is a subset of a separable Banach space, we can define for $p \ge 1$ "(x_n) converges in p-th mean to x" ("L^p - $\lim_{n\in\mathbb{N}} x_n = x$")

$$\lim_{n\to\infty} \int_\Omega \|x_n(\omega) - x(\omega)\|^p \, d\,P(\omega) = 0.$$

Note that by Egoroff's Theorem (see [33] for a proof for random variables with values in Polish spaces) almost uniform and almost sure convergence are equivalent. If $x = $ a.s. - $\lim_{n\in\mathbb{N}} x_n$, then $x = $ P - $\lim_{n\in\mathbb{N}} x_n$ ([33]); conversely, if $x = $ P - $\lim_{n\in\mathbb{N}} x_n$, then there is a subsequence (x_{n_k}) of (x_n) with $x=$ a.s.-$\lim_{k\in\mathbb{N}} x_{n_k}$ ([7]). We even have $x = $ P - $\lim_{n\in\mathbb{N}} x_n$ if and only if every subsequence (x_{n_k}) of (x_n) has a subsequence $(x_{n_{k_i}})$ such that $x = $ a.s.-$\lim_{i\in\mathbb{N}} x_{n_{k_i}}$ (e.g. [2], applied to the real-valued function $d(x(.), x_n(.))$).

This result shows that almost-sure convergence does not have the property (1.7), since otherwise convergence in probability would imply almost-sure convergence, which is well-known not to be true. For this reason, we replaced (1.7) by the weaker condition (1.6) in Definition 1.3. It follows from well-known properties of the modes of convergence considered here (see [33], [6]) that almost-sure convergence, almost uniform convergence, convergence in probability, and convergence in p-th mean fulfill the requirements *of Definition 1.3 and can therefore be taken as examples in those of our results which will involve a convergence as in Definition 1.3. Convergence in probability and in p-th mean also have the property (1.7).*

Definition 1.5: Let ρ_X and $\bar{\rho}_X$ be convergences on $S(X)$. ρ_X will be called "stronger" than $\bar{\rho}_X$ ("$\bar{\rho}_X \leq \rho_X$") if for all sequences $(x_n) \in D(\rho_X)$ and all $x \in \rho_X((x_n))$ we have $(x_n) \in D(\bar{\rho}_X)$ and $x \in \bar{\rho}_X((x_n))$.

In other words, $\bar{\rho}_X \leq \rho_X$ *if* $x = \rho_X - \lim\limits_{n\in\mathbb{N}} x_n$ *implies* $x = \bar{\rho}_X\text{-}\lim\limits_{n\to\infty} x_n$.

E.g., almost-sure convergence is stronger than convergence in probability.

We refer to section 5 for a brief discussion of a mode of convergence that has not been considered in Example 1.4, namely convergence in distribution.

2.) *THE GENERAL CONCEPT FOR THE APPROXIMATION OF SOLUTIONS OF RANDOM OPERATOR EQUATIONS*

The aim of this section is to develop a general concept for the approximation of solutions of random operator equations with not necessarily almost-surely unique solutions. For deterministic equations, such concepts have been developed by Anselone and Ansorge [1], Stummel [3o] and Vainikko [31]. We will use Stummel's approach. Since we want to approximate random solutions of random equations with respect to different modes of convergence, we will formulate our definitions and results in spaces of measurable functions equipped with convergences in the sense of Definition 1.3, where we will frequently use the more suggestive notation of (1.9).

We emphasize that by working a priori in spaces of random variables, all solutions of random equations involved will automatically be random solutions. This seems to be more natural than to use deterministic results for each $\omega \in \Omega$ and to try to prove the measurability of the resulting approximate solutions, which may or may not work; especially, this will generally not work if we have to pick subsequences (as it will frequently be necessary below): If we do that by some deterministic compactness argument for each $\omega \in \Omega$, there will be no way that we can pick the same subsequence for all $\omega \in \Omega$. Thus, if we use compactness arguments, we have to use them directly in spaces of measurable functions. This is a major reason for our working in spaces of random variables with a convergence.

A recent survey about the approximate solution of random integral equations can be found in the paper [5], which contains most of the references mentioned in this paragraph. Random variants of iterative methods for fixed point problems involving operators that fulfill contraction conditions were given first in [3],[17], [8]. In [5] and [23] results of this type are used for approximating solutions of other random equations. Approximation methods for obtaining (least-squares) solutions for linear operator equations in Hilbert space were discussed in [19] and [12], where the latter paper specializes in projection methods. The paper [25] contains a general concept based on Stummel's approach for the approximate solution of random equations, but covers only the case of almost-surely uniquely solvable equations and of a.s.-convergence. In [26] other modes of convergence are allowed, but since an "inverse stability" condition is used there, only the case of (locally) uniquely solvable equations can be treated (cf.[3o, p. 291]).

The aim of this section is to develop the concepts of [25],[26] along the lines of [3o] in such a way that non-uniquely solvable equations involving random operators on stochastic domains can be treated. By using a suitable compactness condition for random

operators we can prove convergence results for solution sets of random equations. Thus we need a definition for convergence of sets of random variables (in the sense of Kuratowski).
Throughout this section let ρ_X and ρ_Y be convergences (in the sense of Definition 1.3) on S(X) and S(Y), respectively.

<u>Definition 2.1:</u> *Let (Z_n) be a sequence of nonempty subsets of S(X).*

a) $\rho_X - \operatorname{Liminf}_{n\in\mathbb{N}} Z_n := \bigcup_{(x_n)\in A} \rho_X((x_n))$, where

$A := \{(x_n) \in S(X)^{\mathbb{N}} \,/\, x_n \in Z_n \text{ for all } n \in \mathbb{N}\}$.

b) $\rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} Z_n := \bigcup_{(x_k)\in B} \rho_X((x_k))$, where

$B := \{(x_k) \in S(X)^{\mathbb{N}}$ / there are $n_1 < n_2 < n_3 < \ldots$ such that $x_k \in Z_{n_k}$ for all $k \in \mathbb{N}\}$.

c) (Z_n) is called "ρ_X-convergent to $Z \in S(X)$" iff $\rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} Z_n = \rho_X - \operatorname{Liminf}_{n\in\mathbb{N}} Z_n = Z$. We then write $Z = \rho_X - \operatorname{Lim}_{n\in\mathbb{N}} Z_n$.

d) (Z_n) is called "ρ_X-compact" iff for all $(x_k) \in S(X)^{\mathbb{N}}$ with $x_k \in Z_{n_k}$ for all $k \in \mathbb{N}$, where $n_1 < n_2 < n_3 < \ldots.$ are arbitrary integers, we have $\rho_X - \operatorname{Limsup}_{k\in\mathbb{N}} \{x_k\} \neq \emptyset$.

<u>Remark 2.2:</u> *First we try to explain Definition 2.1. An $x \in S(X)$ is in $\rho_X - \operatorname{Liminf}_{n\in\mathbb{N}} Z_n$ iff there is a sequence (x_n) with $x = \rho_X - \lim_{n\in\mathbb{N}} x_n$ and $x_n \in Z_n$ for all $n \in \mathbb{N}$.*

An $x \in S(X)$ is in $\rho_x - \operatorname{Limsup}_{n\in\mathbb{N}} Z_n$ iff there is a subsequence

(Z_{n_k}) of (Z_n) such that $x = \rho_X - \lim_{k\to\infty} x_{n_k}$ where $x_{n_k} \in Z_{n_k}$ for all $k \in \mathbb{N}$ is suitably chosen. Thus, the inclusion

$$(2.1) \qquad \rho_X - \operatorname*{Liminf}_{n\in\mathbb{N}} Z_n \subseteq \rho_X - \operatorname*{Limsup}_{n\in\mathbb{N}} Z_n$$

is obvious.

The definition of ρ_X-compactness of (Z_n) means the following: If (Z_{n_k}) is a subsequence of (Z_n) and $(x_k) \in S(X)^{\mathbb{N}}$ is such that $x_k \in Z_{n_k}$ for all $k \in \mathbb{N}$, then there is a subsequence (x_{k_i}) of (x_k) which is ρ_X-convergent in $S(X)$.

If $\bar\rho_X$ is another convergence on $S(X)$ with $\bar\rho_X \le \rho_X$, then it follows directly from the Definitions 1.5 and 2.1, that

$\rho_X - \operatorname*{Liminf}_{n\in\mathbb{N}} Z_n \subseteq \bar\rho_X - \operatorname*{Liminf}_{n\in\mathbb{N}} Z_n$ and

$\rho_X - \operatorname*{Limsup}_{n\in\mathbb{N}} Z_n \subseteq \bar\rho_X - \operatorname*{Limsup}_{n\in\mathbb{N}} Z_n$.

If moreover every $\bar\rho_X$ - convergent sequence contains a subsequence that is ρ_X - convergent to the same (set of) limits, we have $\rho_X - \operatorname*{Limsup}_{n\in\mathbb{N}} Z_n = \bar\rho_X - \operatorname*{Limsup}_{n\in\mathbb{N}} Z_n$. Thus, the following holds in this case:
If (Z_n) is ρ_X-convergent with $Z := \rho_X - \operatorname*{Lim}_{n\in\mathbb{N}} Z_n$, then (Z_n) is also $\bar\rho_X$-convergent and $Z = \bar\rho_X - \operatorname*{Lim}_{n\in\mathbb{N}} Z_n$. Finally, for such ρ_X, $\bar\rho_X$ a sequence (Z_n) is ρ_X-compact if and only if it is $\bar\rho_X$-compact. An

important example where these conditions on ρ_X, $\bar{\rho}_X$ are fulfilled are ρ_X: = a.s.-convergence, $\bar{\rho}_X$: = convergence in probability (see Example 1.4). Thus, (Z_n) is a.s.-compact iff it is P-compact; a.s.-$\operatorname{Limsup}_{n\in\mathbb{N}} Z_n$ = P-$\operatorname{Limsup}_{n\in\mathbb{N}} Z_n$, and if (Z_n) is a.s.-convergent, then it is P-convergent to the same limit set.

Now, let C, C_n $(n \in \mathbb{N})$ be multifunctions from Ω into X and $T: \operatorname{Gr} C \to Y$, $T_n: \operatorname{Gr} C_n \to Y$ $(n \in \mathbb{N})$ be $\mathcal{A}\times\mathcal{B}(X)$-measurable. By (1.1), $T(\omega,x(\omega))$ need not be defined for all $\omega \in \Omega$, if $x \in S(C)$. By $T(.,x(.))$ we will denote any $z \in S(Y)$ such that $z = T(.,\bar{x}(.))$a.s., where $\bar{x} = x$ a.s. and $\bar{x}(\omega) \in C(\omega)$ for all $\omega \in \Omega$; $T_n(.,x_n(.))$ with $x_n \in S(C_n)$ is defined analogously. Thus, $T(.,x(.))$ is a set of random variables any two of which are almost surely equal. If we use expressions like "$\rho_Y - \lim_{n\in\mathbb{N}} T_n(.,x_n(.)) = T(.,x(.))$" we mean "$\rho_Y - \lim_{n\in\mathbb{N}} Z_n = z$" (see(1.9)), where z_n and z represent $T_n(.,x_n(.))$ and $T(.,x(.))$, respectively, in the sense described above. Because of (1.8), this definition is independent of the special choices of (z_n) and z.

<u>*Definition 2.3:*</u>

a) (T_n) will be called "(ρ_X,ρ_Y)-convergent to T" iff for all $(x_n) \in S(X)^{\mathbb{N}}$ with $x_n \in S(C_n)$ for all $n \in \mathbb{N}$ and all $x \in S(C)$ with $x = \rho_X - \lim_{n\in\mathbb{N}} x_n$ we have $T(.,x(.)) = \rho_Y\text{-}\lim_{n\in\mathbb{N}} T_n(.,x_n(.))$.

b) T and (T_n) are called "(ρ_X, ρ_Y)-consistent" iff for all $x\in S(C)$ there is a sequence (x_n) with $x_n \in S(C_n)$ for all $n \in \mathbb{N}$ such that $x = \rho_X - \lim_{n\in\mathbb{N}} x_n$ and $T(.,x(.)) = \rho_Y - \lim_{n\in\mathbb{N}} T_n(.,x_n(.))$.

c) (T_n) is called "(ρ_X,ρ_Y)-stable" iff for all ρ_X-convergent *sequences* (x_n), $(\bar{x}_n)$ *with* $x_n \in S(C_n)$ *and* $\bar{x}_n \in S(C_n)$ *for all* $n \in \mathbb{N}$ *we have: If* $(T_n(.,x_n(.)))$ *is* ρ_Y*-convergent and* $\rho_X - \lim_{n\in\mathbb{N}} x_n = \rho_X - \lim_{n\in\mathbb{N}} \bar{x}_n$ *(i.e.,* $\rho_X((x_n)) = \rho_X((\bar{x}_n))$*), then* $(T_n(.,\bar{x}_n(.)))$ *is* ρ_Y*-convergent and* $\rho_Y\text{-}\lim_{n\in\mathbb{N}} T_n(.,x_n(.)) = \rho_Y - \lim_{n\in\mathbb{N}} T_n(.,\bar{x}_n(.))$.

d) (T_n) *is called "*(ρ_X, ρ_Y)*-inversely stable" iff for all sequences* (x_n), $(\bar{x}_n)$ *with* $x_n \in S(C_n)$ *and* $\bar{x}_n \in S(C_n)$ *for all* $n \in \mathbb{N}$ *we have: If* (x_n) *is* ρ_X*-convergent and* $T_n(.,x_n(.))$ *and* $T_n(.,\bar{x}_n(.))$ *are* ρ_Y*-convergent to the same limit (set), then* $(\bar{x}_n)$ *is* ρ_X*-convergent and* $\rho_X - \lim_{n\in\mathbb{N}} \bar{x}_n = \rho_X - \lim_{n\in\mathbb{N}} x_n$.

The properties defined in Definition 2.3 are straightforward adaptations of corresponding concepts in [3o] to our situation.

<u>*Example 2.4:*</u> *For simplicity, we assume here* $C_n = C$ *for all* $n \in \mathbb{N}$.

a) *If* $\lim_{n\to\infty} T_n(\omega,x) = T(\omega,x)$ *for all* $x \in C(\omega)$ *holds almost surely, then* T *and* (T_n) *are* $(\rho_X$*,a.s.)-consistent for any convergence* ρ_X *on* $S(X)$*. To see this, let* $x \in S(C)$ *be arbitrary and let* $x_n := x$ *for all* $n \in \mathbb{N}$*. By assumption, outside a set of measure 0,* $\lim_{n\to\infty} T_n(\omega,x_n(\omega)) = T(\omega,x(\omega))$*, thus* $\text{a.s.-}\lim_{n\in\mathbb{N}} T_n(.,x_n(.)) = T(.,x(.))$*. Obviously,* $\rho_X\text{-}\lim_{n\in\mathbb{N}} x_n = x$*. Obviously, under our assumptions* T *and* (T_n) *are also* (ρ_X,P)*-consistent. A generalization for* ρ_X *= a.s.-convergence will be given in Theorem 4.5.*

b) *Let* $h\colon \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ *be such that* $h(\omega,.)$ *is continuous in* $t=0$ *a.s. and that the following holds almost surely:*

For all $n \in \mathbb{N}$ and $x,y \in C(\omega)$, $d(T_n(\omega,x), T_n(\omega,y)) \leq h(\omega,d(x,y))$. Then (T_n) is (a.s., a.s.)-stable. This follows easily from the definition of stability and the conditions. Note that a special case of such operators T_n are operators that fulfill almost surely a Lipschitz- or Hölder-condition uniformly in n.

These two results are of some importance since they allow to check consistency and stability "realization-wise": We do not have to check the conditions of the definitions for random variables $x \in S(C)$, but if suffices to check conditions for fixed elements $x \in C(\omega)$ for almost all $\omega \in \Omega$.

As one would expect, the following result links stability, consistency, and convergence as in the deterministic case:

Proposition 2.5: *Let T and (T_n) be (ρ_X,ρ_Y)-consistent. Then (T_n) is (ρ_X, ρ_Y)-convergent to T if and only if (T_n) is (ρ_X,ρ_Y)-stable.*

Proof: *follows immediately from Definition 2.3.*

Remark 2.6: *This result can be used in the following way: Let for all $n \in \mathbb{N}$, $y_n \in S(Y)$, and assume that (y_n) is ρ_Y-convergent to y. Let $L_n := \{x_n \in S(C_n) \,/\, T_n(.,x_n(.)) = y_n\}$, $L := \{x \in S(C) \,/\, T(.,x(.)) = y\}$ and assume that for all $n \in \mathbb{N}$, $L_n \neq \emptyset$. If T and (T_n) are (ρ_X,ρ_Y)-consistent and (T_n) is (ρ_X,ρ_Y)-stable, then $\rho_X\text{-}\operatorname{Liminf}_{n\in\mathbb{N}} L_n \subseteq L$.*

However, we cannot conclude from these assumptions that $\rho_X - \operatorname{Liminf}_{n\in\mathbb{N}} L_n \neq \emptyset$. In other words: ρ_X-limits of solutions of the "approximate equations" $T_n(.,x_n(.)) = y_n$ are solutions of the "exact equation" $T(.,x(.)) = y$; however, the existence of such *limits cannot be guaranteed at this moment. If we require inverse stability, we can guarantee the existence of such limits, but by requiring this, we impose uniqueness on the solutions of the exact equation:*

Theorem 2.7: Let T and (T_n) be (ρ_X,ρ_Y)-consistent and (T_n) be (ρ_X,ρ_Y)-inversely stable. Let $(y_n) \in S(Y)^{\mathbb{N}}$ be ρ_Y-convergent to y, and x_n $(n \in \mathbb{N})$ and x be random solutions of $T_n(.,x_n(.)) = y_n$ and $T(.,x(.)) = y$, respectively, which are assumed to exist. Then $x = \rho_X - \lim_{n\in\mathbb{N}} x_n$. If $\bar{x}$ is another random solution of $T(.,x(.)) = y$, then $x = \bar{x}$ a.s.

Proof: follows easily from Definition 2.3; see also [26].

In section 4 we will give a sufficient condition for (a.s.,a.s.)-inverse stability.

To obtain similar convergence results for non-uniquely solvable equations, we need the following concepts:

Definition 2.8:

a) (T_n) is called "ρ_Y-compact" iff for all $(x_n) \in S(X)^{\mathbb{N}}$ with $x_n \in S(C_n)$ for all $n \in \mathbb{N}$, $(\{T_n(.,x_n(.))\})$ is compact (in the sense of Definition 2.1 d).

b) T and (T_n) are called "(ρ_X,ρ_Y)-closed" iff for all $n_1 < n_2 < n_3 < \ldots$ and sequences (x_{n_k}) with $x_{n_k} \in S(C_{n_k})$ for all $k \in \mathbb{N}$ we have: If $x = \rho_X - \lim_{k\in\mathbb{N}} x_{n_k}$ and $y = \rho_Y - \lim_{k\in\mathbb{N}} T_{n_k}(.,x_{n_k}(.))$, then $x \in S(C)$ and $T(.,x(.))=y$ a.s.

c) T and (T_n) are called "(ρ_X,ρ_Y)-regular" iff for all

$n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ and all sequences (x_{n_k}) with $x_{n_k} \in S(C_{n_k})$ for all $k \in \mathbb{N}$ we have: If $y = \rho_Y\text{-}\lim_{k\in\mathbb{N}} T_{n_k}(.,x_{n_k}(.))$, then there exists an $x \in S(C)$ with $T(.,x(.)) = y$ a.s. and $x \in \rho_X - \operatorname{Limsup}_{k\in\mathbb{N}} \{x_{n_k}\}$.

Proposition 2.9: Let $S(C) = \rho_X - \operatorname{Lim}_{n\in\mathbb{N}} S(C_n)$ and (T_n) be (ρ_X,ρ_Y)-convergent to T. Then we have:

a) T and (T_n) are (ρ_X,ρ_Y)-closed.

b) If $(S(C_n))$ is ρ_X-compact, then (T_n) is ρ_Y-compact.

Proof:

a) Let $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$, (x_{n_k}) such that $x_{n_k} \in S(C_{n_k})$ for all $k \in \mathbb{N}$ be arbitrary, but fixed, $x = \rho_X\text{-} \lim_{k\in\mathbb{N}} x_{n_k}$, $y = \rho_Y - \lim_{k\in\mathbb{N}} T_{n_k}(.,x_{n_k}(.))$; if one of these limits fails to exists, nothing remains to be shown. We assume without loss of generality that $\mathbb{N}\setminus\{n_1, n_2, n_3, \ldots\}$ is infinite; this can be achieved by picking a subsequence of (n_k), which can be done without changing the definitions of x and y because of (1.5). By definition and our assumption on $S(C)$, we have $x \in \rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} S(C_n) = S(C) = \rho_X - \operatorname{Liminf}_{n\in\mathbb{N}} S(C_n)$. Thus, there exists a sequence $(\bar{x}_n)$ with $\bar{x}_n \in S(C_n)$ for all $n \in \mathbb{N}$ and $x = \rho_X - \lim_{n\in\mathbb{N}} \bar{x}_n$. Let

$$\bar{\bar{x}}_n := \begin{cases} x_n & \text{if } n \in \{n_1, n_2, n_3, \ldots\} \\ \bar{x}_n & \text{otherwise.} \end{cases}$$

Since $\{n_1, n_2, n_3, \ldots\}$ and $\mathbb{N}\setminus \{n_1, n_2, n_2, \ldots\}$ are infinite, we can apply (1.6) and obtain

$$(2.2) \qquad x = \rho_X - \lim_{n\in\mathbb{N}} \bar{\bar{x}}_n .$$

Since for all $n \in \mathbb{N}$, $\bar{\bar{x}}_n \in S(C_n)$, *and since* (T_n) *is* (ρ_X,ρ_Y)-*convergent to* T, *(2.2) implies*

$$(2.3) \qquad T(.,x(.)) = \rho_Y - \lim_{n\in\mathbb{N}} T_n(.,\bar{\bar{x}}(.)).$$

Because of (1.5) and the definition of $(\bar{\bar{x}}_n)$, *(2.3) implies*

$$(2.4) \qquad T(.,x(.)) = \rho_Y - \lim_{k\in\mathbb{N}} T_{n_k}(.,x_{n_k}(.)).$$

Together with (1.8), (2.4) implies

$$(2.5) \qquad y = T(.,x(.)) \text{ a.s.}$$

Thus T *and* (T_n) *are* (ρ_X,ρ_Y)-*closed.*

b) *Let* $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$, (x_{n_k}) *be such that* $x_{n_k} \in S(C_{n_k})$ *for all* $k \in \mathbb{N}$. *It suffices to show that*

$$(2.6) \qquad \rho_Y - \operatorname{Limsup}_{k\in\mathbb{N}} \{T_{n_k}(.,x_{n_k}(.))\} \neq \emptyset.$$

Since $(S(C_n))$ *is by assumption* ρ_X-*compact, we have*

$$(2.7) \qquad \rho_X - \operatorname{Limsup}_{k\in\mathbb{N}} \{x_{n_k}\} \neq \emptyset.$$

Thus, there is an infinite subset K *of* $\{n_1, n_2, n_3, \ldots\}$ *(which, as in part a, we can choose such that* $\mathbb{N} \setminus K$ *is infinite) and an* $x \in S(X)$ *such that*

$$(2.8) \qquad x = \rho_X - \lim_{k\in K} x_k .$$

As in part a, we conclude that $x \in S(C)$ *and construct a sequence* $(\bar{\bar{x}}_n)$ *such that*

$$(2.9) \qquad \bar{\bar{x}}_n \in S(C_n) \text{ for all } n \in \mathbb{N}$$

and

$$(2.10) \qquad x = \rho_X - \lim_{n\in\mathbb{N}} \bar{\bar{x}}_n$$

with

(2.11) $\quad \bar{\bar{x}}_n = x_n \quad \text{if } n \in K.$

Because of (2.9), we can conclude from the convergence of (T_n) *to* T *and from (2.1o) that*

(2.12) $$T(.,x(.)) = \rho_Y - \lim_{n\in\mathbb{N}} T_n(.,\bar{\bar{x}}_n(.)).$$

As in part a, we obtain from (2.11) and (2.12) that

(2.13) $$T(.,x(.)) = \rho_Y - \lim_{k\in K} T_k(.,x_k(.)).$$

Thus, $T(.,x(.)) \in \rho_Y - \operatorname{Limsup}_{k\in\mathbb{N}} \{T_{n_k}(.,x_{n_k}(.))\}$,

so that (2.6) holds.

□

Corollary 2.1o: *Let* $S(C) = \rho_X - \operatorname{Lim}_{n\in\mathbb{N}} S(C_n)$, T *and* (T_n) *be* (ρ_X,ρ_Y)*-consistent and* (T_n) *be* (ρ_X,ρ_Y)*-stable. Then we have:*

a) T *and* (T_n) *are* (ρ_X,ρ_Y)*-closed.*

b) If $(S(C_n))$ *is* ρ_X*-compact, then* (T_n) *is* ρ_Y*-compact.*

Proof: Follows from Propositions 2.5 and 2.9

□

The main result of this section, Theorem 2.11, will be a convergence result for solutions of equations involving operators T *and* (T_n) *that are closed and compact and right-hand sides that are assumed to be regular in the sense of Definition 2.8. Sufficient conditions for the assumptions on the operators were already given in Propositions 2.9 and 2.1o (see also Example 2.4); sufficient conditions for the regularity assumption will be given in Remark 2.12.*

In addition to C, C_n, T, T_n, *let* $\mathcal{A}\times\mathcal{B}(X)$*-measurable mappings* $U: Gr\, C \to Y$ *and* $U_n: Gr\, C_n \to Y$ $(n \in \mathbb{N})$ *be given.*

We consider the random operator equations

(2.14) $\qquad T(\omega,x) = U(\omega,x)$

and

(2.14 n) $\qquad T_n(\omega,x) = U_n(\omega,x).$

We define

(2.15) $\qquad L: = \{x \in S(C)/T(.,x(.)) = U(.,x(.))\}$

and for all $n \in \mathbb{N}$

(2.15 n) $\qquad L_n: = \{x_n \in S(C_n)/T_n(.,x_n(.)) = U_n(.,x_n(.))\}$,

the sets of all random solutions of (2.14) and (2.14 n), respectively. See the remarks preceding Definition 2.3 concerning the notation. We view (2.14 n) as approximations to (2.14).

Theorem 2.11: *Let* L *and* L_n $(n \in \mathbb{N})$ *be as in (2.15) and (2.15 n), respectively. Assume that* U *and* (U_n) *are* (ρ_X,ρ_Y)*-regular, that* T *and* (T_n) *are* (ρ_X,ρ_Y)*-closed and that* (T_n) *is* ρ_Y*-compact.*

If for all $n \in \mathbb{N}$

(2.16) $\qquad L_n \neq \emptyset$

then

(2.17) $\qquad \emptyset \neq \rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} L_n \subseteq L.$

Proof: *Let* (x_n) *be such that*

(2.18) $\qquad x_n \in L_n$ *for all* $n \in \mathbb{N}$.

By the compactness assumption, $\rho_Y - \operatorname{Limsup}_{n\in\mathbb{N}} \{T_n(.,x_n(.))\} \neq \emptyset$.

Let $z \in \rho_Y - \operatorname{Limsup}_{n\in\mathbb{N}} \{T_n(.,x_n(.))\}$. *Then there is a subsequence* (x_{n_k}) *of* (x_n) *with* $z = \rho_Y - \lim_{k\in\mathbb{N}} T_{n_k}(.,x_{n_k}(.)) = \rho_Y - \lim_{k\in\mathbb{N}} U_{n_k}(.,x_{n_k}(.))$

because of (2.18). The regularity assumption now implies that

$\rho_X - \operatorname{Limsup}_{k\in\mathbb{N}} \{x_{n_k}\} \neq \emptyset$.

Since $\rho_X - \operatorname{Limsup}_{k\in\mathbb{N}} \{x_{n_k}\} \subseteq \rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} \{x_n\} \subseteq \rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} L_n$,

this implies

(2.19) $$\rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} L_n \neq \emptyset.$$

Now, let $x \in \rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} L_n$ *be arbitrary, but fixed. It suffices to show that*

(2.2o) $$x \in L,$$

since together with (2.19), this implies (2.17).

By the choice of x, there are a sequence $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ *and a sequence* (x_k) *with*

(2.21) $$x_k \in L_{n_k} \quad \text{for all } k \in \mathbb{N}$$

such that

(2.22) $$x = \rho_X - \lim_{k\in\mathbb{N}} x_k.$$

Let $y \in \rho_Y - \operatorname{Limsup}_{k\in\mathbb{N}} \{T_{n_k}(.,x_k(.))\}$ *(which is non-empty because of the compactness assumption; note that* (x_k) *is a subsequence of some sequence of elements of* $L_n \subseteq S(C_n)$ *because of (2.16)!).*

Because of Definition 2.1 b, (1.5), and the closedness assumption we have

(2.23) $$x \in S(C) \text{ and } T(.,x(.)) = y.$$

Because of (2.21) and the definition of y,

$y \in \rho_Y - \operatorname{Limsup}_{k\in\mathbb{N}} \{U_{n_k}(.,x_k(.))\}$. *By Definition 2.1 b and the regularity assumption, this implies the existence of a sequence* $k_1 < k_2 < k_3 < \ldots \in \mathbb{N}$ *and an* $\bar{x} \in S(C)$ *such that*

(2.24) $$U(.,\bar{x}(.)) = y \text{ and } \bar{x} \in \rho_Y - \operatorname{Limsup}_{j\in\mathbb{N}} \{x_{k_j}\}.$$

Because of (1.5), (2.22) and (2.24) imply that

(2.25) $x = \bar{x}$ a.s.

This implies together with (2.23) and (2.24) that (2.2o) holds.

Remark 2.12: *We interpret Theorem 2.11 in the following way: Under the conditions stated, there is a sequence of random solutions of the "approximate problems"(2.14 n) that has a ρ_X-convergent subsequence; all limits of ρ_X-convergent (sub)sequences of random solutions of the approximate problems are random solutions of the "exact problem" (2.14). As noted above, various sufficient conditions for the assumptions on T and (T_n) in Theorem 2.11 were already given above; other sufficient conditions (especially for the compactness assumption) will be given in Section 4. The regularity assumption on U and (U_n) is fulfilled in two important special cases: One is the case of "random fixed point problems"*

(2.26) $T(\omega,x) = x$

and their approximations in the form

(2.26 n) $T_n(\omega,x) = x.$

Here, $X = Y$, U: Gr $C \to X$ and U_n: Gr $C_n \to X$ are defined by $U(\omega,x) := x$ and $U_n(\omega,x) := x$, respectively. Obviously, U and (U_n) are $(\rho_X, \bar{\rho}_X)$-regular for arbitrary convergences with $\rho_X \leq \bar{\rho}_X$; e.g., U and (U_n) are (P, a.s.)-regular. Because of the relationship between a.s.-convergence and convergence in probability (see Example 1.4), U and (U_n) are also (a.s., P)-regular.

Another case where the regularity assumption is fulfilled is the case of equations of the form (1.3) and their approximations by equations of the form

(1.3 n) $T_n(\omega,x) = y_n(\omega)$

under the following assumptions: $y = \rho_Y\text{-}\lim_{n\in\mathbb{N}} y_n$, $(S(C_n))$ is ρ_X-compact and $\rho_X\text{-}\operatorname{Limsup}_{n\in\mathbb{N}} S(C_n) \subseteq S(C)$. Here U: Gr $C \to Y$ and U_n: Gr $C_n \to Y$ are defined by $U(\omega,x) := y$, $U_n(\omega,x) := y_n$.

From this the following results follow:

Corollary 2.13: Let $y: \Omega \to Y$, $y_n: \Omega \to Y$ $(n \in \mathbb{N})$ be measurable, $y = \rho_Y - \lim_{n\in\mathbb{N}} y_n$, $S(C) = \rho_X - \operatorname{Lim}_{n\in\mathbb{N}} S(C_n)$, $(S(C_n))$ be ρ_X-compact, T and (T_n) be (ρ_X,ρ_Y)-consistent and (T_n) be (ρ_X,ρ_Y)-stable. Let $L := \{x \in S(C)/T(.,x(.)) = y\}$ and for all $n \in \mathbb{N}$, $L_n := \{x_n \in S(C_n)/T_n(.,x_n(.)) = y_n\}$ and assume that $L_n \neq \emptyset$ for all $n \in \mathbb{N}$. Then $\emptyset \neq \rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} L_n \subseteq L$.

Proof: follows from Theorem 2.11, Corollary 2.1o, and Remark 2.12.

□

Corollary 2.14: Let $X = Y$, $\rho_X \leq \bar{\rho}_X$ be convergences on X, $S(C) = \rho_X - \operatorname{Lim}_{n\in\mathbb{N}} S(C_n)$, $(S(C_n))$ be ρ_X- compact, T and (T_n) be $(\rho_X,\bar{\rho}_X)$-consistent and (T_n) be $(\rho_X,\bar{\rho}_X)$-stable. Let $L := \{x \in S(C)/T(.,x(.)) = x\}$ and for all $n \in \mathbb{N}$, $L_n := \{x_n \in S(C_n)/T_n(.,x_n(.)) = x_n\}$ and assume that $L_n \neq \emptyset$ for all $n \in \mathbb{N}$. Then $\emptyset \neq \rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} L_n \subseteq L$.
The result holds also if ρ_X denotes a.s.-convergence and $\bar{\rho}_X$ denotes convergence in probability.

Proof: follows from Theorem 2.11, Corollary 2.1o, and Remark 2.12.

□

Remark 2.15: In Theorem 2.11 and Corollaries 2.13 and 2.14, there appears the condition "$L_n \neq \emptyset$ for all $n \in \mathbb{N}$". This condition has to be checked by applying the results about existence of random solutions quoted in section 1. The conditions of Theorem 2.11 can be modified in various ways. E.g., it can be seen from the proof that the compactness of (T_n) has only been used to conclude that

ρ_Y - $\mathop{Limsup}_{k\in \mathbb{N}} \{T_{n_k}(.,x_{n_k}(.))\} \neq \emptyset$, *where* $x_{n_k} \in L_{n_k}$, *whereas from Definition 2.8 a we could conclude this for* $x_{n_k} \in S(C_{n_k})$, *which is much more than we need, but all we can usually check. Thus, the following variant of Theorem 2.11 can be proved analogously to Theorem 2.11:*

Let the assumptions of Theorem 2.11 with the exception of the compactness of (T_n) *be fulfilled; for each* $n \in \mathbb{N}$, *let* $\emptyset \neq \bar{L}_n \subseteq L_n$. *Assume that for each sequence* (x_n) *with* $x_n \in \bar{L}_n$ *for each* $n \in \mathbb{N}$, $(\{T_n(.,x_n(.))\})$ *is* ρ_Y*-compact. Then* $\emptyset \neq \rho_X - \mathop{Limsup}_{n\in \mathbb{N}} \bar{L}_n \subseteq L$.

This modified compactness may sometimes be easier to check, if $\bar{L}_n$ *is a set of solutions of (2.14 n) fulfilling additional requirements like assuming only finitely many values (such solutions will be considered in section 3 in connection with "discretization schemes"); sufficient for such a modified compactness condition will be that* $(\{T_n(.,x_n(.))\})$ *is* ρ_Y*-compact for all* (x_n) *with* $x_n \in D_n$ *for all* $n \in \mathbb{N}$, *where* $D_n \subseteq S(C_n)$ *is a set characterized by the same additional requirements as* $\bar{L}_n$ *and is chosen in such a way that* $\bar{L}_n = L_n \cap D_n$.

In general, it cannot be guaranteed that $\rho_X - \mathop{Liminf}_{n\in \mathbb{N}} L_n \neq \emptyset$ *in Theorem 2.11. However, in the case that (2.14) is a.s.-uniquely solvable, the following result holds.*

<u>*Theorem 2.16:*</u> *Let the assumptions of Theorem 2.11 be fulfilled. In addition, assume that for all* x *and* $\bar{x} \in L$, $x = \bar{x}$ *a.s. and that* ρ_X *fulfills (1.7). Then* (L_n) *is* ρ_X*-convergent to* $L \neq \emptyset$.

<u>*Proof:*</u> *Because of Theorem 2.11 and (2.1), it suffices to show that*

(2.27) $$L \subseteq \rho_X - \mathop{Liminf}_{n\in \mathbb{N}} L_n.$$

Let $x \in L$ *and* (x_n) *be such that* $x_n \in L_n$ *for all* $n \in \mathbb{N}$, (x_{n_k}) *be an arbitrary subsequence of* (x_n). *Using the compactness, the closedness and the regularity assumptions, one shows similarly to the proof of Theorem 2.11 that* (x_{n_k}) *has a subsequence* $(x_{n_{k_j}})$ *that is* ρ_X*-convergent to an* $\bar{x} \in L$.

Since $\bar{x} = x$ *a.s. by assumption, it follows from (1.7) that* $x = \rho_X - \lim_{n \in \mathbb{N}} x_n$. *This implies (2.27).*

□

The Corollaries 2.13 and 2.14 can be complemented in an analogous way.

3.) DISCRETIZATION SCHEMES FOR APPROXIMATING RANDOM OPERATOR EQUATIONS

In section 2 we presented an approximation concept for random operator equations. Of course a central question is now how to construct approximations of the form (2.14 n) to a given equation of the form (2.14). We restrict ourselves to the more special equations (1.3) and (1.3 n).

The approximations (1.3 n) should be such that the approximate operators and right-hand sides are more easily computable than the exact ones and that resulting approximate equations are more easily solvable. It seems to be a reasonable approach to construct (1.3 n) in such a way that the underlying probability space $(\Omega, \mathcal{A}, P)$ *is "discretized" to a finitely generated probability space; the operator and right-hand side in (1.3 n) are then defined such that they are constant (in* ω*) on each set of the finite generator. The resulting problem is then a random equation on a finitely*

generated probability space, which can be thought of as a collection of finitely many deterministic problems. More precisely:
Let C *be a multifunction from* Ω *into* X, T: *Gr* $C \to Y$ *be* $\mathcal{A}\times\mathcal{B}(X)$-*measurable,* y: *Gr* $C \to Y$ *measurable. Let* $n \in \mathbb{N}$ *be arbitrary, but fixed.*
Let $s_n \in \mathbb{N}$, $A_{n1}, \ldots, A_{ns_n} \in \mathcal{A}$ *be pairwise disjoint with*

$\bigcup_{i=1}^{s_n} A_{ni} = \Omega$ *and* $P(A_{ni}) > 0$ *for all* $i \in \{1, \ldots, s_n\}$. *By* $\mathcal{A}_n$ *we denote the* σ-*algebra on* Ω *generated by* $\{A_{n1}, \ldots, A_{ns_n}\}$. *For each* $i \in \{1, \ldots, s_n\}$, *let* C_{ni} *be a non-empty subset of* X, $T_{ni}: C_{ni} \to Y$ *and* $y_{ni} \in Y$. *Now, let* C_n *be the multifunction from* Ω *into* X *defined by*

$$(3.1) \qquad C_n(\omega) := C_{ni} \quad \text{if } \omega \in A_{ni},$$

$y_n: \Omega \to Y$ *be defined by*

$$(3.2) \qquad y_n(\omega) := y_{ni} \quad \text{if } \omega \in A_{ni}$$

and T_n: *Gr* $C_n \to Y$ *be defined by*

$$(3.3) \qquad T_n(\omega,x) := T_{ni}(x) \text{ if } \omega \in A_{ni},\ x \in C_{ni}.$$

The collection $(\mathcal{A}_n, C_n, T_n, y_n)_{n \in \mathbb{N}}$ *is called a "discretization scheme (for (1.3))".*
Note that $(\Omega, \mathcal{A}_n, P/_{\mathcal{A}_n})$ *is a complete probability space and that the generating sets* A_{ni} $(i \in \{1,\ldots,s_n\})$ *are precisely the atoms of* $\mathcal{A}_n$.
If such a discretization scheme is given, we use

$$(1.3\ n) \qquad T_n(\omega,x) = y_n(\omega)$$

as approximation for (1.3), where T_n *and* y_n *are given by (3.3) and (3.2), respectively. Note that (1.3 n) reduces to the* s_n

deterministic equations

$$(3.4\ i)\qquad T_{ni}(x) = y_{ni} \quad \text{if } \omega \in A_{ni} \quad (i \in \{1,\ldots,s_n\}).$$

If all of the equations (3.4 i) are solvable, then one can construct a random solution of (1.3 n) in the following obvious way: For $i \in \{1,\ldots,s_n\}$, *let* $x_{ni} \in C_{ni}$ *be a solution of (3.4 i) and let* $x\colon \Omega \to X$ *be defined by*

$$(3.5)\qquad x(\omega) := x_{ni} \quad \text{if } \omega \in A_{ni}.$$

Then x solves (1.3 n) and x is $\mathcal{A}_n - \mathcal{B}(X)$*-measurable and hence also* $\mathcal{A} - \mathcal{B}(X)$*-measurable. However, note that not all random solutions (with respect to* $\mathcal{A}$*) of (1.3 n) need to be of the form (3.5), i.e., constant on each* A_{ni}*; a random solution of (1.3 n) could jump between different solutions of (3.4 i) on* A_{ni} *in a measurable way. In this context we refer to Remark 2.15. Sometimes it may be easier to check e.g. a compactness condition only for* $(\{T_n(.,x_n(.))\})$*, where* $x_n \in D_n \subseteq S(C_n)$ *and* D_n *consists of all functions that are constant on each* A_{ni}*. Then the modified version of Theorem 2.11 outlined in Remark 2.15 has to be used, yielding a result only about* $\rho_X - \operatorname{Limsup}_{n\in\mathbb{N}} \bar{L}_n$*, where* $\bar{L}_n$ *is the set of solutions of (1.3 n) that are of the form (3.5). This is not too bad, however, since the simple solutions of the form (3.5) are the intrinsic reason for constructing discretization schemes anyway!*

Note that by construction of a discretization scheme, T_n *(as defined by (3.3)) is* $\mathcal{A}_n \times \mathcal{B}(X)$*-measurable if for all* $i \in \{1,\ldots,s_n\}$, $C_{ni} \in \mathcal{B}(X)$ *and* T_{ni} *is* $\mathcal{B}(X) - \mathcal{B}(Y)$*-measurable (these are reasonable assumptions, which are fulfilled e.g. if all* C_{ni} *are closed or open and all* T_{ni} *are continuous). This follows from the fact that for each* $B \in \mathcal{B}(Y)$, $\{(\omega,x) \in \operatorname{Gr} C_n \,/\, T_n(\omega,x) \in B\} = \bigcup_{i=1}^{s_n} [A_{ni} \times (C_{ni} \cap T_{ni}^{-1}(B))]$*. Under those* *assumptions, T_n is (since $\mathcal{A}_n \subseteq \mathcal{A}$) also $\mathcal{A}\times \mathcal{B}(X)$-measurable. Thus the general measurability assumptions of section 2 are fulfilled for such T_n. It is easy to see that even without any assumptions about the C_{ni} and T_{ni}, T_n (as defined by 3.3) is always a random operator with stochastic domain C_n in the sense of Definition 1.2 with respect to $(\Omega, \mathcal{A}_n)$ and also with respect to $(\Omega, \mathcal{A})$.*

The idea of a discretization scheme goes back to [27] and was developed further in [25], [26]. In [27] and [25] a method for constructing approximation schemes using conditional expectations is outlined. We do not pursue this line of research further here. Instead, we address the question under which conditions a discretization scheme can be constructed in such a way that properties of the random operator to be approximated like continuity or compactness are preserved and that the approximate operators have properties that were relevant in section 2 like consistency and stability. By $C(X,Y)$ we denote $\{f: X \to Y \,/\, f \text{ continuous}\}$.

<u>*Theorem 3.1:*</u> *Let $h: X \times \mathbb{R}_0^+ \to \mathbb{R}$ be such that for all $x \in X$, $h(x,.)$ is continuous in 0 and $h(x,0) = 0$, and K be a non-empty subset of $C(X,Y)$. Let $T: \Omega \times X \to Y$ be a random operator such that the following holds for all $\omega \in \Omega$:*

$$T(\omega,.) \in K \tag{3.6}$$

and

$$d(T(\omega,x), T(\omega,z)) \le h(x, d(x,z)) \quad \text{for all } x,z \in X. \tag{3.7}$$

Then the following holds for each $n \in \mathbb{N}$:
There are $s_n \in \mathbb{N}$, pairwise disjoint sets $A_{n1},\dots,A_{ns_n} \in \mathcal{A}$ with $\bigcup_{i=1}^{s_n} A_{ni} = \Omega$ and operators $T_{n1},\dots,T_{ns_n} \in K$ such that with

$$\begin{aligned} T_n: \Omega \times X &\to Y \\ (\omega,x) &\to T_{ni}(X) \quad \text{if } \omega \in A_{ni} \end{aligned} \tag{3.8}$$

we have: T_n is a random operator and (3.7) holds with T_n instead of T. Furthermore, for all $(\omega,x) \in \Omega \times X$ we have

$$(3.9) \qquad \lim_{n\to\infty} T_n(\omega,x) = T(\omega,x).$$

T and (T_n) are (ρ_X,ρ_Y)-consistent, where ρ_X is any convergence on S(X) and ρ_Y denotes a.s.-convergence or convergence in probability. If there exists $\bar{h}: \mathbb{R}_0^+ \to \mathbb{R}$ continuous in 0 with $\bar{h}(0) = 0$ such that $h(x,t) \le \bar{h}(t)$ for all $x \in X$, then (T_n) is also (a.s., a.s.)-stable, and (T_n) is (a.s., a.s.)-convergent to T.

Proof: Let Z a countable dense subset of the Polish space X. For all $z \in Z$ let $i_z: C(X,Y) \to Y$ be defined by $i_z(f) := f(z)$. Since Z is dense, $\{i_z/z \in Z\}$ separates the points of C(X,Y) (by continuity!). Let $\mathcal{T}$ be the weak topology on C(X,Y) generated by $\{i_z/z \in Z\}$, i.e., the coarsest topology on C(X,Y) such that all i_z are continuous. It is well-known that $(C(X,Y),\mathcal{T})$ is homeomorphic to a subspace of $\bigtimes_{z\in Z} Y$ (equipped with the product topology) and hence metrizable and separable. Let

$$(3.10) \qquad \tilde{K} := \{f \in K/d(f(x),f(z)) \le h(x,d(x,z)) \text{ for all } x,z\in X\}.$$

If we equip $\tilde{K}$ with the subspace topology $\mathcal{T} \cap \tilde{K}$, $\tilde{K}$ becomes a metrizable separable (and hence second countable) space. Let $\tilde{\mathcal{B}}$ be the σ-algebra on $\tilde{K}$ generated by the sets of $\mathcal{T} \cap \tilde{K}$ and $\tilde{T}: \Omega \to C(X,Y)$ be defined by $\tilde{T}(\omega) := T(\omega,.)$. Because of our assumptions,

$$(3.11) \qquad \tilde{T}(\Omega) \subseteq \tilde{K}.$$

We show that

$$(3.12) \qquad \tilde{T} \text{ is } \mathcal{A}\text{-}\tilde{\mathcal{B}}\text{-measurable}.$$

To this end, let $C \in \mathcal{T} \cap \tilde{K}$; then C has the form $C = O \cap \tilde{K}$ with $O \in \mathcal{T}$. Since $(C(X,Y),\mathcal{T})$ is second countable and hence Lindelöf, O is a countable union of finite intersections of sets of the form $i_z^{-1}(O_z)$ with $z \in Z$, O_z open in Y. Thus, in order to show that $\tilde{T}^{-1}(C) \in A$ (which proves (3.11)) it suffices to show that for all $z \in Z$ and open $O_z \subseteq Y$,

$$(3.13) \qquad \{\omega \in \Omega / \tilde{T}(\omega) \in i_z^{-1}(O_z)\} \in A.$$

Let $z \in Z$, $O_z \subseteq Y$ be open. By definition of $\tilde{T}$ and since T is a random operator, $\{\omega \in \Omega/\tilde{T}(\omega) \in i_z^{-1}(O_z)\} = \{\omega \in \Omega/T(\omega,z) \in O_z\} \in A$., so that (3.13) and thus (3.12) holds.

We now apply Criterion 5 of [14] to the measurable mapping $\tilde{T}$ from Ω into the separable metric space $\tilde{K}$ and conclude:

For each $n \in \mathbb{N}$, there is an $A - \tilde{B}$ -measurable map $\tilde{T}_n: \Omega \to \tilde{K}$ such that $\tilde{T}_n(\Omega)$ is finite and such that for all $\omega \in \Omega$, $(\tilde{T}_n(\omega))$ converges to $\tilde{T}(\omega)$ with respect to the topology $\mathcal{T} \cap \tilde{K}$.

Let $n \in \mathbb{N}$ be fixed, s_n be the number of elements of $\tilde{T}_n(\Omega) =: \{T_{n1},\dots,T_{ns_n}\} \subseteq \tilde{K}$. For $i \in \{1,\dots,s_n\}$, let

$$(3.14) \qquad A_{ni} := \{\omega \in \Omega/\tilde{T}_n(\omega) = T_{ni}\}.$$

Because of (3.12), we have $A_{ni} \in A$ for all $i \in \{1,\dots,s_n\}$.

Let $T_n: \Omega \times X \to Y$ be as in (3.8); then $T_n(\omega,.) = \tilde{T}_n(\omega)$ and thus $T_n(\omega,.) \in \tilde{K}$ for all $\omega \in \Omega$. Because of (3.10), this implies that $T_n(\omega,.) \in K$ and that (3.7) holds with T_n instead of T.

Let $x \in X$ and $O \subseteq Y$ be open, $I_{x,O} := \{i \in \{1,\dots,s_n\}/T_{ni}(x) \in O\}$. Then $\{\omega \in \Omega/T_n(\omega,x) \in O\} = \bigcup_{i \in I_{x,O}} A_{ni} \in A$. Thus T_n is a random operator.

Let $\omega \in \Omega$ be arbitrary, but fixed. Since $(\tilde{T}_n(\omega))$ converges to $\tilde{T}(\omega)$ with respect to $\mathcal{T} \cap \tilde{K}$, we have that for all $z \in Z$, $(i_z(\tilde{T}_n(\omega))) \to i_z(\tilde{T}(\omega))$, i.e.,

$$(3.15) \qquad (T_n(\omega,z)) \to T(\omega,z) \text{ for all } z \in Z.$$

Now, let also $x \in X$ be arbitrary, but fixed. We show that (3.9) holds. To this end, let $\varepsilon > 0$ be arbitrary. Since Z is dense, there is (because of the properties of h) a $z \in Z$ with

$$(3.16) \qquad h(z,d(x,z)) < \frac{\varepsilon}{3};$$

we fix such a $z \in Z$. Because of (3.15), there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$(3.17) \qquad d(T_n(\omega,z),\ T(\omega,z)) < \frac{\varepsilon}{3}$$

holds. Since T and T_n fulfill (3.7), we obtain from (3.16) and (3.17) that for all $n \geq n_0$, $d(T_n(\omega,x),T(\omega,x)) \leq d(T_n(\omega,x),T_n(\omega,z))+$ $+\ d(T_n(\omega,z),T(\omega,z))+d(T(\omega,z),T(\omega,x)) \leq h(x,d(x,z))+\frac{\varepsilon}{3} + h(x,d(x,z)) < \varepsilon$. Thus, (3.9) holds.

The consistency follows now from Example 2.4 a. Under the additional assumption involving $\bar{h}$, the stability follows from Example 2.4 b, the convergence from Proposition 2.5.

□

<u>Remark 3.2:</u> This result says that under its conditions, a consistent approximation by random operators defined via a discretization scheme always exists. Moreover, if each realization of T has a certain property (described by the set K) such as compactness, contractivity or differentiability, the approximations T_n share this property.

Note that Theorem 3.1 is only applicable to continuous random operators (see (3.6)) for which in each point, the continuity is uniform in ω (see (3.7)). If the continuity is also uniform in X (i.e., if the function $\bar{h}$ exists), then the approximation can be guaranteed to be stable and hence convergent (with respect to a.s.-convergence); this is especially the case for Lipschitzian random operators with a Lipschitz constant independent of ω. In (3.14), there may be A_{ni} with $P(A_{ni}) = 0$; if we add those to

one of the other sets in (3.14) with positive probability and redefine the corresponding T_{ni} *in an obvious way, we can fulfill the condition* $P(A_{ni}) > 0$ *for* $i \in \{1,...,s_n\}$ *(with a smaller* s_n*) required in the definition of a discretization scheme; however, in the statement of Theorem 3.1 "for all* $\omega \in \Omega$*" has to be replaced by "for almost all* $\omega \in \Omega$*" throughout.*
By combining the construction of Theorem 3.1 with a finitely-valued approximating sequence (y_n) *for* y *(more precisely: by taking all intersections of the sets in (3.14) with the sets where* y_n *is constant) the existence of a discretization scheme for (1.3) can be asserted for the case that all* $T(\omega,.)$ *are defined on all of* X*.*

Extensions to the case of stochastic domains should be possible. However, since Theorem 3.1 (though its proof is in principle constructive) does not give a concrete method for constructing a discretization scheme, we do not pursue this any further.

We add in passing that frequently random operators $T: \Omega \times X \to Y$ *have the special form*

$$T(\omega,x) := \hat{T}(z(\omega),x) \quad (\omega \in \Omega,\ x \in X), \tag{3.18}$$

where z *is a random variable from* Ω *into a separable metric space* Z *and* $\hat{T}: Z \times X \to Y$ *is such that* $T(.,x)$ *is continuous for all* $x \in X$*. In this case, a result analogous to Theorem 3.1 can be proven without continuity requirements for* $T(z,.)$*; this is done simply by approximating* z *by a pointwise convergent sequence of finitely-valued random variables* $z_n: \Omega \to Z$ *(see Criterion 5 in [14]) and setting* $T_n(\omega,x) := \hat{T}(z_n(\omega),x)$*. We will consider such operators also in section 4.*

4.) CONDITIONS FOR CONVERGENCE AND COMPACTNESS OF RANDOM OPERATORS AND THEIR DOMAINS WITH RESPECT TO A.S.-AND P-CONVERGENCE

In this section, we will give sufficient conditions for various assumptions of the results of section 2 to hold. We start with conditions that guarantee that $S(C) = \rho_X\text{-}\underset{n\in\mathbb{N}}{Lim}\, S(C_n)$ *as needed in Proposition 2.9 and Corollary 2.1o.*

For any $F \in \mathcal{P}(X)$ *and* $x \in X$, *we denote by* $d(x,F) := \inf_{z\in F} d(x,z)$.

Theorem 4.1: *Let* C *and* C_n $(n \in \mathbb{N})$ *be measurable multifunctions from* Ω *into* X.

a) $S(C) \subseteq a.s.\text{-}\underset{n\in\mathbb{N}}{Liminf}\, S(C_n)$ *if and only if there is an* $N \in \mathcal{A}$ *with* $P(N) = 0$ *such that for all* $\omega \in \Omega\backslash N$ *and all* $x \in C(\omega)$, $\lim_{n\to\infty} d(x,C_n(\omega)) = 0$.

b) *Assume that there is an* $N \in \mathcal{A}$ *with* $P(N) = 0$ *such that for all* $\omega \in \Omega\backslash N$ *and all* $x \in X$, $\lim_{n\to\infty} d(x,C_n(\omega)) = d(x,C(\omega))$.

Then $S(C) \subseteq a.s.\text{-}\underset{n\in\mathbb{N}}{Liminf}\, S(C_n) \subseteq a.s.\text{-}\underset{n\in\mathbb{N}}{Limsup}\, S(C_n) \subseteq S(\bar{C})$,

where $\bar{C}(\omega)$ *denotes the closure of* $C(\omega)$ *in* X.

c) *Let* C *be closed-valued. Under the assumptions of b,* $S(C) = a.s.\text{-}\underset{n\in\mathbb{N}}{Lim}\, S(C_n)$ *and* $S(C) = P\text{-}\underset{n\in\mathbb{N}}{Lim}\, S(C_n)$.

Proof:

a) *Let* $S(C) \subseteq a.s.\text{-}\underset{n\in\mathbb{N}}{Liminf}\, S(C_n)$.

Since $(\Omega,\mathcal{A},P)$ *is by assumption complete,* $(\Omega,\mathcal{A})$ *admits the Souslin operation (see [32]), we can apply the Corollary of [18, p.4o8] to conclude that there exists a sequence* $(x^k) \in S(C)^{\mathbb{N}}$ *such that*

(4.1) $\overline{\{x^k(\omega)/k \in \mathbb{N}\}} \supseteq C(\omega)$ for all $\omega \in \Omega$.

By assumption, for each $k \in \mathbb{N}$ there is a sequence $(x_n^k)_{n \in \mathbb{N}}$ with

(4.2) $x_n^k \in S(C_n)$ for all $n \in \mathbb{N}$

and

(4.3) $\lim_{n\to\infty} x_n^k(\omega) = x^k(\omega)$ a.s.;

let $\bar{N} := \bigcup_{k \in \mathbb{N}} [\{\omega \in \Omega / x^k(\omega) \notin C(\omega)\} \cup \bigcup_{n \in \mathbb{N}} \{\omega \in \Omega / x_n^k(\omega) \notin C_n(\omega)\}]$,

$N := \bar{N} \cup \bigcup_{k \in \mathbb{N}} \{\omega \in \Omega / (x_n^k(\omega))_{n \in \mathbb{N}}$ does not converges to $x^k(\omega)\}$.

Because of (1.1)and (4.3), $P(N) = 0$.

Now let $\omega \in \Omega \setminus N$ and $x \in C(\omega)$ be arbitrary, but fixed. Because of (4.1), there is a sequence (x^m) in $\{x^k(\omega)/k \in \mathbb{N}\}$ with

(4.4) $\lim_{m\to\infty} d(x^m, x) = 0.$

Since all x^m are in $\{x^k(\omega)/k \in \mathbb{N}\}$ and $\omega \notin N$, it follows from (4.2) and (4.3) that for all $m \in \mathbb{N}$, there is a sequence $(x_n^m)_{n \in \mathbb{N}}$ with

(4.5) $x_n^m \in C_n(\omega)$ for all $n \in \mathbb{N}$

and

(4.6) $\lim_{n\to\infty} d(x_n^m, x^m) = 0.$

We define sequences (m_j), (n_j) of integers as follows by induction: Let $m_0 = n_0 = 1$; for all $j \in \mathbb{N}$, let $m_j > m_{j-1}$ be such that

(4.7) $d(x^{m_j}, x) < 2^{-j-1}$,

which is possible because of (4.4), and $n_j > n_{j-1}$ be such that

(4.8) $d(x_n^{m_j}, x^{m_j}) < 2^{-j-1}$ for all $n \geq n_j$, *which is possible because of (4.6).*

We consider the sequence

$$(4.9) \qquad (x_n) := ((x_n^{m_j})_{n_j \le n < n_{j+1}})_{j \in \mathbb{N}_0},$$

i.e., the sequence

$(x_1^1, x_2^1, \ldots , x_{n_2-1}^1, x_{n_2}^{m_2}, \ldots , x_{n_3-1}^{m_2}, x_{n_3}^{m_3}, \ldots)$. *Because of (4.5),*

$$(4.10) \qquad x_n \in C_n(\omega) \quad \text{for all } n \in \mathbb{N}$$

and thus

$$(4.11) \qquad d(x, C_n(\omega)) \le d(x_n, x) \text{ for all } n \in \mathbb{N}.$$

Because of (4.7), (4.8), and (4.9), for all $n \in \mathbb{N}$,

$d(x_n, x) \le 2^{-j}$, *where* $j \in \mathbb{N}$ *is such that* $n_j \le n < n_{j+1}$.

As $n \to \infty$, *also* $j \to \infty$, *so that* $\lim_{n\to\infty} d(x_n, x) = 0$.

Together with (4.11), this implies $\lim_{n\to\infty} d(x, C_n(\omega)) = 0$.

For the converse, assume the existence of a set $N \in A$ *with* $P(N) = 0$ *such that*

$$(4.12) \qquad \lim_{n\to\infty} d(x, C_n(\omega)) = 0 \text{ for } \omega \in \Omega\backslash N,\ x \in C(\omega).$$

For each $n \in \mathbb{N}$, *let* $(x_n^k)_{k \in \mathbb{N}} \in S(X)^{\mathbb{N}}$ *be such that*

$x_n^k(\omega) \in C_n(\omega)$ *for all* $\omega \in \Omega, n, k \in \mathbb{N}$ *and that*

$$(4.13) \qquad \overline{\{x_n^k(\omega)/k \in \mathbb{N}\}} \supseteq C_n(\omega) \quad \text{for all } \omega \in \Omega;$$

the existence of such a sequence follows as above from [18].

Let $x \in S(C)$ *be arbitrary, but fixed. It suffices to show that*

$$(4.14) \qquad x \in \text{a.s.} - \operatorname*{Liminf}_{n\in\mathbb{N}} S(C_n).$$

For all $n,k \in \mathbb{N}$, *let*

$$(4.15) \qquad \tilde{A}_n^k := \{\omega \in \Omega / d(x(\omega), x_n^k(\omega)) \le d(x(\omega), C_n(\omega)) + \tfrac{1}{n}\};$$

we show that

(4.16) $\tilde{A}_n^k \in A$ for all $n,k \in \mathbb{N}$.

To see this, note that for all $n \in \mathbb{N}$ *and* $\omega \in \Omega$, $d(x(\omega), C_n(\omega)) = d(x(\omega), \overline{C_n(\omega)})$. *Since the multifunction* $\bar{C}_n$ *is measurable, which follows from [15], we can conclude from [9, Lemma 6] that* $d(x(.), C_n(.))$ *is measurable for all* $n \in \mathbb{N}$. *Since also* $d(x(.), x_n^k(.))$ *is measurable, (4.16) follows.*

Now let for all $n,k \in \mathbb{N}$

(4.17) $$A_n^k := \tilde{A}_n^k \setminus \bigcup_{i=1}^{k-1} \tilde{A}_n^i.$$

Obviously, $A_n^k \cap A_n^j = \emptyset$ *for all* $n \in \mathbb{N}$ *and* $j \neq k$, $\bigcup_{k \in \mathbb{N}} A_n^k = \bigcup_{k \in \mathbb{N}} \tilde{A}_n^k$ *for all* $n \in \mathbb{N}$;

(4.18) $A_n^k \in A$ for all $n,k \in \mathbb{N}$

holds because of (4.16). Because of (4.13), $\bigcup_{k \in \mathbb{N}} \tilde{A}_n^k = \Omega$ *for all* $n \in \mathbb{N}$, *so that we have*

(4.19) $$\bigcup_{k \in \mathbb{N}} A_n^k = \Omega \quad \text{for all } n \in \mathbb{N}.$$

Thus it is possible to define for all $n \in \mathbb{N}$

(4.2o) $$x_n : \Omega \to X, \quad \omega \to x_n^k(\omega) \quad \text{for } \omega \in A_n^k.$$

Since the x_n^k *are measurable, we can conclude from (4.18) that all* x_n *are measurable; by construction, for all* $n \in \mathbb{N}$ *and* $\omega \in \Omega$, $x_n(\omega) \in C_n(\omega)$, *so that*

(4.21) $x_n \in S(C_n)$ for all $n \in \mathbb{N}$.

It follows from (4.2o), (4.17), and (4.15) that

(4.22) $$d(x(\omega),x_n(\omega)) \le d(x(\omega),C_n(\omega)) + \frac{1}{n} \text{ for all } n\in\mathbb{N}, \omega\in\Omega.$$

Since $x(\omega) \in C(\omega)$ *a.s., it follows from (4.12) and (4.22) that* $(x_n) \to x$ *a.s.; together with (4.21), this implies (4.14). This completes to proof of a.*

b) *Since for* $x \in C(\omega)$, $d(x,C(\omega)) = 0$, *part a) is applicable; it remains to be shown that*

(4.23) $$\text{a.s. - } \operatorname*{Limsup}_{n\in\mathbb{N}} S(C_n) \subseteq S(\bar{C}).$$

Let $x \in$ *a.s. -* $\operatorname*{Limsup}_{n\in\mathbb{N}} S(C_n)$. *By definition, there exist sequences* $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ *and* (x_k) *with* $x_k \in S(C_{n_k})$ *for all* $k \in \mathbb{N}$ *such that* $x = \text{a.s. - } \lim_{k\in\mathbb{N}} x_k$, *i.e.,* $\lim_{k\to\infty} d(x_k(\omega), x(\omega)) = 0$ *a.s.; thus,* $\lim_{k\to\infty} d(x(\omega), C_k(\omega)) = 0$ *a.s. Together with the assumption, this implies* $d(x(\omega),C(\omega)) = 0$ *a.s, i.e.,* $x(\omega) \in \overline{C(\omega)}$ *a.s.; since x is measurable,* $x \in S(\bar{C})$. *This implies (4.23).*

c) *This follows immediately from part b) and Remark 2.2.*

□

For non-empty subsets E,F of X, let

(4.24) $$D(E,F) := \max \{\sup_{x\in E} d(x,F), \sup_{x\in F} d(x,E)\};$$

restricted to the bounded closed sets, D is a metric (the "Hausdorff-metric"); if E or F are unbounded, D(E,F) may be infinite.

<u>*Theorem 4.2:*</u> *Let* C, C_n $(n \in \mathbb{N})$ *be measurable multifunctions*

from Ω *into X, C be closed-valued. Assume that*

(4.25) $$\lim_{n\to\infty} P(\{\omega \in \Omega / D(C(\omega), C_n(\omega)) \geq \varepsilon\}) = 0 \text{ for all } \varepsilon > 0.$$

Then $S(C) = P - \underset{n\in\mathbb{N}}{\mathrm{Lim}}\, S(C_n)$.

Proof: Let $x \in S(C)$ be arbitrary, but fixed. As in the proof of Theorem 4.1 a, we construct a sequence (x_n) with (4.21) and (4.22). Since for all $n \in \mathbb{N}$ we have $d(x(\omega), C_n(\omega)) \leq D(C(\omega), C_n(\omega))$ a.s., we obtain for all $n \in \mathbb{N}$ that

(4.26) $$d(x(\omega), x_n(\omega)) \leq D(C(\omega), C_n(\omega)) + \frac{1}{n} \quad \text{a.s.;}$$

together with (4.25), this implies that $x = \underset{n\in\mathbb{N}}{P\text{-}\lim}\, x_n$.

Thus, $x \in \underset{n\in\mathbb{N}}{P\text{-}\mathrm{Liminf}}\; S(C_n)$, so that

(4.27) $$S(C) \subseteq \underset{n\in\mathbb{N}}{P\text{-}\mathrm{Liminf}}\; S(C_n).$$

Now, let $x \in \underset{n\in\mathbb{N}}{P\text{-}\mathrm{Limsup}}\; S(C_n)$. By definition, there exist sequences $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ and (x_k) with $x_k \in S(C_{n_k})$ for all $k \in \mathbb{N}$ such that

(4.27) $$x = \underset{k\in\mathbb{N}}{P\text{-}\lim}\, x_k.$$

Since for all $\omega \in \Omega$, $d(x(\omega), C(\omega)) \leq d(x(\omega), x_k(\omega)) +$

$+ d(x_k(\omega), C_{n_k}(\omega)) + D(C_{n_k}(\omega), C(\omega))$ for all $k \in \mathbb{N}$, it follows from (4.25), (4.27) and the fact that $x_k(\omega) \in C_{n_k}(\omega)$ a.s. that

(4.28) $$P(\{\omega \in \Omega / d(x(\omega), C(\omega)) \geq \tfrac{1}{i}\}) = 0 \text{ for all } i \in \mathbb{N}$$

and hence

$$(4.29) \qquad d(x(\omega),C(\omega)) = 0 \quad a.s.;$$

since x is measurable, (4.29) implies that $x \in S(C)$, since C is closed-valued. Thus,

$$(4.3o) \qquad P - \operatorname*{Limsup}_{n\in\mathbb{N}} S(C_n) \subseteq S(C).$$

The assertion follows now from (4.27) and (4.3o).

□

Remark 4.3: *Theorem 4.1 a generalizes Lemma 3 of [25] considerably. One can think of Theorem 4.1 as a result about convergence of measurable selectors of C_n to measurable selectors of C. For $X = \mathbb{R}^m$, such results have been given in [28] for a.s.-convergence and convergence in probability and in [29] for convergence in distribution. In both papers (like in our Theorem 4.1), the functions $\omega \to d(x,C_n(\omega))$ play an important role. Note that in [28] the basic concept of convergence is a.s.-convergence of sequences of multifunctions (C_n), while our basic concept is a.s.-convergence of sequences of sets of selectors $(S(C_n))$. The following is easy to see, if one uses the results of [28], especially Theorem 4.3 there: If C, C_n $(n \in \mathbb{N})$ are closed-valued measurable multifunctions from Ω into $\mathbb{R}^m$ and $(C_n) \to C$ a.s. in the sense of [28], then $S(C) = a.s.\text{-} \operatorname*{Lim}_{n\in\mathbb{N}} S(C_n)$. It is not clear if the converse holds; here one also has to take into account that not all sets $Z \subseteq S(X)$ are selector sets of a measurable multifunction, i.e., there need not exist a multifunction C with $Z = S(C)$. This shows that a.s.-convergence in the sense of [28] may not even be definable if $Z=a.s.\text{-}\operatorname*{Lim}_{n\in\mathbb{N}} Z_n$ with Z_n, $Z \subseteq S(X)$. In this sense, our concept of convergence is weaker and more general than the concept in [28]. The essential*

difference is that in our concept, the "exceptional sets" that appear are allowed to depend on the selector considered, which gives us the flexibility we need in that respect for our purpose. It follows from these remarks that Theorem 4.1 can be thought of as a generalization of some of the results of [28] from $\mathbb{R}^m$ *(which was essential in the proofs there!) to a Polish space X. We note that after this paper had been drafted, another such generalization (see [24]) had been brought to our attention. That paper contains two main results. In Theorem 1 there, which is proved by a method similar to that used in the proof of [25, Lemma 3], the author assumes completeness of the values of all multifunctions considered and does not permit exceptional sets of measure zero; at least in one of the two parts of that theorem, exceptional sets cannot be added easily in the same way as they appear in our Theorem 4.1: Note again that in our* "$S(C) \subseteq$ a.s. $- \operatorname{Liminf}_{n\in\mathbb{N}} S(C_n)$"*, the exceptional sets may depend on the selectors! This is not the case in Theorem 2 of [24], where in addition the multifunctions considered are assumed to be compact-valued. Under this stronger assumption, the author obtains a stronger conclusion, namely uniform convergence on certain sets. Thus, our results are not directly comparable to those of [24].*
It should also be noted that we do not always assume that C_n *and C are closed-valued. In Theorem 4.2, we do not assume that* C_n *and C have bounded values. In [28], various possible definitions for convergence in probability of measurable multifunctions are given, one of them being essentially our (4.25). There no results about convergence in probability of selectors are stated, so that our Theorem 4.2 can be thought of as a continuation of that paper in this respect.*

<u>*Example 4.4:*</u> *Let X be a separable Banach space,* $r: \Omega \to \mathbb{R}^+$, $\bar{x}: \Omega \to X$ *and for all* $n \in \mathbb{N}$, $r_n: \Omega \to \mathbb{R}^+$ *and* $\bar{x}_n: \Omega \to X$ *be measur-*

able, and let for all $\omega \in \Omega$ *and* $n \in \mathbb{N}$, $C_n(\omega) := \{x \in X / d(x, \bar{x}_n(\omega)) \leq \leq r_n(\omega)\}$ *and* $C(\omega) := \{x \in X / d(x, \bar{x}(\omega)) \leq r(\omega)\}$. *It is easy to see that* C *and* C_n $(n \in \mathbb{N})$ *are measurable closed-valued multifunctions and that for all* $n \in \mathbb{N}$, $\omega \in \Omega$, $x \in X$ *we have:*

$$(4.31) \qquad d(x, C_n(\omega)) = \max\{0, d(x, \bar{x}_n(\omega)) - r_n(\omega)\},$$

$$(4.32) \qquad d(x, C(\omega)) = \max\{0, d(x, \bar{x}(\omega)) - r(\omega)\},$$

$$(4.33) \qquad D(C(\omega), C_n(\omega)) = d(\bar{x}(\omega), \bar{x}_n(\omega)) + |r(\omega) - r_n(\omega)|.$$

Note that (4.31)-(4.33) and also the conclusions below need not be true in general Polish spaces, as can be seen e.g. in discrete separable metric spaces.
Because of (4.31)-(4.33) and Theorems 4.1 and 4.2, the following results are true:

a) *If* $(\bar{x}_n) \to \bar{x}$ *a.s. and* $(r_n) \to r$ *a.s., then* $S(C) = \text{a.s.-}\underset{n \in \mathbb{N}}{\mathrm{Lim}}\, S(C_n)$.

b) *If* $\bar{x} = P - \lim_{n \in \mathbb{N}} \bar{x}_n$ *and* $r = P - \lim_{n \in \mathbb{N}} r_n$, *then*

$$S(C) = P - \underset{n \in \mathbb{N}}{\mathrm{Lim}}\, S(C_n).$$

Now we turn to sufficient conditions for (ρ_X, ρ_X)*-convergence of* (T_n) *to* T *(where* ρ_X *denotes a.s.-convergence or convergence in probability), which is needed in Proposition 2.9; together with the results of Theorems 4.1 and 4.2, we thus will obtain results that guarantee the closedness of* T *and* (T_n) *as needed in Theorem 2.11. Together with a compactness result of below, we obtain sufficient conditions for Proposition 2.9 b to be applicable, which yields the compactness of* (T_n) *as required in Theorem 2.11.*

Theorem 4.5: Let C, $C_n (n \in \mathbb{N})$ be multifunctions from Ω into X, T: Gr $C \to Y$, T_n: Gr $C \to Y$ $(n \in \mathbb{N})$ be $\mathcal{A} \times \mathcal{B}(X)$-measurable.

a) Assume there is an $N \in \mathcal{A}$ with $P(N) = 0$ such that for all $\omega \in \Omega \backslash N$ we have: For all $x \in C(\omega)$ and $(x_n) \to x$ with $x_n \in C_n(\omega)$ for all $n \in \mathbb{N}$, $(T_n(\omega, x_n)) \to T(\omega, x)$. Then (T_n) is (a.s., a.s.)-convergent to T.
If in addition $S(C) \subseteq$ a.s. - $\operatorname{Liminf}_{n \in \mathbb{N}} S(C_n)$, then T and (T_n) are also (a.s.,a.s.)-consistent and (T_n) is (a.s.,a.s.)-stable.

b) Assume there is an $N \in \mathcal{A}$ with $P(N) = 0$ such that the following holds for all $\omega \in \Omega \backslash N$:

$$(4.34) \quad \begin{cases} \text{For all } x \in C(\omega) \text{ and all sequences } n_1 < n_2 < n_3 < \dots, \\ (x_k) \to x \text{ with } x_k \in C_{n_k}(\omega) \text{ for all } k \in \mathbb{N}, \\ (T_{n_k}(\omega, x_{n_k})) \to T(\omega, x). \end{cases}$$

Then (T_n) is (P,P)-convergent to T. If in addition $S(C) \subseteq P$ - $\operatorname{Liminf}_{n \in \mathbb{N}} S(C_n)$, then T and (T_n) are also (P,P)-consistent and (T_n) is (P,P)-stable. If even $S(C) \subseteq$ a.s. - $\operatorname{Liminf}_{n \in \mathbb{N}} S(C_n)$, then (4.34) needs to be assumed only for $(n_k) = (1,2,3,\dots)$.

Proof:

a) The (a.s.,a.s.)-convergence of (T_n) to T is obvious. If $S(C) \subseteq$ a.s. - $\operatorname{Liminf}_{n \in \mathbb{N}} S(C_n)$, then for all $x \in S(C)$, there is a sequence (x_n) with $x_n \in S(C_n)$ for all $n \in \mathbb{N}$ and $x =$ a.s. - $\lim_{n \in \mathbb{N}} x_n$. By the first part, $T(.,x(.)) =$ $=$ a.s. - $\lim_{n \in \mathbb{N}} T_n(.,x_n(.))$. Thus, T and (T_n) are (a.s.,a.s.)-

consistent. The stability follows from Proposition 2.5.

b) Let $x \in S(C)$, (x_n) *be such that* $x_n \in S(C_n)$ *for all* $n \in \mathbb{N}$ *and* $x = P\text{-}\lim_{n\in\mathbb{N}} x_n$. *We have to show that*

$$T(.,x(.)) = P\text{-}\lim_{n\in\mathbb{N}} T_n(.,x_n(.)). \tag{4.35}$$

Let $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ *be arbitrary; there exists a subsequence* $(n_{k_i})_{i\in\mathbb{N}}$ *of* $(n_k)_{k\in\mathbb{N}}$ *such that* $x = \text{a.s.-}\lim_{i\in\mathbb{N}} x_{n_{k_i}}$ *([2]; see Example 1.4); together with (4.34), we obtain*

$$T(.,x(.)) = \text{a.s.-}\lim_{i\in\mathbb{N}} T_{n_{k_i}}(.,x_{n_{k_i}}(.)). \tag{4.36}$$

Since (n_k) *was arbitrary, (4.36) implies (4.35) ([2]). This completes the proof of convergence.*

If $S(C) \subseteq P\text{-}\operatorname{Liminf}_{n\in\mathbb{N}} S(C_n)$, *we obtain (P,P)-consistency and (P,P)-stability analogously to the proof of part a).*

Let $S(C) \subseteq \text{a.s.-}\operatorname{Liminf}_{n\in\mathbb{N}} S(C_n)$ *and (4.34) hold for* $(n_k) = (1,2,3,\ldots)$. *It suffices to show that (4.34) holds for all sequences* $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$.

Let $n_1 < n_2 < n_3 < \ldots \in \mathbb{N}$ *and* (x_{n_k}) *be such that for all* $k \in \mathbb{N}$, $x_{n_k} \in C_{n_k}(\omega)$ *and* $(x_{n_k}) \to x \in C(\omega)$ *for all* $\omega \in \Omega\backslash N$.

Because of Theorem 4.1 a, $\lim_{n\to\infty} d(x,C_n(\omega)) = 0$ *holds for all* $\omega \in \Omega\backslash\bar{N}$, *where* $\bar{N} \in \mathcal{A}$ *with* $P(\bar{N}) = 0$ *and* $\bar{N}$ *does not depend on* x, (x_{n_k}), *or* (n_k). *For simplicity, we denote the set* $N \cup \bar{N}$ *again by* N. *Let* $\omega \in \Omega\backslash N$ *be arbitrary. Then there is a sequence* $(\bar{x}_n)$ *with* $(\bar{x}_n) \to x$ *and* $\bar{x}_n \in C_n(\omega)$.

For all $n \in \mathbb{N}$, *let*

(4.37) $$\tilde{x}_n := \begin{cases} x_n & \text{if } n \in \{n_1, n_2, n_2, \dots\} \\ \bar{x}_n & \text{otherwise.} \end{cases}$$

Then $(\tilde{x}_n) \to x$ *and* $\tilde{x}_n \in C_n(\omega)$ *for all* $n \in \mathbb{N}$. *By assumption,* $T(\omega,x) = \lim_{n\to\infty} T_n(\omega, \tilde{x}_n)$. *Since* $(x_{n_k})_{k \in \mathbb{N}}$ *is a subsequence of* $(\tilde{x}_n)$, $T(\omega,x) = \lim_{k\to\infty} T_{n_k}(\omega, x_{n_k})$. *Thus, (4.34) holds (with a new set* N, *that is independent of* $x, (x_{n_k})$, *or* (n_k)*).*

This completes the proof.

Remark 4.6: *We consider operators of the type (3.18); let* $\hat{T}: Z \times X \to Y$ *be jointly continuous, where* Z *is a separable metric space,* z *and* z_n $(n \in \mathbb{N})$ *be random variables from* Ω *into* Z, $T(\omega,x) := \hat{T}(z(\omega),x)$ *and for all* $n \in \mathbb{N}$, $T_n(\omega,x) := \hat{T}(z_n(\omega),x)$ *for all* $\omega \in \Omega$, $x \in X$. *The following facts are easy to prove:*

If $z = \text{a.s.-}\lim_{n\in\mathbb{N}} z_n$, *then* (T_n) *is (a.s.,a.s.)-convergent to* T.

If $z = P - \lim_{n\in\mathbb{N}} z_n$, *then* (T_n) *is (P,P)-convergent to* T.

From Example 2.4 a we can conclude that T *and* (T_n) *are also (a.s,a.s.)-consistent (or (P,P)-consistent, respectively), so that by Proposition 2.5,* (T_n) *is (a.s.,a.s.)-stable (or (P,P)-stable), respectively.*

We now give a sufficient condition for inverse stability as needed in Theorem 2.7.

Proposition 4.7: *Let* C, $C_n (n \in \mathbb{N})$ *be multifunctions from* Ω *into* X, $T: Gr\, C \to Y$, $T_n: Gr\, C_n \to Y$ $(n \in \mathbb{N})$ *be* $\mathcal{A} \times \mathcal{B}(X)$*-measurable. Assume that there is an* $N \in \mathcal{A}$ *with* $P(N) = 0$ *such that for all* $\omega \in \Omega\backslash N$ *we have:*

There is a function $\alpha(\omega,.):\mathbb{R}_0^+ \to \mathbb{R}_0^+$ *with* $\lim_{t\to 0} \alpha(\omega,t) = \alpha(\omega,0) = 0$ *such that for all* $n \in \mathbb{N}$ *and all* $x, \bar{x} \in C_n(\omega)$ *we have*

$$d(x,\bar{x}) \le \alpha(\omega, d(T_n(\omega,x), T_n(\omega,\bar{x}))). \tag{4.38}$$

Then (T_n) *is (a.s., a.s.)-inversely stable.*

Proof: Let (x_n), $(\bar{x}_n)$ be sequences such that $x_n \in S(C_n)$ and $\bar{x}_n \in S(C_n)$ for all $n \in \mathbb{N}$ and assume that (x_n), $(T_n(.,x_n(.)))$ and $(T_n(.,\bar{x}_n(.)))$ are a.s.-convergent with a.s.-$\lim_{n\in\mathbb{N}} T_n(.,x_n(.)) =$ $=$ a.s. $-\lim_{n\in\mathbb{N}} T_n(.,\bar{x}_n(.))$.

For all $\omega \in \Omega\backslash N$ we have because of (4.38):

$$d(x_n(\omega), \bar{x}_n(\omega)) \le \alpha(\omega, d(T_n(\omega,x_n(\omega)), T_n(\omega,\bar{x}_n(\omega))));$$

because of our assumptions, the right-hand side tends to 0 as $n \to \infty$, so that $(d(x_n(.),\bar{x}_n(.))) \to 0$ a.s.; thus, $(\bar{x}_n)$ is a.s.-convergent and a.s.-$\lim_{n\in\mathbb{N}} x$ = a.s.-$\lim_{n\in\mathbb{N}} \bar{x}_n$.

This completes the proof.

□

Remark 4.8: The condition (4.38) is "inverse" to the sufficient condition for stability used in Example 2.4 b. Note that the common philosophy of most of the results given so far in this section is the following: We wanted to give sufficient conditions for properties that are defined via measurable selectors in terms of assumptions not involving measurable selectors but only members of almost all realizations of the measurable multifunctions involved.

We now turn to compactness conditions for sequences of sets of *random variables or of random operators. While for convergence in distribution, Prohorov's Theorem (e.g. [6]) provides a comfortable basis for such considerations, the situation is less satisfactory for a.s.-convergence and convergence in probability. The following result gives a sufficient condition for compactness of a set of random variables with respect to the modes of convergence that we are interested in here.*

Surprisingly, we could not find such a result in the literature; only for real-valued random variables, such a result can be found in [2o] without proof. We thank Anton Wakolbinger for valuable discussion concerning the following result.

<u>Theorem 4.9:</u> *Let $E \subseteq S(X)$ and assume that for all $\varepsilon > 0$, there are $n(\varepsilon) \in \mathbb{N}$, $A_1, \ldots, A_{n(\varepsilon)} \in \mathcal{A}$ and a compact set $K_\varepsilon \subseteq X$ that the following hold:*

$$P\left(\bigcup_{i=1}^{n(\varepsilon)} A_i\right) \geq 1-\varepsilon, \tag{4.39}$$

$$\begin{cases} \text{For all } x \in E \text{ and all } \omega \in \bigcup_{i=1}^{n(\varepsilon)} A_i, \\ d(x(\omega), K_\varepsilon) < \varepsilon. \end{cases} \tag{4.40}$$

$$\begin{cases} \text{For all } x \in E \text{ and } i \in \{1,\ldots,n(\varepsilon)\} \text{ there exists} \\ u_{i,x} \in X \text{ such that for all } \omega \in A_i,\ d(x(\omega), u_{i,x}) \leq \varepsilon. \end{cases} \tag{4.41}$$

Then every sequence of elements of E has a subsequence that converges in probability, almost surely and almost uniformly to some element of $S(X)$.

<u>Proof:</u> *Let $D: S(X) \times S(X) \to \mathbb{R}$ be defined by*

$$D(x,y) := \int_\Omega \frac{d(x(\omega),y(\omega))}{1+d(x(\omega),y(\omega))}\, d\,P(\omega). \tag{4.42}$$

It is easy to see (and well-known at least for the case $X = \mathbb{R}$) that

$$(4.43) \qquad x = P - \lim_{n\in\mathbb{N}} x_n \iff \lim_{n\to\infty} D(x_n,x) = 0$$

for $x \in S(X)$, $(x_n) \in S(X)^{\mathbb{N}}$.

D is a pseudometric, $(S(X),D)$ is complete; this follows from the fact that each sequence that is Cauchy in probability is P-convergent (see e.g. Theorem 1.4.18.3 and its proof in [33]; note that a different metric is used there). Thus it suffices to show that

(4.44) $\qquad$ *E is totally bounded in $(S(X),D)$.*

To see this, let $\varepsilon > 0$ be arbitrary, but fixed. Let $A_1,\dots,A_{n(\varepsilon)}$ and K_ε be as in the assumptions of the Theorem; we assume without loss of generality that $A_i \cap A_j = \emptyset$ for $i \neq j \in \{1,\dots,n(\varepsilon)\}$. Since K_ε is compact, there exist $k(\varepsilon) \in \mathbb{N}$ and $x_1,\dots x_{k(\varepsilon)} \in K_\varepsilon$ such that for all $z \in K_\varepsilon$, there exists $j \in \{1,\dots, k(\varepsilon)\}$ with $d(z,x_j) \le \varepsilon$.

Let $J: = \{f:\{1,\dots,n(\varepsilon)\} \to \{1,\dots k(\varepsilon)\}\}$, and for all $f \in J$, let

$$(4.45) \qquad z_f:\ \Omega \to X, \quad \omega \to \begin{cases} x_{f(i)} & \text{if } \omega \in A_i,\ 1 \le i \le n(\varepsilon) \\ a & \text{otherwise,} \end{cases}$$

where a is an arbitrary fixed element of X. $\{z_f/f \in J\}$ is a finite subset of $S(X)$. To prove (4.44), it suffices to show that for each $x \in E$, there exists an $f \in J$ with

$$(4.46) \qquad D(x,z_f) \le 5\varepsilon,$$

since then $\{z_f(f \in J\}$ is proven to be a "finite 5ε-net".

Let $x \in E$ be arbitrary, but fixed. For each $i \in \{1,\dots,n(\varepsilon)\}$, we choose an $\omega_i \in A_i$. Because of (4.4o) and the properties of

$\{x\ ,\ldots,x_{k(\varepsilon)}\}$, for all $i \in \{1,\ldots,n(\varepsilon)\}$, there is a $j \in \{1,\ldots,k(\varepsilon)\}$ such that

$$d(x(\omega_i),x_j) \le 2\varepsilon. \tag{4.47}$$

Let $f \in J$ be such that with $j = f(i)$, (4.47) holds; if z_f is defined as in (4.45), it follows from (4.47) that

$$d(x(\omega_i),\ z_f(\omega_i)) \le 2\varepsilon \quad \text{for all } i \in \{1,\ldots,n(\varepsilon)\}. \tag{4.48}$$

Together with (4.41), this implies

$$d(x(\omega),\ z_f(\omega)) \le 4\varepsilon \quad \text{for all } \omega \in \bigcup_{i=1}^{n(\varepsilon)} A_i. \tag{4.49}$$

Because of (4.49) and (4.39) we have:

$$D(x,z_f) \le \sum_{i=1}^{n(\varepsilon)} \int_{A_i} \frac{d(x(\omega),z_f(\omega))}{1+d(x(\omega),z_f(\omega))}\, d\,P(\omega) + P(\Omega\setminus \bigcup_{i=1}^{n(\varepsilon)} A_i) \le$$

$\le 4\varepsilon + \varepsilon = 5$; thus, (4.46) holds. By the remarks of above, this implies that every sequence in E contains a subsequence that converges in probability to some element in $S(X)$; by Example 1.4, this subsequence contains another subsequence converging almost surely and almost uniformly.

□

Remark 4.1o: The conditions (4.39) - (4.41) seem to be quite natural; (4.39) - (4.4o) remind of the usual tightness conditions in Prohorov's Theorem (see e.g. [6], [16]) for weak compactness of probability measures. Note that (4.4o) is in one respect weaker than the usual tightness requirement, which would involve "$x(\omega) \in K_\varepsilon$". However, since convergence in probability implies convergence in distribution ([6]), it follows from Prohorov's Theorem that under the assumptions of Theorem 4.9, for every $\varepsilon > 0$ there is a compact set $\bar{K}_\varepsilon \subseteq X$ such that for all $x \in E$,

$P(\{\omega/x(\omega) \in \bar{K}_\varepsilon\}) \geq 1-\varepsilon$. In another sense, (4.4o) is stronger than a tightness assumption, since $d(x(\omega), K_\varepsilon) < \varepsilon$ is assumed to hold for all ω in a set with probability $\geq 1-\varepsilon$ that is independent of $x \in E$.

In addition to (4.39) - (4.4o), we need the condition (4.41) that prevents the elements of E from oscillating too much on each A_i. It can be seen from (4.69) and (4.7o) in conjunction with (4.57) that a condition of this kind is in fact necessary. See also Remark 4.14.

If E consists of random variables that are constant on each set of a fixed discretization of $(\Omega,\mathcal{A},P)$ (see section 3), then (4.41) is trivially fulfilled.

Theorem 4.9 can be used to prove compactness of $(S(C_n))$ as needed e.g. in Proposition 2.9; just take $E := \bigcup_{n \in \mathbb{N}} S(C_n)$.

Now we want to give a sufficient condition for compactness of a sequence of random operators (T_n) that have the special form (3.18). For this purpose, we need the following Lemma:

Lemma 4.11: *Let Z be metric space, $K \subseteq Z$ be compact, $C \subseteq X$, $\hat{T}: K \times C \to Y$ be such that*

(4.5o) $\{\hat{T}(.,x)/x \in C\}$ *is equicontinuous on K*

and

(4.51) $\hat{T}(z,C)$ *is relatively compact for all $z \in K$.*

Then $\hat{T}(K \times C)$ is relatively compact.

Proof: *Let $((z_n,x_n))$ be an arbitrary sequence in $K \times C$. It suffices to prove that $(\hat{T}(z_n,x_n))$ has a convergent subsequence. Since K is compact, there is a subsequence (z_{n_i}) of (z_n)*

converging to a $\bar{z} \in K$. *Since* $\hat{T}(\bar{z},C)$ *is relatively compact, there is a subsequence* $(x_{n_{i_j}})$ *of* (x_{n_i}) *such that* $(\hat{T}(\bar{z},x_{n_{i_j}}))$ *converges to some* $y \in Y$. *We will show that*

$$(4.52) \qquad (\hat{T}(z_j,x_j)) \to y,$$

where j *abbreviates* n_{i_j} *from now on. Let* $\varepsilon > 0$ *be arbitrary.*

Because of (4.5o) and the compactness of K, $\{\hat{T}(.,x)/x \in C\}$ *is uniformly equicontinuous on* K. *Thus, there is a* $\delta > 0$ *such that for all* $z_1,z_2 \in K$ *with* $d(z_1,z_2) < \delta$ *and all* $x \in C$, $d(\hat{T}(z_1,x),\ \hat{T}(z_2,x)) < \varepsilon$.

For such a $\delta > 0$, *let* $j_0 \in \mathbb{N}$ *be such that for all* $j \geq j_0$, $d(z_j,\bar{z}) < \delta$ *and* $d(\hat{T}(\bar{z},x_j),y) < \varepsilon$. *Then we have for all* $j \geq j_0$: $d(\hat{T}(z_j,x_j),y) \leq d(\hat{T}(z_j,x_j),\ \hat{T}(\bar{z},x_j)) + d(\hat{T}(\bar{z},x_j),y) < 2\varepsilon$, *where the first term is less than* ε *because of the uniform equicontinuity and the definition of* δ. *Thus, (4.52) holds.*

□

<u>*Remark 4.12:*</u> *The assumption (4.5o) cannot be dispensed with, as the following example shows: Let* $X = Y = Z = \ell^2$, C *can be the unit ball,* $K := \{(z_n) \in Z / \frac{1}{2n} \leq z_n \leq \frac{1}{n}$ *for all* $n \in \mathbb{N}\}$, $\hat{T}: K \times C \to Y$ *be defined by* $\hat{T}(z,x) := \|z\|^{-2} \cdot \langle x,z \rangle \cdot z$. $\hat{T}$ *is jointly continuous. For all* $z \in K$, $\hat{T}(z,C)$ *is a bounded one-dimensional set and thus relatively compact, so that (4.51) holds. For all* $n \in \mathbb{N}$, *let* $z_n := \frac{1}{n} e_n \in K$, $x_n := e_n \in C$, *where* e_n *is the n-th unit vector. Then,* $(\hat{T}(z_n,x_n)) = (e_n)$, *which has no convergent subsequence. Thus,* $\hat{T}(K \times C)$ *is not relatively compact.*

Now, let (in addition to X *and* Y*)* Z *be a Polish space,* $C \subseteq X$, $\hat{T}: Z \times C \to Y$; *let* z *and* z_n $(n \in \mathbb{N})$ *be measurable mappings from* Ω *into* Z, *and let for all* $\omega \in \Omega$ *and* $x \in C$, (4.53) $T_n(\omega,x) := \hat{T}(z_n(\omega),x)$ for all $n \in \mathbb{N}$.

With this notation, we have the following compactness result for (T_n), where we identify a $C \subseteq X$ with the constant map $\omega \to C(\omega) := C$.

Theorem 4.13: Let T_n ($n \in \mathbb{N}$) be defined by (4.53) and assume that

(4.54) $\{\hat{T}(.,x)/x \in C\}$ is uniformly equicontinuous on Z,

(4.55) $\hat{T}(z,C)$ is relatively compact for all $z \in Z$,

and that

(4.56)
$$\begin{cases} \text{for all } \varepsilon > 0, \text{ there exist pairwise disjoint} \\ B_1,\dots,B_{k(\varepsilon)} \in \mathcal{A} \text{ with } P(\bigcup_{i=1}^{k(\varepsilon)} B_i) \geq 1-\varepsilon \text{ such that} \\ \text{for all } z \in Z,\ x \in S(C), \text{ and } i \in \{1,\dots,k(\varepsilon)\} \text{ there} \\ \text{is a } u_i \in Y \text{ such that for all } \omega \in B_i, \\ d(\hat{T}(z,x(\omega)), u_i) < \varepsilon \text{ holds.} \end{cases}$$

Furthermore, assume that

(4.57) $z = P - \lim_{n\in\mathbb{N}} z_n$.

Then, (T_n) is ρ_Y-compact, where ρ_Y denotes convergence in probability, a.s.-convergence or almost uniform convergence.

Proof: It suffices to show that for each $(x_n) \in S(C)^{\mathbb{N}}$, $(T_n(.,x_n(.)))$ is ρ_Y-compact; we will show this by proving that the set $E := \{T_n(.,x_n(.))\}$ fulfills the assumptions of Theorem 4.9. To this end, let $(x_n) \in S(C)^{\mathbb{N}}$ and $\varepsilon > 0$ be arbitrary, but fixed. By Egoroff's Theorem ([33]), there exists a subsequence of (z_n)

that converges to z almost uniformly; since we want to construct a subsequence of $(T_n(\cdot,x_n(\cdot)))$, *we may assume without loss of generality that*

$$(4.58) \qquad z = \text{a.u.} - \lim_{n\in\mathbb{N}} z_n .$$

Let $\delta > 0$ *be such that*

$$(4.59) \qquad \begin{cases} \text{for all } y_1,\ y_2 \in Z \text{ with } d(y_1,\ y_2) < \delta \text{ and all} \\ x \in C \text{ we have } d(\hat{T}(y_1,x),\ \hat{T}(y_2,x)) < \frac{\varepsilon}{2} . \end{cases}$$

Such a $\delta > 0$ *exists because of (4.54). Because of (4.58), there exists an* $\Omega' \subseteq \Omega$ *such that* $P(\Omega') \geq 1 - \frac{\varepsilon}{4}$ *and* $(z_n) \to z$ *uniformly on* Ω'. *Thus, there exists an* $n_0 \in \mathbb{N}$ *such that*

$$(4.60) \qquad d(z_n(\omega),z(\omega)) < \frac{\delta}{2} \text{ for all } n \geq n_0 \text{ and } \omega \in \Omega'.$$

Since the distribution of a single random variable is tight (see [6,Theorem 1.4]), there exist compact sets $L_0,\ L_1,\ \ldots\ ,\ L_{n_0-1}$ *such that with* $\Omega_0 := \{\omega \in \Omega / z(\omega) \in L_0\}$ *and* $\Omega_i := \{\omega \in \Omega / z_i(\omega) \in L_i\}$ $(1 \leq i \leq n_0-1)$ *we have*

$$(4.61) \qquad P(\Omega_i) \geq 1 - \frac{\varepsilon}{4n_0} \text{ for all } i \in \{0,\ldots,n_0-1\}.$$

Let

$$(4.62) \qquad \Omega'' := \Omega' \cap \bigcap_{i=0}^{n_0-1} \Omega_i$$

and

$$(4.63) \qquad L_\varepsilon := \bigcup_{i=0}^{n_0-1} L_i .$$

Then

$$(4.64) \qquad P(\Omega'') \geq 1 - \frac{\varepsilon}{2}$$

and $L_\varepsilon \subseteq Z$ *is compact; thus there exist* $r' \in \mathbb{N}$ *and* $\hat{z}_1,\dots,\hat{z}_{r'} \in Z$ *such that*

$$(4.65) \qquad L_\varepsilon \subseteq \bigcup_{i=1}^{r'} B(\tfrac{\delta}{2}, \hat{z}_i),$$

where $B(\frac{\delta}{2}, \hat{z}_i) := \{y \in Z / d(y,\hat{z}_i) \le \frac{\delta}{2}\}$.

For $i \in \{1,\dots,r'\}$ *and* $j \in \{1,\dots,n_0-1\}$, *let*

$$(4.66) \qquad A_{io}' := z^{-1}(B(\tfrac{\delta}{2}, \hat{z}_i)) \cap \Omega''$$

and

$$(4.67) \qquad A_{ij}' := z_j^{-1}(B(\tfrac{\delta}{2}, \hat{z}_i)) \cap \Omega'' .$$

For all $j \in \{0,\dots n_0-1\}$, $\bigcup_{i=1}^{r'} A_{ij}' = \Omega''$; *this follows from* *(4.62), (4.63) and (4.65). Now, let* $A_1',\dots,A_r'$ *be all non-empty sets of the form* $\bigcap_{j=0}^{n_0-1} A'_{i(j)j}$. *Then*

$$(4.68) \qquad A_1' \cup \dots \cup A_r' = \Omega'',$$

and (because of the measurability of z *and* (z_n)*),* $\{A_1', \dots, A_r'\} \subseteq \mathcal{A}$. *Because of the definition of these sets, we have for all* $i \in \{1,\dots,r\}$:

$$(4.69) \qquad \begin{cases} \text{There is a } z_{io} \in Z \text{ such that for all } \omega \in A_i', \\ d(z(\omega), z_{io}) < \frac{\delta}{2} . \end{cases}$$

$$(4.70) \qquad \begin{cases} \text{For all } n \in \{1,\dots,n_0-1\} \text{ there is a } z_{in} \in Z \text{ such} \\ \text{that for all } \omega \in A_i',\ d(z_n(\omega), z_{in}) < \frac{\delta}{2} . \end{cases}$$

The z_{io} *and* z_{in} *can be taken from the set* $\{\hat{z}_1,\dots, \hat{z}_r\}$ *because* of (4.66) and (4.67). From (4.69) we obtain together with (4.6o):

$$(4.71) \qquad \begin{cases} \text{With } z_{io} \text{ as in (4.69), we have} \\ d(z_n(\omega), z_{io}) < \delta \text{ for all } n \geq n_0 \text{ and } \omega \in A_i'. \end{cases}$$

Because of (4.62) and (4.63), $z(\omega) \in L_\varepsilon$ and $z_n(\omega) \in L_\varepsilon$ for all $\omega \in \Omega''$ and $n \in \{1, \ldots, n_0-1\}$; together with (4.6o), this implies

$$(4.72) \qquad d(z_n(\omega), L_\varepsilon) < \frac{\delta}{2} \text{ for all } n \in \mathbb{N} \text{ and } \omega \in \Omega''.$$

Now, let

$$(4.73) \qquad K_\varepsilon := \overline{\hat{T}(L_\varepsilon \times C)}.$$

Because of Lemma 4.11, $K_\varepsilon \subseteq Y$ is compact.
Let $k := k(\frac{\varepsilon}{2})$ and $B_1, \ldots, B_k$ be chosen as in (4.56) for $\frac{\varepsilon}{2}$ instead of ε. By $A_1, \ldots, A_m$ we denote all distinct non-empty sets of the form $B_s \cap A_j'$, where $s \in \{1,\ldots,k\}$, $j \in \{1,\ldots,r\}$. Then because of (4.64) and (4.56),

$$(4.74) \qquad P(\bigcup_{i=1}^{m} A_i) \geq 1-\varepsilon.$$

Let $i \in \{1,\ldots,m\}$ be arbitrary and let $s \in \{1,\ldots,k\}$, $j \in \{1,\ldots,r\}$ be such that

$$(4.75) \qquad A_i = B_s \cap A_j'.$$

For $n \in \{1,\ldots,n_0-1\}$, let $u_{in} \in Y$ be such that for all $\omega \in A_i$ we have

$$(4.76) \qquad d(\hat{T}(z_{jn}, x_n(\omega)), u_{in}) < \frac{\varepsilon}{2}.$$

For $n \geq n_0$ and $j \in \{1,\ldots r\}$, let $z_{jn} := z_{jo}$ (see(4.69)) and let $u_{in} \in Y$ be such that for all $\omega \in A_i$, (4.76) holds. Such u_{in} exist because of (4.56) and (4.75).

For all $n \in \mathbb{N}$ *and* $\omega \in A_i$, *we have:*

$d(\hat{T}(z_n(\omega), x_n(\omega)), u_{in}) \leq d(T(z_{jn}, x_n(\omega)), u_{in}) +$

$+ d(\hat{T}(z_n(\omega), x_n(\omega)), \hat{T}(z_{jn}, x_n(\omega))) < \frac{\varepsilon}{2} + d(\hat{T}(z_n(\omega), x_n(\omega)),$

$\hat{T}(z_{jn}, x_n(\omega)))$ *because of (4.76). Because of (4.59), (4.75), (4.7o), and (4.71) (together with* $z_{jn} := z_{jo}$ *for* $n \geq n_0$*), the second expression is also less than* $\frac{\varepsilon}{2}$*, so that*

$$(4.77) \qquad d(\hat{T}(z_n(\omega), x_n(\omega)), u_{in}) < \varepsilon \quad \text{for all } n \in \mathbb{N} \text{ and } \omega \in A_i.$$

Now let $n \in \mathbb{N}$ *and* $\omega \in \bigcup_{i=1}^{m} A_i$ *be arbitrary. Because of (4.72), there exists a* $v \in L_\varepsilon$ *with* $d(z_n(\omega), v) < \frac{\varepsilon}{2}$. *Because of (4.59), this implies* $d(\hat{T}(z_n(\omega), x_n(\omega)), \hat{T}(v, x_n(\omega))) < \varepsilon$; *since* $\hat{T}(v, x_n(\omega)) \in K_\varepsilon$, *this in turn implies*

$$(4.78) \qquad d(\hat{T}(z_n(\omega), x_n(\omega)), K_\varepsilon) < \varepsilon \quad \text{for all } n \in \mathbb{N} \text{ and } \omega \in \bigcup_{i=1}^{m} A_i.$$

Since (4.78) is valid for all $n \in \mathbb{N}$ *and* $\omega \in \bigcup_{i=1}^{m} A_i$, *we have altogether shown the following:*

For all $\varepsilon > 0$, *there are sets* $A_1, \ldots, A_{m(\varepsilon)} \in \mathcal{A}$ *with (4.74) and a compact set* $K_\varepsilon \subseteq Y$ *such that for all* $\omega \in \bigcup_{i=1}^{m(\varepsilon)} A_i$, *(4.78) holds.*

Furthermore, for all $n \in \mathbb{N}$ *and* $i \in \{1, \ldots, m(\varepsilon)\}$ *there exists a* $u_{in} \in Y$ *such that (4.77) holds. Thus we can conclude from Theorem 4.9 that* $\{\hat{T}_n(z_n(.), x_n(.))\} = \{T_n(., x_n(.))\}$ *is compact with respect to convergence in probability and hence also with respect to almost-sure and almost uniform convergence (cf. Example 1.4). This completes the proof.*

□

Remark 4.14: *The uniform equicontinuity in (4.54) can be replaced by equicontinuity if Z is compact; condition (4.56) is certainly hard to fulfill if we really need it for all $x \in S(C)$. In this context it is advantageous to use the modified version of Theorem 2.11 as outlined in Remark 2.15, since then (4.56) has to be checked only for $x \in D_n$ (see Remark 2.15 for the notation). If D_n is the set of elements of S(C) that are constant on all sets of a fixed discretization of Ω which is independent of n (see section 3), then (4.56) is trivially fulfilled.*

We think that a condition like (4.56) has to be assumed in Theorem 4.13, since for obtaining P-compactness, one certainly has to bound the oscillation on a finite discretization of Ω. This belief comes from the fact that for real-valued random variables, the analogue of (4.41) is necessary for P-compactness (see [2o]).

5.) CONCLUDING REMARKS

We have introduced many assumptions in this paper that we needed for our general convergence results in section 2; for these assumptions, we have given various sufficient conditions. The following table will help to locate the most important places where these concepts are needed and where sufficient conditions are given; all numbers refer to Theorems, Remarks, etc.:

We formulated our results for convergences fulfilling (1.4),(1.5), (1.6), and (1.8). Especially, we concentrated on convergence in probability, almost-sure and almost uniform convergence. Since two random variables may have the same distribution without being equal anywhere, (1.8) rules out convergence in distribution. It can be seen from the proofs that in some of our results (1.8) is in fact not necessary. Hopefully, we can pursue our concept for convergence in distribution in a subsequent paper. One obstacle is the following: If x_1 and $x_2 \in S(X)$ have the same distribution, then

Concept	*defined in*	*needed in*	*sufficient conditions*
$S(C) \subseteq \text{Liminf}\, S(C_n)$	2.1	4.5	4.1
$S(C) = \text{Lim}\, S(C_n)$	2.1	2.9, 2.1o, 2.13, 2.14	4.1, 4.2, 4.4, 4.9
(T_n) converges to T	2.3	2.9	2.5, 3.1, 4.5, 4.6
(T_n) and T consistent	2.3	2.5, 2.7, 2.1o, 2.13, 2.14	2.4a, 3.1, 4.5, 4.6
(T_n) stable	2.3	2.1o, 2.13, 2.14	2.4b, 2.5, 3.1, 4.5, 4.6
(T_n) inversely stable	2.3	2.7	4.7
(T_n) compact	2.8	2.11	2.9, 2.1o, 4.13
$S(C_n)$ compact	2.1	2.9, 2.1o, 2.13, 2.14	4.9
(T_n) and T closed	2.8	2.11	2.9, 2.1o
(U_n) regular	2.8	2.11	2.12

$T(.,x_1(.))$ and $T(.,x_2(.))$ need not have the same distribution. To see this, take P to be Lebesque-measure on $\Omega := [0,1]$, $\hat{T}(x,y) := x.y$ for all $x,y \in X := [0,1]$. For all $\omega \in \Omega$, let

$$x_1(\omega) := \begin{cases} 0 & \omega \leq \frac{1}{2} \\ 1 & \omega > \frac{1}{2} \end{cases}$$

and

$$x_2(\omega) := \begin{cases} 1 & \omega < \frac{1}{2} \\ 0 & \omega \geq \frac{1}{2} \end{cases}$$

x_1 and x_2 have the same distribution. With $T(\omega,x) := \hat{T}(x_1(\omega),x)$ for $\omega \in \Omega$, $x \in X$, we have $T(.,x_1(.)) = x_1$ and $T(.,x_2(.)) = 0$, which do not have the same distribution.
Thus, the obvious idea to replace a.s.-equality by equality of the distribution in (1.8) does not work.
If one uses convergence in distribution, all compactness assumptions will be more comfortable than in this paper, since our Theorem 4.9 is replaced by Prohorov's Theorem. In this context, until recently there was the following additional obstacle: If a sequence of random variables is proven by Prohorov's Theorem to converge in distribution to a probability measure, is this measure the distribution of a random variable on the same probability space? This obstacle has been removed in [13].

Note that of course all our results that yield convergence in probability imply convergence in distribution. However, an adequate theory for convergence in distribution should also use weaker assumptions that do not imply convergence in probability. Thus, more work in this context seems to be justified.

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