Random operator equations and their approximation

- Heinz Engl's nonrandom contributions -

W. Römisch

Humboldt-University Berlin
Institute of Mathematics

www.math.hu-berlin.de/~romisch

DFG Research Center MATHEON
Mathematics for key technologies

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Introduction

Differential and integral equations with random coefficients and random right-hand sides are now within reach of efficient computational methods.

The latter require a combination of discretization and sampling techniques and a specific theoretical justification.

New sampling methods based on randomized lattice rules (specific Quasi-Monte Carlo methods) are available, which led to a breakthrough in high-dimensional numerical integration and to lifting the curse of dimension.

(recent work of Sloan/Kuo/Joe and Dick/Pillichshammer)

A general approximation theory for random operator equations is already available since about 25 years and has been developed by **Heinz Engl** and his collaborators.

Recent work on random partial differential equations

Simultaneous stochastic Galerkin and FE methods

Elliptic partial differential equation with random coefficients

$$-\nabla(a(\xi, z(\omega))\nabla x(\xi,\omega)) = f(\xi) \ (\xi \in D), \ x(\xi,\omega) = 0 \ (\xi \in \partial D).$$

where $D \subset \mathbb{R}^m$, $z = (z_j)_{j\in\mathbb{N}}$ is uniformly distributed in $[0,1]^{\mathbb{N}}$,

$$a(\xi, z) = \bar{a}(\xi) + \sum_{j=1}^{\infty} z_j \psi_j(\xi) \ (\xi \in D) \text{ (Karhunen-Loeve expansion).}$$

Variational formulation:

$$X = H_0^1(D) \subset H = L_2(D) \subset X^* = H^{-1}(D), \ \|v\|_X = \|\nabla v\|_H$$

$$\int_D a(\xi, z(\omega))\langle \nabla x(\xi), \nabla v(\xi)\rangle d\xi = \int_D f(\xi)v(\xi)d\xi \quad (\forall v \in X).$$

s-term N-sample QMC scheme: $a(\xi, z_N^s) = \bar{a}(\xi) + \sum_{j=1}^{s} z_j \psi_j(\xi)$
FE method: Replace X by a finite element subspace X_h.

(Kuo/Schwab/Sloan: SIAM J. Num. Anal. 2012)

Random operator equations: Existence

Let (X, d) be a separable complete metric space, Y and Z separable metric spaces and $0 \in Y$ fixed. All metric spaces are endowed with their Borel σ-fields. Consider the random operator equation

$$T(x, z(\omega)) = 0 \quad (\omega \in \Omega),$$

where $T : X \times Z \to Y$ is a mapping and z is a Z-valued random variable given on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Question: (Existence of a random solution (Hanš 56))
Does there exist a measurable map $x : \Omega \to X$ such that

$$T(x(\omega), z(\omega)) = 0, \text{ } \mathbb{P}\text{-almost surely}$$

if the equation $T(x, z(\omega)) = 0$ is solvable for all $\omega \in \Omega$?

Note that measurability of $T(x, \cdot)$ with respect to z $(\forall x \in X)$ together with $\mathbb{P}$ -a.s. solvability of $T(x, z(\omega)) = 0$ is not sufficient!

Theorem: (Engl 78, Nowak 78)
Assume that $S(\omega) = \{x \in X : T(x, z(\omega)) = 0\} \neq \emptyset$ $\mathbb{P}$ -a.s.
If $T(\cdot, z(\cdot))^{-1}(0) \in \mathcal{F} \times \mathcal{B}(X)$, there exists a random solution.
<u>Proof:</u>
There is $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 1$ and $S(\omega) \neq \emptyset$, $\forall \omega \in A$.

$$\mathrm{Gr}\, S = \{(\omega, x) : x \in S(\omega)\} = T(\cdot, z(\cdot))^{-1}(0) \cap \{A \cap B : B \in \mathcal{F}\} \times \mathcal{B}(X)$$

Apply measurable selection theorems for set-valued maps with measurable graph on the completion of $\{A \cap B : B \in \mathcal{F}\}$ (Saint-Beuve, Leese, Himmelberg 74/75) and modify a measurable selection on a set of measure 0. $\square$

The condition $T(\cdot, z(\cdot))^{-1}(0) \in \mathcal{F} \times \mathcal{B}(X)$ is implied by (a) or (b):
(a) T is Borel measurable.
(b) T is a Carathéodory mapping, i.e., $T(\cdot, z)$ is continuous for all $z \in Z$ and $T(x, \cdot)$ is measurable for all $x \in X$.

Extensions due to Engl:
(i) Stochastic domains, i.e., $T(\cdot, z(\omega)) : C(\omega) \to Y$, where $C : \Omega \rightrightarrows X$ is a closed-valued measurable multifunction.
(ii) Set-valued operators T, i.e., $T : X \times Z \rightrightarrows Y$.

RANDOM FIXED POINT THEOREMS

Heinz W. Engl*
Johannes-Kepler-Universität

I. INTRODUCTION

The study of random operator equations was initiated by the Prague school of probabilists around Špaček and Hanš in the 1950's. As it seems to be a current trend to use stochastic models rather than deterministic ones it is not surprising that the interest in random operator equations has been revived in the last years. The basic questions one might ask about random operator equations contain of course all problems which are interesting for deterministic operator equations, such as existence, uniqueness, stability and approximation of solutions. But the randomization leads to several new questions such as the measurability of solutions and their statistical properties. In this paper we deal with the question of measurability of fixed points of single- and multivalued random operators on randomly varying domains of definition. A complete survey about the "state of the art" in this area up to late 1976 can be found in [1].

Let throughout this paper (unless stated otherwise) X be a real separable Banach space, (Ω, A, μ) a σ-finite measure space. We will use the words "stochastic" and "random" interchangeably also if μ is not a probability measure. B denotes the σ-algebra on X generated by the open sets. By 2^X we denote

A Borel probability measure $\mu_X \in \mathcal{P}(X)$ is called weak solution of the random operator equation

$$T(x, z(\omega)) = 0 \quad (\omega \in \Omega)$$

iff there exists $\mu \in \mathcal{P}(X \times Z)$ such that

$$\mu\, T^{-1} = \delta_0\,, \quad \mu_X = \mu\, p_X^{-1}\,, \quad \mathcal{L}(z) = \mu\, p_Z^{-1},$$

where $\mathcal{L}(z) = \mathbb{P}\, z^{-1} : \mathcal{B}(Z) \to [0,1]$ is the probability distribution (or law) of z, p_X and p_Z are the projections from $X \times Z$ to X and Z, respectively, and $\delta_0 \in \mathcal{P}(Y)$ denotes the Dirac measure placing unit mass at $0 \in Y$.

Remark:

If $x : \Omega \to X$ is a random solution, $\mathcal{L}(x)$ is a weak solution (by putting $\mu = \mathcal{L}(x, z)$).

A weak solution of a random operator equation is a random solution on some probability space.

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APPROXIMATE SOLUTIONS OF NONLINEAR RANDOM OPERATOR EQUATIONS: CONVERGENCE IN DISTRIBUTION

HEINZ W. ENGL AND WERNER RÖMISCH

For nonlinear random operator equations where the distributions of the stochastic inputs are approximated by sequences of random variables converging in distribution and where the underlying deterministic equations are simultaneously approximated, we prove a result about tightness and convergence in distribution of the approximate solutions. We apply our result to a random differential equation under Peano conditions and to a random Hammerstein integral equation and its quadrature approximations.

1. Introduction. In [**15**], we developed a theory of convergence of approximate solutions of random operator equations using concepts like consistency, stability, and compactness in sets of measurable functions. The results of that paper are valid for rather general notions of convergence including almost-sure convergence and convergence in probability, but excluding convergence in distribution. Of course, all the results in [**15**] that guarantee e.g. almost-sure convergence of approximate solutions imply their convergence in distribution. However, an adequate theory for convergence in distribution should also use weaker assumptions on the way the "stochastic inputs" (operator, right-hand side) are approximated that do not imply e.g. almost-sure convergence of the "stochastic outputs" (approximate solutions). It is shown in the concluding remarks of [**15**] that it is not possible to carry over the theory developed there to the case of convergence in distribution in a straightforward way.

In this paper, we prove a result about convergence in distribution of approximate solutions of random operator equations in fixed-point form; the conditions needed are such that they do not imply stronger modes of convergence for the approximate solutions: The stochastic quantities entering into the equation are approximated with respect to convergence in distribution only. Note that convergence in distribution is often sufficient for approximating statistical characteristics of the solution, since if (x_n) converges to x in distribution, then $(E(f(x_n))) \to E(f(x))$ for all bounded continuous real functions f, where E denotes the expected value (see [**6**, p. 23]).

Random operator equations: Approximations

Let X, Y and Z be complete separable metric (Polish) spaces, $T : X \times Z \to Y$ be Borel measurable, z be a Z-valued random variable (on $(\Omega, \mathcal{F}, \mathbb{P})$) and $0 \in Y$ be fixed. We consider

$$T(x, z(\omega)) = 0 \quad (\omega \in \Omega).$$

In addition, we consider the approximate random operator equations

$$T_n(x, z_n(\omega)) = 0 \quad (\omega \in \Omega_n; n \in \mathbb{N}),$$

where for each $n \in \mathbb{N}$, $X_n \subset X$, $Z_n \subset Z$, $T_n : X_n \times Z_n \to Y$ be Borel measurable and z_n be a Z_n-valued random variable (on $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$).

Let (x_n) be a sequence of random solutions to the approximate random operator equations.

Motivation: Approximation procedures for solving random equations require a '*discretization*' of T and an *approximation* ('sampling', 'estimation') of z.

Weak convergence in $\mathcal{P}(X)$: (μ_n) converges weakly to μ iff

$$\lim_{n\to\infty} \int_X f(x)\mu_n(dx) = \int_X f(x)\mu(dx) \qquad \forall f \in C_b(X,\mathbb{R}).$$

The topology of weak convergence is metrizable if X is separable. Weak compactness is characterized by uniform tightness due to Prokhorov's theorem.

Problem: Find conditions on (T_n) and T that imply weak convergence of $(\mathcal{L}(x_n))$ if $(\mathcal{L}(z_n))$ converges weakly to $\mathcal{L}(z)$.

A sequence (T_n) converges discretely to T iff

(i) $d(x,X_n) = \inf_{y\in X_n} d_X(x,y) \to 0 \quad (\forall x \in X)$,
$d(z,Z_n) = \inf_{v\in Z_n} d_Z(z,v) \to 0 \quad (\forall z \in Z)$.

(ii) For all $(x,z) \in X \times Z$ and sequences $(x_n,z_n) \in X_n \times Z_n$ such that $x_n \to x$ in X and $z_n \to z$ in Z it holds

$$T(x_n,z_n) \to T(x,z) \quad (\text{in } Y).$$

(Stummel, Reinhardt, Vainikko)

Theorem:

Let the following conditions be satisfied:

(a) $\bigcup_{n\in\mathbb{N}}[T_n(\cdot, z_n)]^{-1}(K)\cap B$ is relatively compact in X for each $z_n \in Z_n$, $n \in \mathbb{N}$, bounded $B \subset X$ and compact $K \subset Y$.

(b) $\{T_n(x, \cdot) : x \in B \cap X_n, n \in \mathbb{N}\}$ is equicontinuous on K for each bounded $B \subset X$ and compact $K \subset Z$.

(c) (T_n) converges discretely to T.

(d) $(\mathcal{L}(z_n))$ converges weakly to $\mathcal{L}(z)$.

(e) The set $\{\mathcal{L}(x_n) : n \in \mathbb{N}\}$ is stochastically bounded, i.e., for each $\varepsilon > 0$ there exists a bounded Borel set $B_\varepsilon \subset X$ such that

$$\inf_{n\in\mathbb{N}} \mathcal{L}(x_n)(B_\varepsilon) \geq 1 - \varepsilon\,.$$

Then the set $\{\mathcal{L}(x_n) : n \in \mathbb{N}\}$ is relatively compact with respect to the weak topology and each weak limit of a subsequence is a weak solution of the random operator equation $T(x, z(\omega)) = 0 \quad (\omega \in \Omega)$.

Elliptic PDEs with random coefficients

We consider the random elliptic PDE

$$-\nabla(a(\xi, z(\xi,\omega))\nabla x(\xi)) = f(\xi)\ (\xi \in D),\ x(\xi) = 0\ (\xi \in \partial D).$$

where $D \subset \mathbb{R}^m$ is a bounded polyhedron, z a Z-valued random variable, where Z is a bounded subset of $L_\infty(D)$, $X = H_0^1(D)$ and $Y = H^{-1}(D) = X^*$.

Variational formulation:

$$X = H_0^1(D) \subset H = L_2(D) \subset X^* = H^{-1}(D),\ \|x\|_X = \|\nabla x\|_{L_2^m}$$

$$\int_D a(\xi, z)\langle\nabla x(\xi), \nabla v(\xi)\rangle d\xi = \int_D f(\xi)v(\xi)d\xi \quad (\forall v \in X).$$

Let a be continuous, $C_{\min} \leq a(\xi, z) \leq C_{\max}$, $\forall(\xi, z) \in D \times Z$. Let X_n be a finite element subspace of X such that $d(x, X_n) \to 0$ $\forall x \in X$. Assume $(\mathcal{L}(z_n))$ converges weakly to $\mathcal{L}(z)$.
Then the sequence $(\mathcal{L}(x_n))$ converges weakly to a weak solution.

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WEAK CONVERGENCE OF APPROXIMATE SOLUTIONS OF STOCHASTIC EQUATIONS WITH APPLICATIONS TO RANDOM DIFFERENTIAL AND INTEGRAL EQUATIONS*

Heinz W. Engl

Institut für Mathematik
Johannes-Kepler-Universität
A-4o4o Linz
Austria

and

Werner Römisch

Sektion Mathematik
Humboldt-Universität
DDR-1o86 Berlin
German Democratic Republic

Dedicated to the memory of

A.T.Bharucha-Reid

ABSTRACT:

In this paper, we considerably extend our earlier result about convergence in distribution of approximate solutions of random operator equations, where the stochastic inputs and the underlying deterministic equation are simultaneously approximated. As a by-product, we obtain convergence results for approximate solutions of equations between spaces of probability measures. We apply our results to random Fredholm integral equations of the second kind and to a random non-linear elliptic boundary value problem.

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Approximate the integral

$$I(G(z)) = \int_Z G(z)dz = \lim_{s\to\infty} \int_{[0,1]^s} G(z_1,\ldots,z_s,0,\ldots)dz_1\cdots dz_s$$

by a lattice rule $Q_{s,N}(\triangle,\cdot)$ with random shift $\triangle$ in $[0,1]^s$ and N points in s dimensions, and for each shifted lattice point solve the approximate elliptic problem

$$\int_D a(\xi,z(\omega))\langle\nabla x_h(\xi),\nabla v(\xi)\rangle d\xi = \int_D f(\xi)v(\xi)d\xi \quad (\forall v\in X_h).$$

in a finite element subspace X_h (on $D\subset\mathbb{R}^m$) with $M_h = O(h^{-m})$ degrees of freedom with linear cost $O(M_h)$.

Theorem: (Kuo/Schwab/Sloan 12)
The convergence rate of the scheme is

$$\left(\mathbb{E}\left[|I(G(z)) - Q_{s,N}(\triangle;G(x_h))|^2\right]\right)^{\frac{1}{2}} \le C\left(s^{-1} + N^{-1+\delta} + h^{\tau}\right),$$

where $0\le\tau = t+t'\le 2$, N is prime, $\delta\in(0,\frac{1}{2}]$, $f\in H^{-1+t}(D)$, $G\in H^{-1+t'}(D)$, $p=\frac{2}{3}$ and $\sum_{j\in\mathbb{N}}\|\psi_j\|^p_{L_\infty(D)} < \infty$.

Conclusions and thanks

- Recently the numerical analysis of random differential equations became a very active field of research. In particular, combinations of generalized Wiener or Karhunen-Loéve expansions with multi-level Monte Carlo and Quasi-Monte Carlo methods became popular.
- There is a pre-history of existence and approximation results for random operator equations.
- Heinz Engl contributed basic results to both existence and approximation approaches of random equations.

- **Many thanks** Heinz for several years of valuable collaboration.
- **Many thanks** Heinz for all your support and for the lovely visits to Austria.

Congratulations Heinz to your 60th birthday !

Thank you for your attention !

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