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Airline network revenue management by multistage stochastic programming

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Abstract A multistage stochastic programming approach to airline network revenue management is presented. The objective is to determine seat protection levels for all itineraries, fare classes, points of sale of the airline network and all dcps of the booking horizon such that the expected revenue is maximized. While the passenger demand and cancelation rate processes are the stochastic inputs of the model, the stochastic protection level process represents its output and allows to control the booking process. The stochastic passenger demand and cancelation rate processes are approximated by a finite number of tree structured scenarios. The scenario tree is generated from historical data using a stability-based recursive scenario reduction scheme. Numerical results for a small hub-and-spoke network are reported.

Keywords Airline revenue management · Seat inventory control · Multistage stochastic programming · Scenario tree generation

1 Introduction

Revenue management in the airline industry refers to strategies for controlling the sale of seats according to the passenger demand in a flight network in order to maximize

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revenue. Revenue management started with the pioneering work of Littlewood (1972) and became standard in airline industries. For introductions and overviews we refer to Weatherford (1998), McGill and van Ryzin (1999), Pak and Piersma (2002), Klein and Petrick (2003), Van Ryzin and Talluri (2003), Talluri and van Ryzin (2004).

While earlier approaches to revenue management used linear programming models based on the average demand, more recently, probabilistic optimization approaches are suggested due the stochastic nature of the passenger demand and of the entire booking process. In Talluri and van Ryzin (1999) a randomized linear programming approach is proposed, where a deterministic model is solved for a sequence of demand samples. The averages of the resulting dual multipliers are then used as bid prices to control the booking process. The authors of Talluri and van Ryzin (1999) showed that their method outperforms the deterministic approach. In Bertsimas and de Boer (2005), a combination of a stochastic gradient algorithm and of approximate dynamic programming ideas is used to improve initial booking limits. In Higle and Sen (2005), De Boer et al. (2002), Cooper and Homen-de-Mello (2006), Chen and Homen-de-Mello (2006) two-stage stochastic programs are proposed to deal with the stochastic character of the booking process. In De Boer et al. (2002) a simple recourse model is used, where the LP relaxation is replaced by an equivalent problem based on Wets (1983). Compared to other approaches, e.g., Wollmer (1986) based on expected marginal revenue, this model does not require additional integer variables to deal with the stochastic passenger demand. The authors of Higle and Sen (2005) propose a two-stage model within a bid price approach where the capacity constraints in the first stage uses leg based seat allocations. The seats allocated to itineraries are then considered in the second stage. The simulation experiments provide higher revenues in most cases than probabilistic nonlinear programs as formulated, e.g., in Talluri and van Ryzin (1998). In Cooper and Homen-de-Mello (2006) a hybrid method is suggested where the second stage corresponds to the solution of a Markov decision problem. In Chen and Homen-de-Mello (2006) two-stage and multistage stochastic programs are considered. Due to the non-convexity of the multi-stage program (and its continuous relaxation), solving two-stage stochastic programs (similar to De Boer et al. 2002) on a rolling horizon is suggested.

The multistage stochastic programming approach to revenue management is so far only proposed in our earlier work Möller et al. (2004) and in the recent paper DeMiguel and Mishra (2006). In DeMiguel and Mishra (2006) a different model for network revenue management is considered by making optimal decisions on sales instead of seat protection levels and by excluding cancelations. This leads to a simpler and linear programming model. The focus of DeMiguel and Mishra (2006) is on testing different strategies for generating scenario trees (Monte Carlo sampling, principal components sampling, moment matching and bootstrapping methods), where the branching structure of the tree is prescribed. The authors of DeMiguel and Mishra (2006) test in-sample and out-of-sample stability for evaluating scenario trees and the performance on a small and a large flight network. They show that their multistage stochastic programming approach outperforms the deterministic approach and that the performance is also better than the approach of Talluri and van Ryzin (1999) if Monte Carlo sampling with a sufficiently high number of scenario branchings is employed.

In the present paper, we continue and extend our earlier work Möller et al. (2004) on multistage stochastic programming models in network revenue management into several directions. As in Möller et al. (2004) seat protection levels are determined and the cancelation process is taken into account allowing for overbooking in all time periods before departure. The disjunctive constraints describing the dynamics and constraints of the booking process are incorporated and reformulated by introducing auxiliary binary variables. After approximating the underlying stochastic process, the model is solved by mixed-integer linear programming algorithms. A new method (see Heitsch and Römisch 2005) for generating scenario trees as approximate representation of the passenger demand and cancelation rate process is used. It is based on a recent stability result for multistage stochastic programs in Heitsch et al. (2006) and does not impose conditions on the underlying probability distribution. The method starts with a certain number of possible scenarios for the passenger demand and cancelation rate process. It generates clusters of scenarios and branchings using a recursive scenario reduction procedure such that the maximal expected revenue of the original problem is approximated. Due to the multi-dimensionality of the multivariate passenger demand and cancelation rate process (containing various statistical dependencies between booking classes, dcps and legs), the generation technique for scenario trees is of great significance. The latter effect was also observed in the computational studies of DeMiguel and Mishra (2006). Our approach is tested on a single hub-and-spokes airline network and a variety of different starting scenario sets.

In the last 2 years, airline revenue management was challenged by increasing low fare competition which involved dismantling of booking class restrictions. The consequential change in passenger booking demand has required changes in the modeling assumptions. However, we believe, that this development will not affect all markets. In particular, the large network carriers which dominate long-haul routes will have to manage a combination of unrestricted low fare markets and more traditional markets, where rules and regulations cause different passenger demand patterns. Hence, we feel certain, that our model still meets the requirements of practice.

Our paper is organized as follows. First we describe the network revenue management problem and introduce a stochastic model which is refined in the sequel. Next, we discuss the approximation of the stochastic input process by scenario trees and describe a stability-based scenario tree generation method. The tree structure is used to reformulate the problem in node representation. Then, the incorporation of cancelations is motivated by an example. In the next section, we present numerical results which suggest the applicability of the approach (at least) to small networks. Finally, concluding comments are given.

2 Multistage stochastic programming model

2.1 Problem description

We consider a flight network consisting of I origin-destination itineraries (ODI), J fare classes, K points of sale (POS), L legs and $M(l)$ compartments in each leg $l = 1, \ldots, L$. The booking horizon is subdivided into T booking intervals. The interval bounds $t = 0, \ldots, T$ are called data collection points (dcp). The subscripts i, j, k, t are used to denote the itinerary, fare class, point of sale and dcp, respectively.

The stochastic input parameters are the unconstrained passenger demand $d_{i,j,k,t} \in \mathbb{Z}$ and the cancelation rates $\gamma_{i,j,k,t} \in [0, 1)$. Let d_t and γ_t denote the vectors $(d_{i,j,k,t})_{i,j,k}$ and $(\gamma_{i,j,k,t})_{i,j,k}$, respectively, containing all itineraries, fare classes and points of sale. The passenger demand and the cancelation rates are represented by a discrete time stochastic process $\xi = (\xi_0, \xi_1, \ldots, \xi_T)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\xi_t := (d_t, \gamma_t)$. For each $t \in \{0, \ldots, T\}$ we denote by $\mathcal{F}_t \subseteq \mathcal{F}$ the σ-field generated by the random vector $(\xi_0, \ldots, \xi_t)$. The σ-fields $\mathcal{F}_t$, $t = 0, \ldots, T$, represent a filtration, i.e., it holds $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ and without loss of generality $\mathcal{F}_T = \mathcal{F}$.

The decision variables are the protection levels $P_t = (P_{i,j,k,t})_{i,j,k}$, $t = 0, \ldots, T-1$, for the next booking interval $(t, t+1]$. These protection levels restrict the number of sold and uncanceled tickets of each itinerary, fare class and point of sale up to dcp $t+1$. The decision variables P_t form a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, too. We require, that this process is adapted to the filtration of σ-fields $\mathcal{F}_t$, $t = 0, \ldots, T$, i.e., the decision P_t at dcp t only depends on the information available until t (non-anticipativity).

The products, i.e., the tickets for each itinerary, fare class and point of sale, are assigned to the resources, i.e., the capacities $\mathcal{C}_{l,m}$ of the compartments $m = 1, \ldots, M(l)$ on the legs $l = 1, \ldots, L$, by a matrix $A = (a_{ijk,lm})$. Let $\mathcal{I}_l$ denote the set of itineraries containing leg l and $\mathcal{J}_m(l)$ the set of fare classes belonging to compartment m on leg l. The entries $a_{ijk,lm}$ of A belong to $\{0, 1\}$, where $a_{ijk,lm} = 1$ if $i \in \mathcal{I}_l$ and $j \in \mathcal{J}_m(l)$ and $a_{ijk,lm} = 0$ else. Let $\mathcal{C}$ denote the vector $(\mathcal{C}_{l,m})_{l,m}$ of capacities.

2.2 Stochastic model

The objective of the stochastic network revenue management model consists in determining protection levels P_t such that the expected revenue is maximized, i.e.,

$$\max_{(P_0,\ldots,P_{T-1})} \mathbb{E}\left[\sum_{t=0}^{T}(\langle f_t^b, b_t\rangle - \langle f_t^c, c_t\rangle)\right] \tag{1}$$

where $f_t^b = (f_{i,j,k,t}^b)_{i,j,k}$ and $f_t^c = (f_{i,j,k,t}^c)_{i,j,k}$ denote the vectors of fares and refunds, respectively, $b_t = (b_{i,j,k,t})_{i,j,k}$ and $c_t = (c_{i,j,k,t})_{i,j,k}$ denote the number of bookings and cancelations, respectively, during the booking interval $(t-1, t]$, the scalar product $\langle f_t^b, b_t\rangle$ is given as usual by

$$\langle f_t^b, b_t\rangle = \sum_{i,j,k} f_{i,j,k,t}^b b_{i,j,k,t}.$$

and $\langle f_t^c, c_t\rangle$ accordingly.

The number of tickets sold depends on the passenger demand and on the protection levels. Bookings will be made as long as the passenger demand is not satisfied and the protection levels allow bookings, respectively. Hence, we have the constraints

$$b_t = \min\left\{ P_{t-1} - \sum_{\tau=0}^{t-1} b_\tau + \sum_{\tau=0}^{t} c_\tau, d_t \right\}, \quad t = 1, \ldots, T, \ \mathbb{P}\text{-a.s.}, \tag{2}$$

which describe the dynamics of the booking and cancelation process. We note that the vectors $b_0 \in \mathbb{Z}$ and $c_0 \in \mathbb{Z}$ contain the bookings and cancelations, respectively, made before the optimization horizon starts.

Since the number of uncanceled seats in all compartment on each leg may not exceed the physical capacity of the compartments, we arrive at the capacity constraints

$$\sum_{i\in\mathcal{I}_l} \sum_{j\in\mathcal{J}_m(l)} \sum_{k=1}^{K} P_{i,j,k,T-1} \le \mathcal{C}_{l,m}, \quad m = 1, \ldots, M(l), \quad l = 1, \ldots, L, \quad \mathbb{P}\text{-a.s.},$$

or more compactly

$$A\ P_{T-1} \le \mathcal{C} \quad \mathbb{P}\text{-a.s.} \tag{3}$$

The latter constraint is required only for the last booking interval $(T-1, T]$ to allow overbookings in the preceding booking intervals without additional efforts like overbooking rules. We note that dcp $T-1$ is usually very close to departure. Since the constraint (2) implies $\sum_{\tau=0}^{T}(b_\tau - c_\tau) \le P_{T-1}$ $\mathbb{P}$-a.s. (3) refers indeed to the time of departure.

Furthermore, some variables of the optimization problem have to satisfy integrality and non-negative conditions.

$$b_t, P_t \in \mathbb{Z}^{I\times J\times K} \quad b_t, P_t \ge 0 \quad t = 1, \ldots, T, \ \mathbb{P}\text{-a.s.} \tag{4}$$

Finally, we require that the state and decision variables of the stochastic program are non-anticipative, i.e.,

$$b_t \text{ and } P_t \text{ are } \mathcal{F}_t\text{-measurable.} \tag{5}$$

The non-anticipativity constraint (5) expresses how the information flow evolves over time. If the stochastic input process has only a finite number of scenarios, the constraint (5) may be modeled by finite linear equality constraints in various ways, see (Ruszczyński and Shapiro, 2003, Chap. 3.6) and Römisch and Schultz (2001).

2.3 Reformulation of the optimization model

We denote by $B_{i,j,k,t} := \sum_{\tau=0}^{t} b_{i,j,k,t}$ the number of cumulative bookings and by $C_{i,j,k,t}$ the number of cumulative cancelations. The number of cumulative cancelations $C_{i,j,k,t}$ is then set to $C_{i,j,k,t} := \lfloor \gamma_{i,j,k,t} B_{i,j,k,t} + 0.5 \rfloor$, where $\lfloor \alpha \rfloor \in \mathbb{Z}$ means the lower integer part of $\alpha \in \mathbb{R}$. The number of cancelations in $(t-1, t]$ is given by $c_t = C_t - C_{t-1}$. The initial cumulative bookings and cancelations in dcp 0 are denoted by $\bar{B}^0$ and $\bar{C}^0$, respectively.

To derive an approximation for the equation in (2), we start from

$$b_t = \min\{P_{t-1} - B_{t-1} + \gamma_t(B_{t-1} + b_t), d_t\}$$

and obtain the equation

$$b_t = \min\left\{(1-\gamma_t)^{-1}P_{t-1} - B_{t-1}, d_t\right\}.$$

To have $b_t \in \mathbb{Z}$, we introduce an auxiliary variable $B_t^{\text{aux}} \in \mathbb{Z}$ by setting $B_t^{\text{aux}} = \lfloor (1-\gamma_t)^{-1}P_{t-1} + 0.5 \rfloor$ and replace the equation in (2) by

$$b_t = \min\left\{B_t^{\text{aux}} - B_{t-1}, d_t\right\},$$

which is equivalent to

$$B_t \le B_t^{\text{aux}} \tag{6}$$

$$b_t \le d_t \tag{7}$$

$$(6) \text{ or } (7) \text{ are active.} \tag{8}$$

The disjunctive constraints (8) may be modeled by introducing binary auxiliary variables (see Nemhauser and Wolsey 1988, Sect. I.4). For this purpose we introduce vectors of binary variables $\tilde{z}_t \in \{0,1\}^{I\times J\times K}$ as well as vectors of slack variables $z_t^d \in \mathbb{Z}^{I\times J\times K}$, and $z_t^P \in \mathbb{Z}^{I\times J\times K}$. The conditions (6)–(8) are then replaced by the (in)equalities

$$B_t + z_t^P = B_t^{\text{aux}} \quad b_t + z_t^d = d_t \quad 0 \le z_t^d \le \tilde{z}_t d_t \quad 0 \le z_t^P \le (1-\tilde{z}_t)\varkappa\,,$$

where $\varkappa$ is a sufficiently large positive constant.

The stochastic network revenue management model now reads

$$\max_{(P_0,\ldots,P_{T-1})} \mathbb{E}\left[\sum_{t=1}^{T}(\langle f^b, b_t\rangle - \langle f^c, c_t\rangle)\right]$$

subject to the dynamics of the cumulative bookings

$$B_0 := \bar{B}^0 \quad B_t = B_{t-1} + b_t \quad \mathbb{P}\text{-a.s.}$$

the protection level conditions

$$B_t + z_t^P = \left\lfloor (1-\gamma_t)^{-1}P_{t-1} + 0.5 \right\rfloor \quad \mathbb{P}\text{-a.s.}$$

the passenger demand constraints

$$b_t + z_t^d = d_t \quad \mathbb{P}\text{-a.s.}$$

the disjunctive constraints for the number of bookings

$$0 \le z_t^d \le \tilde{z}_t d_t \quad 0 \le z_t^P \le (1 - \tilde{z}_t)\varkappa \quad \mathbb{P}\text{-a.s.}$$

the approximations of the cumulative cancelations

$$C_t := \lfloor \gamma_t B_t + 0.5 \rfloor \quad \mathbb{P}\text{-a.s.}$$

the number of cancelations

$$c_t = C_t - C_{t-1} \quad \text{where } C_0 := \bar{C}^0 \quad \mathbb{P}\text{-a.s.}$$

the capacity constraints

$$A \ P_{T-1} \le \mathcal{C} \quad \mathbb{P}\text{-a.s.}$$

the integrality and non-negativity constraints

$$B_t, C_t, P_t, z_t^d \in \mathbb{Z}^{I \times J \times K}, \ \tilde{z}_t \in \{0,1\}^{I \times J \times K}, \ b_t, c_t \ge 0 \quad \mathbb{P}\text{-a.s.}$$

as well as the non-anticipativity constraints

$$b_t, P_t, \tilde{z}_t \text{ and } z_t^d \text{ are } \mathcal{F}_t\text{-measurable.}$$

2.4 Approximation of the stochastic process by scenario trees

The first step of solving the stochastic revenue management model consists in approximating the discrete-time stochastic process $\xi = (\xi_0, \xi_1, \ldots, \xi_T)$ by a stochastic process $\tilde{\xi} = (\tilde{\xi}_0, \tilde{\xi}_1, \ldots, \tilde{\xi}_T)$ having a finite number of scenarios with known probabilities. The stability result for multistage stochastic programs in Heitsch et al. (2006) implies that the process $\tilde{\xi}$ should be close to ξ in the sense of the L_1-distance (9) and of the filtration distance (10), where

$$\|\xi - \tilde{\xi}\|_1 := \mathbb{E}[|\xi - \hat{\xi}|] \tag{9}$$

$$D_{\mathrm{f}}(\xi, \hat{\xi}) := \sup_{x \in \mathcal{B}_\infty} \sum_{t=1}^{T-1} \|\mathbb{E}[x_t|\mathcal{F}_t(\xi)] - \mathbb{E}[x_t|\mathcal{F}_t(\tilde{\xi})]\|_1, \tag{10}$$

$\mathcal{B}_\infty = \{x : \Omega \to \mathbb{R}^{(T+1)d_x} : x \text{ is measurable}, |x(\omega)| \le 1, \forall \omega \in \Omega\}$ (with d_x denoting the dimension of decisions at each $t \in \{0, 1, \ldots, T\}$), and $\mathcal{F}_t(\xi)$ and $\mathcal{F}_t(\tilde{\xi})$ are σ-fields generated by $(\xi_0, \ldots, \xi_t)$ and by $(\tilde{\xi}_0, \ldots, \tilde{\xi}_t)$, respectively. Here, $\mathbb{E}[\cdot\,|\mathcal{G}]$ denotes the conditional expectation with respect to some σ-subfield $\mathcal{G}$ of $\mathcal{F}$. Since $\mathcal{F}_t(\xi)$ increases with respect to t, the same property has to required for $\mathcal{F}_t(\tilde{\xi}), t = 0, 1, \ldots, T$. Such properties can only be achieved if $\tilde{\xi}$ consists of a sufficiently large number of scenarios and of scenario branchings, i.e., $\tilde{\xi}$ is representable as a scenario tree.

Formally, a scenario tree consists of nodes and arcs, where the nodes at period t correspond to possible values of $\tilde{\xi}_t$, $t = 0, \ldots, T$, and the arcs describe which

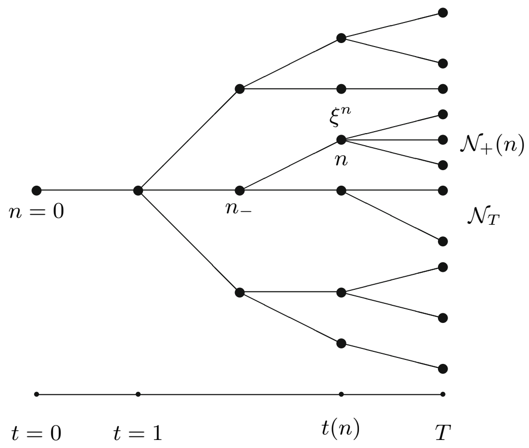

Fig. 1 Scenario tree with $T = 4$, $N = 21$ and 11 leaves

nodes are connected to scenarios. Let $\mathcal{N} := \{0, 1, \ldots, N\} \subset \mathbb{N}$ denote the set of all nodes, where $n = 0$ corresponds to the root node at $t = 0$ and $t(n)$ denotes the time period belonging to node n. By $\mathcal{N}_t$ we denote the set $\{n \in \mathcal{N} : t(n) = t\}$ for each $t = 0, \ldots, T$. Each node $n \in \mathcal{N}_t$, $t \in \{1, \ldots, T\}$, is connected with the unique *predecessor* node n_- at $t - 1$ by an arc. To each node $n \in \mathcal{N}_t$ with $t \in \{0, \ldots, T - 1\}$ a nonempty set $\mathcal{N}_+(n) \subset \mathcal{N}_{t+1}$ of *successors* is associated. By path(n) we denote the set $\{0, \ldots, (n_-)_-, n_-, n\}$ of nodes from the root to node n. Hence, each *scenario* corresponds to path(n) for some *leaf* $n \in \mathcal{N}_T$. The number of scenarios or leaves is denoted by S. We say that the *first stage* begins at time $t = 0$ and that $t \in \{1, \ldots, T\}$ marks the beginning of a new *stage* if there exists $n \in \mathcal{N}_{t-1}$ such that $\mathcal{N}_+(n)$ is not a singleton. We refer to Fig. 1 for a scenario tree instance with four stages. With the given scenario probabilities π^n, $n \in \mathcal{N}_T$, we associate a probability π^n to each node $n \in \mathcal{N}$ by the recursion $\pi^n = \sum_{m \in \mathcal{N}_+(n)} \pi^m$. Hence, we obtain $\sum_{n \in \mathcal{N}_t} \pi^n = 1$ for each $t = 0, \ldots, T$ and, in particular, $\pi^0 = 1$. In the following, we use the notation $\{\xi^n\}_{n \in \mathcal{N}}$ for the scenario tree representing the approximate stochastic input process.

2.5 Generation of scenario trees

Potential users of multistage stochastic programming models are often able to generate a (large) number of scenarios with given probabilities. Such scenarios may be obtained, e.g., by simulating from stochastic models that are calibrated to historical data or by using the past observations obtained under comparable circumstances directly as scenarios and by assigning them identical probabilities. In many practical cases, however, such sets of scenarios are not tree-structured except the appearance of the root node that corresponds to the presently available or initial information. We refer to the discussion in Dupačová et al. (2000, Sect. 3) for further information and relevant references. Presently available approaches to scenario tree generation are based on the use of bounds Kuhn (2005), on Monte Carlo Shapiro (2003) or Quasi-Monte Carlo methods Pennanen (2006), on moment matching Høyland and Wallace (2001), on metric distances of distributions Pflug (2001) and on stability arguments Heitsch and Römisch (2005). Most of them make use of a prescribed branching structure. Some of these methods have been implemented in DeMiguel and Mishra (2006) and tested on several instances of network revenue management models.

We will briefly describe the approach of Heitsch and Römisch (2005) which starts with a stochastic process $\hat{\xi}$ having a finite number of scenarios $\xi^s = (\xi_0^s, \xi_1^s, \dots, \xi_T^s)$ with probabilities p_s, $s = 1, \dots, S$, and being defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as the original process ξ. It is assumed that $\hat{\xi}$ consists of scenarios with common root, i.e., $\xi_0^1 = \dots = \xi_0^S = \xi_0^*$, and has the property that the mean or L_1-distance (9) and the filtration distance (10) are small. It is shown in Heitsch and Römisch (2005, Example 5.3) that, for example, sampling from some discrete probability distribution leads to suitable approximations $\hat{\xi}$ satisfying the above conditions for sufficiently large sample sizes.

Starting from $\hat{\xi}$ the approach of Heitsch and Römisch (2005) determines adaptively a stochastic process ξ_{tr} on $(\Omega, \mathcal{F}, \mathbb{P})$, whose scenarios have tree form, and satisfies the condition

$$\|\hat{\xi} - \xi_{\text{tr}}\|_1 \le \varepsilon \tag{11}$$

where $\varepsilon > 0$ is a prescribed tolerance. We denote by $I_t \subset \{1, \dots, S\}$ the index set of realizations of ξ_{tr} at $t \in \{0, \dots, T\}$ and by $I_{t,i}$ the index set of scenarios coinciding with scenario $i \in I_t$ at t. In particular, the set I_0 is a singleton.

It is shown in Heitsch and Römisch (2005, Sect. 5) that, if $\|\xi - \hat{\xi}\|_1$ is sufficiently small, there exists a constant $K > 0$ such that the estimate

$$\begin{aligned} |v(\xi) - v(\xi_{\text{tr}})| &\le K(\|\xi - \hat{\xi}\|_1 + D_{\text{f}}(\xi, \hat{\xi}) + \|\hat{\xi} - \xi_{\text{tr}}\|_1 + D_{\text{f}}(\hat{\xi}, \xi_{\text{tr}})) \\ &\le K(\|\xi - \hat{\xi}\|_1 + D_{\text{f}}(\xi, \hat{\xi}) + \varepsilon + g(\varepsilon)) \end{aligned}$$

holds for the optimal values $v(\xi)$ and $v(\xi_{\text{tr}})$ of a multistage stochastic program with inputs ξ and ξ_{tr}, respectively, in the right-hand side of linear constraints. The function g has the property that $g(\varepsilon)$ tends to 0 as $\varepsilon \to 0$. The estimate is valid if the stochastic programming model is linear (without integrality requirements). Although the underlying optimization model for revenue management is mixed-integer (due to the disjunctive constraints even if the integrality constraints are relaxed), we consider the preceding estimate as a justification of our tree generation process.

Next we describe an algorithm to construct the process ξ_{tr} starting from $\hat{\xi}$ and such that (11) for a given tolerance ε is satisfied. To this end, let further tolerances ε_t for each period $t = 1, \dots, T$ be given such that $\sum_{t=1}^{T} \varepsilon_t \le \varepsilon$ holds. For each $t = 1, \dots, T$ we define clusters $\mathcal{C}_t$ of scenarios, i.e., partitions of the index set $I := \{1, \dots, S\}$, and processes $\hat{\xi}^t$ such that $\xi_{\text{tr}} := \hat{\xi}^T$, where the scenarios of ξ_{tr} and their probabilities are given by the structure of the final partition $\mathcal{C}_T$. The algorithm may be described as follows.

Step 0: Set $\hat{\xi}^0 := \hat{\xi}$ and $\mathcal{C}_0 = \{I\}$, i.e., the first cluster contains all scenarios of $\hat{\xi}$.

Step 1: Determine disjoint index sets $I_1 := I_1^1$ and J_1^1 such that $I_1 \cup J_1^1 = I$ and

$$J_1^1 = \cup_{i \in I_1^1} J_{1,i}^1,\ J_{1,i}^1 := \{j \in J_1^1 : i = i_1^1(j)\},\ i_1^1(j) \in \arg\min_{i \in I_1} |\hat{\xi}_1^{1,i} - \hat{\xi}_1^{1,j}|.$$

We define $\hat{\xi}^1 = \{\hat{\xi}_\tau^1\}_{\tau=1}^T$ via its scenarios $\hat{\xi}^{1,i}$, $i \in I$, by setting

$$\hat{\xi}_\tau^{1,i} = \begin{cases} \xi_\tau^{\alpha_1(i)}, & \tau = 1, \\ \xi_\tau^i, & \text{otherwise,} \end{cases}$$

where scenario $\hat{\xi}^{1,i}$ appears with probability p_i, $i \in I$, and the mapping $\alpha_1 : I \to I_1$ is given by

$$\alpha_1(j) = \begin{cases} i_1^1(j), & j \in J_1^1, \\ j, & \text{otherwise.} \end{cases}$$

The index sets I_1 and J_1^1 are determined such that the estimate

$$\sum_{i \in I_1} \sum_{j \in J_{1,i}^1} p_j |\xi_1^j - \xi_1^i| = \sum_{j \in J_1^1} p_j \min_{i \in I_1} |\xi_1^i - \xi_1^j| \le \varepsilon_1$$

holds. Set $\mathcal{C}_1 = \{\alpha_1^{-1}(i) : i \in I_1\}$.

Step t: Let $\mathcal{C}_{t-1} = \{C_{t-1}^1, \ldots, C_{t-1}^{K_{t-1}}\}$. Determine disjoint index sets I_t^k and J_t^k such that $I_t^k \cup J_t^k = C_{t-1}^k$, $k = 1, \ldots, K_{t-1}$ and

$$I_t := \cup_{k=1}^{K_{t-1}} I_t^k,\ J_t^k = \cup_{i \in I_t^k} J_{t,i}^k,\ J_{t,i}^k := \{j \in J_t^k : i = i_t^k(j)\},$$

$$i_t^k(j) \in \arg\min_{i \in I_t^k} |\hat{\xi}_t^{t-1,i} - \hat{\xi}_t^{t-1,j}|.$$

We define $\hat{\xi}^t = \{\hat{\xi}_\tau^t\}_{\tau=1}^T$ via its scenarios $\hat{\xi}^{t,i}$, $i \in I$, by setting

$$\hat{\xi}_\tau^{t,i} = \begin{cases} \xi_\tau^{\alpha_t(i)}, & \tau \le t, \\ \xi_\tau^i, & \text{otherwise,} \end{cases}$$

where scenario $\hat{\xi}^{t,i}$ appears with probability p_i, $i \in I$, and the mapping $\alpha_t : I \to I_t$ is given by

$$\alpha_t(j) = \begin{cases} i_t^k(j), & j \in J_t^k, \\ j, & \text{otherwise.} \end{cases}$$

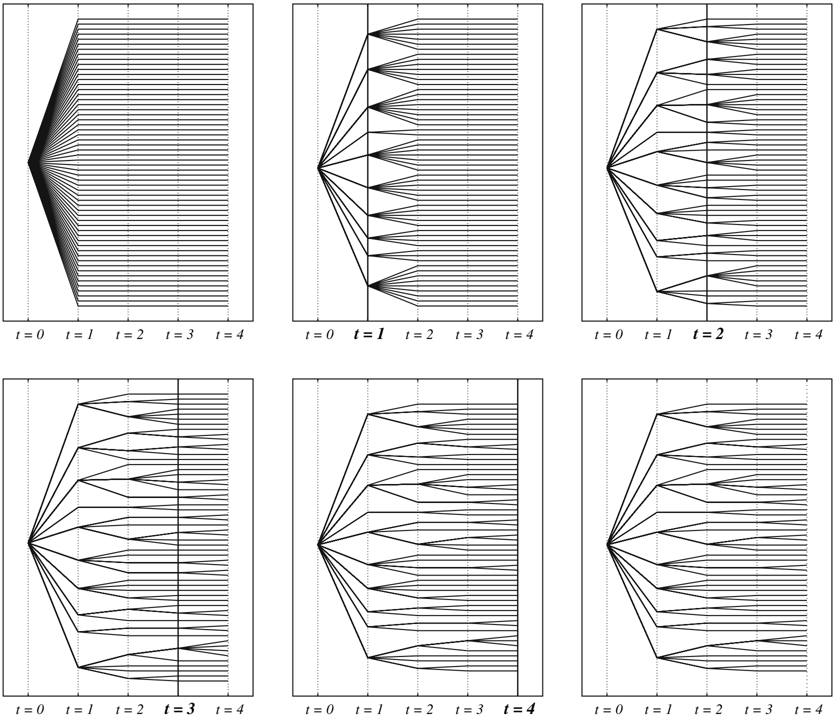

Fig. 2 Illustration of the above algorithm for an example with $T = 4$ starting with a scenario fan containing $N = 58$ scenarios

The index sets I_t^k and J_t^k, $k = 1, \ldots, K_{t-1}$, are determined such that

$$\sum_{k=1}^{K_{t-1}} \sum_{j \in J_t^k} p_j \min_{i \in I_t^k} |\xi_t^i - \xi_t^j| \leq \varepsilon_t.$$

Set $\mathcal{C}_t = \{\alpha_t^{-1}(i) : i \in I_t^k,\ k = 1 \ldots, K_{t-1}\}$.

Step T+1: Let $\mathcal{C}_T = \{C_T^1, \ldots, C_T^{K_T}\}$. Determine a stochastic process ξ_{tr} having the K_T scenarios $\hat{\xi}^{T,k}$ where $\hat{\xi}_t^{T,k} := \xi_t^{\alpha_t(i)}$ if $i \in C_T^k$, $k = 1, \ldots, K_T$, $t = 1, \ldots, T$.

It is shown in (Heitsch and Römisch, 2005, Theorem 4.4) that then the estimate

$$\|\hat{\xi} - \xi_{\text{tr}}\|_1 \leq \sum_{t=1}^{T} \varepsilon_t \leq \varepsilon$$

and, hence, (11) is valid.

The above algorithm is illustrated in Fig. 2. The first and last picture show the original scenario set $\hat{\xi}$ and the final scenario tree ξ_{tr}, respectively. Picture i corresponds to the situation after Steps $i-1$, $i = 2, \ldots, 5$.

2.6 Stochastic programming model in node representation

After rewriting the stochastic programming model in node representation, where the node index is denoted by the superscript n, it consists in maximizing the total expected revenue

$$\max_{(P^n_{i,j,k})} \sum_{n=1}^{N} \pi^n \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} \left[f^b_{i,j,k,t(n)} b^n_{i,j,k} - f^c_{i,j,k,t(n)} c^n_{i,j,k} \right] \tag{12}$$

subject to the dynamics for the cumulative bookings

$$B^0_{i,j,k} = \bar{B}^0_{i,j,k} \qquad B^n_{i,j,k} = B^{n-}_{i,j,k} + b^n_{i,j,k} \quad (n \in \mathcal{N} \setminus \{0\}) \tag{13}$$

the protection level conditions

$$B^n_{i,j,k} + z^{P,n}_{i,j,k} = \left\lfloor (1 - \gamma^n_{i,j,k})^{-1} P^{n-}_{i,j,k} + 0.5 \right\rfloor \tag{14}$$

the passenger demand constraints

$$b^n_{i,j,k} + z^{d,n}_{i,j,k} = d^n_{i,j,k} \quad (n \in \mathcal{N} \setminus \{0\}) \tag{15}$$

the disjunctive constraints for the number of bookings ($\varkappa > 0$ sufficiently large)

$$0 \le z^{d,n}_{i,j,k} \le \tilde{z}^n_{i,j,k} d^n_{i,j,k} \qquad 0 \le z^{P,n}_{i,j,k} \le (1 - \tilde{z}^n_{i,j,k}) \varkappa \quad (n \in \mathcal{N} \setminus \{0\}) \tag{16}$$

the approximations of the cumulative cancelations

$$C^n_{i,j,k} = \left\lfloor \gamma^n_{i,j,k} B^n_{i,j,k} + 0.5 \right\rfloor \quad (n \in \mathcal{N} \setminus \{0\}) \tag{17}$$

the number of cancelations

$$c^n_{i,j,k} = C^n_{i,j,k} - C^{n-}_{i,j,k} \quad (n \in \mathcal{N} \setminus \{0\}), \quad C^0_{i,j,k} = \bar{C}^0_{i,j,k} \tag{18}$$

the leg-capacity limits for all $m = 1, \ldots, M(l)$ and $l = 1, \ldots, L$

$$\sum_{i \in \mathcal{I}_l} \sum_{j \in \mathcal{J}_m(l)} \sum_{k=1}^{K} P^n_{i,j,k} \le \mathcal{C}_{l,m} \quad (n \in \mathcal{N}_{T-1}) \tag{19}$$

and the non-negativity and integrality constraints

$$B^n_{i,j,k}, C^n_{i,j,k}, P^n_{i,j,k} \in \mathbb{Z},\ \tilde{z}^n_{i,j,k} \in \{0, 1\},\ b^n_{i,j,k}, c^n_{i,j,k} \ge 0. \tag{20}$$

All constraints except (19) have to be satisfied for all $i = 1, \dots, I$, $j = 1, \dots, J$ and $k = 1, \dots, K$. The non-anticipativity constraints are satisfied by construction.

Compared to Möller et al. (2004) the optimization model takes into account the disjunctive constraints. The cumulative bookings $B^n_{i,j,k}$ play the role of a major state variable.

Altogether, the optimization model (12)–(20) represents a large-scale structured mixed-integer linear program. It contains $4IJKN$ continuous variables, $IJK(N + 1 - S) + 2IJKN$ integer variables, $IJKN$ binary variables and $7IJKN + \sum_{n\in\mathcal{N}_{T-1}} \sum_{l=1}^{L} M(l)$ constraints. Here, S denotes the number of scenarios in the scenario tree (see also Sect. 2.4).

After solving the optimization model, the (deterministic) protection levels $P^0_{i,j,k}$ of dcp $t_0 = 0$ may be taken as decision variables to control the booking requests. As current inventory systems cannot handle $P^0_{i,j,k}$ directly, these decision variables could be utilized by a separate component to which all booking requests are directed from the inventory system (so called "seamless operational mode"). A new scenario tree may then be generated having its root node at the next dcp $t_0 + 1$ and the multistage stochastic program is resolved to get the protection level decisions for the next booking interval. This procedure may be continued until the booking horizon ends (moving horizon). Alternatively, the information from the state variables B_{t_0+1} and C_{t_0+1} may be used to approximate a decision from the solution tree. If there is some $n \in \mathcal{N}_{t_0+1}$ with $B_{t_0+1} = B^n$ and $C_{t_0+1} = C^n$, then the protection levels P^n may be used to control the booking process during the booking interval $(t(n), t(n) + 1]$. Otherwise, some information on the probability distribution (like averages, quantiles etc.) of the relevant protection levels based on the difference between (B_{t_0+1}, C_{t_0+1}) and $(B^n, C^n)_{t(n)=t_0+1}$ could be taken to compute approximate protection levels at $t_0 + 1$.

2.7 Motivation for incorporating cancelations

Cancelations are incorporated into the optimization model due to practical limitations. In particular, cancelations are considered to allow booking and cancelation scenarios where the number of cancelations may exceed the number of bookings during some booking intervals. If in such a case the passenger demand and the cancelation rates would instead be replaced by a reduced demand

$$\tilde{d}^n_{i,j,k} := (1 - \gamma^n_{i,j,k}) \sum_{m\in\text{path}(n)\setminus\{0\}} d^m_{i,j,k} - (1 - \gamma^{n_-}_{i,j,k}) \sum_{m\in\text{path}(n_-)\setminus\{0\}} d^m_{i,j,k}, \tag{21}$$

which are possibly negative and if we allow negative numbers of bookings $\tilde{b}^n_{i,j,k}$, then each $\tilde{b}^n_{i,j,k}$ has to satisfy the condition

$$\tilde{b}^n_{i,j,k} \geq -\left(\gamma^n_{i,j,k} \sum_{m\in\text{path}(n)\setminus\{0\}} b^m_{i,j,k} - \gamma^{n_-}_{i,j,k} \sum_{m\in\text{path}(n_-)\setminus\{0\}} b^m_{i,j,k}\right) \tag{22}$$

Table 1 Parameter describing the scenario tree structure

n	0	1	2	3	4
n_-	–	0	0	1	2
π^n	1.0	0.7	0.3	0.7	0.3
$t(n)$	0	1	1	2	2

Table 2 Passenger demand, cancelation rates and reduced demand

$d^1_{1,1,1} = 0$	$\gamma^1_{1,1,1} = 0$	$\Rightarrow \tilde{d}^1_{1,1,1} = 0$
$d^1_{1,2,1} = 140$	$\gamma^1_{1,2,1} = 0.2$	$\Rightarrow \tilde{d}^1_{1,2,1} = 112$
$d^3_{1,1,1} = 50$	$\gamma^3_{1,1,1} = 0$	$\Rightarrow \tilde{d}^3_{1,1,1} = 50$
$d^3_{1,2,1} = 110$	$\gamma^3_{1,2,1} = 0.2$	$\Rightarrow \tilde{d}^3_{1,2,1} = 88$
$d^2_{1,1,1} = 0$	$\gamma^2_{1,1,1} = 0$	$\Rightarrow \tilde{d}^2_{1,1,1} = 0$
$d^2_{1,2,1} = 140$	$\gamma^2_{1,2,1} = 0.2$	$\Rightarrow \tilde{d}^2_{1,2,1} = 112$
$d^4_{1,1,1} = 160$	$\gamma^4_{1,1,1} = 0$	$\Rightarrow \tilde{d}^4_{1,1,1} = 160$
$d^4_{1,2,1} = 140$	$\gamma^4_{1,2,1} = 0.3$	$\Rightarrow \tilde{d}^4_{1,2,1} = 84$

where the quantities are defined as in Sect. 2.6. Hence, $\tilde{b}^n_{i,j,k}$ may not be smaller than the maximally possible number of cancelations. But, the information on the cancelation rates $\gamma^n_{i,j,k}$ and the number of bookings $b^n_{i,j,k}$ are no longer available if we use $\tilde{d}^n_{i,j,k}$ and $\tilde{b}^n_{i,j,k}$ in the model. Therefore, the condition above can not be ensured. Hence, the solution may require more cancelations as possible based on the passenger demand and on the cancelation rates. The following two-stage, three-period example demonstrates this effect.

Example $N = 5$, $T = 2$, $I = 1$, $J = 2$, $K = 1$, $L = 1$, $M(1) = 1$, $C_{1,1} = 250$, $f^b_{1,1,1,1} = f^c_{1,1,1,2} = f^b_{1,1,1,1} = f^c_{1,1,1,2} = 900.00$, $f^b_{1,2,1,1} = f^c_{1,2,1,2} = f^b_{1,2,1,1} = f^c_{1,2,1,2} = 600.00$, $\bar{B}^0_{1,1,1} = \bar{B}^0_{1,2,1} = 0$. The structure of the scenario tree is described by the parameters given in Table 1. The passenger demand $d^n_{i,j,k}$, the cancelation rates $\gamma^n_{i,j,k}$ and the reduced demand $\tilde{d}^n_{i,j,k}$ are summarized in Table 2.

An optimal solution of the problem is given by $P^0_{1,1,1} = 0$, $P^0_{1,2,1} = 112$, $P^1_{1,1,1} = 50$, $P^1_{1,2,1} = 200$, $P^2_{1,1,1} = 160$ and $P^2_{1,2,1} = 90$. The optimality is implied by the facts that in both scenarios (0, 1, 3) and (0, 2, 4), respectively, the airplane is operating at full capacity as well as that in both scenarios the number of sold high fare tickets is maximal. The corresponding values of $\tilde{b}^n_{i,j,k}$ are: $\tilde{b}^1_{1,1,1} = 0$, $\tilde{b}^1_{1,2,1} = 112$, $\tilde{b}^2_{1,1,1} = 0$, $\tilde{b}^2_{1,2,1} = 112$, $\tilde{b}^3_{1,1,1} = 50$, $\tilde{b}^3_{1,2,1} = 88$, $\tilde{b}^4_{1,1,1} = 160$ and $\tilde{b}^4_{1,2,1} = -22$. Based on the cancelation rate $\gamma^2_{1,2,1} = 0.2$, there have to be 140 bookings when moving from node 0 to node 1 in order to achieve $\tilde{b}^2_{1,2,1} = 112$. On the other hand, $\gamma^4_{1,2,1} = 0.3$ allows only 14 additional cancelations for $i = 1$, $j = 2$ and $k = 1$ when moving from node 2 to node 4. Thus, $\tilde{b}^4_{1,2,1}$ has to be greater than or equal to -14 which contradicts to the solution $\tilde{b}^4_{1,2,1} = -22$. Hence, the optimal solution requires more cancelations than we can expect from the cancelation rates $\gamma^2_{1,2,1}$ and $\gamma^4_{1,2,1}$.

Table 3 Dimensions

I	J	K	L	$M(l)$ $(l = 1, \ldots, 6)$	T
12	6	1	6	2	13

The effect occurs since the information about the cancelation rate is not available and negative values for the reduced number of bookings (including cancelations) $\tilde{b}^n_{i,j,k}$ are allowed. Because the inequality

$$0 \geq -\left(\gamma^n_{i,j,k} \sum_{m \in \text{path}(n)\setminus\{0\}} b^m_{i,j,k} - \gamma^{n_-}_{i,j,k} \sum_{m \in \text{path}(n_-)\setminus\{0\}} b^m_{i,j,k}\right)$$

holds, the effect does not occur if $\tilde{b}^n_{i,j,k} \geq 0$ is still required provided that a feasible solution exists. But, the latter condition excludes solutions where the number of cancelations exceeds the number of bookings in some booking interval. While neglecting the ratio of bookings and cancelations provides larger feasible sets in general, the feasible set may be reduced by the condition $\tilde{b}^n_{i,j,k} \geq 0$ in particular cases which leads to lower revenues if cancelation rates are ignored. In our example, $\tilde{b}^n_{i,j,k} \geq 0$ provides a solution that neglects the 14 cancelations in fare class 2 when moving from node 2 to node 4. Thus, the canceled seats will be sold again in fare class 2 and not in fare class 1 as possible. This results in a loss of an amount of $0.3 \cdot 14 \cdot (900 - 600)$ in the objective function. The possible impact of taking cancelations into account was already observed earlier, e.g., in Subramanian et al. (1999).

3 Numerical results

Computational tests are carried out for a single hub-and-spoke flight network illustrated in Fig. 3. The dimensions of the corresponding revenue management problem are summarized in Table 3. For each dcp $t = 0, \ldots, T$ the days to departure (d) are listed in Table 4.

The compartments are denoted by “B” ($m = 1$) and “E” ($m = 2$), respectively. The capacities of compartments B and E comprise 24 and 216 seats, respectively, on all legs. The fare classes are “B1” ($j = 1$), “B2” ($j = 2$), “E1” ($j = 3$), “E2” ($j = 4$),

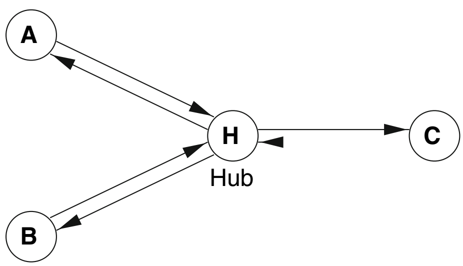

Fig. 3 Hub-and-Spoke flight network

Table 4 Data collection points and days to departure

dcp	0	1	2	3	4	5	6	7	8	9	10	11	12	13
d	182	126	84	56	35	21	14	10	7	5	3	2	1	0

"E3" ($j = 5$) and "E4" ($j = 6$). As proposed in De Boer et al. (2002), Chen and Homen-de-Mello (2006), Weatherford et al. (1993) the booking process is modeled by a non-homogeneous Poisson process (NHPP). This process allows to model the uncertainty of the total number of booking requests as well as the variability of the arrival intensity for each fare class over time, simultaneously De Boer et al. (2002). The total number of cumulative booking requests over the booking horizon G_{ijk} is assumed to have a Gamma distribution. The arrival pattern of the booking requests $\beta_{ijk}(t)$ is assumed to have a Beta distribution. The arrival intensity of the booking requests $\lambda_{ijk}(t)$ is then given by

$$\lambda_{ijk}(t) = \beta_{ijk}(t)\ G_{ijk} \quad G_{ijk} \sim \text{Gamma}(p_{ijk}, g_{ijk}).$$

As in De Boer et al. (2002) we assume in this example that the cumulative booking requests are independent for each i, j and k. The density function of the (standard) Gamma distribution with shape and scale parameters $p > 0$ and $g > 0$, respectively, is

$$f_{\text{Gamma}(p,g)}(x) := \frac{(x/g)^{p-1} e^{-x/g}}{g\Gamma(p)} = \frac{(1/g)^p}{\Gamma(p)} x^{p-1} e^{-x/g} \quad 0 \le x < +\infty,$$

where $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$ denotes the Gamma function.

For $t \in [0, T]$ and parameter $a > 0$, $b > 0$ the density function $\beta(t)$ of the Beta distribution is defined by

$$\beta(t) := \frac{1}{T B(a,b)} \left(\frac{t}{T}\right)^{a-1} \left(1 - \frac{t}{T}\right)^{b-1} \quad 0 \le t \le T,$$

where $B(a, b) := \int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ denotes the beta function.

The cumulative booking requests $D_{ijk}(t)$ until some $t \in [0, T]$ now are

$$D_{ijk}(t) = \int_0^t \lambda_{ijk}(\tau) d\tau = G_{ijk} \int_0^t \beta_{ijk}(\tau) d\tau$$

For each itinerary i, fare class j and point of sale k we generate samples for the Gamma distribution. The parameters of the Gamma distribution and the fares are given in Table 5. Since the arrival intensity samples G_{ijk} do not reflect possible cancelations so far, they are scaled by $(1 - \gamma_{i,j,k,T})^{-1}$, where for all j, k, t, the

Table 5 Parameters of the Gamma distribution and fares

ODI	Fare class	POS	p	g	Mean	dcp	Fare
AH, HA,	B1	–	3.0	1.5	4.5	1–13	500
BH, HB	B2	–	3.0	1.5	4.5	1–13	340
	E1	–	10.0	1.2	12.0	1–13	200
	E2	–	40/3	1.2	16.0	1–13	160
	E3	–	22.0	1.0	22.0	1–13	130
	E4	–	30.0	1.0	30.0	1–13	100
CH, HC	B1	–	2.0	1.5	3.0	1–13	500
	B2	–	2.0	1.5	3.0	1–13	340
	E1	–	5.0	1.2	6.0	1–13	200
	E2	–	20/3	1.2	8.0	1–13	160
	E3	–	11.0	1.0	11.0	1–13	130
	E4	–	15.0	1.0	15.0	1–13	100
AHB, BHA	B1	–	2.0	1.5	3.0	1–13	800
	B2	–	2.0	1.5	3.0	1–13	540
	E1	–	7.5	1.2	9.0	1–13	320
	E2	–	10.0	1.2	12.0	1–13	260
	E3	–	16.5	1.0	16.5	1–13	210
	E4	–	22.5	1.0	22.5	1–13	160
AHC, CHA	B1	–	3.0	1.5	4.5	1–13	800
BHC, BHA	B2	–	3.0	1.5	4.5	1–13	540
	E1	–	15.0	1.2	18.0	1–13	320
	E2	–	20.0	1.2	24.0	1–13	260
	E3	–	33.0	1.0	33.0	1–13	210
	E4	–	45.0	1.0	45.0	1–13	160

Table 6 Parameters of the Beta distribution

ODI	Fare class	POS	a	b
All	B1	–	12.0	1.5
	B2	–	8.0	2.0
	E1	–	6.0	2.0
	E2	–	4.0	3.0
	E3	–	3.0	4.0
	E4	–	2.0	4.0

cancelation rates $\gamma_{i,j,k,t}$ are set to 0.1 if $i = 1, 2$, 0.05 if $i = 3, 4$ and 0.0 if $i = 5, 6$, respectively. For each dcp t, these scaled samples are multiplied with the value of the cumulative distribution function of the Beta distribution at t. The parameters of the Beta distribution are given in Table 6. In this way, scenarios for the entire flight network are generated.

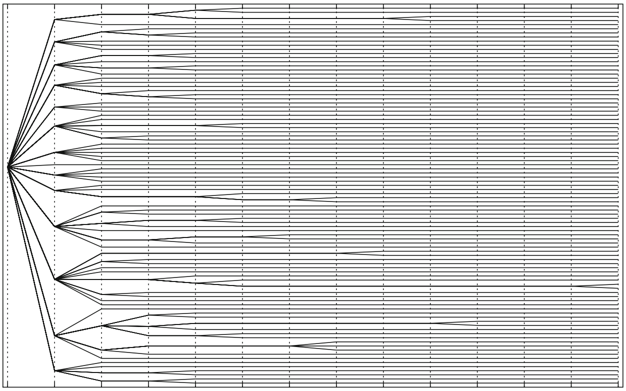

Fig. 4 Input scenario tree

Table 7 SP model dimensions

S	N	Number of cont. variables	Number of binary variables	Number of constraints
92	1,017	506,016	73,224	513,660

The booking request scenario tree is generated by the algorithm described in Sect. 2.5 starting with initial sets consisting of $S = 100, 200, 300$ and 400 scenarios, respectively. For each S three sets of initial scenarios are generated as outlined above. The parameters and tolerances needed for the generation algorithm in Sect. 2.5 are set to $r = 1$, $\varepsilon = 0.30\,\varepsilon_{\max}$, and $\varepsilon_t \approx Cq^{t+1}$, $t = 1, \ldots, 13$, with $q := 0.65$ in all examples. Here, the normalization constant $\varepsilon_{\max}$ is defined as the smallest L_1-distance between the initial scenario set (with identical weights $\frac{1}{S}$) and one of its scenarios endowed with unit mass. The constant $C > 0$ is choosen such that the condition $\sum_{t=1}^{T} \varepsilon_t \leq \varepsilon$ is satisfied. The resulting scenario tree for example 1 of Table 8 is illustrated in Fig. 4. In this example, branchings occur at the beginning of 11 of the 13 booking intervals, i.e., the optimization model has 11 stages. No branchings appear in the two last but one intervals. The dimensions of this tree and of the corresponding optimization model are given in Table 7.

The computations are performed on a Linux-PC equipped with a 2.4 GHz Intel Pentium 4 processor. The program input consists of the data of the flight network and of the generated scenario tree. The integrality constraints in (20) are ignored, so that only the binary variables $\tilde{z}^n_{i,j,k}$ are integer. The constant $\varkappa$ is chosen as $\varkappa := 5\max_{m=1,\ldots,M(l),\ l=1,\ldots,L} \mathcal{C}_{l,m}$. The mixed-integer linear program was solved using CPLEX 9.1 where the MIP gap is set to be 0.005. The program output consists in the optimal protection levels. The optimal values and the computing times are summarized in Table 8. The optimal values are very similar in all examples except Example 5. This observation suggests that at least for the considered network the generation of scenario trees starting with 100 initial scenarios provides sufficiently good results at reasonably fast computing times.

Table 8 Computational results for different samples sizes

Example	Initial Scenarios	Scenario tree Nodes	Scenario tree Scenarios	Optimal value	Computing time (h:mm:ss)
1	100	1,018	92	210,393	0:03:29.8
2	100	1,022	93	210,128	0:03:38.4
3	100	1,016	90	210,271	0:02:44.1
4	200	2,006	181	210,001	0:10:03.4
5	200	2,017	183	214,477	0:11:49.9
6	200	2,025	183	210,485	0:11:29.6
7	300	3,020	272	210,517	0:28:41.8
8	300	2,998	271	210,436	0:24:18.6
9	300	3,008	273	210,327	0:29:58.6
10	400	4,007	364	209,916	1:05:31.3
11	400	4,045	367	210,108	3:23:55.7
12	400	4,018	363	210,470	0:39:42.3

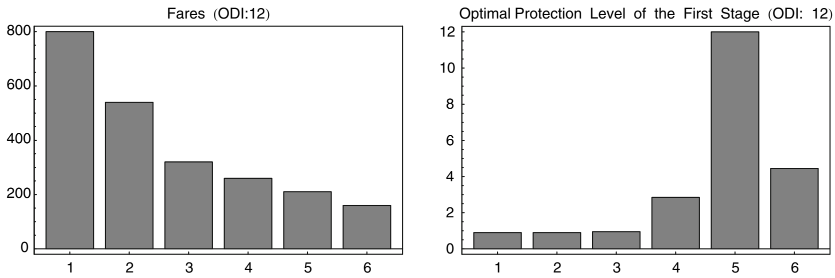

Fig. 5 Fares and optimal first stage protection levels for ODI C-H-B ($i = 12$)

Figure 5 shows the fares and initial protection levels for ODI C-H-B from example 1. Figure 6 illustrates the scenario trees for the cumulative passenger demand and for the protection levels of selected fare classes from example 1. Each picture also contains the mean value and the 5 and 95% quantiles. The passenger demand for classes with essentially different fares arrives during different time intervals with different intensity. For example, the demand of the low fare class 6 arrives earlier than the that of the high fare class 3 of the same compartment and is is essentially higher. The same effect can be observed for fare class 3 compared with the highest class 1. As expected the protection levels of the low fare class 6 restrict the number of tickets for sale compared to the passenger demand while in fare classes 1 and 3 the protection levels are similar to the cumulative passenger demand.

Finally, the sum of the protection levels in both compartments of leg HC from Example 1 are shown in Fig. 7. Recall that the capacity of compartments 1 and 2 are 24 and 215 passengers, respectively. The figures illustrate that the mean values of the sum of the protection levels are close to the compartment capacity. The mean

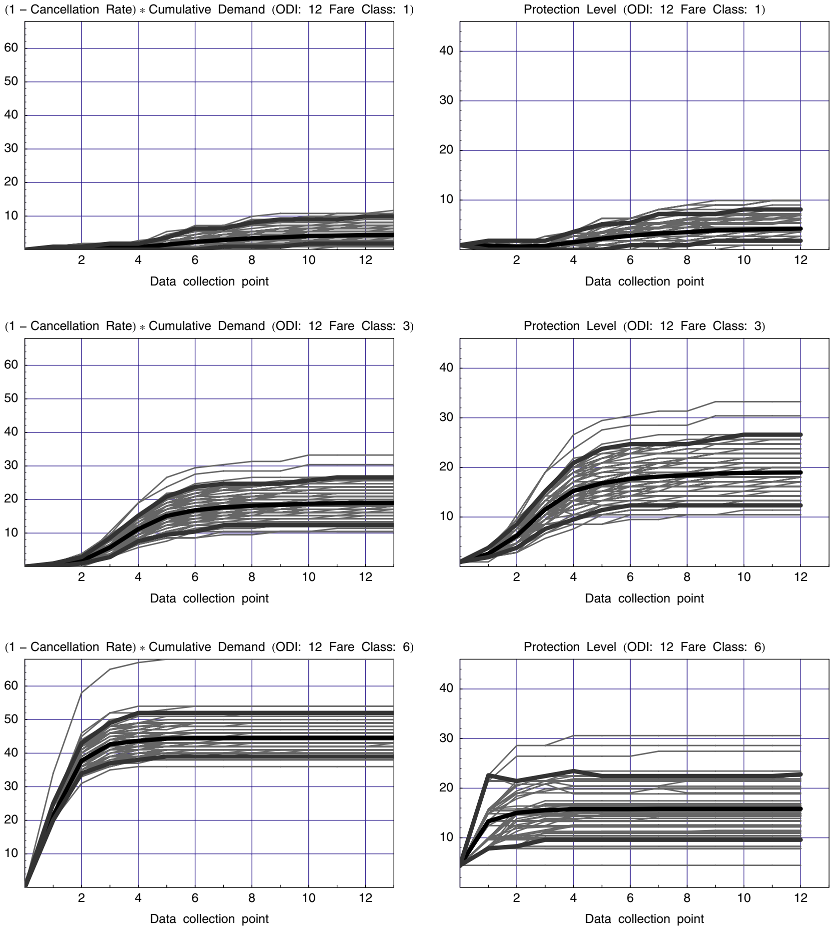

Fig. 6 Cumulative passenger demand and protection levels for selected fare classes of ODI C-H-B ($i = 12$)

value of about 22 passenges for compartment 1 is due to the fact that the demand and cancelation rates are choosen such the resulting bookings and cancelations just meet the compartment capacity and since sums of protection levels above this capacity are truncated by the protection level constraints.

The results and computing times are reasonable and encourage the use of our solution approach at least for small airline networks.

4 Extension to bid prices

The Lagrange multipliers of the capacity constraints may serve as approximate bid prices for the itineraries of the network. However, in our stochastic network revenue management model capacity constraints are only required for the last time period,

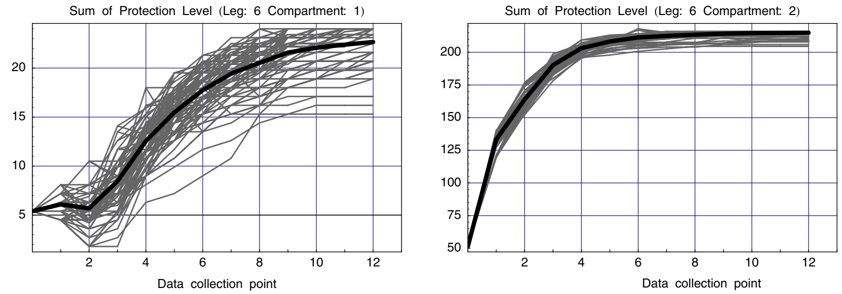

Fig. 7 Sum of protection level for leg HC ($l = 6$)

i.e., at dcp $t = T - 1$, in order to allow for overbookings in earlier time periods. Hence, introducing capacity constraints at the dcp's $t = 0, \dots, T - 2$ would lead to approximate bid prices, but also to the loss of the overbooking option. In addition, the dimension of the dual problem increases essentially. As a compromise, capacity constraints might be introduced only at $t = 0$ or for a few time periods $t = 0, \dots, t_0$ and the model be resolved with t_0 as starting point. In this way, overbooking is still possible in the remaining dcp's $t = t_0, \dots, T - 1$, approximate bid prices are available and the size of the dual problem remains reasonably small.

5 Conclusions and outlook

We propose a model for airline network revenue management that allows for cancelations and overbookings, provides optimal seat protection levels and represents a mixed-integer multistage stochastic program. The booking controls resulting from our optimization approach are not yet in practical use for controlling booking requests, though actual developments in inventory and revenue management systems set the stage for it. The stochastic passenger demand and cancelation process is approximated by a scenario tree with possible branchings in dcps. The scenario tree is generated by a stability-based recursive reduction and bundling technique which allows to handle multi-dimensional and multivariate stochastic processes. When solving real-life airline network revenue management models, the initial scenario set should be based directly on historical passenger demand data, which has to be adjusted subject to a suitable demand model (unconstraining) in order to minimize or at least reduce spiral-down effects (cf. Cooper et al. 2006). The node representation of the revenue management model corresponds to a large scale, structured mixed-integer linear program which is solved by standard MILP software (CPLEX). The numerical results and running times confirm the applicability of our approach to small networks. Future work will focus on the decomposition of the optimization problem into smaller subproblems for each itinerary, fare class and point of sale by Lagrangian relaxation. Preliminary numerical results encourage the applicability of such a Lagrangian decomposition approach to real-life flight networks. The model may be extended to compute approximate bid prices by introducing additional capacity constraints as discussed in the previous section.

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