On Weak Convergence of Approximate Solutions of Random Operator Equations

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1. Introduction

This paper deals with random operator equations, suitable solution concepts and their approximate solution. In section 2, we consider the concepts of a "random solution" and of a "weak solution". An existence theory for random solutions is well-developed (see e.g. [6], [10] and Theorem 2.2). But, as observed in [7] and [8], random solutions cannot be expected as limits of approximate solutions of random equations if the stochastic inputs (entering into the equation) are approximated in the sense of weak convergence of probability distributions. What can be expected is that these limits are weak solutions, a concept which was introduced in [8]. (Note that our Def. 2.1b is an adaptation to the type of random equation we consider in this paper.) Roughly speaking, a weak solution is the probability distribution of a random solution on some probability space (Remark 2.5). Moreover, we present a general existence and uniqueness result for weak solutions (Theorem 2.6).
In section 3, we review a general result ([8, Theorem 4.6]) on weak convergence of approximate solutions of random operator equations (to a weak solution of the original equation), where the underlying deterministic equation and the stochastic inputs are approximated simultaneously (the latter w.r.t. weak convergence). A result of this kind provides a theoretical basis for the use of approximation procedures for solving random equations which often consist of a "discretization" of the equation and an approximation ("simulation") of the stochastic input (see several papers in [3], and e.g. [4], [7]).
For details, proofs and applications of the general results we refer to [12].
Let us now fix some notations. For a metric space X we denote by $\mathfrak{B}(X)$ the σ-algebra of Borel subsets of X and by $\mathfrak{P}(X)$ the

set of all probability measures defined on $(X, \mathcal{B}(X))$ equipped with the topology of weak convergence ([5]). For $u \in X$ let $\delta_u \in \mathcal{P}(X)$ denote the unit mass at u. If x is an X-valued random variable (defined on some probability space $(\Omega,\mathcal{A},P)$), we denote by $D(x) := P \circ x^{-1} \in \mathcal{P}(X)$ its probability distribution.

2. Random operator equations: solution concepts and existence results

In the following, let X,Y and Z be separable metric spaces, z a Z-valued random variable (defined on some probability space $(\Omega,\mathcal{A},P)$) and T a mapping from ZxX into Y. We will be concerned with the random operator equation

$$T(z(\omega),x) = 0 \quad (\omega \in \Omega), \qquad (2.1)$$

where $0 \in Y$ is some fixed element in Y.

In the sequel, we consider the following solution concepts for equation (2.1). The first one is the classical concept of a "random solution" (see e.g. [2]), and the second that of a "weak solution" (introduced in [8]).

Definition 2.1:

a) A random variable $x:\Omega \longrightarrow X$ is called a "random solution" of (2.1) iff $T(z(\omega),x(\omega)) = 0$ holds P-almost surely.

b) A probability measure $\mu_X \in \mathcal{P}(X)$ is called a "weak solution" of (2.1) iff there exists a $\mu \in \mathcal{P}(ZxX)$ such that

$$\mu T^{-1} = \delta_0 \ , \quad D(z) = \mu p_Z^{-1} \quad \text{and} \quad \mu_X = \mu p_X^{-1} \ ,$$

where p_X and p_Z denote the coordinate projections from ZxX onto X and Z, respectively.

There exists a well-developed existence theory for random solutions of (2.1) (e.g. [6], [10]) based on measurable selection theorems (cf. [15]). The first result we state is an immediate consequence of [10, Theorem 1].

Theorem 2.2:

Let X be complete and $T:ZxX \longrightarrow Y$ Borel measurable, i.e., measurable with respect to $\mathcal{B}(ZxX)$ and $\mathcal{B}(Y)$. Assume that there exists $B \in \mathcal{B}(Z)$ such that $D(z)(B) = 0$ and $T(\bar{z},x) = 0$ is solvable, for all $\bar{z} \in Z \setminus B$.

Then (2.1) has a random solution.

Remark 2.3:

An example in [10] shows that Theorem 2.2 would not remain true if T is only individually Borel measurable with respect to the first and second variable. Note that $T:ZxX \longrightarrow Y$ is Borel measurable if for each $\bar{z} \in Z$ $T(\bar{z},\cdot)$ is continuous and for each $x \in X$ $T(\cdot,x)$ is Borel measurable.

Lemma 2.4:

Let $T:ZxX \longrightarrow Y$ be Borel measurable and $x:\Omega \longrightarrow X$ a random solution of (2.1). Then $\mu_X = D(x)$ is a weak solution of (2.1).

Proof:

Putting $\mu := D(z,x) = P\circ(z(\cdot),x(\cdot))^{-1} \in \mathcal{P}(ZxX)$ it remains to show that $\mu T^{-1} = \delta_0$. For $B \in \mathcal{B}(Y)$ we have

$$\mu(T^{-1}(B)) = P(\{\omega \in \Omega : (z(\omega),x(\omega)) \in T^{-1}(B)\})$$
$$= P(\{\omega \in \Omega : T(z(\omega),x(\omega)) \in B\}) = \delta_0(B) \ . \ \square$$

Remark 2.5:

However, an example shows (see [12]) that a weak solution need not be the distribution of a random solution (on $(\Omega,\mathcal{A},P)$). A weak solution μ_X of (2.1) may be interpreted as follows:

There exists a probability space $(\bar{\Omega},\bar{\mathcal{A}},\bar{P})$ and random variables $\bar{z}:\bar{\Omega} \longrightarrow Z$, $\bar{x}:\bar{\Omega} \longrightarrow X$ (on $(\bar{\Omega},\bar{\mathcal{A}},\bar{P})$) such that $T(\bar{z}(\omega),\bar{x}(\omega)) = 0$ holds $\bar{P}$-almost surely, and $D(z) = D(\bar{z})$, $\mu_X = D(\bar{x})$.

Thus a weak solution is the probability distribution of a random solution on <u>some</u> probability space (with fixed input distribution D(z)).

Furthermore, it can be shown ([12]) that the conditions $\mu T^{-1} = \delta_0$ and $D(z) = \mu p_Z^{-1}$ (in 2.1b) hold if and only if $\mu \tilde{T}^{-1} = \delta_0 \times D(z)$, where $\delta_0 \times D(z)$ is the usual product measure on YxZ and the mapping $\tilde{T}:ZxX \longrightarrow YxZ$ is defined by $\tilde{T}(\bar{z},x) := (T(\bar{z},x),\bar{z})$, for all $(\bar{z},x) \in ZxX$ (Note that $\tilde{T}$ is Borel measurable if T is Borel measurable.).

This observation is useful in the proof of the following existence and uniqueness result for weak solutions of (2.1).

Theorem 2.6:

Let X, Y and Z be complete separable metric spaces and $T:ZxX \longrightarrow Y$ Borel measurable.

a) There exists a weak solution of (2.1) iff $\tilde{T}^{-1}(B) \neq \emptyset$ for every $B \in \mathcal{B}(YxZ)$ with $\delta_0 \times D(z)(B) > 0$ (where $\tilde{T}$ is defined as in Remark 2.5).

b) Assume that there exists $B \in \mathfrak{B}(Z)$ such that $D(z)(B) = 0$ and $T(\tilde{z},x) = 0$ has a unique solution for every $\tilde{z} \in Z \setminus B$. Then (2.1) has a unique weak solution.

<u>Proof:</u>

a) follows from [9,Theorem 2.5] and b) is proved in [12].

<u>Remark 2.7:</u>

Theorem 2.6a implies that a necessary condition for the existence of a weak solution of (2.1) is that there is no Borel subset B of Z such that $D(z)(B) > 0$ and $\{x \in X: T(\tilde{z},x) = 0\}$ is empty for every $\tilde{z} \in B$.

3. <u>Approximate solution of random operator equations</u>

Throughout this section, let X, Y and Z be complete separable metric spaces, X_n $(n \in \mathbb{N})$ subsets of X and $0 \in Y$ some fixed element. Let $T: Z \times X \longrightarrow Y$ and $T_n: Z \times X_n \longrightarrow Y$ $(n \in \mathbb{N})$ be Borel measurable mappings, z and z_n $(n \in \mathbb{N})$ Z-valued random variables (defined on some probability spaces $(\Omega,\mathcal{A},P)$ and $(\Omega_n,\mathcal{A}_n,P_n)$ $(n \in \mathbb{N})$, respectively).

We consider the random operator equation

$$T(z(\omega),x) = 0 \qquad (\omega \in \Omega) \tag{3.1}$$

and its "approximations"

$$T_n(z_n(\omega),x) = 0 \qquad (\omega \in \Omega_n\ ,\ n \in \mathbb{N}). \tag{3.2}$$

Assume for the following that random solutions $x_n: \Omega_n \longrightarrow X_n$ of the approximate equations (3.2) exist for all $n \in \mathbb{N}$.

Now, our aim is to look for sufficient conditions on T and (T_n) that imply weak convergence of the sequence $(D(x_n))$ (in $\mathfrak{P}(X)$) to a weak solution of (3.1) if $(D(z_n))$ converges weakly to $D(z)$.

The following concepts turn out to be useful in this context.

<u>Definition 3.1:</u>

Let S and S' be metric spaces, $S_n \subseteq S$ $(n \in \mathbb{N})$ and $A: S \longrightarrow S'$, $A_n: S_n \longrightarrow S'$ $(n \in \mathbb{N})$.

a) We say that (A_n) is "discretely convergent" to A iff

(i) $\inf\{d(s,\tilde{s}): \tilde{s} \in S_n\} \longrightarrow 0$, for all $s \in S$ (d being the metric in S), and

(ii) for all $s \in S$, $s_n \in S_n$ $(n \in \mathbb{N})$, with $s_n \longrightarrow s$ (in S),

we have that $A_n s_n \longrightarrow As$ (in S') (see e.g. [11], [13]).

b) (A_n) is called "collectively regular" iff $\bigcup_{n \in \mathbb{N}} A_n^{-1}(K) \cap B$ is relatively compact in S for each bounded $B \subseteq S$ and compact $K \subseteq S'$.

<u>Remark 3.2:</u>

It is shown in [8, Lemma 3.7] that the collective regularity of (A_n) can be characterized in terms of the concept of "a-regularity" which is well-known in the literature on approximation methods for (deterministic) equations (e.g. [1], [11], [14]).

<u>Lemma 3.3:</u>

Let (T_n) be as above and assume that

a) for all $\tilde{z} \in Z$, $(T_n(\tilde{z},\cdot))$ is collectively regular, and

b) $\{T_n(\cdot,x) : x \in B \cap X_n\ ,\ n \in \mathbb{N}\}$ is equicontinuous on K, for each bounded $B \subseteq X$ and compact $K \subseteq Z$.

Then $\bigcup_{n \in \mathbb{N}} T_n^{-1}(\{0\}) \cap K \times B$ is relatively compact in $Z \times X$, for each bounded $B \subseteq X$ and compact $K \subseteq Z$.

<u>Proof:</u> ([12], similar as the proof of [8, Prop. 4.5])

<u>Theorem 3.4:</u>

Let T, (T_n), z and (z_n) be as above and let for all $n \in \mathbb{N}$ x_n be a random solution of (3.2) for the index n. Assume that

a) conditions a) and b) of Lemma 3.3 are satisfied,

b) (T_n) converges discretely to T (jointly in both variables),

c) $(D(z_n))$ converges weakly to $D(z)$,

d) $(D(x_n))$ is "stochastically bounded", i.e., for all $\varepsilon > 0$ there exists a bounded Borel set B_ε in X such that

$$\inf_{n \in \mathbb{N}} D(x_n)(B_\varepsilon) \geq 1 - \varepsilon .$$

Then $(D(x_n))$ is relatively compact with respect to the topology of weak convergence in $\mathfrak{P}(X)$ and every weak limit of $(D(x_n))$ is a weak solution of (3.1). If furthermore the weak solution of (3.1) is unique, $(D(x_n))$ converges weakly to this limit.

<u>Proof:</u> (See [8, Theorem 4.6] using the methodology of that paper; see also [12] for a short proof using Lemma 3.3 and Prohorov's and Rubin's theorems ([5, p. 34 and 37]).)

Applications of Theorem 3.4 to approximation schemes for (linear and nonlinear) random integral equations and (ordinary and partial) random differential equations are given in [7], [8] .

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