Progress in high-dimensional numerical integration and its application to stochastic optimization

W. Römisch

Humboldt-University Berlin
Department of Mathematics

www.math.hu-berlin.de/~romisch

DFG Research Center MATHEON
Mathematics for key technologies

Short course, Rutgers University, May 21–23, 2013

Part III: QMC algorithms for solving stochastic optimization problems: Challenges and solutions

Contents:

Introduction

Aim: Apply randomized Quasi-Monte Carlo methods, in particular, randomly shifted lattice rules to optimization models containing high-dimensional integrals.

Example: Option pricing (Wang-Sloan 11)
Consider the pricing of a path-dependent option with payoff $g(S_{t_1}, \dots, S_{t_d})$ where S_{t_j} are the prices of the underlying asset at times t_j, $j = 1, \dots, d$. Suppose the prices are considered at equally spaced times $t_j = j\frac{T}{d}$, where T is the expiration date, and the asset price follows a *geometric Brownian motion*

$$dS_t = rS_t dt + \sigma S_t dB_t \,,$$

where r is the risk-free interest rate, σ the volatility and B_t the standard Brownian motion (normal with zero mean and $\mathbb{E}[B_t B_s] = \min\{t, s\}$).
The analytical solution of the (scalar linear) stochastic differential equation is

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t\right).$$

The value of the option at $t = 0$ is $\mathbb{E}[\exp(-rT) g(S_{t_1}, \dots, S_{t_d})]$.

Consider Asian call options based on the geometric or arithmetic average of the underlying asset. With the strike price K at time T their terminal payoffs are

$$g(S_{t_1}, \ldots, S_{t_d}) = \max\left\{0, \prod_{t=1}^{d} S_{t_j}^{\frac{1}{d}} - K\right\} \quad \text{or} \quad = \max\left\{0, \frac{1}{d}\sum_{j=1}^{d} S_{t_j} - K\right\}.$$

If Σ denotes the covariance matrix of the normal random vector $(B_{t_1}, \ldots, B_{t_d})^\top$ and A is a matrix satisfying $\Sigma = A\,A^\top$, the random vector $(z_1, \ldots, z_d)^\top$ such that

$$(B_{t_1}, \ldots, B_{t_d})^\top = A(z_1, \ldots, z_d)^\top$$

is standard normal with independent components. For the first case it holds

$$\prod_{t=1}^{d} S_{t_j}^{\frac{1}{d}} = \exp\left(m + \frac{\sigma}{T}\sum_{k=1}^{d} A_k z_k\right)$$

with $A_k = \sum_{j=1}^{d} a_{jk}$, $A = (a_{jk})$ and $m = \log S_0 + \frac{T(d-1)}{2d}(r - \frac{\sigma^2}{2})$. Hence, the value of the option at $t = 0$ is

$$\exp(-rT)\int_{\mathbb{R}^d} \max\left\{0, \exp\left(m + \frac{\sigma}{T}\sum_{k=1}^{d} A_k z_k\right) - K\right\}\rho_d(z)dz$$

with the d-dimensional standard normal density ρ_d.

Example: (Optimization problem with random constraints)

We consider the linear optimization problem with random constraints

$$\min\{c^\top x : T(\xi)x = h(\xi), x \in X\},$$

where X is a polyhedron in $\mathbb{R}^m$, $T(\xi)$ a random matrix and $h(\xi)$ a random vector. The model is inappropriate to find a deterministic decision !

Idea: Introduce a compensation or recourse variable $y \geq 0$, a recourse matrix W, a (possibly random) recourse cost vector $q(\xi)$, replace the constraint "$T(\xi)x = h(\xi)$" by "$Wy = h(\xi) - T(\xi)x$" and select a random recourse decision $y(\xi)$ with minimal recourse costs "$q(\xi)^\top y(\xi)$". Adding the expected recourse costs to the original cost term $c^\top x$ leads to the two-stage stochastic optimization model

$$\min\{c^\top x + \int_{\mathbb{R}^d} \inf\{q(\xi)^\top y : Wy = h(\xi) - T(\xi)x, y \geq 0\}\rho_d(\xi)d\xi : x \in X\},$$

where ρ_d is the density of the underlying random vector ξ on $\mathbb{R}^d$.

Challenge: In both examples the integrands do not belong to the tensor product Sobolev space (after transformation to $[0, 1]^d$).

Two-stage linear stochastic optimization

We consider the linear two-stage stochastic program

$$\min\Big\{\int_{\Xi} f(x,\xi)P(d\xi) : x \in X\Big\},$$

where f is extended real-valued defined on $\mathbb{R}^m \times \mathbb{R}^d$ given by

$$f(x,\xi) = \langle c,x\rangle + \Phi(q(\xi), h(\xi) - T(\xi)x),\ (x,\xi) \in X \times \Xi,$$

$c \in \mathbb{R}^m$, $X \subseteq \mathbb{R}^m$ and $\Xi \subseteq \mathbb{R}^d$ are convex polyhedral, W is an $(r,\overline{m})$-matrix, P is a Borel probability measure on Ξ, and the vectors $q(\xi) \in \mathbb{R}^{\overline{m}}$, $h(\xi) \in \mathbb{R}^r$ and the (r,m)-matrix $T(\xi)$ are affine functions of ξ, Φ is the second-stage optimal value function

$$\Phi(u,t) = \inf\{\langle u,y\rangle : Wy = t, y \geq 0\} \quad ((u,t) \in \mathbb{R}^{\overline{m}} \times \mathbb{R}^r),$$

Let $\operatorname{pos} W = W(\mathbb{R}^{\overline{m}}_+)$, $\mathcal{D} = \{u \in \mathbb{R}^{\overline{m}} : \{z \in \mathbb{R}^r : W^\top z \leq u\} \neq \emptyset\}$.

Assumptions:

(A1) $h(\xi) - T(\xi)x \in \operatorname{pos} W$ and $q(\xi) \in \mathcal{D}$ for all $(x,\xi) \in X \times \Xi$.

(A2) $\int_{\Xi} \|\xi\|^2 P(d\xi) < \infty$.

Proposition:
(A1) and (A2) imply that the two-stage stochastic program represents a convex minimization problem with respect to the first stage decision x with polyhedral constraints.

Lemma: (Walkup-Wets 69, Nožička-Guddat-Hollatz-Bank 74)
Φ is finite, polyhedral and continuous on the $(\overline{m}+r)$-dimensional polyhedral cone $\mathcal{D} \times \operatorname{pos} W$ and there exist $(r, \overline{m})$-matrices C_j and $(\overline{m}+r)$-dimensional polyhedral cones $\mathcal{K}_j$, $j = 1, ..., \ell$, such that

$$\bigcup_{j=1}^{\ell} \mathcal{K}_j \; = \; \mathcal{D} \times \operatorname{pos} W \quad \text{and} \quad \operatorname{int} \mathcal{K}_i \cap \operatorname{int} \mathcal{K}_j = \emptyset \, , \; i \neq j,$$

$$\Phi(u,t) \; = \; \langle C_j u, t \rangle, \quad \text{for each} \quad (u,t) \in \mathcal{K}_j, \; j = 1, ..., \ell.$$

The function $\Phi(u, \cdot)$ is convex on $\operatorname{pos} W$ for each $u \in \mathcal{D}$, and $\Phi(\cdot, t)$ is concave on $\mathcal{D}$ for each $t \in \operatorname{pos} W$. The intersection $\mathcal{K}_i \cap \mathcal{K}_j$, $i \neq j$, is either equal to $\{0\}$ or contained in a $(\overline{m}+r-1)$-dimensional subspace of $\mathbb{R}^{\overline{m}+r}$ if the two cones are adjacent.

Challenge: The integrand $f(x, \cdot)$ is not in the tensor product Sobolev space.

The ANOVA decomposition of multivariate functions

Idea: Decompositions of f may be used, where most of the terms are smooth, but hopefully only some of them relevant.

Let $D = \{1, \ldots, d\}$ and $f \in L_{1,\rho}(\mathbb{R}^d)$ with $\rho(\xi) = \prod_{j=1}^{d} \rho_j(\xi_j)$, where

$$f \in L_{p,\rho}(\mathbb{R}^d) \quad \text{iff} \quad \int_{\mathbb{R}^d} |f(\xi)|^p \rho(\xi) d\xi < \infty \quad (p \geq 1).$$

Let the projection P_k, $k \in D$, be defined by

$$(P_k f)(\xi) := \int_{-\infty}^{\infty} f(\xi_1, \ldots, \xi_{k-1}, s, \xi_{k+1}, \ldots, \xi_d) \rho_k(s) ds \quad (\xi \in \mathbb{R}^d).$$

Clearly, $P_k f$ is constant with respect to ξ_k. For $u \subseteq D$ we write

$$P_u f = \Big(\prod_{k \in u} P_k \Big)(f),$$

where the product means composition, and note that the ordering within the product is not important because of Fubini's theorem. The function $P_u f$ is constant with respect to all x_k, $k \in u$.

ANOVA-decomposition of f:

$$f = \sum_{u \subseteq D} f_u \,,$$

where $f_\emptyset = I_d(f) = P_D(f)$ and recursively

$$f_u = P_{-u}(f) - \sum_{v \subset u} f_v$$

or (due to Kuo-Sloan-Wasilkowski-Woźniakowski 10)

$$f_u = \sum_{v \subseteq u} (-1)^{|u|-|v|} P_{-v} f = P_{-u}(f) + \sum_{v \subset u} (-1)^{|u|-|v|} P_{u-v}(P_{-u}(f)),$$

where P_{-u} and P_{u-v} mean integration with respect to ξ_j, $j \in D \backslash u$ and $j \in u \backslash v$, respectively. The second representation motivates that f_u is essentially as smooth as $P_{-u}(f)$.

If f belongs to $L_{2,\rho}(\mathbb{R}^d)$, its ANOVA terms $\{f_u\}_{u \subseteq D}$ are orthogonal in $L_{2,\rho}(\mathbb{R}^d)$.

We set $\sigma^2(f) = \|f - I_d(f)\|_{L_2}^2$ and $\sigma_u^2(f) = \|f_u\|_{L_2}^2$, and have

$$\sigma^2(f) = \|f\|_{L_2}^2 - (I_d(f))^2 = \sum_{\emptyset \neq u \subseteq D} \sigma_u^2(f) \,.$$

The dimension distribution and effective dimension of a function

Owen's superposition (truncation) dimension distribution of f: Probability measure ν_S (ν_T) defined on the power set of D

$$\nu_S(s) := \sum_{|u|=s} \frac{\sigma_u^2(f)}{\sigma^2(f)} \qquad \left(\nu_T(s) = \sum_{\max\{j:j\in u\}=s} \frac{\sigma_u^2(f)}{\sigma^2(f)}\right) \ (s \in D).$$

Effective superposition (truncation) dimension $d_S(\varepsilon)$ ($d_T(\varepsilon)$) of f is the $(1-\varepsilon)$-quantile of ν_S (ν_T):

$$d_S(\varepsilon) = \min\left\{s \in D : \sum_{|u|\le s} \sigma_u^2(f) \ge (1-\varepsilon)\sigma^2(f)\right\} \le d_T(\varepsilon)$$

$$d_T(\varepsilon) = \min\left\{s \in D : \sum_{u\subseteq\{1,\ldots,s\}} \sigma_u^2(f) \ge (1-\varepsilon)\sigma^2(f)\right\}$$

It holds

$$\max\left\{\Big\|f - \sum_{|u|\le d_S(\varepsilon)} f_u\Big\|_{2,\rho}, \Big\|f - \sum_{u\subseteq\{1,\ldots,d_T(\varepsilon)\}} f_u\Big\|_{2,\rho}\right\} \le \sqrt{\varepsilon}\,\sigma(f).$$

(Caflisch-Morokoff-Owen 97, Owen 03, Wang-Fang 03)

ANOVA decomposition of two-stage integrands

Assumptions:

(A1), **(A2)** and

(A3) P has a density of the form $\rho(\xi) = \prod_{j=1}^{d} \rho_j(\xi_j)$ ($\xi \in \mathbb{R}^d$) with continuous marginal densities ρ_j, $j \in D$.

Proposition:

(A1) implies that the function $f(x, \cdot)$, where

$$f_x(\xi) := f(x, \xi) = \langle c, x \rangle + \Phi(q(\xi), h(\xi) - T(\xi)x) \quad (x \in X, \xi \in \Xi)$$

is the two-stage integrand, is continuous and piecewise linear-quadratic. For each $x \in X$, $f(x, \cdot)$ is linear-quadratic on each polyhedral set

$$\Xi_j(x) = \{\xi \in \Xi : (q(\xi), h(\xi) - T(\xi)x) \in \mathcal{K}_j\} \quad (j = 1, \ldots, \ell).$$

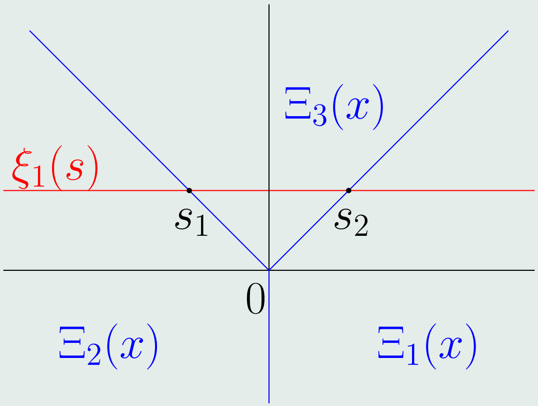
It holds $\operatorname{int} \Xi_j(x) \neq \emptyset$, $\operatorname{int} \Xi_j(x) \cap \operatorname{int} \Xi_i(x) = \emptyset$, $i \neq j$, and the sets $\Xi_j(x)$, $j = 1, \ldots, \ell$, decompose Ξ. Furthermore, the intersection of two adjacent sets $\Xi_i(x)$ and $\Xi_j(x)$, $i \neq j$, is contained in some $(d-1)$-dimensional affine subspace.

(Griebel-Kuo-Sloan 13)

To compute projections $P_k f$ for $k \in D$, let $\xi_i \in \mathbb{R}$, $i = 1, \ldots, d$, $i \neq k$, be given. We set $\xi^k = (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_d)$ and

$$\xi_k(s) = (\xi_1, \ldots, \xi_{k-1}, s, \xi_{k+1}, \ldots, \xi_d) \in \mathbb{R}^d \quad (s \in \mathbb{R}).$$

We fix $x \in X$ and consider the one-dimensional affine subspace $\{\xi_k(s) : s \in \mathbb{R}\}$:



Example with $d = 2 = p$, where the polyhedral sets are cones

It meets the nontrivial intersections of two adjacent polyhedral sets $\Xi_i(x)$ and $\Xi_j(x)$, $i \neq j$, at finitely many points s_i, $i = 1, \ldots, p$ if all $(d-1)$-dimensional subspaces containing the intersections do not parallel the kth coordinate axis.

The $s_i = s_i(\xi^k)$, $i = 1, \ldots, p$, are affine functions of ξ^k. It holds

$$s_i = -\sum_{l=1, l\neq k}^{p} \frac{g_{il}}{g_{ik}} \xi_l + a_i \quad (i = 1, \ldots, p)$$

for some $a_i \in \mathbb{R}$ and $g_i \in \mathbb{R}^d$ belonging to an intersection of polyhedral sets.

Proposition:

Let $k \in D$, $x \in X$. Assume (A1)–(A3) and that all $(d-1)$-dimensional affine subspaces containing nontrivial intersections of adjacent sets $\Xi_i(x)$ and $\Xi_j(x)$ do not parallel the kth coordinate axis.

Then the kth projection $P_k f$ has the explicit representation

$$P_k f(\xi^k) = \sum_{i=1}^{p+1} \sum_{j=0}^{2} p_{ij}(\xi^k; x) \int_{s_{i-1}}^{s_i} s^j \rho_k(s) ds,$$

where $s_0 = -\infty$, $s_{p+1} = +\infty$ and $p_{ij}(\cdot; x)$ are polynomials in ξ^k of degree $2-j$, $j = 0, 1, 2$, with coefficients depending on x, and is continuously differentiable. $P_k f$ is infinitely differentiable if the marginal density ρ_k belongs to $C^\infty(\mathbb{R})$.

Theorem:

Let $x \in X$, assume (A1)–(A3) and that the following geometric condition (GC) be satisfied: All $(d-1)$-dimensional affine subspaces containing nontrivial intersections of adjacent sets $\Xi_i(x)$ and $\Xi_j(x)$ do not parallel any coordinate axis. Then the ANOVA approximation

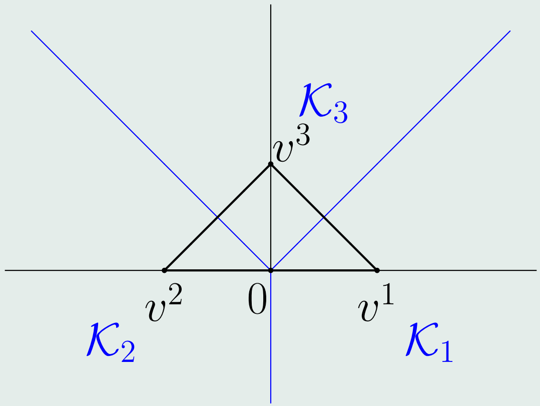
$$f_{d-1} := \sum_{|u| \leq d-1} f_u \quad \text{i.e.} \quad f = f_{d-1} + f_D$$

of f is infinitely differentiable if all densities ρ_k, $k \in D$, belong to $C_b^\infty(\mathbb{R})$.

Here, the subscript b means that all derivatives of functions belonging to that space are bounded on $\mathbb{R}$.

Example: Let $\bar{m} = 3$, $d = 2$, P denote the two-dimensional standard normal distribution, $h(\xi) = \xi$, q and W be given such that (A1) is satisfied and the dual feasible set is

$$\{z \in \mathbb{R}^2 : -z_1 + z_2 \leq 1, z_1 + z_2 \leq 1, -z_2 \leq 0\}.$$



Dual feasible set, its vertices v^j and the normal cones $\mathcal{K}_j$ to its vertices

The function Φ and the integrand are of the form

$$\Phi(t) = \max_{i=1,2,3} \langle v^i, t\rangle = \max\{t_1, -t_1, t_2\} = \max\{|t_1|, t_2\}$$
$$f(\xi) = \langle c, x\rangle + \Phi(\xi - Tx) = \langle c, x\rangle + \max\{|\xi_1 - [Tx]_1|, \xi_2 - [Tx]_2\}$$

and the convex polyhedral sets are $\Xi_j(x) = Tx + \mathcal{K}_j$, $j = 1, 2, 3$.
The ANOVA projection $P_1 f$ is in C^∞, but $P_2 f$ is not differentiable.

QMC quadrature error estimates

If the assumptions of the theorem are satisfied, the two-stage integrand $f = f_x$ (for fixed $x \in X$) allows the representation $f = f_{d-1} + f_D$ with f_{d-1} belonging to $\mathbb{F}_d$. This implies

$$\Big| \int_{[0,1]^d} f(\xi)d\xi - \frac{1}{n}\sum_{j=1}^{n} f(\xi^j)\Big| \leq e(Q_{n,d})\|f_{d-1}\|_\gamma + \Big| \int_{[0,1]^d} f_D(\xi)d\xi - \frac{1}{n}\sum_{j=1}^{n} f_D(\xi^j)\Big|$$

$$\leq e(Q_{n,d})\|f_{d-1}\|_\gamma + \|f_D\|_{L_2} + \Big(\frac{1}{n}\sum_{j=1}^{n} |f_D(\xi^j)|^2\Big)^{\frac{1}{2}}$$

where $\|\cdot\|_\gamma$ is the weighted tensor product Sobolev space norm.

As f_D is (Lipschitz) continuous and if the ξ^j, $j = 1, \ldots, n$, are properly selected, the last term in the above estimate may be assumed to be bounded by $2\|f_D\|_{L_2}$.

Hence, if the effective superposition dimension satisfies $d_S(\varepsilon) \leq d-1$, i.e., $\|f_D\|_{L_2} \leq \sqrt{\varepsilon}\sigma(f)$ holds for some small $\varepsilon > 0$, the first term $e(Q_{n,d})\|f_{d-1}\|_\gamma$ dominates and the convergence rate of $e(Q_{n,d})$ becomes most important.

Challenge: How important is the geometric condition (GC) ?

Partial answer: If P is normal with nonsingular covariance matrix, (GC) is satisfied for almost all covariance matrices. Namely, it holds

Proposition: Let $x \in X$, (A1), (A2) be satisfied, $\operatorname{dom}\Phi = \mathbb{R}^r$ and P be a normal distribution with nonsingular covariance matrix Σ. Then the infinite differentiability of the ANOVA approximation f_{d-1} of f is a generic property, i.e., it holds in a residual set (countable intersection of open dense subsets) in the metric space of orthogonal (d,d)-matrices Q (endowed with the norm topology) appearing in the spectral decomposition $\Sigma = Q^\top D\, Q$ of Σ (with a diagonal matrix D containing the eigenvalues of Σ).

Challenge: For which two-stage stochastic programs is $\|f_D\|_{L_{2,\rho}}$ small, i.e., the effective superposition dimension $d_S(\varepsilon)$ of f is less than $d-1$ or even much less?

Partial answer: In case of a (log)normal probability distribution P the effective dimension depends on the choice of the matrix A in the decomposition $\Sigma = A\,A^\top$ of thenonsingular covariance matrix Σ.

Dimension reduction in case of (log)normal distributions

Let P be the normal distribution with mean μ and nonsingular covariance matrix Σ. Let A be a matrix satisfying $\Sigma = A\,A^\top$. Then η defined by $\xi = A\eta + \mu$ is standard normal.

A universal principle is principal component analysis (PCA). Here, one uses $A = (\sqrt{\lambda_1}u_1, \ldots, \sqrt{\lambda_d}u_d)$, where $\lambda_1 \geq \cdots \geq \lambda_d > 0$ are the eigenvalues of Σ in decreasing order and the corresponding orthonormal eigenvectors u_i, $i = 1, \ldots, d$. Wang-Fang 03, Wang-Sloan 05 report an enormous reduction of the effective truncation dimension in financial models if PCA is used.

A problem-dependent principle may be based on the following equivalence principle (Papageorgiou 02, Wang-Sloan 11).

Proposition: Let A be a fixed $d \times d$ matrix such that $A\,A^\top = \Sigma$. Then it holds $\Sigma = B\,B^\top$ if and only if B is of the form $B = A\,Q$ with some orthogonal $d \times d$ matrix Q.

Idea: Determine Q for given A such that the effective truncation dimension is minimized (Wang-Sloan 11).

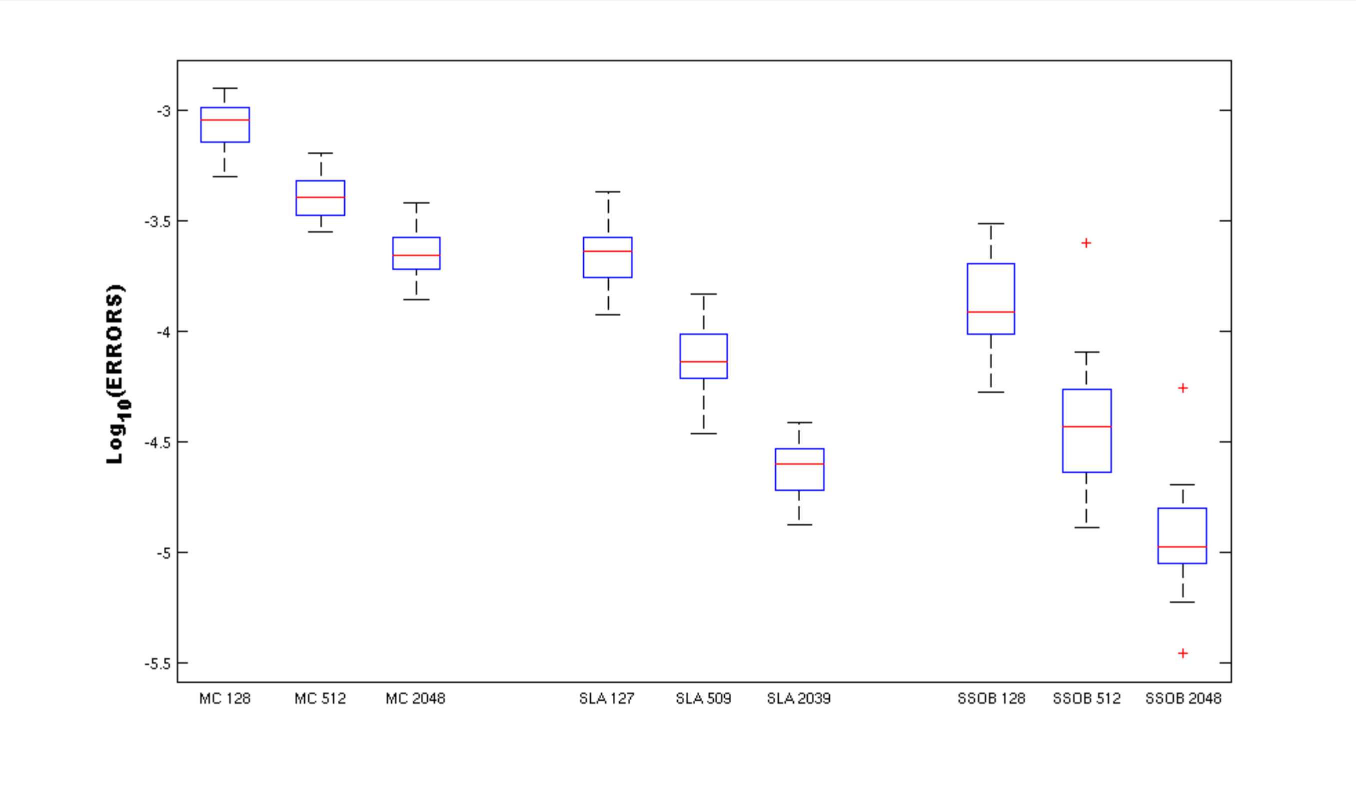
Some computational experience

We considered a two-stage production planning problem for maximizing the expected revenue while satisfying a fixed demand in a time horizon with $d = T = 100$ time periods and stochastic prices for the second-stage decisions. It is assumed that the probability distribution of the prices ξ is log-normal. The model is of the form

$$\max \left\{ \sum_{t=1}^{T} \left(c_t^\top x_t + \int_{\mathbb{R}^T} q_t(\xi)^\top y_t P(d\xi) \right) : Wy + Vx = h, y \geq 0, x \in X \right\}$$

The use of PCA for decomposing the covariance matrix has led to effective truncation dimension $d_T(0.01) = 2$. As QMC methods we used a randomly scrambled Sobol sequence (SSobol)(Owen, Hickernell) with $n = 2^7$, 2^9, 2^{11} and a randomly shifted lattice rule (Sloan-Kuo-Joe) with $n = 127, 509, 2039$, weights $\gamma_j = \frac{1}{j^3}$ and for MC the Mersenne-Twister. 10 runs were performed for the error estimates and 30 runs for plotting relative errors.

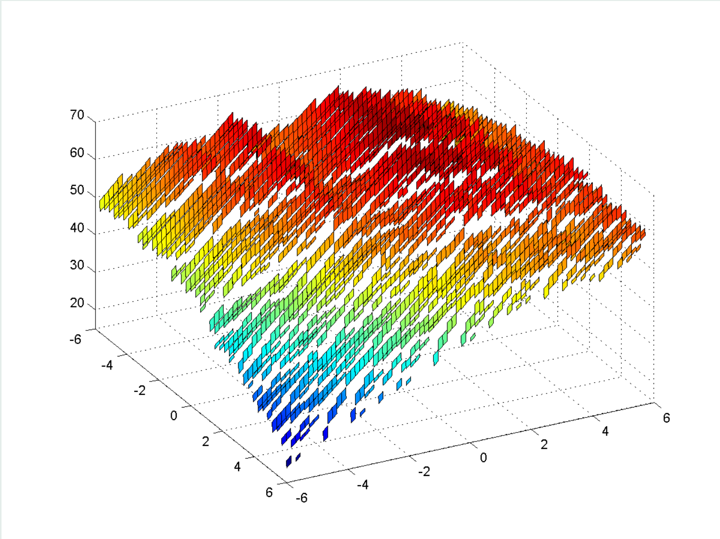
Average rate of convergence for QMC: $O(n^{-0.9})$ and $O(n^{-0.8})$.
Instead of $n = 2^7$ SSobol samples one would need $n = 10^4$ MC samples to achieve a similar accuracy as SSobol.



$\log_{10}$ of the relative errors of MC, SLA (randomly shifted lattice rule) and SSOB (scrambled Sobol' points)

Conclusions

- Our analysis provides a theoretical basis for applying QMC methods accompanied by dimension reduction techniques to two-stage stochastic programs.
- The analysis also applies to sparse grid quadrature techniques.
- The results are extendable and will be extended to mixed-integer two-stage models, to multi-stage situations, and to other models in stochastic optimization.



Second-stage optimal value function of an integer program (van der Vlerk)

References

R. E. Caflisch, W. Morokoff and A. Owen: Valuation of mortgage backed securities using Brownian bridges to reduce effective dimension, *Journal of Computational Finance* 1 (1997), 27–46.

J. Dick, F. Pillichshammer: *Digital Nets and Sequences*, Cambridge University Press, Cambridge 2010.

J. Dick, F. Y. Kuo, I. H. Sloan: High-dimensional integration – the Quasi-Monte Carlo way, *Acta Numerica* (2014), 1–153.

M. Griebel, F. Y. Kuo and I. H. Sloan: The smoothing effect of integration in $\mathbb{R}^d$ and the ANOVA decomposition, *Mathematics of Computation* 82 (2013), 383-400.

H. Heitsch, H. Leövey and W. Römisch, Are Quasi-Monte Carlo algorithms efficient for two-stage stochastic programs?, *Stochastic Programming E-Print Series* 5-2012 (`www.speps.org`) and submitted.

T. Homem-de-Mello: On rates of convergence for stochastic optimization problems under non-i.i.d. sampling, *SIAM Journal on Optimization* 19 (2008), 524-551.

F. Y. Kuo: Component-by-component constructions achieve the optimal rate of convergence in weighted Korobov and Sobolev spaces, *Journal of Complexity* 19 (2003), 301-320.

F. Y. Kuo, I. H. Sloan, G. W. Wasilkowski, H. Woźniakowski: On decomposition of multivariate functions, *Mathematics of Computation* 79 (2010), 953–966.

F. Y. Kuo, I. H. Sloan, G. W. Wasilkowski, B. J. Waterhouse: Randomly shifted lattice rules with the optimal rate of convergence for unbounded integrands, *Journal of Complexity* 26 (2010), 135–160.

A. B. Owen: Randomly permuted (t, m, s)-nets and (t, s)-sequences, in: *Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing*, Lecture Notes in Statistics, Vol. 106, Springer, New York, 1995, 299–317.

A. B. Owen: The dimension distribution and quadrature test functions, *Statistica Sinica* 13 (2003), 1–17.

A. B. Owen: Multidimensional variation for Quasi-Monte Carlo, in J. Fan, G. Li (Eds.), International Conference on Statistics, World Scientific Publ., 2005, 49–74.

T. Pennanen, M. Koivu: Epi-convergent discretizations of stochastic programs via integration quadratures, *Numerische Mathematik* 100 (2005), 141–163.

I. H. Sloan and H. Woźniakowski: When are Quasi Monte Carlo algorithms efficient for high-dimensional integration, *Journal of Complexity* 14 (1998), 1–33.

I. H. Sloan, F. Y. Kuo and S. Joe: Constructing randomly shifted lattice rules in weighted Sobolev spaces, *SIAM Journal Numerical Analysis* 40 (2002), 1650–1665.

X. Wang and K.-T. Fang: The effective dimension and Quasi-Monte Carlo integration, *Journal of Complexity* 19 (2003), 101–124.

X. Wang and I. H. Sloan: Why are high-dimensional finance problems often of low effective dimension, *SIAM Journal Scientific Computing* 27 (2005), 159–183.

X. Wang and I. H. Sloan: Low discrepancy sequences in high dimensions: How well are their projections distributed ? *Journal of Computational and Applied Mathematics* 213 (2008), 366–386.

X. Wang and I. H. Sloan, Quasi-Monte Carlo methods in financial engineering: An equivalence principle and dimension reduction. *Operations Research* 59 (2011), 80–95.