ON THE “STANDARD” CONDITION FOR NONCOMPACT HOMOGENEOUS EINSTEIN SPACES

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Abstract. A nonflat Einstein solvmanifold $(\mathcal{S}, g)$ is said to be of standard type if in the associated metric Lie algebra $\mathfrak{s}$, the orthogonal complement $\mathfrak{a}$ of the derived algebra is abelian. It is an open question whether the standard condition is automatically satisfied for all nonflat Einstein solvmanifolds. We derive certain properties of the metric Lie algebra $\mathfrak{s}$ of a nonflat Einstein solvmanifold $(\mathcal{S}, g)$ under the assumption $\dim[\mathfrak{a}, \mathfrak{a}] \leq 1$. In particular, we obtain some new sufficient conditions which imply standard type.

1. Introduction

The classification of noncompact homogeneous Einstein spaces is an important open problem. Since Ricci flat homogeneous spaces are flat [2], the interesting case is that of negative Einstein constant; equivalently: the nonflat noncompact case. A conjecture by D.V. Alekseevskii [1] says that for any nonflat homogeneous Einstein space G/K the isotropy group K must be a maximal compact subgroup of G. In case G is a linear group, the latter property implies that G/K is isometric to a Riemannian solvmanifold $(\mathcal{S}, g)$ where $\mathcal{S}$ is a simply connected solvable Lie group and g a left invariant metric on $\mathcal{S}$. Alekseevskii's conjecture is open so far; however, Yu.G. Nikonorov has recently proved some interesting partial results which support it [10].

All known examples of noncompact homogeneous Einstein manifolds do fall into the class of Riemannian solvmanifolds; even more, they all are of “standard type”; i.e., they satisfy the condition that in the associated metric Lie algebra $\mathfrak{s}$, the orthogonal complement $\mathfrak{a} := [\mathfrak{s}, \mathfrak{s}]^\perp$ of the derived algebra is abelian. Classical examples are the symmetric spaces of noncompact type and the classical bounded homogeneous domains. For other examples and for classifications in certain cases see, e.g., [1], [5], [11], [9], [6], [8], [7].

J. Heber's groundbreaking habilitation [8] contains several deep results concerning the structure of the moduli space of Einstein solvmanifolds. One of his results says that if a solvable Lie group carries a standard left invariant Einstein metric then it cannot carry a nonstandard one. Moreover, a given solvable Lie group can carry at most one standard Einstein metric up to isometry and scaling. Heber also shows that the standard Einstein manifolds correspond to an open subspace in the moduli space of all Einstein solvmanifolds. On this open subset, he obtains additional structural results which even allow him to outline a program for classifying all standard Einstein solvmanifolds.

In view of these results, it is particularly intriguing that the question whether nonstandard Einstein solvmanifolds exist at all is still open. A solution of this question would

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have very interesting implications. A general nonexistence theorem for nonstandard Einstein solvmanifolds would imply that those of Heber's results which a priori hold only in the "standard" case actually hold for all Einstein solvmanifolds. On the other hand, a construction of counterexamples would reveal the existence of a completely new type of Einstein spaces.

There are only some partial results in the direction of a nonexistence theorem. In [8], Lemma 4.7, Heber lists several sufficient conditions each of which does imply standard type: Nonpositive sectional curvature [3], nonnegative Killing form (e.g., in the case of purely real roots) or Killing form of index at most one [8]; moreover, the standard condition is trivially satisfied if $\mathfrak{a}$ is of dimension one.

The purpose of the present paper is to extend this list of sufficient conditions. We consider the case of Einstein solvmanifolds $(\mathcal{S}, g)$ with the property that $\dim[\mathfrak{a}, \mathfrak{a}] = 1$ (in some sense the least violent violation of the standard condition). Although we are not able to rule out the possible existence of Einstein manifolds with this property, we obtain the following partial results:

Theorem 3.1. *In any nonflat Einstein solvmanifold* $(\mathcal{S}, g)$ *with* $\dim[\mathfrak{a}, \mathfrak{a}] = 1$*, the orthogonal projection of* $[\mathfrak{a}, \mathfrak{a}]$ *to the center of* $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ *is zero. Moreover, there exists a subspace* $\mathfrak{a}' \subset \mathfrak{a}$ *of codimension two in* $\mathfrak{a}$ *such that* $[\mathfrak{a}', \mathfrak{a}] = 0$.

Corollary 3.2. *Let* $(\mathcal{S}, g)$ *be a nonflat Einstein solvmanifold. If* $\dim[\mathfrak{a}, \mathfrak{a}] \leq 1$ (*e.g., if* $\mathfrak{n}$ *is of codimension two in* $\mathfrak{s}$) *and if* $\mathfrak{n}$ *is abelian, then* $(\mathcal{S}, g)$ *is of standard type.*

These results may also be viewed as practical hints about where *not* to search for possible counterexamples to the standard condition.

2. Preliminaries

Throughout this paper, we consider a solvable metric Lie algebra $(\mathfrak{s}, \langle\ ,\ \rangle)$; that is, $\mathfrak{s}$ is a solvable Lie algebra and $\langle\ ,\ \rangle$ is a euclidean inner product on $\mathfrak{s}$. Let $\mathcal{S}$ denote the simply connected Lie group with Lie algebra $\mathfrak{s}$, endowed with the left invariant metric g whose restriction to $\mathfrak{s} = T_e\mathcal{S}$ is $\langle\ ,\ \rangle$. We identify elements of $\mathfrak{s}$ with left invariant vector fields on $\mathcal{S}$.

Notation and Remarks 2.1.

(i) Let $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$ be the derived algebra; note that $\mathfrak{n}$ is nilpotent. By $\mathfrak{a} := \mathfrak{n}^\perp$ we denote the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{s}$. We do *not* assume $\mathfrak{a}$ to be abelian.

(ii) For $A \in \mathfrak{a}$, we let $F_A := \mathrm{ad}_A|_{\mathfrak{n}} : \mathfrak{n} \to \mathfrak{n}$.

(iii) For $X \in \mathfrak{n}$, we define a skewsymmetric map $j_X : \mathfrak{a} \to \mathfrak{a}$ by requiring

$$\langle j_X A, A' \rangle = \langle X, [A, A'] \rangle$$

for all $A, A' \in \mathfrak{a}$.

(iv) The mean curvature vector $H \in \mathfrak{a}$ is given by $\langle H, U \rangle = \mathrm{tr}\, \mathrm{ad}_U$ for all $U \in \mathfrak{s}$.

(v) The Killing form $B : \mathfrak{s} \times \mathfrak{s} \to \mathbb{R}$ is defined by $B(U, V) := \mathrm{tr}(\mathrm{ad}_U \circ \mathrm{ad}_V)$ for all $U, V \in \mathfrak{s}$.

(vi) We choose an orthonormal basis $\{E_1, \ldots, E_n\}$ of $\mathfrak{s}$.

The Ricci tensor associated with the left invariant metric g on $\mathcal{S}$, viewed as a symmetric bilinear map ric : $\mathfrak{s} \times \mathfrak{s} \to \mathbb{R}$, is given by the following formula (and its polarized version):

Lemma 2.2 ([4], 7.38)**.** *Let $U \in \mathfrak{s}$. Then*

$$\mathrm{ric}(U,U) = -\langle [H,U],U\rangle - 1/2\ B(U,U) - 1/2\ \|\mathrm{ad}_U\|^2 + 1/4 \textstyle\sum_{i,j} \langle [E_i\,,E_j],U\rangle^2.$$

Here $\|\mathrm{ad}_U\|^2$ denotes the usual Euclidean squared norm $\mathrm{tr}({}^t\mathrm{ad}_U \circ \mathrm{ad}_U)$.

We will also need the following Lemma by J. Heber and its corollary below, which is also implicitly contained in [8].

Lemma 2.3 ([8], 4.1)**.** *Assume that $\langle\ ,\ \rangle$ is an Einstein metric on $\mathfrak{s}$; that is,* $\mathrm{ric} = c\cdot\langle\ ,\ \rangle$ *for some $c \in \mathbb{R}$. Then*

$$\mathrm{tr}(\mathrm{ad}_{\varphi(H)} \circ {}^t\varphi) + 1/2 \sum_i B\big({}^t\varphi\varphi(E_i), E_i\big) \le 0$$

for every derivation φ of $\mathfrak{s}$, and equality holds if and only if ${}^t\varphi$ is also a derivation of $\mathfrak{s}$.

Corollary 2.4. *If $\langle\ ,\ \rangle$ is Einstein and φ is a derivation of $\mathfrak{s}$ that vanishes on $\mathfrak{a}$, then ${}^t\varphi$ is also a derivation of $\mathfrak{s}$.*

Proof. We have $\varphi(H) = 0$ since $H \in \mathfrak{a}$; thus the first term in the inequality of Lemma 2.3 vanishes. We claim that the second term vanishes, too. In fact, by $\varphi|_{\mathfrak{a}} = 0$ we can assume that the E_i run through an orthonormal basis of $\mathfrak{n}$ alone; moreover, we need to consider only the traces of the operators $\mathrm{ad}_{{}^t\varphi\varphi(E_i)} \circ \mathrm{ad}_{E_i}$ restricted to $\mathfrak{n}$ since $\mathrm{Im}(\mathrm{ad}_{\mathfrak{s}}) = \mathfrak{n}$. But if $E_i \in \mathfrak{n}$ then $\mathrm{ad}_{E_i}|_{\mathfrak{n}}$ is a nilpotent derivation of the nilpotent Lie algebra $\mathfrak{n}$, and $\mathrm{ad}_{{}^t\varphi\varphi(E_i)}|_n$ is a derivation of $\mathfrak{n}$; by considering the grading of $\mathfrak{n}$ given, e.g., by the lower central series, one easily concludes that the trace of the composition of these two derivations vanishes. □

3. Implications in case $\dim[\mathfrak{a},\mathfrak{a}] = 1$

As mentioned in the Introduction, all known examples of solvable metric Lie algebras $(\mathfrak{s}, \langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle$ is Einstein and the corresponding solvmanifold $(\mathcal{S}, g)$ is nonflat, are of standard type; that is, the orthogonal complement $\mathfrak{a}$ of the derived algebra $\mathfrak{n}$ is abelian. It is an open question whether this condition is automatically satisfied in the above situation. In the following, we derive some consequences of the specialized assumption that $[\mathfrak{a},\mathfrak{a}]$ is of dimension one and $\langle\ ,\ \rangle$ is Einstein with $(\mathcal{S}, g)$ nonflat. Although this will not answer the question whether this constellation is at all possible, we are at least able to exclude certain types of possible counterexamples to the standard condition. This provides an addition to Heber's earlier list of conditions which imply standard type (see [8], Lemma 4.7). Our main result is the following.

Theorem 3.1. *Assume that the solvmanifold $(\mathcal{S}, g)$ associated with the solvable metric Lie algebra $(\mathfrak{s}, \langle\ ,\ \rangle)$ is Einstein and nonflat. Furthermore, suppose that* $\dim[\mathfrak{a},\mathfrak{a}] = 1$, *where $\mathfrak{a} = \mathfrak{n}^\perp$, $\mathfrak{n} = [\mathfrak{s},\mathfrak{s}]$. Then the following holds:*

(i) *There is a subspace $\mathfrak{a}' \subset \mathfrak{a}$ of codimension two in $\mathfrak{a}$ such that $[\mathfrak{a},\mathfrak{a}'] = 0$.*
(ii) $[\mathfrak{a},\mathfrak{a}]$ *is orthogonal to the center $\mathfrak{z}_{\mathfrak{n}}$ of $\mathfrak{n}$.*

Corollary 3.2. *Assume that* $(\mathcal{S}, g)$ *is Einstein and nonflat. If* $\dim[\mathfrak{a}, \mathfrak{a}] \le 1$ (*for example, if* $\mathfrak{n}$ *has codimension two in* $\mathfrak{s}$) *and if* $\mathfrak{n}$ *is abelian, then* $(\mathcal{S}, g)$ *is of standard type.*

The remainder of this section is devoted to proving the above theorem. In the following, we will always assume that $(\mathcal{S}, g)$ is Einstein and nonflat. Note that the Einstein constant c is necessarily negative here because, as is well known, homogeneous Einstein manifolds with positive or vanishing Einstein constant are compact or flat, respectively (the first assertion follows from the Bonnet-Myers Theorem, the second was shown in [2]).

We continue to use Notation 2.1. The following lemma holds regardless of $\dim[\mathfrak{a}, \mathfrak{a}]$.

Lemma 3.3. *We have* $\bigcap_{A\in\mathfrak{a}} \ker F_A = \{0\}$.

Proof. Let $X \in \mathfrak{n}$ be in the kernel of F_A for all $A \in \mathfrak{a}$. Then ad_X vanishes on $\mathfrak{a}$. By Corollary 2.4, ${}^t\mathrm{ad}_X$ is also a derivation of $\mathfrak{s}$ and hence (by restriction) of $\mathfrak{n}$. Considering the lower central series of $\mathfrak{n}$ once more, one easily sees that the transpose of an inner derivation ad_X of $\mathfrak{n}$ cannot be a derivation unless ad_X vanishes. So X lies in the center of $\mathfrak{n}$ and also of $\mathfrak{s}$. In the formula for $\mathrm{ric}(X, X)$ of Lemma 2.2, the first three terms now vanish since $\mathrm{ad}_X = 0$ on $\mathfrak{s}$. The fourth term is nonnegative, thus $\mathrm{ric}(X, X) \ge 0$. But this contradicts $c < 0$. □

Proof of Theorem 3.1(i). Let $\dim[\mathfrak{a}, \mathfrak{a}] = 1$ and suppose that $[\mathfrak{a}, \mathfrak{a}']$ were nontrivial for each subspace $\mathfrak{a}'$ of codimension two in $\mathfrak{a}$. Let $X \in \mathfrak{n}$ be a vector which generates $[\mathfrak{a}, \mathfrak{a}]$, and let $A \in \mathfrak{a}$ be an eigenvector of j_X^2. Define $\mathfrak{a}'$ as the orthogonal complement in $\mathfrak{a}$ of $\mathrm{span}\{A, j_X A\}$. Then $\mathfrak{a}'$ commutes with A and $j_X A$. Since $\mathfrak{a}'$ is of codimension at most two, $[\mathfrak{a}, \mathfrak{a}']$ is nonzero by assumption. Hence there exist vectors $B, C \in \mathfrak{a}'$ such that $[B, C] = X$. The Jacobi identity now implies $F_A X = [A, [B, C]] = [[A, B], C] + [B, [A, C]] = 0$. Since the eigenvectors of j_X^2 span the whole space $\mathfrak{a}$, we have shown that $F_A X = 0$ for all $A \in \mathfrak{a}$. But this contradicts Lemma 3.3 because of $X \ne 0$. □

Notation and Remarks 3.4. In the following, we always suppose that $\dim[\mathfrak{a}, \mathfrak{a}] = 1$.

(i) Let $\mathfrak{a}' \subset \mathfrak{a}$ be as in Theorem 3.1(i), and let $\{A_1, A_2\} \subset \mathfrak{a}$ be an orthonormal basis of the orthogonal complement of $\mathfrak{a}'$ in $\mathfrak{a}$.

(ii) Let $X := [A_1, A_2] \in \mathfrak{n}$. Note that X generates $[\mathfrak{a}, \mathfrak{a}]$; in particular, $X \ne 0$. Denote by $X^{\mathfrak{z}}$ the orthogonal projection of X to the center $\mathfrak{z}_{\mathfrak{n}}$ of $\mathfrak{n}$.

(iii) Let $\mathfrak{w} \subset \mathfrak{n}$ be the smallest subspace which contains $X^{\mathfrak{z}}$ and is invariant under F_A for each $A \in \mathfrak{a}$. Note $\mathfrak{w} \subset \mathfrak{z}_{\mathfrak{n}}$ because $\mathfrak{z}_{\mathfrak{n}}$ is invariant under the derivations F_A of $\mathfrak{n}$.

(iv) Let $\mathfrak{u} \subset \mathfrak{w}$ be the orthogonal complement of $\mathrm{span}\{X^{\mathfrak{z}}\}$ in $\mathfrak{w}$, and let $m := \dim \mathfrak{u}$.

(v) For $A \in \mathfrak{a}$, we denote by $D_A : \mathfrak{n} \to \mathfrak{n}$ the symmetric part of F_A. By $D_A^{\mathfrak{u}}$ we denote the restriction $D_A|_{\mathfrak{u}}$, followed by orthogonal projection to $\mathfrak{u}$.

(vi) Let $a_i := \mathrm{tr}\, F_i \in \mathbb{R}$ for $i = 1, 2$. Note that $a_i = \langle H, A_i \rangle$.

Together with Lemma 2.2 and Theorem 3.1(i), the following Lemma is the key of the proof of Theorem 3.1(ii).

Lemma 3.5.

(i) $F_{A'}(\mathfrak{w}) = 0$ *for all* $A' \in \mathfrak{a}'$.

(ii) $({}^tF_{A_i} + a_i \,\mathrm{Id})X^{\mathfrak{z}} \perp \mathfrak{z}_{\mathfrak{n}}$ *for* $i = 1, 2$.

(iii) $\mathfrak{u}$ *is invariant under* F_{A_i} *for* $i = 1, 2$.
(iv) *If* $X^{\mathfrak{z}} \neq 0$ *then* $\dim \mathfrak{w} \geq 2$*; in particular,* $m \geq 1$.

Proof. (i) Let $A' \in \mathfrak{a}'$. By Corollary 2.4, ${}^t\mathrm{ad}_{A'}$ is a derivation. Since ${}^t\mathrm{ad}_{A'}$ vanishes on $\mathfrak{a}$, it follows that ${}^t\mathrm{ad}_{A'}$ commutes with ad_A for each $A \in \mathfrak{a}$. In particular, $\mathrm{ad}_{A'}$ is a normal derivation; thus $F_{A'}$ is a normal derivation of $\mathfrak{n}$. Moreover, the Jacobi identity implies $F_{A'}(X) = 0$ since $[A_1, A_2] = X$ and A' commutes with A_1 and A_2. Being a normal derivation, $F_{A'}$ must therefore annihilate the $\mathfrak{z}_{\mathfrak{n}}$-component $X^{\mathfrak{z}}$ of X as well. It follows that $F_{A'}$ annihilates $\mathfrak{w}$ since it commutes with each F_A.

(ii) Let $Z \in \mathfrak{z}_{\mathfrak{n}}$ be arbitrary. Since the Killing form B vanishes on $\mathfrak{s} \times \mathfrak{n}$ and since $\mathfrak{a} \perp [\mathfrak{s}, \mathfrak{s}]$, it follows from Lemma 2.2 that

$$\begin{aligned} 2\,\mathrm{ric}(A_2, Z) &= -\langle [H, A_2], Z\rangle - \langle \mathrm{ad}_{A_2}, \mathrm{ad}_Z\rangle \\ &= -\langle [a_1 A_1, A_2], Z\rangle - \langle [A_1, A_2], F_{A_1} Z\rangle = -\langle a_1 X, Z\rangle - \langle {}^tF_{A_1} X, Z\rangle. \end{aligned}$$

But $\mathrm{ric}(A_2, Z)$ must vanish by the Einstein condition. This implies the statement for $i = 1$; in case $i = 2$ it is shown analogously.

(iii) Since $\mathfrak{w}$ is invariant under F_{A_i} by definition, we only need to check that $F_{A_i}(\mathfrak{u})$ is orthogonal to $X^{\mathfrak{z}}$. But (ii) already implies that ${}^tF_{A_i}(X^{\mathfrak{z}})$ is orthogonal to $\mathfrak{u}$.

(iv) Lemma 2.2 implies

$$\mathrm{ric}(X^{\mathfrak{z}}, X^{\mathfrak{z}}) = -\langle [H, X^{\mathfrak{z}}], X^{\mathfrak{z}}\rangle - 1/2\, \|\mathrm{ad}_{X^{\mathfrak{z}}}\|^2 + \alpha,$$

where $\alpha \geq 1/2 \sum_{i=1}^2 |\,{}^tF_{A_i}(X^{\mathfrak{z}})|^2$. In fact, for an adapted choice of the orthonormal basis $\{E_1, \ldots, E_n\}$, the latter expression corresponds to those summands in the last term of the sum in Lemma 2.2 where $E_i \in \mathrm{span}\{A_1, A_2\}$ and $E_j \in \mathfrak{n}$ or vice versa. If $X^{\mathfrak{z}} \neq 0$ then $\mathrm{ric}(X^{\mathfrak{z}}, X^{\mathfrak{z}}) < 0$ since the Einstein constant c is negative. Using the above statements (i) and (ii), we derive

$$\begin{aligned} 0 &> \textstyle\sum_{i=1}^2 \big(-a_i \langle F_{A_i} X^{\mathfrak{z}}, X^{\mathfrak{z}}\rangle - \frac{1}{2}\, |F_{A_i}(X^{\mathfrak{z}})|^2\big) + \alpha \\ &\geq \textstyle\sum_{i=1}^2 \big(a_i^2 |X^{\mathfrak{z}}|^2 - \frac{1}{2} |F_{A_i}(X^{\mathfrak{z}})|^2 + \frac{1}{2} |\,{}^tF_{A_i}(X^{\mathfrak{z}})|^2\big). \end{aligned}$$

Now if $\dim \mathfrak{w} \geq 2$ were false, then $X^{\mathfrak{z}}$ would be an eigenvector of each F_{A_i}. In particular, we would have $|\,{}^tF_{A_i}(X^{\mathfrak{z}})|^2 \geq |F_{A_i}(X^{\mathfrak{z}})|^2$. Thus the right hand side in the above inequality would be nonnegative, which is a contradiction. □

Proof of Theorem 3.1(ii). Suppose that statement (ii) of the Theorem were false; that is, $X^{\mathfrak{z}} \neq 0$. We will derive a contradiction from this using 2.2 and 3.5.

Let $i \in \{1, 2\}$. In the expression for $\mathrm{ric}(A_i, A_i)$ from Lemma 2.2, the first and fourth terms vanish. Therefore,

$$\begin{aligned} \mathrm{ric}(A_i, A_i) &= -1/2\, B(A_i, A_i) - 1/2\, \|\mathrm{ad}_{A_i}\|^2 = -1/2\, \langle \mathrm{ad}_{A_i}, {}^t\mathrm{ad}_{A_i} + \mathrm{ad}_{A_i}\rangle \\ &= -\langle F_{A_i}, D_{A_i}\rangle - 1/2\, \|\mathrm{ad}_{A_i}|_{\mathfrak{a}}\|^2 = -\|D_{A_i}\|^2 - 1/2\, |X|^2. \end{aligned}$$

Recall that $\langle D_{A_i} X^{\mathfrak{z}}, X^{\mathfrak{z}}\rangle = \langle {}^tF_{A_i} X^{\mathfrak{z}}, X^{\mathfrak{z}}\rangle = -a_i |X^{\mathfrak{z}}|^2$ by Lemma 3.5(ii). Therefore, $\|D_{A_i}\|^2 \geq a_i^2 + \|D^{\mathfrak{u}}_{A_i}\|^2 \geq a_i^2 + (\mathrm{tr}\, D^{\mathfrak{u}}_{A_i})^2/m$. Summing over $i = 1, 2$, we conclude that

$$2mc \leq -m(a_1^2 + a_2^2) - (\mathrm{tr}\, D^{\mathfrak{u}}_{A_1})^2 - (\mathrm{tr}\, D^{\mathfrak{u}}_{A_2})^2 - m|X|^2. \tag{1}$$

On the other hand, consider $\operatorname{ric}(Z,Z)$ for $Z \in \mathfrak{u}$. Just as we did in the above proof of Lemma 3.5(iv) for $X^{\mathfrak{z}}$, we see that

$$\operatorname{ric}(Z,Z) \geq \sum_{i=1}^{2}\bigl(-a_i\langle F_{A_i}Z, Z\rangle - \tfrac{1}{2}|F_{A_i}Z|^2 + \tfrac{1}{2}|\,{}^tF_{A_i}Z|^2\bigr).$$

Now let Z run through an orthonormal basis of $\mathfrak{u}$ and consider the sum of the corresponding inequalities. Then the sum involving the last two of the three terms on the right hand side is nonnegative because $\mathfrak{u}$ is invariant under F_{A_i} by Lemma 3.5(iii). Therefore we obtain

$$mc \geq -\sum_{i=1}^{2} a_i \operatorname{tr} D^{\mathfrak{u}}_{A_i}\,,$$

which by the Cauchy-Schwarz inequality implies

$$2mc \geq -a_1^2 - a_2^2 - (\operatorname{tr} D^{\mathfrak{u}}_{A_1})^2 - (\operatorname{tr} D^{\mathfrak{u}}_{A_2})^2. \tag{2}$$

From the inequalities (1) and (2) we conclude

$$0 \geq (m-1)(a_1^2 + a_2^2) + m|X|^2,$$

which is impossible since $m \geq 1$ by Lemma 3.5(iv), and $X \neq 0$. □

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