

ISOSPECTRAL MANIFOLDS WITH DIFFERENT LOCAL GEOMETRIES

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ABSTRACT. We construct several new classes of isospectral manifolds with different local geometries. After reviewing a theorem by Carolyn Gordon on isospectral torus bundles and presenting certain useful specialized versions (Chapter 1) we apply these tools to construct the first examples of isospectral four-dimensional manifolds which are not locally isometric (Chapter 2). Moreover, we construct the first examples of isospectral left invariant metrics on compact Lie groups (Chapter 3). Thereby we also obtain the first continuous isospectral families of globally homogeneous manifolds and the first examples of isospectral manifolds which are simply connected and irreducible. Finally, we construct the first pairs of isospectral manifolds which are conformally equivalent and not locally isometric (Chapter 4).

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Introduction

In this work we construct several new classes of isospectral manifolds with different local geometries. More precisely, we obtain

- the first examples of four-dimensional isospectral manifolds which are not locally isometric (Chapter 2),
- the first examples of isospectral left invariant metrics on compact Lie groups (Chapter 3) and the first examples of isospectral manifolds which are simply connected and irreducible,
- the first examples of conformally equivalent manifolds which are isospectral and not locally isometric (Chapter 4).

The spectrum of a closed Riemannian manifold is the eigenvalue spectrum of the associated Laplace operator acting on functions, counted with multiplicities; two manifolds are said to be isospectral if their spectra coincide. Spectral geometry deals with the mutual influences between the spectrum of a Riemannian manifold and its geometry (see the books [1], [4], [11] for an introduction to spectral geometry).

Which geometric properties are determined by the spectrum? Y. Colin de Verdière showed that generically the Laplace spectrum determines the spectrum of lengths of closed geodesics [12]. Moreover, the spectrum determines a sequence of so-called heat invariants, the first few of which are the dimension, the volume, and the total scalar curvature (see, e.g., [4], [18]). A few Riemannian manifolds are known to be completely characterized by their spectra. For example, S. Tanno proved this for the round spheres in dimensions up to six [45], using heat invariants; for round spheres of arbitrary dimension it is only known that the spectrum on functions together with the spectrum on 1-forms characterizes them completely, as was shown by V. Patodi [36]. Moreover, there are several rigidity and compactness results in special situations. S. Tanno [46] showed that every round sphere is infinitesimally spectrally rigid, that is, one cannot continuously deform the round metric without changing the spectrum. C. Croke and V. Sharafutdinov proved infinitesimal spectral rigidity for metrics of negative sectional curvature [14]. B. Osgood, R. Phillips, and P. Sarnak showed that the set of metrics on a surface which are isospectral to a given metric is always compact in the C^∞ -topology [34]; the same holds for bounded plane domains with respect to their Dirichlet spectrum [35]. A similar result, although restricted to isospectral metrics within a fixed conformal class, was shown for closed manifolds of dimension three ([10], [8]). Moreover, R. Brooks, P. Perry, and P. Petersen showed in [7] that on a three-dimensional manifold every Riemannian metric which is close enough to a metric of constant curvature has the property that the corresponding set of isospectral metrics is compact. H. Pesce proved a compactness result in the case of a fixed Riemannian covering: For any Riemannian manifold (M, g) the set of discrete subgroups

$\Gamma < \text{Isom}(M, g)$ for which the compact quotient manifold $(\Gamma \backslash M, g)$ is isospectral to a given such manifold is compact in the set of discrete subgroups of $\text{Isom}(M, g)$ [38].

The general questions in these contexts, namely, whether “most” Riemannian metrics are infinitesimally spectrally rigid, or whether each isospectral set of metrics is compact in some appropriate topology, are still open.

On the other hand, many examples of isospectral manifolds have been constructed, mainly during the last two decades. Note that the study of such examples is the only possibility of finding geometric properties which are *not* determined by the spectrum. The first example of isospectral manifolds was given in 1964 by J. Milnor: a pair of flat tori in dimension sixteen [33] (by now there are also examples of isospectral flat tori in dimension four [13]). This was the first proof of the fact that the spectrum does not determine the isometry class of a Riemannian manifold. In 1980, M.-F. Vignéras discovered examples of isospectral Riemann surfaces and of isospectral hyperbolic manifolds in dimension three, the latter showing that the fundamental group is not spectrally determined ([47]; see also P. Buser’s book [9] on the spectral theory of Riemann surfaces). The first examples of *continuous* families of isospectral metrics were found by Carolyn Gordon and Edward Wilson in 1984 [25]; these were locally homogeneous metrics, induced by left invariant ones, on compact quotients of nilpotent or solvable Lie groups. In 1985 T. Sunada established a general isospectrality principle [43] which, either in its original or certain generalized versions, came to be known as “the Sunada method”. One generalized version, established by Carolyn Gordon and Dennis DeTurck [16] in 1987, is the following:

Let (M, g) be a Riemannian manifold and $G < \text{Isom}(M, g)$. Suppose Γ_1, Γ_2 are two discrete cocompact subgroups of G which act freely and properly discontinuously on M with compact quotients. If the quasi-regular representations of G on $L^2(\Gamma_1 \backslash G)$ and $L^2(\Gamma_2 \backslash G)$ are unitarily equivalent, then $\Gamma_1 \backslash M$ and $\Gamma_2 \backslash M$, each endowed with the metric induced by g , are isospectral.

This theorem not only covered most of the isospectral examples known at that time, but also led (via another generalization by P. Bérard ([2],[3]) to the case of orbifolds) to the famous first examples of bounded plane domains with the same Dirichlet (and Neumann) spectrum; these were found in 1991 by C. Gordon, D. Webb, and S. Wolpert [24]. Thereby, M. Kac’s question of 1966, “Can one hear the shape of a drum?” [32], was finally answered negatively. Note, however, that these domains have nonsmooth boundaries; the answer to Kac’s question in the smoothly bounded case is still open.

There is a generic converse to the Sunada theorem in the case where the covering manifold M is compact: H. Pesce proved that there exists an open and dense set of metrics g on each closed manifold M such that all possible pairs of isospectral quotient manifolds of the form $(\Gamma_1 \backslash M, g)$, $(\Gamma_2 \backslash M, g)$ must necessarily arise from the Sunada construction [39].

By the very principle of the Sunada method described above, the isospectral manifolds which arise from it always have a common Riemannian covering. In particular, they are always locally isometric. Their geometries can be distinguished only by global properties; for example, by the continuously changing mass of certain

homology classes [17] or the changing distance of certain geometrically distinguished families of geodesic loops [41].

We now come to the history of isospectral manifolds which are not locally isometric. In 1991, Zoltan Szabó discovered the first pairs of such manifolds (see [44], published much later); these were manifolds with boundary, diffeomorphic to the product of an eight-dimensional ball and a three-dimensional torus, arising as domains in quotients of certain harmonic manifolds. Motivated by Szabó's examples, and related to them, were the first pairs of isospectral manifolds without boundary which Carolyn Gordon gave in 1992 ([20], [21]); these were pairs of two-step nilmanifolds with different underlying group structures. Her isospectrality proof for these examples revealed another general principle which is quite different from Sunada's and does not imply local isometry of the resulting isospectral manifolds:

If a torus acts on two Riemannian manifolds freely and isometrically with totally geodesic fibers, and if the quotients of the manifolds by any subtorus of codimension at most one are isospectral when endowed with the submersion metric, then the original two manifolds are isospectral.

Using this principle, C. Gordon and E. Wilson [27] generalized Z. Szabó's examples and obtained continuous multiparameter families of isospectral, locally non-isometric metrics on products of $(m \geq 5)$ -dimensional balls with $(r \geq 2)$ -dimensional tori. These arise as domains in certain Riemannian nilmanifolds whose Ricci tensors have in general different eigenvalues. Next, it was observed during a workshop in Grenoble in 1997 that the boundaries of these manifolds are again isospectral (by the same principle) and not locally isometric [22]. Among the isospectral families discovered in this way are some where the maximal scalar curvature changes during the deformation, which shows that the range of the scalar curvature is not spectrally determined (although the total scalar curvature is). Independently, Z. Szabó showed, in a more special setting, that the boundaries of his original examples are isospectral (see again [44]). Among his examples is a pair of isospectral metrics one of which is homogeneous while the other is not even locally homogeneous; thus (local) homogeneity is not encoded in the spectrum. By embedding the torus factor of the manifolds given in [22] into a compact Lie group and extending the metrics in such a way that the above isospectrality principle for torus bundles still applied, the author constructed, also in 1997, the first examples of simply connected, closed isospectral manifolds [42]; note that non-simple connectivity was another feature always present in the Sunada type examples of isospectral closed manifolds.

In [42] the author also used the new examples to show that the individual terms in the linear combination $5 \int \text{scal}^2 - 2 \int \|\text{Ric}\|^2 + 2 \int \|R\|^2$, which is one of the heat invariants, are not spectrally determined; more precisely, the corresponding heat invariant for the Laplace operator acting on 1-forms, which is a different linear combination of the same terms, changes during the isospectral deformations given in [42]. In particular, these manifolds are not isospectral on 1-forms. Before that, examples of manifolds which are isospectral on functions but not on 1-forms had been given by A. Ikeda [31] (lens spaces, see also [30], Carolyn Gordon [19] (Heisenberg manifolds, see also [26]), and Ruth Gornet ([28], [29]) who also constructed the first continuous families with this property. Note that none of those examples arose from the Sunada method, but used special constructions. In fact, the Sunada setting — except for a certain further generalization of it established by H. Pesce

[37] which also explains Ikeda's examples — always implies isospectrality not only on functions, but also on all p -forms [16]. Still, in the above examples by Ikeda, Gordon, and Gornet, the isospectral manifolds do have a common Riemannian covering. It is not hard to see that integrals of functions which are induced on manifolds of the same volume by isometry invariant functions on a common covering manifold (such as scal^2 , $\|\text{Ric}\|^2$, etc.) must always be the same. Thus isospectral manifolds with a common Riemannian covering always share the same heat invariants, also for the Laplace operator on p -forms, even if they are not isospectral on p -forms. So it was the first time in [42] that heat invariants alone were used to prove non-isospectrality on 1-forms.

Recently Carolyn Gordon and Zoltan Szabó gave a version of the isospectral torus bundle theorem for the case where the fibers are not necessarily totally geodesic, imposing certain other restrictions instead. In particular, they obtained by this approach the first continuous families of negatively curved isospectral manifolds with boundary [23], contrasting with the above rigidity result by Croke and Sharafutdinov [14] for the case of closed manifolds.

Thereby we finish our account of the previously known examples of isospectral manifolds, and turn now to the description of the contents of the present work.

In the preliminary Chapter 1 we first review Carolyn Gordon's above principle of isospectral torus bundles with totally geodesic fibers (Theorem 1.3) and give a slightly more special version (Theorem 1.6) in which we assume the submersion quotients to be not only isospectral, but isometric, and formulate this condition in terms of bundle connection forms with which the metrics are associated. The formulation becomes quite simple in the case of trivial bundles; i.e., products of the base manifold and a torus (Proposition 1.8). We then review the examples given in [22] and [42] and interpret them as applications of Proposition 1.8 and Theorem 1.6, respectively.

Recall that the isospectral manifolds from [22] were diffeomorphic to $S^{m-1} \times T^2$ with $m \geq 5$, and thus at least six-dimensional. Examples of isospectral, locally non-isometric manifolds in lower dimensions were not known until now. In Chapter 2 we use the point of view developed in Chapter 1 on these previous examples in order to drop an unnecessary property of their metrics and apply Proposition 1.8 in a systematic way to construct isospectral, locally non-isometric metrics in dimension four, namely, on $S^2 \times T^2$ (see Example 2.6). We show that in all isospectral pairs arising from Proposition 1.8 in which the base manifold is — as here — two-dimensional, the associated scalar curvature functions share the same range and the same integrals of each of their powers (Theorem 2.11), which contrasts with the properties of the higher-dimensional examples. Nevertheless, in one of our pairs of isospectral metrics on $S^2 \times T^2$ the preimages of the maximum of the associated scalar curvature functions have different dimensions (Proposition 2.10), which shows that the manifolds are not locally isometric. In many examples the metrics can also be distinguished by the integral of $(\Delta \text{scal})^2$ (see Remark 2.12(iii)).

In Chapter 3 we apply Theorem 1.6 / Proposition 1.8 to the case of compact Lie groups with left invariant metrics (Proposition 3.1 / Corollary 3.2). The isospectrality result formulated in Proposition 3.1 can be shown not only "geometrically" by deducing it from Theorem 1.6 or 1.3, but also by purely algebraic methods involving

the expression of the Laplace operator associated to a left invariant metric in terms of the right-regular representation of the Lie group. We use Proposition 3.1 and Corollary 3.2 to construct continuous isospectral families of left invariant metrics on $\mathrm{SO}(m \geq 5) \times T^2$, $\mathrm{Spin}(m \geq 5) \times T^2$, $\mathrm{SU}(m \geq 3) \times T^2$, $\mathrm{SO}(n \geq 8)$, $\mathrm{Spin}(n \geq 8)$, and $\mathrm{SU}(n \geq 6)$ (see the examples 3.3 and 3.7–3.10 in Section 3.2). These are not only the first examples of isospectral left invariant metrics on compact Lie groups in general, but among them are also the first examples of *irreducible* simply connected isospectral manifolds. Moreover, they are the first examples of continuous families of globally homogeneous isospectral manifolds. We also obtain the first examples of continuous isospectral families of manifolds of positive Ricci curvature. In Section 3.3 we prove that the left invariant metrics in our isospectral examples are not locally isometric; more precisely, the norm of the associated Ricci tensors, $\|\mathrm{Ric}\|^2$ (which is a constant function on each of the manifolds) changes during the deformations (see Proposition 3.15 and Theorem 3.14). In particular, a certain heat invariant for the Laplace operator on 1-forms also changes during the deformations (Corollary 3.17). Continuous isospectral families of left invariant metrics of the type constructed in Section 3.2 can occur arbitrarily close to bi-invariant metrics while still enjoying the non-isometry properties established in Section 3.3 (see Remark 3.12(i)). In contrast to this, we prove in Section 3.4 a rigidity result for bi-invariant metrics: Any continuous isospectral family of left invariant metrics which contains a bi-invariant metric must be trivial (Theorem 3.19).

In Chapter 4 we formulate a canonical generalization of Theorem 1.6 which in turn can be viewed as a special form of C. Gordon’s and Z. Szabó’s above-mentioned isospectrality theorem for principal torus bundles whose fibers are not assumed to be totally geodesic (see Theorem 4.3 and Remark 4.4). Here again, we also give a simpler formulation for the case of trivial bundles (Proposition 4.5). We then use Proposition 4.5 to construct the first examples of conformally equivalent isospectral manifolds with different local geometries (Example 4.6). The non-isometry proof consists of showing that in our pairs of conformally equivalent isospectral manifolds the respective preimages of the maximal scalar curvature constitute a pair of isospectral, globally homogeneous submanifolds of the type studied in Chapter 3, whose Ricci tensors were already shown to have different norms (Proposition 4.7). Note that the first examples of isospectral, conformally equivalent manifolds were constructed by Robert Brooks and Carolyn Gordon [6] in 1990 using the Sunada method as formulated in [16]; in particular, those manifolds were locally isometric. Our new examples are the first ones which show that even within a fixed conformal class the local geometry is not determined by the spectrum.

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1. Constructions of isospectral, locally non-isometric manifolds

In this chapter we present the tools which we will use in Chapters 2 and 3 to construct new examples of isospectral manifolds which are not locally isometric.

The starting point is a general theorem by Carolyn Gordon concerning torus bundles with totally geodesic fibers; see Theorem 1.3 in Section 1.1. We formulate somewhat more special versions (Theorem 1.6 and Proposition 1.8 in Section 1.2) which account for almost all previous applications of Theorem 1.3 and also for most of the new examples of isospectral manifolds which we present in this work. In Section 1.3 we review some previously known examples which we interpret as applications of Theorem 1.6 or Proposition 1.8.

1.1 Isospectral torus bundles with totally geodesic fibers.

Definition 1.1. Let (M, g) be a closed Riemannian manifold, and let Δ_g be the Laplacian acting on functions by

$$(\Delta_g f)(p) := - \sum_{i=1}^n \frac{d^2}{dt^2} \Big|_{t=0} f(c_i(t)) \quad \text{for } p \in M,$$

where the c_i are geodesics starting in p such that $\{\dot{c}_1(0), \dots, \dot{c}_n(0)\}$ is an orthonormal basis for $T_p M$. The discrete sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$ of the eigenvalues of Δ_g , counted with the corresponding multiplicities, is called the *spectrum* of (M, g) ; we will denote it by $\text{spec}(M, g)$ or $\text{spec}(\Delta_g)$. If $\mathcal{H} \subseteq L^2(M, g)$ is a subspace of functions such that $C^\infty(M) \cap \mathcal{H}$ is invariant under Δ_g , we will denote the corresponding spectrum of eigenvalues by $\text{spec}(\mathcal{H})$. Two closed Riemannian manifolds are called *isospectral* if their spectra coincide.

All previously known examples of closed isospectral manifolds which are isospectral and not locally isometric, except for some recent examples by Gordon and Szabó [23] (see Remark 4.4 in Section 4.1), arise from the following theorem by Carolyn Gordon [21] which we present below.

Notation 1.2. By a *torus*, we always mean a compact connected abelian Lie group. If a torus H acts smoothly and freely by isometries on a closed Riemannian manifold (M, g) , then there is a unique Riemannian metric, denoted g^H , on the quotient manifold M/H such that the canonical projection $\pi_H : (M, g) \rightarrow (M/H, g^H)$ becomes a Riemannian submersion.

Theorem 1.3 ([Go3]). *Let H be a torus, and let (M, g) and (M', g') be two principal H -bundles such that the Riemannian metrics g, g' are invariant under the action of H . Assume:*

- (i) *The fibers of the action of H are totally geodesic submanifolds of (M, g) , resp. of (M', g') .*
- (ii) *For any closed subgroup W of H which is either H itself or a subtorus of codimension 1 in H , the manifolds $(M/W, g^W)$ and $(M'/W, g'^W)$ are isospectral.*

Then (M, g) and (M', g') are isospectral.

Proof. We consider the unitary representation of H on the Hilbert space $\mathcal{H} := L^2(M, g)$, defined by $(zf)(x) = f(zx)$ for all $f \in \mathcal{H}$, $z \in H$, $x \in M$. Write $H = \mathfrak{h}/\mathcal{L}$, where \mathfrak{h} is isomorphic to some \mathbb{R}^r , and let \mathcal{L}^* be the dual lattice. Since H is abelian, \mathcal{H} decomposes as the orthogonal sum $\bigoplus_{\mu \in \mathcal{L}^*} \mathcal{H}_\mu$ with $\mathcal{H}_\mu = \{f \in \mathcal{H} \mid zf = e^{2\pi i \mu(Z)} f \text{ for all } z \in H\}$, where Z denotes any representative for z in \mathfrak{h} . In particular, this implies the following coarser decomposition:

$$(1) \quad \mathcal{H} = \mathcal{H}_0 \oplus \bigoplus_W (\mathcal{H}_W \ominus \mathcal{H}_0),$$

where W runs through the set of all closed connected subgroups of codimension 1 in H , and \mathcal{H}_W is the sum of all \mathcal{H}_μ such that $\mu \in \mathcal{L}^*$ and $T_e W \subseteq \ker \mu$. Note that \mathcal{H}_W is just the space of W -invariant functions in \mathcal{H} . Let $\mathcal{C}_W := C^\infty(M) \cap \mathcal{H}_W$ and $\mathcal{C}_0 := C^\infty(M) \cap \mathcal{H}_0$. Since the action of H is by isometries and therefore commutes with Δ_g , the spaces \mathcal{C}_W and \mathcal{C}_0 are invariant under Δ_g . Note that π_H^* is a linear bijection from $C^\infty(M/H)$ to \mathcal{C}_0 . Since π_H is a Riemannian submersion with totally geodesic fibers by (i), π_H^* intertwines the corresponding Laplacians. Thus $\text{spec}(\mathcal{H}_0) = \text{spec}(M/H, g^H)$. Assumption (ii) for $W = H$ implies, with the obvious analogous notations for (M', g') , that $\text{spec}(\mathcal{H}_0) = \text{spec}(\mathcal{H}'_0)$.

Now let W be a closed connected subgroup of codimension 1 in H . Then π_W^* is a linear bijection from $C^\infty(M/W)$ to \mathcal{C}_W . Note that assumption (i) implies that also the W -orbits are totally geodesic, since the induced metric on the H -orbits is invariant and thus flat. This, together with assumption (ii) for W implies, by the same argument as before, that $\text{spec}(\mathcal{H}_W) = \text{spec}(\mathcal{H}'_W)$. From (1) we now conclude that $\text{spec}(M, g) = \text{spec}(M', g')$. \square

Remarks 1.4.

(i) Note that Theorem 1.3 is trivial if the torus H is one-dimensional: In that case, condition (ii) on subtori of codimension 1 is equivalent to the assertion of the theorem.

(ii) We remark that *all continuous* isospectral families (M, g_t) of principal torus bundles with totally geodesic fibers must necessarily arise from Theorem 1.3. In fact, if H is the fiber of M , then for each closed subgroup W of H the fibers of the W -action are again totally geodesic in (M, g_t) , and thus the spectrum of $(M/W, g_t^W)$ is contained in the discrete set $\text{spec}(M, g_t)$ which is independent of t by assumption. Hence $\text{spec}(M/W, g_t^W)$ must itself be independent of t . Therefore also condition (ii) of Theorem 1.3 is satisfied, which means that the isospectral family (M, g_t) arises from this theorem.

(iii) Although the isospectral manifolds arising from Theorem 1.3 are in general not locally isometric, certain well-known families of isospectral, locally isometric manifolds too can be viewed as applications of this theorem. For example, there are isospectral families $(\Phi_t(\Gamma) \backslash G, g)$ of two-step nilmanifolds [15] (where G is a simply connected two-step nilpotent Lie group, g is a left invariant metric, Γ is a cocompact discrete subgroup of G , and Φ_t a family of so-called almost inner automorphisms of G). These arose originally in the Sunada type context (see the Introduction and [25], [16]), but can as well be viewed as arising from Theorem 1.3 (or the more special Theorem 1.6 below). This is not only clear from (ii) above, but in fact the

isospectrality proof given in [15] for these particular families was already similar in vein to the proof of Theorem 1.3.

(iv) The observation in (ii) illustrates that Theorem 1.3 is in fact a quite general result. Its “broadness” does not make it very obvious how to find explicit applications — except for the ones which motivated its discovery in the first place (i.e., the two-step nilmanifolds from [21]) and their closely related companions from [22] and [42] (see Examples 1.11 and 1.14).

Therefore, we formulate in this work several specialized versions of Theorem 1.3 (namely, Theorem 1.6 and Proposition 1.8 below, as well as Proposition 3.1 and Corollary 3.2 in Chapter 3). Using these we will be able to give interesting new applications.

(v) In almost all known applications of Theorem 1.3, the pairs of quotient manifolds $(M/W, g^W)$ and $(M'/W, g'^W)$ are not only isospectral, but isometric. The only exceptions of this are certain pairs of two-step nilmanifolds constructed in [21] and [27]. The isospectrality of the (non-isometric) quotient manifolds follows there from explicit knowledge of their spectra. We will not review these examples in the present work. All other known applications of Theorem 1.3 are actually applications of Theorem 1.6 in the following section which is somewhat more special and does imply isometric quotient manifolds, as we will see in the proof.

1.2 A first specialization and its application to products.

Notation and Remarks 1.5. Let H be a torus and $\mathfrak{h} := T_e H$. When we call a metric on a torus *invariant* we will always mean left invariant (which is here the same as bi-invariant). Now let H be equipped with a fixed invariant metric. Let M be a principal H -bundle over a Riemannian manifold (N, h) . Each fiber canonically inherits an H -invariant metric from the given metric on H .

- (i) For $Z \in \mathfrak{h}$ we denote the corresponding vector field $p \mapsto \frac{d}{dt}|_{t=0} p \cdot \exp(tZ)$ on M by Z again. Note that Z is H -invariant since H is abelian.
- (ii) A connection form ω on M is an \mathfrak{h} -valued, H -invariant 1-form on M such that $\omega(Z) = Z$ for all $Z \in \mathfrak{h}$. For any connection form ω on M and any $Z \in \mathfrak{h}$, we define the 1-form $\omega_Z := \langle \omega(\cdot), Z \rangle$ on M , where $\langle \cdot, \cdot \rangle$ denotes the scalar product induced on \mathfrak{h} by the metric on H .
- (iii) For any connection form ω on M , we denote by g_ω the unique H -invariant Riemannian metric on M which satisfies:
 - 1.) g_ω induces the given invariant metric on each fiber.
 - 2.) The projection $\pi_H : (M, g_\omega) \rightarrow (N, h)$ is a Riemannian submersion.
 - 3.) The ω -horizontal distribution $\ker \omega$ is g_ω -orthogonal to the fibers.
 Note that in particular, $g_\omega(X, Z) = \langle \omega(X), Z \rangle = \omega_Z(X)$ for all $X \in TM$ and $Z \in \mathfrak{h}$.
- (iv) If M is the trivial bundle $N \times H$ and λ is an \mathfrak{h} -valued 1-form on N then we write $g_\lambda := g_\omega$, where ω is the unique connection form extending $\lambda|_{TN \times \{0\}}$; that is, $\omega(X, Z) = \lambda(X) + Z$ for all $(X, Z) \in T_{(p,z)}(N \times H) \cong T_p N \times \mathfrak{h}$. In particular, g_λ has the properties 1.), 2.) from (iii) above, and the vector $(X, -\lambda(X))$ is horizontal for all $X \in TN$. For each $Z \in \mathfrak{h}$ we define the 1-form $\lambda_Z := \langle \lambda(\cdot), Z \rangle$ on N .

- (v) If $F : M \rightarrow M$ is a gauge transformation, that is, a bundle automorphism which induces the identity on N , then it is obvious from the definitions that F is an isometry from $(M, g_{F^*\omega})$ to (M, g_ω) for any connection form ω on M . If α and ω are two connection forms such that $\alpha = \omega + df$ for some $f \in C^\infty(M, \mathfrak{h})$, then the gauge transformation $F : p \mapsto p \cdot \exp(f(p))$ satisfies $\alpha = F^*\omega$; thus (M, g_α) and (M, g_ω) are isometric. Analogously, if two \mathfrak{h} -valued 1-forms λ, μ on N differ by df for some $f \in C^\infty(N, \mathfrak{h})$, then the associated metrics g_λ and g_μ on $N \times H$ are isometric.

Theorem 1.6. *Let (N, h) be a closed Riemannian manifold and H be a torus equipped with an invariant metric. Let M be a principal H -bundle over (N, h) , and let ω, ω' be two connection forms on M . Assume:*

- (*1) *For every $Z \in \mathfrak{h}$ there exists a bundle automorphism $F_Z : M \rightarrow M$ which induces an isometry on the base manifold (N, h) and satisfies $\omega'_Z = F_Z^*\omega_Z$.*

Then (M, g_ω) and $(M, g_{\omega'})$ are isospectral.

Proof. We show that (M, g_ω) and $(M, g_{\omega'})$ satisfy the hypotheses of Theorem 1.3. Denote by ∇^ω the Levi-Civita connection of g_ω . Let $X \in TM$ be arbitrary and extend it to an H -invariant vector field on M . Then $\omega(X) : M \rightarrow \mathfrak{h}$ is H -invariant; moreover, for each $Z \in \mathfrak{h}$ the flows of X and Z commute, whence $[X, Z] = 0$. Thus

$$g_\omega(\nabla_Z^\omega X, X) = Z(g_\omega(X, X)) - g_\omega(Z, \nabla_X^\omega X) = Z\langle \omega(X), X \rangle - g_\omega(Z, \nabla_X^\omega X) = 0 - 0 = 0.$$

Since $X \in TM$ was arbitrary, we conclude $\nabla_Z^\omega X = 0$ for each $Z \in \mathfrak{h}$. Therefore the H -orbits are totally geodesic in (M, g_ω) . The first condition of Theorem 1.3 is thus satisfied for g_ω , and similarly for $g_{\omega'}$.

It remains to check condition (ii) of Theorem 1.3. For $W = H$, there is nothing to show since $(M/H, g_\omega^H)$ and $(M/H, g_{\omega'}^H)$ both equal (N, h) by the choice of g_ω and $g_{\omega'}$. Let W be a closed subgroup of codimension 1 in H . Choose $Z \perp T_e W$ in $\mathfrak{h} \setminus \{0\}$, and let $F_Z : M \rightarrow M$ be a bundle automorphism as in (*1). We claim that F_Z induces an isometry from $(M/W, g_{\omega'}^W)$ to $(M/W, g_\omega^W)$, where g_ω^W and $g_{\omega'}^W$ are the submersion metrics induced by g_ω and $g_{\omega'}$. Since F_Z commutes with the H -action and induces an isometry on the base manifold (N, h) , we only need to check that for any ω' -horizontal vector X , the vector $F_{Z*}X$ is ω -horizontal up to an error tangent to the W -orbits; in other words, $F_{Z*}X$ is g_ω -orthogonal to Z . But $g_\omega(F_{Z*}X, Z) = \langle \omega(F_{Z*}X), Z \rangle = \omega_Z(F_{Z*}X) = \omega'_Z(X) = \langle \omega'(X), Z \rangle = 0$ since $\omega'_Z = F_Z^*\omega_Z$ and $\omega'(X) = 0$. \square

Remarks 1.7.

(i) For any closed subgroup W of codimension 1 in H , let \mathcal{C}_W denote the space of smooth functions on M which are invariant under W . Recall that $\pi_W^* : C^\infty(M/W) \rightarrow \mathcal{C}_W$ intertwines the Laplacians associated with g_ω and g_ω^W (or $g_{\omega'}$ and $g_{\omega'}^W$) because, as we have seen in the proof of Theorem 1.6, the fibers of π_W are totally geodesic for g_ω and $g_{\omega'}$. We also saw in the proof of Theorem 1.6 that F_Z induces an isometry from $(M/W, g_{\omega'}^W)$ to $(M/W, g_\omega^W)$, where $Z \in \mathfrak{h} \setminus \{0\}$ is orthogonal to $T_e W$, and F_Z is chosen as in (*1). The pullback of this isometry intertwines

the corresponding Laplacians on M/W . Combining these intertwining maps, we conclude that

$$\Delta_{g_{\omega'}}|_{C_W} = (F_Z^* \circ \Delta_{g_\omega} \circ F_Z^{*-1})|_{C_W}.$$

This last fact can of course also be derived directly from the assumptions on F_Z , without even introducing the quotient manifolds M/W . We do not present this alternative argument here because we will do so later in a more general situation, namely, in the proof of Theorem 4.3 in Chapter 4 (see also Remark 4.4(i)) which is a generalization of Theorem 1.6.

(ii) If in the context of Theorem 1.6 there exists a bundle automorphism $F : M \rightarrow M$ which satisfies (*1) for each $Z \in \mathfrak{h}$, then we have $\omega' = F^*\omega$; hence $F : (M, g_{\omega'}) \rightarrow (M, g_\omega)$ is an isometry. In order to obtain nontrivial pairs of isospectral manifolds from Theorem 1.6 it is thus crucial that (*1) be satisfied without there being a choice of the F_Z independent of Z . Such examples do exist; see Section 1.3 below and Chapters 2 and 3.

In the following proposition we specialize Theorem 1.6 to the case of products. We use Notation 1.5(iv).

Proposition 1.8. *Let (N, h) be a closed Riemannian manifold and H be a torus with Lie algebra $\mathfrak{h} := T_e H$, equipped with an invariant metric. Let λ, λ' be two \mathfrak{h} -valued 1-forms on N which satisfy:*

$$(*2) \quad \lambda'_Z \in \text{Isom}(N, h)^*(\lambda_Z) \text{ for each } Z \in \mathfrak{h}.$$

Then $(N \times H, g_\lambda)$ and $(N \times H, g_{\lambda'})$ are isospectral.

Proof. Let ω, ω' be the connection forms on the trivial H -bundle $N \times H$ which extend λ and λ' , respectively; thus $g_\lambda = g_\omega$ and $g_{\lambda'} = g_{\omega'}$. We check that ω and ω' satisfy condition (*1) of Theorem 1.6. Let $Z \in \mathfrak{h}$. By (*2), there exists $f_Z \in \text{Isom}(N, h)$ such that $\lambda'_Z = f_Z^*(\lambda_Z)$. Define $F_Z := (f_Z, \text{Id}) : N \times H \rightarrow N \times H$. Then F_Z is obviously a bundle isomorphism which induces the isometry f_Z on (N, h) and by the definition of ω, ω' satisfies $\omega'_Z = F_Z^*(\omega_Z)$. \square

Remark 1.9. In analogy with Remark 1.7(ii) we observe that if $\lambda' \in \text{Isom}(N, h)^*\lambda$, then $(N \times H, g_\lambda)$ and $(N \times H, g_{\lambda'})$ are isometric. But there do exist examples where this is not the case although (*2) is satisfied; see Example 1.11 in the following section, and various new examples in the Chapters 2 and 3.

1.3 Review of some previously known examples.

In this section we will explain some previously known examples of closed isospectral, locally non-isometric manifolds from the point of view of Theorem 1.6 and Proposition 1.8.

The first class of examples (Example 1.11) concerns manifolds diffeomorphic to $S^{m-1} \times T^r$ with $m \geq 5$ and $r \geq 2$. Continuous isospectral families of locally non-isometric metrics on such manifolds were constructed in [22]; independently, Z. Szabó [44] found pairs of isospectral, locally non-isometric metrics on $S^{4k-1} \times T^3$

with $k \geq 2$. These manifolds arise as the boundaries of certain isospectral, locally non-isometric manifolds with boundary which were constructed in [27] and [44], respectively.

The second class of examples (Example 1.14) concerns manifolds diffeomorphic to $S^{m-1} \times S$, where $m \geq 5$ and S is a compact Lie group of rank at least two. The author constructed in [42] continuous families of isospectral, locally non-isometric metrics on these manifolds, thereby providing, in particular, the first examples of simply connected isospectral manifolds. Otherwise, the assumption that S is simply connected which was made in [42] plays no role in the construction of the isospectral metrics and in the non-isometry arguments. Thus the first class of examples, mentioned above, can actually be viewed as a subclass of the second one.

As we will see, both classes of examples arise from Theorem 1.6; the first class arises even from the more special Proposition 1.8.

We recall that together with the pairs of two-step nilmanifolds constructed in [21] and [27] (see Remark 1.4(v)), and some recent examples by C. Gordon and Z. Szabó [23] (see Remark 4.4 in Chapter 4), these manifolds provide all previously known examples of isospectral, locally non-isometric, closed manifolds.

We start by a definition introduced by C. Gordon and E. Wilson.

Definition 1.10 [27]. Two linear maps $j, j' : \mathbb{R}^r \rightarrow \mathfrak{so}(m)$ are called *isospectral*, denoted $j \sim j'$, if for every $Z \in \mathbb{R}^r$ there exists $A_Z \in \mathrm{O}(m)$ such that $j'_Z = A_Z j_Z A_Z^{-1}$.

Example 1.11: Products of spheres with tori ([22], [44]).

Let $S^{m-1} \subset \mathbb{R}^m$ be the $(m-1)$ -dimensional unit sphere. Let H be a torus with a fixed invariant metric and Lie algebra $\mathfrak{h} \cong \mathbb{R}^r$.

For each linear map $j : \mathfrak{h} \rightarrow \mathfrak{so}(m)$ we define an \mathfrak{h} -valued 1-form λ on S^{m-1} by requiring that

$$(2) \quad \lambda_Z(X) = -\frac{1}{2} \langle j_Z p, X \rangle$$

for each $X \in T_p S^{m-1}$ and $Z \in \mathfrak{h}$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on \mathbb{R}^m . Here, $j_Z \in \mathfrak{so}(m)$ acts on $p \in S^{m-1} \subset \mathbb{R}^m$ by usual multiplication. (Why the factor $-1/2$ is convenient will become clear at the end of the following Remark 1.12.)

Now let $j, j' : \mathfrak{h} \rightarrow \mathfrak{so}(m)$ be such that $j \sim j'$. Let λ, λ' be the associated \mathfrak{h} -valued 1-forms on S^{m-1} . For each $Z \in \mathfrak{h}$ choose $A_Z \in \mathrm{O}(m)$ such that $j'_Z = A_Z j_Z A_Z^{-1}$. Then it follows immediately from (2) that $\lambda'_Z = A_Z^{-1*} \lambda_Z$. Let h be the round standard metric on $N := S^{m-1}$. Since A_Z^{-1} is an isometry of (N, h) , condition (*2) from Proposition 1.8 is satisfied for λ and λ' . We conclude that $(S^{m-1} \times H, g_\lambda)$ and $(S^{m-1} \times H, g_{\lambda'})$ are isospectral, where the metrics g_λ and $g_{\lambda'}$ are associated with λ and h , resp. with λ' and h , as in Notation 1.5(iv).

Remark 1.12.

We explain why the isospectral manifolds from Example 1.11 are exactly those constructed in [22]. In that paper the approach was as follows (up to minor changes of notation).

Let $\mathfrak{v} := \mathbb{R}^m$ and $\mathfrak{h} := \mathbb{R}^r$ be endowed with the euclidean standard metrics, and let $\mathcal{L} \subset \mathfrak{h}$ be a lattice of full rank. For each linear map $j : \mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{v})$ consider the two-step nilpotent Lie algebra $\mathfrak{g}_j := \mathfrak{v} \oplus \mathfrak{h}$ whose Lie bracket is defined by requiring that \mathfrak{h} be central, $[\mathfrak{g}_j, \mathfrak{g}_j] \subseteq \mathfrak{h}$, and $\langle [X, Y], Z \rangle = \langle j_Z X, Y \rangle$ for all $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{h}$. Let G_j be the associated simply connected Lie group, and let g_j be the left invariant metric on G_j which corresponds to the standard scalar product on $\mathfrak{g}_j = \mathfrak{v} \oplus \mathfrak{h}$. The group exponential map $\exp : \mathfrak{g}_j \rightarrow G_j$ is a diffeomorphism which restricts to a linear isomorphism between \mathfrak{h} and $\exp \mathfrak{h} \subset G_j$. Denote by \bar{G}_j the quotient of G_j by the discrete central subgroup $\exp \mathcal{L} \subset \exp \mathfrak{h}$, and denote by \bar{g}_j the left invariant metric on \bar{G}_j induced by g_j . The group exponential map $\overline{\exp} : \mathfrak{g}_j \rightarrow \bar{G}_j$ induces a diffeomorphism from $\mathfrak{v} \times (\mathfrak{h}/\mathcal{L})$ to \bar{G}_j . Define $M_j \subset \bar{G}_j$ as the image of $S^1(\mathfrak{v}) \times (\mathfrak{h}/\mathcal{L})$ under this diffeomorphism, and denote the induced metric on M_j by \bar{g}_j again.

In [22] it was proven that for $j \sim j'$, the Riemannian manifolds (M_j, \bar{g}_j) and $(M_{j'}, \bar{g}_{j'})$ (denoted there by $N(j)$ and $N(j')$) are isospectral. This is exactly the same as what we stated in Example 1.11 above: As we are going to see now, (M_j, \bar{g}_j) is isometric to our above $(S^{m-1} \times H, g_\lambda)$, where we let $H := \mathfrak{h}/\mathcal{L}$ and where λ is associated with j as in (2). More precisely, under the identification of S^{m-1} with $S^1(\mathfrak{v})$, we claim that $\overline{\exp} : \mathfrak{v} \oplus \mathfrak{h} \rightarrow \bar{G}_j$ induces an isometry from $(S^1(\mathfrak{v}) \times (\mathfrak{h}/\mathcal{L}), g_\lambda)$ to (M_j, \bar{g}_j) .

We first consider the metric $\exp^* g_j$ on $\mathfrak{v} \oplus \mathfrak{h}$. Extend λ to an \mathfrak{h} -valued 1-form on \mathfrak{v} by letting $\langle \lambda(X), Z \rangle = -\frac{1}{2} \langle j_Z V, X \rangle$ for each $V \in \mathfrak{v}$ and $X \in T_V \mathfrak{v}$. By the Campbell-Baker-Hausdorff formula and the definition of λ and $[\cdot, \cdot]$ (both associated with j) we have

$$\exp_{*|_{V+W}}(X+Z) = L_{\exp(V+W)*}(X+Z - \frac{1}{2}[V, X]) = L_{\exp(V+W)*}(X + \lambda(X) + Z)$$

for all $V, X \in \mathfrak{v}$ and $W, Z \in \mathfrak{h}$. Thus

$$(\exp^* g_j)|_{V+W}(X_1 + Z_1, X_2 + Z_2) = \langle X_1 + \lambda(X_1) + Z_1, X_2 + \lambda(X_2) + Z_2 \rangle$$

for all $V, X_1, X_2 \in \mathfrak{v}$ and $W, Z_1, Z_2 \in \mathfrak{h}$. This shows that $\exp^* g_j$ induces the given euclidean metric on the \mathfrak{h} -fibers, that $X - \lambda(X)$ is orthogonal to $\mathfrak{h} \subset T_{V+W}(\mathfrak{v} \oplus \mathfrak{h})$, and that $(\exp^* g_j)(X_1 - \lambda(X_1), X_2 - \lambda(X_2)) = \langle X_1, X_2 \rangle$ for all $X, X_1, X_2 \in \mathfrak{v} \subset T_{V+W}(\mathfrak{v} \oplus \mathfrak{h})$.

Note that $\overline{\exp}^* \bar{g}_j = \exp^* g_j$. Thus the above properties hold also for $\overline{\exp}^* \bar{g}_j$, and in particular for its restriction to $S^1(\mathfrak{v}) \times \mathfrak{h}$. By the definition of g_λ this implies that $\overline{\exp}$ induces indeed an isometry from $(S^1(\mathfrak{v}) \times (\mathfrak{h}/\mathcal{L}), g_\lambda)$ to $(\overline{\exp}(S^1(\mathfrak{v}) \times \mathfrak{h}), \bar{g}_j) = (M_j, \bar{g}_j)$.

Remarks 1.13.

(i) Given $H = \mathfrak{h}/\mathcal{L}$ as above, we say that two linear maps $j, j' : \mathfrak{h} \rightarrow \mathfrak{so}(m)$ are *trivially isospectral* if there exist $A \in O(m)$ and $C \in O(\mathfrak{h})$ such that $C(\mathcal{L}) = \mathcal{L}$ and $j'_{CZ} = A j_Z A^{-1}$ for all $Z \in \mathfrak{h}$. (Note that in contrast to the isospectrality condition in Definition 1.10, the map A is assumed to be independent of $Z \in \mathfrak{h}$.) If this is the case, then the corresponding metrics g_λ and $g_{\lambda'}$ on $S^{m-1} \times H$ are obviously isometric; an isometry from g_λ to $g_{\lambda'}$ is induced by (A, C) .

For $m \leq 4$ and $\dim \mathfrak{h} \leq 2$, and also for $m \leq 3$ and arbitrary dimension of \mathfrak{h} , elementary arguments show that isospectrality of two linear maps $j, j' : \mathfrak{h} \rightarrow \mathfrak{so}(m)$ always implies triviality in the above sense. Nontrivial isospectral manifolds of the type described in Example 1.11 can therefore occur only in the case $m + \dim \mathfrak{h} \geq 7$, that is, if $\dim(S^{m-1} \times H) \geq 6$. Dimension six is indeed attained here since for $m = 5$ and $\dim \mathfrak{h} = 2$ there do exist nontrivial families of isospectral metrics; see (ii) below and also Proposition 1.16.

(ii) It was proven in [22], using a result from [27], that for each $m \geq 5$ the above method provides continuous d -parameter families of isospectral metrics on $S^{m-1} \times T^2$, where d is of order at least $O(m^2)$. The proof that the manifolds in these multiparameter-families have pairwise different local geometries is rather abstract in the sense that it does not distinguish the metrics in terms of straightforwardly formulated geometrical quantities.

However, it was also shown in [22] that for some of the isospectral families the maximum of the scalar curvature varies during the deformation. (In particular, the manifolds are not locally isometric.) For a more detailed statement concerning the critical values of the scalar curvature see Proposition 1.16 below which refers to the following Example 1.14. Note that the above Example 1.11 can be considered as a special case of Example 1.14; thus all statements of Proposition 1.16 hold also for the isospectral manifolds of Example 1.11.

Example 1.14: Products of spheres with compact Lie groups ([42]).

Let S be a compact Lie group containing a torus H of dimension at least two, and let $\mathfrak{h} := T_e H \subseteq T_e S$. Let k be a bi-invariant metric on S , and let H be equipped with the metric induced by k . Let $S^{m-1} \subset \mathbb{R}^m$ be the $(m-1)$ -dimensional unit sphere. View $S^{m-1} \times S$ as a principal H -bundle with respect to the left action of H on the second factor.

For each linear map $j : \mathfrak{h} \rightarrow \mathfrak{so}(m)$ we define a connection form ω on the H -bundle $S^{m-1} \times S$ by requiring (using Notation 1.5(ii)) that

$$(3) \quad \omega_Z(X, U) = -\frac{1}{2} \langle j_Z p, X \rangle + k(U, R_{s*} Z)$$

for all $(X, U) \in T_{(p,s)}(S^{m-1} \times S)$ and $Z \in \mathfrak{h}$, where R_s denotes right multiplication by s and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on \mathbb{R}^m . In order to check that ω is indeed a connection form, note that the definition implies (using Notation 1.5(i)) that $\omega(Z_{(p,s)}) = \omega(0, R_{s*} Z) = Z$ for all $Z \in \mathfrak{h}$; furthermore, ω is H -(left) invariant because for all $(X, U) \in T_{(p,s)}(S^{m-1} \times S)$ and all $Z \in \mathfrak{h}$ and $z \in H$ we have

$$\begin{aligned} \omega_Z(X, L_{z*} U) &= -\frac{1}{2} \langle j_Z p, X \rangle + k(L_{z*} U, R_{z s*} Z) = -\frac{1}{2} \langle j_Z p, X \rangle + k(U, R_{s*} \text{Ad}_z^{-1} Z) \\ &= \omega_Z(X, U) \end{aligned}$$

by the bi-invariance of k and the commutativity of H .

Now let $j, j' : \mathfrak{h} \rightarrow \mathfrak{so}(m)$ be such that $j \sim j'$, and let ω, ω' be the associated connection forms on $S^{m-1} \times S$. For each $Z \in \mathfrak{h}$ choose $A_Z \in \text{O}(m)$ such that $j'_Z = A_Z j_Z A_Z^{-1}$. Define $F_Z := (A_Z^{-1}, \text{Id}) : S^{m-1} \times S \rightarrow S^{m-1} \times S$. Obviously F_Z is a bundle automorphism. Moreover, it follows immediately from (3) that $\omega'_Z = F_Z^* \omega_Z$. Finally, define the metric h on the base manifold $N := S^{m-1} \times (H \backslash S)$

as the product of the round standard metric on S^{m-1} and the submersion metric \bar{k} on $H \backslash S$ induced by k . Then F_Z induces the h -isometry $\bar{F}_Z := (A_Z^{-1}, \text{Id})$ on N . Thus ω and ω' satisfy condition (*1) of Theorem 1.6. We conclude that $(S^{m-1} \times S, g_\omega)$ and $(S^{m-1} \times S, g_{\omega'})$ are isospectral, where the metrics g_ω and $g_{\omega'}$ are associated with ω and h , resp. with ω' and h , as in Notation 1.5(iii).

Remark 1.15.

We explain why the isospectral manifolds from Example 1.14 are exactly those constructed in [42]. Actually, S was assumed to be simply connected there, but this did not play a role in any of the arguments, except for the fact that it caused the manifolds $S^{m-1} \times S$ to be simply connected.

Let S, H, \mathfrak{h} , and k be as in Example 1.14. Let $r := \dim \mathfrak{h}$, and let $\mathfrak{v} := \mathbb{R}^m$ be equipped with the euclidean standard metric. In [42] we associated with each linear map $j : \mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{v})$ a metric g_j on $\mathfrak{v} \times S$. Instead of reviewing its original definition here, we recall a characterization of g_j resulting from Lemma 1.7(i) in [42]. Namely, a g_j -orthogonal basis at $(V, s) \in \mathfrak{v} \times S$ is given by

$$(4) \quad \{(X_a, R_{s*}(\frac{1}{2}[V, X_a])) \mid a = 1, \dots, m\} \cup \{(0, R_{s*}U_i) \mid i = 1, \dots, \dim S\},$$

where $\{X_1, \dots, X_m\}$ is an orthonormal basis of \mathfrak{v} , $\{U_1, \dots, U_{\dim S}\}$ is a k -orthonormal basis of $T_e S$, and $[\cdot, \cdot]$ is the Lie bracket on $\mathfrak{g}_j := \mathfrak{v} \oplus \mathfrak{h}$ associated with j as in Remark 1.12; that is, \mathfrak{h} is central, $[\mathfrak{g}_j, \mathfrak{g}_j] \subseteq \mathfrak{h}$, and $\langle [X, Y], Z \rangle = \langle j_Z X, Y \rangle$ for all $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{h}$. Denote the restriction of g_j to $S^1(\mathfrak{v}) \times S \subset \mathfrak{v} \times S$ by g_j again.

In [42] we showed that for $j \sim j'$ the Riemannian manifolds $(S^1(\mathfrak{v}) \times S, g_j)$ and $(S^1(\mathfrak{v}) \times S, g_{j'})$ are isospectral. This is exactly the same as what was stated in Example 1.14 above: As we are going to see now, the metric g_j on $S^1(\mathfrak{v}) \times S = S^{m-1} \times S$ is equal to our above metric g_ω on $S^{m-1} \times S$, where ω is associated with j as in (3).

In fact, it follows from the description of g_j on $\mathfrak{v} \times S$ by the orthonormal bases given in (4) that

- 1.) The metric induced by g_j on the left H -orbits in $S^{m-1} \times S$ is the one inherited from the metric $k|_H$ on H .
- 2.) The projection from $(S^{m-1} \times S, g_j)$ to $(S^{m-1} \times (H \backslash S), h)$ is a Riemannian submersion, where h is as in Example 1.14.
- 3.) The vector $(X, U) \in T_{(p,s)}(S^{m-1} \times S)$ is g_j -orthogonal to the fiber $\{p\} \times Hs$ if and only if for each $Z \in \mathfrak{h}$ we have

$$\begin{aligned} 0 &= g_j((X, U), (0, R_{s*}Z)) \\ &= g_j((X, R_{s*}(\frac{1}{2}[p, X])), (0, R_{s*}Z)) - k(R_{s*}(\frac{1}{2}[p, X]), R_{s*}Z) + k(U, R_{s*}Z) \\ &= 0 - \frac{1}{2}\langle j_Z p, X \rangle + k(U, R_{s*}Z) = \omega_Z(X, U). \end{aligned}$$

From these properties and the definition of g_ω (recall Notation 1.5(iii)) it follows that $g_j = g_\omega$.

Concerning non-isometry criteria for the above manifolds $(S^{m-1} \times S, g_j)$, we showed in [42] the following result, parts of which we will need again in Chapter 3.

Proposition 1.16 ([42]).

In the context of Example 1.14 and Remark 1.15 suppose $m \geq 5$ and $\dim H = 2$, and let $\{Z_1, Z_2\}$ be an orthonormal basis of $\mathfrak{h} \subseteq T_e S$.

- (i) Let j, j' be two linear maps from \mathfrak{h} to $\mathfrak{so}(m)$ such that $j \sim j'$.
 - (a) If $j_{Z_1}^2 + j_{Z_2}^2$ and $j'_{Z_1}{}^2 + j'_{Z_2}{}^2$ have different sets of eigenvalues, then the scalar curvature of g_j and the scalar curvature of $g_{j'}$ on $S^{m-1} \times S$ have different sets of critical values ([42], Proposition 3.5).
 - (b) If $\|j_{Z_1}^2 + j_{Z_2}^2\|^2 \neq \|j'_{Z_1}{}^2 + j'_{Z_2}{}^2\|^2$ (where $\|\cdot\|$ denotes the standard euclidean norm on real $m \times m$ -matrices), then the heat invariants for the Laplacian on 1-forms are not equal for g_j and $g_{j'}$ ([42], Proposition 5.1).
- (ii) There exists a Zariski open subset \mathcal{U} of the space \mathcal{J} of all linear maps from \mathfrak{h} to $\mathfrak{so}(m)$ with the property that every $j \in \mathcal{U}$ is contained in a smooth isospectral family $j(t)$, defined in some open interval around zero, such that $j(0) = j$ and such that $\|j_{Z_1}(t)^2 + j_{Z_2}(t)^2\|^2$ has nonzero derivative at $t = 0$ ([42], Proposition 4.1).
- (iii) From (i) it follows that the isospectral manifolds $(S^{m-1} \times S, g_{j(t)})$ from (ii) are not isospectral on 1-forms and have different sets of critical values for the scalar curvature. In particular, the manifolds are not locally isometric.
- (iv) For $m = 5$, an explicit example of an isospectral family $j(t)$ with the property that $\|j_{Z_1}(t)^2 + j_{Z_2}(t)^2\|^2 \neq \text{const}$ is given by

$$j_{Z_1}(t) := \begin{pmatrix} 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & t-1 & 0 \\ t & 0 & 0 & 0 & -\varphi(t) \\ 0 & 1-t & 0 & 0 & -\psi(t) \\ 0 & 0 & \varphi(t) & \psi(t) & 0 \end{pmatrix}, \quad j_{Z_2}(t) := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\varphi(t) = ((t^4 - 3t^2 + 1)/(1 - 2t))^{1/2}$ and $\psi(t) = ((-t^4 + 4t^3 - 3t^2 - 2t + 1)/(1 - 2t))^{1/2}$ (see [42], Remark 4.3(ii)). This family is defined for $t \in [\frac{1}{2}(1 - \sqrt{5}), \frac{1}{2}(3 - \sqrt{5})]$. The $j(t)$ are pairwise isospectral since $\det(\lambda \text{Id} - (sj_{Z_1}(t) + uj_{Z_2}(t))) = \lambda^5 + (3s^2 + 2u^2)\lambda^3 + (s^2 + u^2)^2\lambda$ is independent of t ; however, $\|j_{Z_1}(t)^2 + j_{Z_2}(t)^2\|^2 = 4t^2 - 4t + 26$ is nonconstant in t .

2. Four-dimensional examples

Recall from Remark 1.13(i) that nontrivial isospectral pairs of linear maps $j : \mathbb{R}^r \rightarrow \mathfrak{so}(m)$ can occur only if $m + r \geq 7$; therefore, all nontrivial examples of isospectral manifolds arising from Example 1.11 are of dimension at least six. The manifolds from Example 1.14 have at least as many dimensions as those from Example 1.11. Finally, the locally non-isometric, two-step nilmanifolds from [21] and [27], too, are based on pairs of isospectral maps j as above; in particular, they are of dimension at least seven. Thus the smallest dimension for which examples of isospectral, locally non-isometric manifolds have been known is six.

In this chapter we will present pairs of isospectral, locally non-isometric metrics in dimension four; more precisely, on $S^2 \times T^2$. The idea is the following: Let h be the standard metric on S^2 . Try to construct nontrivial pairs of \mathbb{R}^2 -valued 1-forms

λ, λ' on S^2 such that $\lambda'_Z \in \text{Isom}(S^2, h)^* \lambda_Z = \text{O}(3)^* \lambda_Z$ for all $Z \in \mathbb{R}^2$. “Nontrivial” means that $\lambda' \notin \text{O}(3)^* \lambda$ (or more strongly, in view of Remark 2.1(i) below, $\lambda' \notin \text{O}(3)^*(\lambda + df)$ for all $f \in C^\infty(S^2, \mathbb{R}^2)$). Then apply Proposition 1.8 to obtain isospectral metrics on $S^2 \times T^2$.

Since we know that this would not be possible here by the approach of Example 1.11 / Remark 1.12, we must drop an unnecessary assumption which was made there. Remember that in Example 1.11 each λ_Z was of the form $X \mapsto -\frac{1}{2} \langle j_Z p, X \rangle$ for $X \in T_p M$; in particular, λ depended *linearly on the coordinates of the basepoint* $p \in S^{m-1} \subset \mathbb{R}^m$ of X . This plays no role for the applicability of Proposition 1.8. The reason why this linearity was present throughout the earlier examples (1.11 and 1.14) lies in the way they were originally discovered, namely, as companions to certain two-step nilmanifolds (recall Remark 1.12; for a different point of view which naturally includes the mentioned linearity see Remark 3.4).

The key of our construction of examples in dimension four is to replace the linear dependence of λ on the basepoint by quadratic dependence. This is what we do in Section 2.1, obtaining explicit nontrivial examples. In Section 2.2 we derive a formula for the associated scalar curvature and show for a specific isospectral pair $g_\lambda, g_{\lambda'}$ given in Section 2.1 that the preimages in $S^2 \times T^2$ of the maximal scalar curvature of g_λ and $g_{\lambda'}$ have different dimensions. In particular, $(S^2 \times T^2, g_\lambda)$ and $(S^2 \times T^2, g_{\lambda'})$ are not locally isometric. Finally, we present a more general result concerning the scalar curvature: Let $(N \times H, g_\lambda), (N \times H, g_{\lambda'})$ be any pair of isospectral manifolds arising from Proposition 1.8, and assume that the base manifold N is two-dimensional. In this situation we prove that the associated scalar curvature functions have the same range, and for each $k \in \mathbb{N}$ the integrals of their k -th powers coincide (Theorem 2.11). If in addition the torus H is also two-dimensional, then the same holds for the L^2 -norms of the Ricci and curvature tensors. This contrasts with the higher-dimensional examples $(S^{m-1} \times T^2, g_\lambda)$ from Example 1.11, where the isospectral manifolds did in general neither share the range of the scalar curvature nor $\int \text{scal}^2$ or $\int \|\text{Ric}\|^2$ (compare Remark 1.13(ii), Proposition 1.16, and [42], Lemma 5.4).

2.1 Isospectral, locally non-isometric metrics on $S^2 \times T^2$.

Before starting the search for suitable pairs of \mathbb{R}^2 -valued 1-forms λ, λ' on S^2 in the “quadratic” category (i.e., with λ, λ' depending quadratically on the basepoint), we make an observation which implies that it would be useless to instead replace the skew-symmetric linear maps j_Z of Example 1.11 by more general linear maps or by constant ones; moreover, improving the dimension of the manifold to three instead of four is not possible by our methods.

Remarks 2.1.

(i) Let H be a torus with an invariant metric and Lie algebra $\mathfrak{h} := T_e H$, and let $\{Z_1, \dots, Z_r\}$ be an orthonormal basis of \mathfrak{h} . Let h be the standard metric on S^{m-1} , and let μ, ν be two \mathfrak{h} -valued 1-forms on S^{m-1} . For $\lambda := \mu - \nu$ suppose that each $\lambda_Z := \langle \lambda(\cdot), Z \rangle$ is of the type $X \mapsto \langle w_Z(p), X \rangle$ for $X \in T_p S^{m-1}$, where w_Z is a vector field on \mathbb{R}^m which has a potential function φ_Z . For example, this is the case if $w_Z(p)$ is constant, or if w_Z is linear and symmetric. Then $\lambda = df$

with $f(p) := \sum_{i=1}^r \varphi_{Z_i}(p)Z_i \in \mathfrak{h}$. Thus $\mu = \nu + df$, and by 1.5(v) the associated metrics g_μ and g_ν on $S^{m-1} \times H$, defined as in Notation 1.5(iv), are isometric.

(ii) If λ, λ' are two \mathfrak{h} -valued 1-forms on S^1 then each of them is the sum of an S^1 -invariant form and an exact one. By 1.5(v) we can thus assume, without changing the isometry classes of g_λ and $g_{\lambda'}$ on $S^1 \times H$, that λ and λ' are S^1 -invariant. But if two such forms satisfy condition (*2) of Proposition 1.8, then it is easy to see that $\lambda = \pm\lambda'$ and thus the associated metrics on $S^1 \times H$ are isometric. This shows that in order to get nontrivial pairs of isospectral metrics on $S^{m-1} \times H$ by using Proposition 1.8 we cannot lower the dimension of S^{m-1} to less than two. Since the dimension of H must also be at least two (recall Remark 1.4(i)), examples in dimension four (which we obtain in this chapter) is the best we can hope for.

(iii) More generally, if in the context of Theorem 1.3 the base manifold M/H is one-dimensional, then any H -invariant metric g for which the H -orbits are totally geodesic must be flat. Thus any isospectral pair of metrics g, g' arising from Theorem 1.3 in this situation would be locally isometric anyway.

Notation and Remarks 2.2.

- (i) Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product on \mathbb{R}^3 , and let $\text{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$ be endowed with the canonically induced scalar product; that is, $\langle (X_1 Y_1)^* \otimes V_1, (X_2 Y_2)^* \otimes V_2 \rangle = \frac{1}{2}(\langle X_1, X_2 \rangle \langle Y_1, Y_2 \rangle + \langle X_1, Y_2 \rangle \langle X_2, Y_1 \rangle) \langle V_1, V_2 \rangle$. The group $O(3)$ acts orthogonally on $\text{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$ in the canonical way; that is, $A \in O(3)$ maps $(XY)^* V$ to $(AX \cdot AY)^* AV$.
- (ii) Denote by $P : \text{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3 \rightarrow \text{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$ the linear map defined by $P : (XY)^* \otimes V \mapsto \frac{1}{3}((XY)^* \otimes V + (XV)^* \otimes Y + (YV)^* \otimes X)$. Obviously P is an orthogonal projection onto its image $\text{Im } P$ which is isomorphic, as a representation space of $O(3)$, to the ten dimensional space $\text{Sym}^3 \mathbb{R}^3$. The projection P^\perp onto the orthogonal complement $\ker P$ of $\text{Im } P$ is given by $P^\perp : (XY)^* \otimes V \mapsto \frac{1}{3}(2(XY)^* \otimes V - (XV)^* \otimes Y - (YV)^* \otimes X)$. Moreover, P is $O(3)$ -equivariant; thus $\text{Im } P$ and $\ker P$ are invariant under the $O(3)$ -action.
- (iii) Denote by $\text{End}_0(\mathbb{R}^3)$ the space of traceless endomorphisms of \mathbb{R}^3 . Define a linear map $\Phi : \text{End}_0(\mathbb{R}^3) \rightarrow \text{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$ by letting

$$\Phi(b) : X \cdot X \mapsto bX \times X$$

for all $X \in \mathbb{R}^3$, where \times denotes the vector product in \mathbb{R}^3 and we interpret $\Phi(b) \in \text{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$ as a linear map from $\text{Sym}^2(\mathbb{R}^3)$ to \mathbb{R}^3 (obtained from the above formula by polarization and linear extension). Obviously Φ maps $\text{End}_0(\mathbb{R}^3)$ to $\ker P$ since $3\langle P(\Phi(b))(X \cdot X), V \rangle = \langle bX \times X, V \rangle + \langle bX \times V, X \rangle + \langle bV \times X, X \rangle = 0$ for all $X, V \in \mathbb{R}^3$. Moreover, Φ is $SO(3)$ -equivariant, where $SO(3)$ acts on $\text{End}_0(\mathbb{R}^3)$ by conjugation.

Lemma 2.3. $\Phi : \text{End}_0(\mathbb{R}^3) \rightarrow \ker P = \text{Im } P^\perp$ is an $SO(3)$ -invariant isomorphism. In particular, $\text{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3 = \text{Im } P \oplus \ker P$ is isomorphic, as a representation space of $SO(3)$, to the sum of $\text{Sym}^3(\mathbb{R}^3)$ and $\text{End}_0(\mathbb{R}^3)$.

Proof. If $\Phi(b) = 0$ for some $b \in \text{End}_0(\mathbb{R}^3)$ then $bX \times X = 0$ and thus $bX \parallel X$ for all $X \in \mathbb{R}^3$. Hence b is a multiple of the identity; from $\text{tr}(b) = 0$ it follows that $b = 0$. Therefore Φ is injective. Since the dimension of $\ker P$ equals

$\dim(\text{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3) - \dim(\text{Im } P) = 18 - 10 = 8 = \dim(\text{End}_0(\mathbb{R}^3))$, it follows that Φ is a vector space isomorphism. \square

Proposition 2.4. *Denote by $S_0(\mathbb{R}^3) \subset \text{End}_0(\mathbb{R}^3)$ the space of symmetric traceless endomorphisms of \mathbb{R}^3 .*

- (i) *There exist pairs of linear maps $c, c' : \mathbb{R}^2 \rightarrow S_0(\mathbb{R}^3)$ such that the following conditions are satisfied:*
 - 1.) *For each $Z \in \mathbb{R}^2$ the elements c_Z and c'_Z of $S_0(\mathbb{R}^3)$ are conjugate by an element of $\text{SO}(3)$.*
 - 2.) *There is no $A \in \text{O}(3)$ such that either $c' = I_A \circ c$ or $-c' = I_A \circ c$, where I_A denotes conjugation by A .*
- (ii) *Let c, c' be as in (i), and let $q := \Phi \circ c, q' := \Phi \circ c' : \mathbb{R}^2 \rightarrow \ker P \subset \text{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$. Then q, q' have the following properties:*
 - 1.) *For each $Z \in \mathbb{R}^2$ the elements q_Z and q'_Z of $\text{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$ belong to the same $\text{SO}(3)$ -orbit.*
 - 2.) *There is no $A \in \text{O}(3)$ such that $q' = A \circ q$.*
- (iii) *An example of a pair $c, c' : \mathbb{R}^2 \rightarrow S_0(\mathbb{R}^3)$ satisfying the conditions in (i) is given by*

$$c_{Z_1} = c'_{Z_1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c_{Z_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad c'_{Z_2} = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix},$$

where $\{Z_1, Z_2\}$ is the standard basis of \mathbb{R}^2 , and we identify elements of $S_0(\mathbb{R}^3)$ with their matrix representation with respect to the standard basis of \mathbb{R}^3 .

Proof. $\Phi : \text{End}_0(\mathbb{R}^3) \rightarrow \ker P$ is an isomorphism of $\text{SO}(3)$ -representation spaces by Lemma 2.3. The element $-\text{Id}$ of $\text{O}(3)$ acts as multiplication by -1 on $\ker P$ and as multiplication by 1 on $\text{End}_0(\mathbb{R}^3)$. Therefore (i) implies (ii). We now show (i).

Equivalence of two elements of $S_0(\mathbb{R}^3)$ modulo conjugation by orthogonal endomorphisms can easily be checked, namely, by deciding whether the characteristic polynomials are equal. In the following we identify elements of $S_0(\mathbb{R}^3)$ with their matrix representation with respect to the standard basis.

Let $a := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and consider all matrices b of the form $\begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{12} & 0 & b_{23} \\ b_{13} & b_{23} & 0 \end{pmatrix}$. Define $c^b : \mathbb{R}^2 \rightarrow S_0(\mathbb{R}^3)$ by $c^b_{Z_1} := a$ and $c^b_{Z_2} := b$, where $\{Z_1, Z_2\}$ is the standard basis of \mathbb{R}^2 . We claim that there are pairs b, b' such that c^b and $c^{b'}$ satisfy the conditions 1.) and 2.) above.

Condition 1.) is equivalent to the characteristic polynomials of $sa + tb$ and $sa + tb'$ being equal for all $(s, t) \in \mathbb{R}^2$. We have

$$\det(\lambda \text{Id} - (sa + tb)) = \lambda^3 - (t^2(b_{12}^2 + b_{13}^2 + b_{23}^2) + s^2)\lambda - st^2(b_{23}^2 - b_{12}^2) - 2t^3b_{12}b_{13}b_{23}.$$

Thus condition 1.) for c^b and $c^{b'}$ is equivalent to the following three equations being satisfied:

$$(5) \quad \begin{aligned} b_{12}^2 + b_{13}^2 + b_{23}^2 &= b_{12}'^2 + b_{13}'^2 + b_{23}'^2, \\ b_{23}^2 - b_{12}^2 &= b_{23}'^2 - b_{12}'^2, \\ b_{12}b_{13}b_{23} &= b_{12}'b_{13}'b_{23}'. \end{aligned}$$

Concerning condition 2.), note that if there does exist $A \in \mathrm{O}(3)$ such that either $c^{b'} = I_A \circ c^b$ or $-c^{b'} = I_A \circ c^b$ then $\pm a = AaA^{-1}$ and $\pm b' = AbA^{-1}$. In particular, we then have $\mathrm{tr}(a^2b^2) = \mathrm{tr}(a^2b'^2)$; that is,

$$b_{12}^2 + 2b_{13}^2 + b_{23}^2 = b'_{12}{}^2 + 2b'_{13}{}^2 + b'_{23}{}^2.$$

Hence for all pairs b, b' which satisfy (5), but not the latter equation, the corresponding pairs $c^b, c^{b'}$ satisfies 1.) and 2.) of (i). It is easy to see that many such pairs exist. One example is given by

$$b := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad b' := \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

The corresponding pair $c^b, c^{b'}$ is just the specific example given in (iii). \square

Notation and Remarks 2.5.

(i) Let \mathbb{R}^2 and \mathbb{R}^3 be equipped with the standard scalar products. With each linear map $q : \mathbb{R}^2 \rightarrow \mathrm{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$ we associate an \mathbb{R}^2 -valued 1-form λ on S^2 by letting $\lambda_Z(X) = \langle q_Z(p \cdot p), X \rangle$ for $X \in T_p S^2$. Here, λ_Z denotes $\langle \lambda(\cdot), Z \rangle$ as usual; moreover, we have interpreted $q_Z \in \mathrm{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$ as a linear map from $\mathrm{Sym}^2 \mathbb{R}^3$ to \mathbb{R}^3 . In other words, each $q_Z \in \mathrm{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$ defines a quadratic vector field $v_Z : p \mapsto q_Z(p \cdot p)$ on \mathbb{R}^3 , and λ_Z is the pullback (by the inclusion $S^2 \hookrightarrow \mathbb{R}^3$) of the 1-form $\langle q_Z(p \cdot p), \cdot \rangle$ which is dual to v_Z .

(ii) Let q and \hat{q} be two linear maps from \mathbb{R}^2 to $\mathrm{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$. For each $Z \in \mathbb{R}^2$ consider the vector field $v_Z : p \mapsto (\hat{q}_Z - q_Z)(p \cdot p)$ on \mathbb{R}^3 . If the image of $\hat{q} - q$ is contained in $\mathrm{Im} P$ then $\langle dv_Z|_p X, Y \rangle = 2\langle (\hat{q}_Z - q_Z)(p \cdot X), Y \rangle$ is symmetric in $X, Y \in \mathbb{R}^3$ for each $p \in \mathbb{R}^3$. This means that v_Z satisfies the integrability conditions and thus has a potential function on \mathbb{R}^3 . Similarly, if the image of $\hat{q} - q$ is contained in $\Phi(\mathrm{Skew}(\mathbb{R}^3))$, where $\mathrm{Skew}(\mathbb{R}^3) \subset \mathrm{End}_0(\mathbb{R}^3)$ denotes the skew-symmetric endomorphisms of \mathbb{R}^3 , then for each $Z \in \mathbb{R}^2$ there exists $a_Z \in \mathbb{R}^3$ such that v_Z is of the form $p \mapsto (p \times a_Z) \times p$, and thus the component of v_Z tangent to S^2 is the same as that of the constant vector field a_Z . In either of the two cases, it follows from Remark 2.1(i) that the \mathbb{R}^2 -valued 1-forms $\lambda, \hat{\lambda}$ which are associated with q, \hat{q} as in (i) differ by df for some $f \in C^\infty(S^2, \mathbb{R}^2)$.

Thus if λ is associated with any linear map $q : \mathbb{R}^2 \rightarrow \mathrm{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$, then there exists $f \in C^\infty(S^2, \mathbb{R}^2)$ such that $\hat{\lambda} := \lambda + df$ is associated with a linear map $\hat{q} : \mathbb{R}^2 \rightarrow \Phi(S_0(\mathbb{R}^3))$. Recall from 1.5(v) (or Remark 2.1(i)) that the associated metrics g_λ and $g_{\hat{\lambda}}$ on $S^2 \times T^2$ are isometric. This is the reason why we are interested only in pairs q, q' with image in $\Phi(S_0(\mathbb{R}^3))$, as in Proposition 2.4.

Example 2.6.

(i) Let h be the standard round metric on S^2 , and let $T^2 := \mathbb{R}^2/\mathcal{L}$, where \mathcal{L} is any uniform lattice in \mathbb{R}^2 . Let T^2 be equipped with the metric induced from the euclidean metric on \mathbb{R}^2 . For any pair of linear maps $q, q' : \mathbb{R}^2 \rightarrow \mathrm{Sym}^2(\mathbb{R}^3)^* \otimes \mathbb{R}^3$ let λ, λ' be the corresponding \mathbb{R}^2 -valued 1-forms on S^2 as in 2.5(i), and let g_λ and $g_{\lambda'}$ be the associated metrics on $S^2 \times T^2$.

If q and q' satisfy condition 1.) of Proposition 2.4(ii), then $(S^2 \times T^2, g_\lambda)$ and $(S^2 \times T^2, g_{\lambda'})$ are isospectral by Proposition 1.8. In fact, the condition implies that for each $Z \in \mathbb{R}^2$ there exists $A_Z \in O(3)$ (even $A_Z \in SO(3)$) such that

$$\lambda'_Z|_p = \langle q'_Z(p \cdot p), \cdot \rangle = \langle A_Z q_Z (A_Z^{-1} p \cdot A_Z^{-1} p), \cdot \rangle = \lambda_Z|_{A_Z^{-1} p} \circ A_Z^{-1} = (A_Z^{-1})^* \lambda_Z|_p$$

for all $p \in S^2$. Hence $\lambda'_Z = (A_Z^{-1})^* \lambda_Z \in \text{Isom}(S^2, h)^* \lambda_Z$. Thus condition (*2) of Proposition 1.8 is satisfied, and consequently the two manifolds are isospectral.

(ii) In the context of (i), consider now the pair $q := \Phi \circ c$ and $q' := \Phi \circ c'$, where c, c' is the explicit pair of linear maps given in Proposition 2.4(iii). By (i), the associated Riemannian manifolds $(S^2 \times T^2, g_\lambda)$ and $(S^2 \times T^2, g_{\lambda'})$ are isospectral. In the next section we will show that they are not locally isometric. More precisely, the preimage of the maximal scalar curvature in the first manifold has greater dimension than the preimage of the maximal scalar curvature in the second manifold (Proposition 2.10).

Remark 2.7. For amusement of the reader, we illustrate the specific pair of metrics $g_\lambda, g_{\lambda'}$ on $S^2 \times T^2$ from Example 2.6(ii) by explicitly writing down the horizontal distributions. Let $\{Z_1, Z_2\}$ be the standard basis of \mathbb{R}^2 , and denote the corresponding vector fields on $S^2 \times T^2$ by Z_1, Z_2 again. Then for each $(p, z) \in S^2 \times T^2$, the g_λ -orthogonal complement in $T_{(p,z)}(S^2 \times T^2)$ to the T^2 -fiber through (p, z) is given by the vectors

$$X - \left\langle \begin{pmatrix} -p_2 p_3 \\ 2p_1 p_3 \\ -p_1 p_2 \end{pmatrix}, X \right\rangle Z_1 - \left\langle \begin{pmatrix} p_1 p_3 - p_2^2 + p_3^2 \\ p_1 p_2 - p_2 p_3 \\ -p_1 p_3 - p_1^2 + p_2^2 \end{pmatrix}, X \right\rangle Z_2,$$

where $X \in T_p S^2$ and $\langle \cdot, \cdot \rangle$ is the standard scalar product in \mathbb{R}^3 . Similarly, the $g_{\lambda'}$ -horizontal distribution is given by the vectors

$$X - \left\langle \begin{pmatrix} -p_2 p_3 \\ 2p_1 p_3 \\ -p_1 p_2 \end{pmatrix}, X \right\rangle Z_1 - \left\langle \begin{pmatrix} -\sqrt{2} p_1 p_2 \\ \sqrt{2} (p_1^2 - p_3^2) \\ \sqrt{2} p_2 p_3 \end{pmatrix}, X \right\rangle Z_2$$

with $X \in T_p S^2, p \in S^2$. This, together with the fact that for both metrics the T^2 -fibers are isometrically embedded and that the projection to the round sphere (S^2, h) is a Riemannian submersion, describes g_λ and $g_{\lambda'}$ completely.

2.2 Curvature properties.

In this section we first compute the scalar curvature of the manifolds $(N \times H, g_\lambda)$ from Notation 1.5(iv) in terms of λ and the scalar curvature of (N, h) (Proposition 2.8). Applying this formula to the specific kind of metrics from the previous section we conclude, in particular, that the manifolds from Example 2.6(ii) are not locally isometric because the preimages of their maximal scalar curvatures have different dimensions (Proposition 2.10). Finally, we make some general observations about the curvature properties of pairs of isospectral manifolds $(N \times H, g_\lambda)$,

$(N \times H, g_{\lambda'})$ arising from Proposition 1.8 in the special case that N is of dimension two (Theorem 2.11 and Remarks 2.12).

In the following we always assume that H is a torus, $\mathfrak{h} = T_e H$, and H is equipped with an invariant metric whose restriction to \mathfrak{h} we denote by $\langle \cdot, \cdot \rangle$. We fix an orthonormal basis $\{Z_1, \dots, Z_r\}$ of \mathfrak{h} .

Proposition 2.8. *Let (N, h) be a closed Riemannian manifold and λ be an \mathfrak{h} -valued 1-form on N . Let g_λ be the associated metric on $N \times H$ as in Notation 1.5(iv). Then we have*

$$\text{scal}_{(p,z)}^{g_\lambda} = \text{scal}_p^h - \frac{1}{4} \|d\lambda|_p\|_h^2$$

for all $(p, z) \in N \times H$, where scal^{g_λ} and scal^h denote the scalar curvature functions associated with g_λ and h , respectively, and $\|\cdot\|_p^2$ denotes the euclidean norm on tensors associated with the scalar product $h|_{T_p N}$.

Note that our choice of the norm $\|\cdot\|_h$ on tensors means, for example, that the canonical volume form v associated to the standard metric h on S^{m-1} satisfies $\|v|_p\|_h^2 = (m-1)!$ for all $p \in S^{m-1}$.

Proof. Let $Z \in \mathfrak{h}$ and X, Y be vector fields on N . We denote by the same names also the corresponding H -invariant vector fields on $N \times H$. By \tilde{X}, \tilde{Y} we denote the associated horizontal vector fields, that is, $\tilde{X}_{(p,z)} = (X_p, -\lambda_p(X)) \in T_{(p,z)}(N \times H)$. Note that $\lambda(X), \lambda(Y)$ commute since they are H -invariant and tangent to the fibers. Thus

$$\begin{aligned} g_\lambda([\tilde{X}, \tilde{Y}], Z) &= -g_\lambda([X, \lambda(Y)], Z) + g_\lambda([Y, \lambda(X)], Z) + g_\lambda([X, Y], Z) \\ &= -X(\lambda_Z(Y)) + Y(\lambda_Z(X)) + \lambda_Z([X, Y]) = -d\lambda_Z(X, Y). \end{aligned}$$

Moreover, $[\tilde{X}, Z] = 0$ since \tilde{X} is H -invariant. By the Koszul formula $\nabla_Z Z = 0$, $\nabla_Z \tilde{X}$ is horizontal, and $g_\lambda(\nabla_Z \tilde{X}, \tilde{Y}) = -\frac{1}{2} g_\lambda([\tilde{X}, \tilde{Y}], Z) = \frac{1}{2} d\lambda_Z(X, Y)$. Since the projection to (N, h) is a Riemannian submersion, $\nabla_{\tilde{X}} \tilde{X}$ is horizontal and thus $0 = Z(g_\lambda(\nabla_{\tilde{X}} \tilde{X}, Z)) = g_\lambda(\nabla_Z \nabla_{\tilde{X}} \tilde{X}, Z)$. Hence $g_\lambda(R(Z, \tilde{X})\tilde{X}, Z) = -g_\lambda(\nabla_{\tilde{X}} \nabla_Z \tilde{X}, Z) = \|\nabla_Z \tilde{X}\|_{g_\lambda}^2 = \frac{1}{4} \|d\lambda_Z(X, \cdot)\|_h^2$. By the flatness of the fibers this implies

$$\text{Ric}^{g_\lambda}(Z, Z) = \frac{1}{4} \|d\lambda_Z\|_h^2.$$

Moreover, from O'Neill's formula and the above formula for the vertical part of $[\tilde{X}, \tilde{Y}]$ it follows that

$$\text{Ric}^{g_\lambda}(\tilde{X}, \tilde{X}) = (\text{Ric}^h(X, X) - \frac{3}{4} \sum_i \|d\lambda_{Z_i}(X, \cdot)\|_h^2) + \frac{1}{4} \sum_i \|d\lambda_{Z_i}(X, \cdot)\|_h^2.$$

Consequently $\text{scal}^{g_\lambda} = \text{scal}^h - \frac{1}{2} \sum_i \|d\lambda_{Z_i}\|_h^2 + \frac{1}{4} \sum_i \|d\lambda_{Z_i}\|_h^2$, which gives the desired formula. \square

Next, we focus on the case $N = S^2$, $H = T^2$, and λ depending quadratically on the basepoint as in Section 2.1. Recall from 2.2(iii) and 2.5(ii) that the relevant 1-forms λ in this category are of the type $\lambda_Z|_p = \langle c_Z p \times p, \cdot \rangle$, where c is a linear map from \mathbb{R}^2 to the space $S_0(\mathbb{R}^3)$ of symmetric traceless endomorphisms of \mathbb{R}^3 .

Lemma 2.9. *Let $c : \mathbb{R}^2 \rightarrow S_0(\mathbb{R}^3)$ be a linear map and λ be the associated \mathbb{R}^2 -valued 1-form on S^2 as given by 2.2(iii) and 2.5(i); that is, $\lambda_Z|_p(X) = \langle c_Z p \times p, X \rangle$ for all $p \in S^2$, $X \in T_p S^2$, and $Z \in \mathbb{R}^2$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on \mathbb{R}^3 . Then for the associated metric g_λ on $S^2 \times T^2$ we have*

$$\text{scal}_{(p,z)}^{g_\lambda} = 2 - \frac{9}{2} \langle c_{Z_1} p, p \rangle^2 - \frac{9}{2} \langle c_{Z_2} p, p \rangle^2,$$

where $\{Z_1, Z_2\}$ is the standard basis of \mathbb{R}^2 .

Proof. Let $p \in S^2$ and $\{X, Y\}$ be an orthonormal basis of $T_p S^2$ with respect to the standard round metric h on S^2 . Then $\|d\lambda|_p\|^2 = 2\|d\lambda|_p(X, Y)\|^2$. By Proposition 2.8 and the fact that the scalar curvature of (S^2, h) equals 2, the lemma will follow if we show that $\|d\lambda_Z|_p(X, Y)\|^2 = 9\langle c_Z p, p \rangle^2$ for all $Z \in \mathbb{R}^2$. We interpret λ_Z as the restriction of a 1-form on \mathbb{R}^3 (defined by the same formula as λ_Z) and extend X, Y to constant vector fields on \mathbb{R}^3 . Then

$$\begin{aligned} d\lambda_Z|_p(X, Y) &= \langle c_Z X \times p + c_Z p \times X, Y \rangle - \langle c_Z Y \times p + c_Z p \times Y, X \rangle \\ &= \langle c_Z X \times p, Y \rangle + 2\langle c_Z p \times X, Y \rangle - \langle c_Z Y \times p, X \rangle \\ &= \langle c_Z X \times p, Y \rangle + 2\langle c_Z p \times X, Y \rangle + \langle Y \times c_Z p, X \rangle + \langle Y \times p, c_Z X \rangle \\ &= 3\langle c_Z p \times X, Y \rangle. \end{aligned}$$

Note that in the third equation we have used the fact that $\text{tr}(c_Z) = 0$ implies $\det(c_Z Y, p, X) + \det(Y, c_Z p, X) + \det(Y, p, c_Z X) = 0$. Thus $\|d\lambda_Z|_p(X, Y)\|^2 = 9 \det(c_Z p, X, Y)^2 = 9\langle c_Z p, p \rangle^2$, as claimed. \square

Proposition 2.10. *Let $c, c' : \mathbb{R}^2 \rightarrow S_0(\mathbb{R}^3)$ be the specific pair of linear maps given in Proposition 2.4(iii), and $g_\lambda, g_{\lambda'}$ be the associated metrics on $S^2 \times T^2$ as in Example 2.6(ii). Then the maximal scalar curvature equals 2 for both metrics. Its preimage under scal^{g_λ} is*

$$(\{p \in S^2 \mid p_1 = -p_3\} \cup \{(\pm 1/\sqrt{2}, 0, \pm 1/\sqrt{2})\}) \times T^2$$

and thus contains a submanifold of codimension one, whereas its preimage under $\text{scal}^{g_{\lambda'}}$ is

$$\{(0, \pm 1, 0)\} \times T^2$$

and thus is of codimension two.

Proof. By Lemma 2.9 we have that scal^{g_λ} , respectively $\text{scal}^{g_{\lambda'}}$, attains its maximum precisely in those points where the function

$$\begin{aligned} \langle c_{Z_1} p, p \rangle^2 + \langle c_{Z_2} p, p \rangle^2 &= (p_1^2 - p_3^2)^2 + (2p_2(p_1 + p_3))^2, \text{ respectively} \\ \langle c'_{Z_1} p, p \rangle^2 + \langle c'_{Z_2} p, p \rangle^2 &= (p_1^2 - p_3^2)^2 + (2\sqrt{2}p_1 p_3)^2, \end{aligned}$$

attains its minimum. Both minima are obviously zero and are attained precisely in the sets given in the statement. \square

We conclude this chapter by some general observations about the behaviour of the scalar curvature functions associated with two isospectral manifolds $(N \times H, g_\lambda)$, $(N \times H, g_{\lambda'})$ arising from Proposition 1.8 in the case that N is of dimension two.

Theorem 2.11. *Let (N, h) be a closed Riemannian manifold and H be a torus with Lie algebra $\mathfrak{h} = T_e H$, equipped with an invariant metric. Let λ, λ' be two \mathfrak{h} -valued 1-forms on N which satisfy condition (*2) of Proposition 1.8, and let $g_\lambda, g_{\lambda'}$ be the associated isospectral metrics on $M = N \times H$. If N is two-dimensional, then the following holds:*

- (i) $\int_M (\text{scal}^{g_\lambda})^k d\text{vol}_{g_\lambda} = \int_M (\text{scal}^{g_{\lambda'}})^k d\text{vol}_{g_{\lambda'}}$ for all $k \in \mathbb{N}$.
- (ii) *The functions scal^{g_λ} and $\text{scal}^{g_{\lambda'}}$ have the same range.*

Proof.

(i) We first consider the case that N is orientable. We choose an orientation and let v be the associated volume form on (N, h) , that is, $v|_p(X, Y) = 1$ for a positively oriented h -orthonormal basis $\{X, Y\}$ of $T_p N$. Then there exist linear (!) maps $\varphi, \psi : \mathfrak{h} \rightarrow C^\infty(N)$ such that $d\lambda_Z = \varphi_Z v$, $d\lambda'_Z = \psi_Z v$ for all $Z \in \mathfrak{h}$. Since $A^*v = \pm v$ for $A \in \text{Isom}(N, h)$, and $\lambda'_Z \in \text{Isom}(N, h)^* \lambda_Z$ by the condition of Proposition 1.8, it follows that $\psi_Z^2 \in \text{Isom}(N, h)^* \varphi_Z^2$ for all $Z \in \mathfrak{h}$. Thereby, for all $s \in \mathbb{R}^r$, $m \in \mathbb{N}$ and any $\text{Isom}(N, h)$ -invariant function f on N we have

$$\int_N f \cdot (\varphi_{s_1 Z_1 + \dots + s_r Z_r}^2)^m d\text{vol}_h = \int_N f \cdot (\psi_{s_1 Z_1 + \dots + s_r Z_r}^2)^m d\text{vol}_h.$$

Expanding this into monomials in the s_i , using the linearity of φ and ψ , we get

$$(6) \quad \int_N f \prod_{i=1}^r \varphi_{Z_i}^{n_i} d\text{vol}_h = \int_N f \prod_{i=1}^r \psi_{Z_i}^{n_i} d\text{vol}_h$$

for all $\text{Isom}(N, h)$ -invariant functions f and all $n_1, \dots, n_r \in \mathbb{N}$ such that $\sum_i n_i$ is even. By Proposition 2.8 and the fact that $\|v|_p\|_h^2 \equiv 2$, we have

$$\int_M (\text{scal}^{g_\lambda})^k d\text{vol}_{g_\lambda} = \text{vol}(H) \cdot \int_N (\text{scal}^h - \frac{1}{2} \sum_{i=1}^r \varphi_{Z_i}^2)^k d\text{vol}_h,$$

and similarly

$$\int_M (\text{scal}^{g_{\lambda'}})^k d\text{vol}_{g_{\lambda'}} = \text{vol}(H) \cdot \int_N (\text{scal}^h - \frac{1}{2} \sum_{i=1}^r \psi_{Z_i}^2)^k d\text{vol}_h.$$

Since scal^h and its powers are $\text{Isom}(N, h)$ -invariant, these two integrals decompose into two sums whose summands are of the type given in (6) and thus match up pairwise. This proves (i) in the case that N is orientable.

If N is not orientable, consider its orientable double covering $\pi : \bar{N} \rightarrow N$ and let $\bar{h} = \pi^* h$, $\bar{\lambda} = \pi^* \lambda$, $\bar{\lambda}' = \pi^* \lambda'$. Clearly, $\bar{\lambda}$ and $\bar{\lambda}'$ again satisfy the condition of Proposition 1.8 since every isometry of (N, h) lifts to an isometry of (\bar{N}, \bar{h}) . With respect to the associated Riemannian metrics $g_{\bar{\lambda}}, g_{\bar{\lambda}'}$ on $\bar{N} \times H$, the projections to $N \times H$ are Riemannian coverings. Since the assertion for $g_{\bar{\lambda}}, g_{\bar{\lambda}'}$ holds by the above arguments, it also follows for $g_\lambda, g_{\lambda'}$ by the fact that the integrals in question equal just one half of the corresponding integrals over $\bar{M} = \bar{N} \times H$.

(ii) From (6) we can also conclude, using the same argument as in the proof of (i), that

$$\int_M (a + b \operatorname{scal}^{g_\lambda})^k d\operatorname{vol}_{g_\lambda} = \int_M (a + b \operatorname{scal}^{g_{\lambda'}})^k d\operatorname{vol}_{g_{\lambda'}}$$

for all $k \in \mathbb{N}$ and all $a, b \in \mathbb{R}$. If we choose $a \geq -\min\{\min \operatorname{scal}^{g_\lambda}, \min \operatorname{scal}^{g_{\lambda'}}\}$ and $b = 1$ then $a + b \operatorname{scal}^{g_\lambda}$ and $a + b \operatorname{scal}^{g_{\lambda'}}$ induce nonnegative functions Φ, Ψ on (N, h) with the property that for each $k \in \mathbb{N}$ their L^k -norms coincide. This implies that Φ and Ψ — and consequently $\operatorname{scal}^{g_\lambda}$ and $\operatorname{scal}^{g_{\lambda'}}$ — have the same maximum. In fact, we could otherwise rescale Φ and Ψ simultaneously such that $\max \Phi < 1$ and $\max \Psi > 1$ (or vice versa). But then it would follow that $\int_N \Phi^k d\operatorname{vol}_h \rightarrow 0$ for $k \rightarrow \infty$, while $\int_N \Psi^k d\operatorname{vol}_h \rightarrow \infty$ for $k \rightarrow \infty$, which is a contradiction. A similar argument, using $a \geq \max \operatorname{scal}^{g_\lambda} = \max \operatorname{scal}^{g_{\lambda'}}$ and $b = -1$ shows that $\operatorname{scal}^{g_\lambda}$ and $\operatorname{scal}^{g_{\lambda'}}$ have the same minimum. \square

Remarks 2.12.

(i) If in Theorem 2.11 the manifold $M = N \times H$ is four-dimensional (that is, if $\dim H = 2$), then we can also conclude $\int_M \|\operatorname{Ric}^{g_\lambda}\|^2 d\operatorname{vol}_{g_\lambda} = \int_M \|\operatorname{Ric}^{g_{\lambda'}}\|^2 d\operatorname{vol}_{g_{\lambda'}}$ and $\int_M \|R^{g_\lambda}\|^2 d\operatorname{vol}_{g_\lambda} = \int_M \|R^{g_{\lambda'}}\|^2 d\operatorname{vol}_{g_{\lambda'}}$. In fact, $5 \int \operatorname{scal}^2 - 2 \int \|\operatorname{Ric}\|^2 + 2 \int \|R\|^2$ is a heat invariant (see [18], Theorem 4.8.18) and thus is the same for g_λ and $g_{\lambda'}$ because of their isospectrality; moreover, $\int \operatorname{scal}^2 - 4 \int \|\operatorname{Ric}\|^2 + \int \|R\|^2$ is a topological invariant in dimension four because of the Gauss-Bonnet-Chern formula (see, e.g., [40], p. 291). Since the vectors $(5, -2, 2)$, $(1, -4, 1)$, and $(1, 0, 0)$ are linearly independent, equality of $\int (\operatorname{scal}^{g_\lambda})^2$ and $\int (\operatorname{scal}^{g_{\lambda'}})^2$ implies equality of the other two pairs of integrals too.

(ii) The results of Theorem 2.11 contrast with the properties of the scalar curvature in isospectral examples with higher dimensional base manifold N . In [22] examples were given of isospectral metrics on $S^{m-1} \times T^2$ (interpretable as arising from Proposition 1.8; see Example 1.11 / Remark 1.12) such that the associated scalar curvature functions have different maxima. In [42] it was shown that in these examples (and in a certain generalization of them) the isospectral metrics have in general different total squared curvatures and different total squared norms of the Ricci tensor. (See Example 1.11, Remark 1.13(ii), Proposition 1.16, and [42], Lemma 5.4.)

(iii) We finally return to the special case where (N, h) is the standard two-dimensional sphere and λ, λ' are quadratic \mathbb{R}^2 -valued 1-forms on S^2 which are associated with linear maps $c, c' : \mathbb{R}^2 \rightarrow S_0(\mathbb{R}^3)$, as in Lemma 2.9. Assume that c, c' satisfy the isospectrality condition 1.) from Proposition 2.4(i). For the associated metrics $g_\lambda, g_{\lambda'}$ on $M = S^2 \times T^2$ it is then possible to show:

$$\int_M (\Delta_{g_\lambda} \operatorname{scal}^{g_\lambda})^2 d\operatorname{vol}_{g_\lambda} \neq \int_M (\Delta_{g_{\lambda'}} \operatorname{scal}^{g_{\lambda'}})^2 d\operatorname{vol}_{g_{\lambda'}} \iff \operatorname{tr}(c_{Z_1}^2 c_{Z_2}^2) \neq \operatorname{tr}(c'_{Z_1}{}^2 c'_{Z_2}{}^2).$$

Recall from the proof of Proposition 2.4(i) that there exist many examples where the latter is the case (one of them being the specific pair c, c' given in 2.4(iii) which was used for Example 2.6(ii)). The proof of the above equivalence statement is quite elementary but somewhat tedious, and we do not present it here.

3. Isospectral left invariant metrics on compact Lie groups

In the first section of this chapter, we will formulate versions of Theorem 1.6 and Proposition 1.8 for the special case of left invariant metrics on compact Lie groups (see Proposition 3.1 and Corollary 3.2, respectively). In Section 3.2 we will then give explicit applications.

We obtain the first examples of left invariant isospectral metrics on compact Lie groups, even continuous families of such metrics. This provides us with the first examples of continuous families of globally homogeneous isospectral metrics. (Note that there have previously been *pairs* of globally homogeneous isospectral metrics, namely, pairs of isospectral flat tori [33], [13]. There also have been continuous families of *locally* homogeneous isospectral manifolds, namely, isospectral families of nil- and solvmanifolds; see, e.g., [25], [41], [29].)

In particular, we will obtain continuous isospectral families of left invariant metrics on $\mathrm{SO}(m \geq 5) \times T^2$, $\mathrm{Spin}(m \geq 5) \times T^2$, $\mathrm{SU}(m \geq 3) \times T^2$, $\mathrm{SO}(n \geq 8)$, $\mathrm{Spin}(n \geq 8)$, and $\mathrm{SU}(n \geq 6)$. Among these are the first examples of simply connected *irreducible* isospectral manifolds. (The first examples of simply connected isospectral manifolds, given in [42], were products; recall Example 1.14.) We also obtain the first continuous families of isospectral manifolds of positive Ricci curvature (see Remark 3.12(ii)).

In Section 3.3 we prove that the isospectral homogeneous manifolds constructed in Section 3.2 are not locally isometric by computing the norm of their Ricci tensors, which turns out to be in general nonconstant during the isospectral deformations. From this we will also conclude, using heat invariants, that the manifolds are not isospectral for the Laplace operator acting on 1-forms.

We will finish this chapter by proving in Section 3.4 that, although the deformations from Section 3.2 can occur arbitrarily close to bi-invariant metrics, a bi-invariant metric itself can never be contained in a nontrivial continuous isospectral family of left invariant metrics on a compact Lie group; in other words, bi-invariant metrics are infinitesimally spectrally rigid within the class of left invariant metrics.

3.1 Application of the torus bundle construction to compact Lie groups.

Proposition 3.1. *Let G be a compact Lie group with Lie algebra $\mathfrak{g} = T_e G$, and let g_0 be a bi-invariant metric on G . Let $H \subset G$ be a torus in G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Denote by \mathfrak{u} the g_0 -orthogonal complement of the centralizer $\mathfrak{z}(\mathfrak{h})$ of \mathfrak{h} in \mathfrak{g} . Let $\lambda, \lambda' : \mathfrak{g} \rightarrow \mathfrak{h}$ be two linear maps with $\lambda|_{\mathfrak{h} \oplus \mathfrak{u}} = \lambda'|_{\mathfrak{h} \oplus \mathfrak{u}} = 0$ which satisfy:*

- (*3) *For every $Z \in \mathfrak{h}$ there exists $a_Z \in G$ such that a_Z commutes with H and $\lambda'_Z = \mathrm{Ad}_{a_Z}^* \lambda_Z$, where $\lambda_Z := g_0(\lambda(\cdot), Z)$ and $\lambda'_Z := g_0(\lambda'(\cdot), Z)$.*

Denote by g_λ and $g_{\lambda'}$ the left invariant metrics on G which correspond to the scalar products $(\mathrm{Id} + \lambda)^ g_0$ and $(\mathrm{Id} + \lambda')^* g_0$ on \mathfrak{g} . Then (G, g_λ) and $(G, g_{\lambda'})$ are isospectral.*

Note that $\mathrm{Id} + \lambda, \mathrm{Id} + \lambda'$ are indeed invertible maps since $\lambda^2 = \lambda'^2 = 0$. Thus the definition of g_λ and $g_{\lambda'}$ makes sense.

We give two proofs of Proposition 3.1. Although these proofs are related to each other, the “geometric” one uses Theorem 1.6, whereas the “algebraic” one is self-contained and uses only the expression for the Laplacian on unimodular Lie groups with a left invariant metric.

Geometric proof of Proposition 3.1.

We want to apply Theorem 1.6 to the torus H , equipped with the restriction of g_0 , and to $M := G$ which we interpret as a principal H -bundle with respect to the right action of H on G . Denote by $\text{pr}_{\mathfrak{h}}$ the g_0 -orthogonal projection from \mathfrak{g} to \mathfrak{h} . Let $\omega := \lambda + \text{pr}_{\mathfrak{h}}$, $\omega' := \lambda' + \text{pr}_{\mathfrak{h}} : \mathfrak{g} \rightarrow \mathfrak{h}$, and extend ω, ω' to left invariant, \mathfrak{h} -valued 1-forms on G . We claim that ω and ω' are invariant under the right action of H . First note that for every $Z \in \mathfrak{h}$, the map ad_Z annihilates $\mathfrak{z}(\mathfrak{h})$ and thus, being g_0 -skew symmetric, maps \mathfrak{g} to \mathfrak{u} . Therefore, if $X \in \mathfrak{g}$ and $z \in H$ we have indeed

$$\omega(R_{z*}X) - \omega(X) = \omega(\text{Ad}_z^{-1}X - X) \in \omega([\mathfrak{g}, \mathfrak{h}]) \subseteq \omega(\mathfrak{u}) = 0$$

by $\lambda|_{\mathfrak{u}} = 0$ and the definition of ω ; analogously for ω' . Moreover, $\lambda|_{\mathfrak{h}} = 0$ implies $\omega|_{\mathfrak{h}} = \omega'|_{\mathfrak{h}} = \text{Id}_{\mathfrak{h}}$. Hence we can view ω, ω' as connection forms on the H -bundle G .

Let $N := G/H$ and h be the submersion metric on N induced by g_0 . The definitions imply immediately that the metrics g_ω and $g_{\omega'}$ (as defined in Notation 1.5(iii)) then equal g_λ and $g_{\lambda'}$, respectively. In order to prove Proposition 3.1 it now suffices to check condition (*1) of Theorem 1.6 for ω and ω' .

Let $Z \in \mathfrak{h}$ and choose $a_Z \in G$ such that a_Z commutes with H and $\lambda'_Z = \text{Ad}_{a_Z}^* \lambda_Z$. Then $F_Z := L_{a_Z} \circ R_{a_Z}^{-1} : G \rightarrow G$ is a bundle automorphism satisfying $\omega'_Z = F_Z^* \omega_Z$. Moreover, since g_0 is bi-invariant, F_Z is an isometry of (G, g_0) and therefore induces an isometry on $(N, h) = (G/H, g_0^H)$. \square

Algebraic proof of Proposition 3.1.

The group G acts on the Hilbert space $L^2(G)$ by $(\rho_x f) = R_x^* f$ for all $f \in L^2(G)$ and $x \in G$. By unimodularity of G this action is unitary. In particular, H acts unitarily on $L^2(G)$ by the restriction of ρ to H . Let $\mathcal{L} := \exp^{-1}(e) \cap \mathfrak{h} \subset \mathfrak{h}$, and denote by \mathcal{L}^* the dual lattice in \mathfrak{h}^* . Then $L^2(G) = \bigoplus_{\mu \in \mathcal{L}^*} \mathcal{H}_\mu$, where

$$\mathcal{H}_\mu := \{f \in L^2(G) \mid \rho_{\exp(Z)} f = e^{2\pi i \mu(Z)} f \text{ for all } Z \in \mathfrak{h}\}.$$

We claim that Δ_{g_λ} and $\Delta_{g_{\lambda'}}$ leave each $\mathcal{C}_\mu := C^\infty(G) \cap \mathcal{H}_\mu$ invariant, and that their spectra on \mathcal{H}_μ coincide. Isospectrality of (G, g_λ) and $(G, g_{\lambda'})$ will then follow.

To prove our claim we first note that if g is any left invariant metric on G and $\{X_1, \dots, X_d\}$ is a g -orthonormal basis of \mathfrak{g} , then Δ_g is given by $-\sum_{i=1}^d X_i^2 - \sum_{i=1}^d \nabla_{X_i} X_i$. The second term is zero here because for each $Y \in \mathfrak{g}$ we have $\langle \sum_{i=1}^d \nabla_{X_i} X_i, Y \rangle = -\sum_{i=1}^d \langle X_i, [X_i, Y] \rangle = \text{tr}(\text{ad}_Y) = 0$ by unimodularity of the compact Lie group G . Thus

$$(7) \quad \Delta_g = -\sum_{i=1}^d X_i^2 = -\sum_{i=1}^d (\rho_* X_i)^2.$$

Now let $\{E_1, \dots, E_d\}$ be a g_0 -orthonormal basis of \mathfrak{g} . Then $\{E_1 - \lambda(E_1), \dots, E_d - \lambda(E_d)\}$ is a left invariant g_λ -orthonormal frame. We assume $\{E_1, \dots, E_d\}$ to be adapted to the g_0 -orthogonal decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{u}$; then in particular E_i

and $\lambda(E_i)$ commute for each i since $\lambda|_{\mathfrak{u}} = 0$. Consequently,

$$\begin{aligned}
\Delta_{g_\lambda}|_{\mathcal{C}_\mu} &= - \sum_{i=1}^d (E_i^2 - 2 E_i \circ \lambda(E_i) + \lambda(E_i)^2)|_{\mathcal{C}_\mu} \\
&= - \sum_{i=1}^d (E_i^2 - 4\pi i \mu(\lambda(E_i)) E_i - 4\pi^2 \mu(\lambda(E_i))^2 \text{Id})|_{\mathcal{C}_\mu} \\
(8) \qquad &= (\Delta_{g_0} + 4\pi i Y_\mu(\lambda) + 4\pi^2 \|Y_\mu(\lambda)\|_{g_0}^2 \text{Id})|_{\mathcal{C}_\mu},
\end{aligned}$$

where $Y_\mu(\lambda) := \sum_{i=1}^d \mu(\lambda(E_i)) E_i$. Note that for each $x \in G$ the map R_x is an isometry with respect to the bi-invariant metric g_0 . Therefore Δ_{g_0} commutes with R_x and hence with ρ_x for each $x \in G$. In particular, \mathcal{C}_μ is invariant under Δ_{g_0} . We claim that \mathcal{C}_μ is also invariant under $Y_\mu(\lambda)$. Let Z_μ be the dual vector to μ with respect to $g_0|_{\mathfrak{h}}$, and denote by ${}^T \lambda : \mathfrak{h} \rightarrow \mathfrak{g}$ the transpose of λ with respect to g_0 . Then

$$Y_\mu(\lambda) = \sum_{i=1}^d g_0(Z_\mu, \lambda(E_i)) E_i = {}^T \lambda(Z_\mu) \subseteq \mathfrak{u}^\perp = \mathfrak{z}(\mathfrak{h})$$

since $\mathfrak{u} \subset \ker \lambda$. But $Y_\mu(\lambda) \in \mathfrak{z}(\mathfrak{h})$ implies that \mathcal{C}_μ is indeed invariant under $Y_\mu(\lambda)$. Thus \mathcal{C}_μ is invariant under Δ_{g_λ} by (8), and analogously under $\Delta_{g_{\lambda'}}$.

Moreover, if we let $a_\mu := a_{Z_\mu}$ be as in (*3), then ρ_{a_μ} too leaves \mathcal{C}_μ invariant since a_μ commutes with H . By $\lambda'_{Z_\mu} = \text{Ad}_{a_\mu}^* \lambda_{Z_\mu}$ and the bi-invariance of g_0 we have

$$Y_\mu(\lambda') = {}^T \lambda'(Z_\mu) = {}^T \text{Ad}_{a_\mu} ({}^T \lambda(Z_\mu)) = \text{Ad}_{a_\mu}^{-1} (Y_\mu(\lambda)).$$

This implies $\|Y_\mu(\lambda)\|_{g_0}^2 = \|Y_\mu(\lambda')\|_{g_0}^2$, and by (8):

$$\Delta_{g_{\lambda'}}|_{\mathcal{C}_\mu} = (\rho_{a_\mu}^{-1} \circ \Delta_{g_\lambda} \circ \rho_{a_\mu})|_{\mathcal{C}_\mu}.$$

Therefore the spectra of Δ_{g_λ} and $\Delta_{g_{\lambda'}}$ on \mathcal{H}_μ coincide, as claimed. \square

We conclude this section with a corollary of Proposition 3.1 which follows as well from Proposition 1.8 and can be regarded as the intersection of both.

Corollary 3.2. *Let K be a compact Lie group with Lie algebra $\mathfrak{k} = T_e K$, and let h be a bi-invariant metric on K . Let H be a torus with Lie algebra $\mathfrak{h} := T_e H$, equipped with an invariant metric. Let $\lambda, \lambda' : \mathfrak{k} \rightarrow \mathfrak{h}$ be two linear maps which satisfy:*

$$(*4) \qquad \lambda'_Z \in \text{Ad}_K^*(\lambda_Z) \text{ for each } Z \in \mathfrak{k}.$$

Then $(K \times H, g_\lambda)$ and $(K \times H, g_{\lambda'})$ are isospectral, where λ, λ' are interpreted as left invariant \mathfrak{h} -valued 1-forms on K , and $g_\lambda, g_{\lambda'}$ are the associated metrics on $K \times H$ as in Notation 1.5(iv).

Proof. The corollary follows immediately from either Proposition 1.8 or Proposition 3.1. In the context of Proposition 1.8, the manifold (K, h) plays the role of

(N, h) , and it suffices to note that the inner automorphisms of K are isometries with respect to h .

In the context of Proposition 3.1, the group $K \times H$, equipped with the product metric, plays the role of (G, g_0) ; we extend λ, λ' to linear maps from $\mathfrak{k} \oplus \mathfrak{h}$ to \mathfrak{h} by letting $\lambda|_{\mathfrak{h}} = \lambda'|_{\mathfrak{h}} = 0$. It suffices to note that now H is central in G and thus $\mathfrak{u} = 0$; hence $\lambda|_{\mathfrak{u}} = \lambda'|_{\mathfrak{u}} = 0$ is trivially satisfied. Condition (*3) is implied by (*4) and the fact that each $a_Z \in K$ commutes with H . \square

3.2 Examples.

We will now exploit Proposition 3.1 to construct the first examples of isospectral left invariant metrics on compact Lie groups, as announced at the beginning of this chapter.

In Subsection 3.2.1, we apply Proposition 3.1 via the more special Corollary 3.2 to obtain continuous isospectral families of left invariant metrics on $\mathrm{SO}(m) \times T^2$ and $\mathrm{Spin}(m) \times T^2$ for $m \geq 5$, and also on $\mathrm{SU}(m) \times T^2$ for $m \geq 3$.

In the subsections 3.2.2 / 3.2.3 we will then use Proposition 3.1 in its general form to find continuous isospectral families of left invariant metrics on the irreducible groups $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$ for $n \geq 8$ and on $\mathrm{SU}(n)$ for $n \geq 6$.

3.2.1 Isospectral deformations on $\mathrm{SO}(m) \times T^2$ ($m \geq 5$), $\mathrm{Spin}(m) \times T^2$ ($m \geq 5$), and $\mathrm{SU}(m) \times T^2$ ($m \geq 3$).

One application of Corollary 3.2 has already been waiting in a barely disguised form, as we are going to see now. The main tool is the existence of nontrivial isospectral families of linear maps $j(t) : \mathbb{R}^2 \rightarrow \mathfrak{so}(m)$ with $m \geq 5$ which is guaranteed by Proposition 1.16(ii) from the last chapter and which were the key tool in Example 1.11 and 1.14.

Example 3.3.

Let $K = \mathrm{SO}(m)$ and $\mathfrak{k} := \mathfrak{so}(m) = T_e K$; assume $m \geq 5$. Consider a bi-invariant metric h on K (unique up to scaling) and the associated Ad_K -invariant scalar product on \mathfrak{k} . Let H be a two-dimensional torus with Lie algebra $\mathfrak{h} = T_e H$, equipped with some invariant metric. Let $\{Z_1, Z_2\}$ be an orthonormal basis of \mathfrak{h} .

Recall from Proposition 1.16(ii) that there exists a Zariski open subset \mathcal{U} of the space \mathcal{J} of linear maps $j : \mathfrak{h} \rightarrow \mathfrak{so}(m) = \mathfrak{k}$ such that for each $j \in \mathcal{U}$ there is a continuous family $j(t)$ in \mathcal{J} , defined in some open neighbourhood of $t = 0$, such that $j(0) = j$ and such that:

- 1.) The maps $j(t)$ are pairwise isospectral in the sense of Definition 1.10.
- 2.) The function $t \mapsto \|j_{Z_1}(t)^2 + j_{Z_2}(t)^2\|^2 = \mathrm{tr}((j_{Z_1}(t)^2 + j_{Z_2}(t)^2)^2)$ is nonconstant in t in every interval around zero.

An explicit example for a family $j(t)$ satisfying 1.) and 2.) in case $m = 5$ was given in Proposition 1.16(iv).

Now let $\{j(t)\}_{t \in (-\varepsilon, \varepsilon)}$ be any continuous family in \mathcal{J} which satisfies 1.) and 2.). Recall that condition 1.) means that for each $Z \in \mathfrak{h}$ the path $t \mapsto j_Z(t)$ is contained

in the $\text{Ad}_{\text{SO}(m)}$ -orbit of $j_Z(0)$. By continuity it follows that $j_Z(t)$ must even be contained in the $\text{Ad}_{\text{SO}(m)}$ -orbit of $j_Z(0)$. Define linear maps $\lambda(t) : \mathfrak{k} \rightarrow \mathfrak{h}$ by letting

$$\lambda_Z(t) := \langle \cdot, j_Z(t) \rangle$$

for each $Z \in \mathfrak{h}$, where $\lambda_Z(t)$ means $\langle (\lambda(t))(\cdot), Z \rangle$ as usual. In other words, $\lambda(t) : \mathfrak{k} \rightarrow \mathfrak{h}$ is the transpose of $j(t) : \mathfrak{h} \rightarrow \mathfrak{k}$ with respect to the given metrics. By the bi-invariance of h and the fact that for any fixed $Z \in \mathfrak{h}$ we have $j_Z(t) \in \text{Ad}_K(j_Z(0))$ for all t , it follows that $\lambda_Z(t) \in \text{Ad}_K^*(\lambda_Z(0))$ for all t . But this just means that the maps $\lambda(t)$ pairwise satisfy condition (*4) of Corollary 3.2. We conclude that the Riemannian manifolds

$$(\text{SO}(m) \times H, g_{\lambda(t)})$$

are isospectral, where $g_{\lambda(t)}$ is the left invariant metric associated with $\lambda(t)$ and h as in Corollary 3.2.

Note that instead of $K = \text{SO}(m)$ we may as well consider its universal covering $\tilde{K} := \text{Spin}(m)$ because $\text{Ad}_{\tilde{K}}$ -orbits in \mathfrak{k} are the same as Ad_K -orbits. Hence our above families $\lambda(t) : \mathfrak{k} \rightarrow \mathfrak{h}$ satisfy condition (*4) of Corollary 3.2 also with respect to \tilde{K} . Thus for each $m \geq 5$ we also get isospectral families

$$(\text{Spin}(m) \times H, g_{\lambda(t)}),$$

where $H \cong T^2$ is as above and $g_{\lambda(t)}$ is the left invariant metric which is associated, as in Corollary 3.2, with $\lambda(t)$ and the bi-invariant metric \tilde{h} on $\text{Spin}(m)$ which is the pullback of the above metric h on $\text{SO}(m)$. With these notations the projection from $(\text{Spin}(m) \times H, g_{\lambda(t)})$ to $(\text{SO}(m) \times H, g_{\lambda(t)})$ is a Riemannian covering; thus isospectrality of the manifolds in the latter family is actually implied by continuity and by the isospectrality of the covering manifolds.

For each t , the norm of the associated Ricci tensor $\text{Ric}^{g_{\lambda(t)}}$ is a constant function on $\text{SO}(m) \times H$ (resp. $\text{Spin}(m) \times H$) since $g_{\lambda(t)}$ is left invariant. We claim that from the above conditions 1.), 2.) on $j(t) = {}^T \lambda(t)$ it follows that:

- (i) $\|\text{Ric}^{g_{\lambda(t)}}\|^2$ is nonconstant in t . In particular, the manifolds $(\text{SO}(m) \times H, g_{\lambda(t)})$ are not pairwise locally isometric.
- (ii) The second heat invariant for the Laplace operator on 1-forms associated with $(\text{SO}(m) \times H, g_{\lambda(t)})$ depends nontrivially on t . In particular, the manifolds are not pairwise isospectral on 1-forms.

The analogous statements hold for $(\text{Spin}(m) \times H, g_{\lambda(t)})$. We postpone the proof of these facts to Section 3.3 (see Theorem 3.14, Proposition 3.15, and Corollary 3.17).

Remark 3.4. The relation between the isospectral metrics on $\text{SO}(m) \times T^2$ from Example 3.3 on the one hand and those on $S^{m-1} \times T^2$ from Example 1.11 on the other hand can be explained by the following general principle.

Let K be a compact Lie group with Lie algebra \mathfrak{k} , equipped with a bi-invariant metric, and suppose that K acts by isometries on a closed Riemannian manifold (N, h) . Each $j_Z \in \mathfrak{k}$ is canonically identified with the Killing vector field $p \mapsto \frac{d}{dt}|_{t=0} \exp(tj_Z)p$ on N ; by taking duals on both sides, each $\lambda_Z \in \mathfrak{k}^*$ is canonically identified with a certain 1-form on N . If λ_Z, λ'_Z belong to the same coadjoint orbit

then the associated 1-forms on N are related by an element of $K \subseteq \text{Isom}(N, h)$. In other words, two linear maps $\lambda, \lambda' : \mathfrak{k} \rightarrow \mathfrak{h} \cong \mathbb{R}^r$ satisfying the isospectrality condition (*4) from Corollary 3.2 produce an associated pair of \mathfrak{h} -valued 1-forms on N which satisfies the isospectrality condition (*2) from Proposition 1.8, and this for each Riemannian manifold (N, h) on which K acts by isometries.

In our above context, $K = \text{SO}(m)$ and (N, h) is the round standard sphere S^{m-1} . Obviously the above point of view opens prospects for examples of isospectral metrics not only on $\text{SO}(m) \times T^2$ or $S^{m-1} \times T^2$, but also on $N \times T^2$ where N is, for example, a Grassmannian, or any other manifold admitting a metric h with respect to which $K = \text{SO}(m \geq 5)$ acts by isometries. In view of Example 3.7 below, the same considerations are also valid for $K = \text{SU}(m \geq 3)$. We will not pursue these ideas in the present work, but concentrate entirely on the Lie groups themselves.

Our next aim is to construct isospectral metrics on $\text{SU}(m) \times T^2$ by analogous methods as those used above for $\text{SO}(m) \times T^2$. As we will see, this is indeed possible for $m \geq 3$. First we need a result analogous to the one cited in Proposition 1.16(ii) (that is, to [42], Proposition 4.1).

Definition 3.5. Two linear maps $j, j' : \mathbb{R}^r \rightarrow \mathfrak{su}(m)$ are called *isospectral*, denoted $j \sim j'$, if for every $Z \in \mathbb{R}^r$ there exists $A_Z \in \text{SU}(m)$ such that $j'_Z = A_Z j_Z A_Z^{-1}$.

Proposition 3.6. *Let $m \geq 3$ and $\{Z_1, Z_2\}$ be the standard basis of \mathbb{R}^2 .*

- (i) *There exists a Zariski open subset \mathcal{U} of the space \mathcal{J} of linear maps $j : \mathbb{R}^2 \rightarrow \mathfrak{su}(m)$ such that for each $j \in \mathcal{U}$ there is a continuous family $j(t)$ in \mathcal{J} , defined in some open neighbourhood of $t = 0$, such that $j(0) = j$ and:

 - 1.) *The maps $j(t)$ are pairwise isospectral in the sense of Definition 3.5.*
 - 2.) *The function $t \mapsto \|j_{Z_1}(t)^2 + j_{Z_2}(t)^2\|^2 = \text{tr}((j_{Z_1}(t)^2 + j_{Z_2}(t)^2)^2)$ is not constant in t in any interval around zero.**
- (ii) *For $m = 3$, an explicit example of an isospectral family $j(t) : \mathbb{R}^2 \rightarrow \mathfrak{su}(3)$ with $\|j_{Z_1}(t)^2 + j_{Z_2}(t)^2\|^2 \neq \text{const}$ is given by*

$$j_{Z_1}(t) := \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad j_{Z_2}(t) := \begin{pmatrix} 0 & t & \sqrt{1-2t^2} \\ -t & 0 & t \\ -\sqrt{1-2t^2} & -t & 0 \end{pmatrix}.$$

This family is defined for $t \in [-1/\sqrt{2}, 1/\sqrt{2}]$. The $j(t)$ are pairwise isospectral since $\det(\lambda \text{Id} - (sj_{Z_1}(t) + uj_{Z_2}(t))) = \lambda^3 + (s^2 + u^2)\lambda$ is independent of t . However, $\|j_{Z_1}(t)^2 + j_{Z_2}(t)^2\|^2 = 8 - 4t^2$ is nonconstant in t .

Proof. Part (ii) can be checked by straightforward computation. As to part (i), the proof of [42], Proposition 4.1, which asserted the analogous statement for $\mathfrak{so}(m \geq 5)$ (see Proposition 1.16(ii)) instead of $\mathfrak{su}(m \geq 3)$ carries over almost verbatim.

First of all, elementary arguments show that two elements of $\mathfrak{su}(m)$ are conjugate by an element of $\text{SU}(m)$ if and only if they have the same characteristic polynomials. Therefore the condition $j \sim j'$ is equivalent to $sj_1 + uj_2$ having the same characteristic polynomial as $sj'_1 + uj'_2$ for all s, u , where we write $j_1 := j_{Z_1}$, $j_2 := j_{Z_2}$. This is in turn equivalent to $\text{tr}((sj_1 + uj_2)^k) = \text{tr}((sj'_1 + uj'_2)^k)$ for all

$k = 1, \dots, m$, or equivalently, for all $k \in \mathbb{N}$. By expanding into monomials in s, u we get that

$$j \sim j' \iff p_{a,b}(j) = p_{a,b}(j') \text{ for all } a, b \in \mathbb{N}_0 \text{ with } a + b > 0,$$

where

$$p_{a,b}(j) := \sum_{\sigma \in \mathfrak{S}_{a,b}} \text{tr}(j_{\sigma(1)} \cdots j_{\sigma(a+b)})$$

and $\mathfrak{S}_{a,b}$ denotes the set of all maps $\sigma : \{1, \dots, a+b\} \rightarrow \{1, 2\}$ which satisfy $\#\sigma^{-1}(1) = a$, $\#\sigma^{-1}(2) = b$.

The algebraic vector field Y on \mathcal{J} , given by $Y(j) = (j_1^5 j_2 - j_2 j_1^5, 0)$ satisfies $dp_{a,b}|_j Y = 0$ for all $j \in \mathcal{J}$. The proof of this fact is purely combinatoric and reads exactly as the proof of [42], Lemma 4.3, except that there we used the exponent 3 instead of 5. This implies that the (locally defined) flow lines of Y consist of pairwise isospectral maps.

For $j \sim j'$, we obviously have $\text{tr}(j_1^4) = \text{tr}(j_1'^4)$ and $\text{tr}(j_2^4) = \text{tr}(j_2'^4)$; hence in this case, the condition $\text{tr}((j_1^2 + j_2^2)^2) = \text{tr}((j_1'^2 + j_2'^2)^2)$ is equivalent to $q(j) = q(j')$, where $q(j) := \text{tr}(j_1^2 j_2^2)$.

We have $dq|_j Y = \text{tr}(j_1^5 j_2 j_1 j_2^2 - j_1 j_2 j_1^5 j_2^2)$. This polynomial does not vanish identically on \mathcal{J} if $m \geq 3$; e.g., for

$$j_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -3i \end{pmatrix} \text{ and } j_2 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

it equals $240 \neq 0$. Therefore, the Zariski open subset $\mathcal{U} := \{j \in \mathcal{J} \mid dq|_j Y \neq 0\}$ has the required properties. \square

We now proceed in analogy with Example 3.3.

Example 3.7.

Let $K := \text{SU}(m)$ and $\mathfrak{k} := \mathfrak{su}(m) = T_e K$; assume $m \geq 3$. Consider a bi-invariant metric h on K (unique up to scaling) and the associated Ad_K -invariant scalar product on \mathfrak{k} . Let H be a two-dimensional torus with Lie algebra $\mathfrak{h} = T_e H$, equipped with some invariant metric. Let $\{Z_1, Z_2\}$ be an orthonormal basis of \mathfrak{h} .

Let $\{j(t)\}_{t \in (-\varepsilon, \varepsilon)}$ be any continuous family of linear maps from \mathfrak{h} to $\mathfrak{su}(m)$ which satisfies conditions 1.) and 2.) of Proposition 3.6(i). Define linear maps $\lambda(t) : \mathfrak{k} \rightarrow \mathfrak{h}$ by letting

$$\lambda_Z(t) := \langle \cdot, j_Z(t) \rangle$$

for each $Z \in \mathfrak{h}$; that is, $\lambda(t) = {}^T j(t)$ with respect to the given metrics on \mathfrak{k} and \mathfrak{h} . From condition 1.) it follows that $\lambda_Z(t) \in \text{Ad}_K^*(\lambda_Z(0))$ for all t . Thus the maps $\lambda(t)$ pairwise satisfy condition (*4) of Corollary 3.2. Hence the Riemannian manifolds

$$(\text{SU}(m) \times H, g_{\lambda(t)})$$

are isospectral, where $g_{\lambda(t)}$ is the left invariant metric associated with $\lambda(t)$ and h as in Corollary 3.2.

In Section 3.3 we will prove that the conditions 1.) and 2.) imply the same properties as in Example 3.3; that is:

- (i) $\|\text{Ric}^{g_{\lambda(t)}}\|^2$ is nonconstant in t . In particular, the manifolds $(\text{SU}(m) \times H, g_{\lambda(t)})$ are not pairwise locally isometric (see Theorem 3.14 and Proposition 3.15).
- (ii) The second heat invariant for the Laplace operator on 1-forms associated with $(\text{SU}(m) \times H, g_{\lambda(t)})$ depends nontrivially on t . In particular, the manifolds are not pairwise isospectral on 1-forms (Corollary 3.17).

3.2.2 Isospectral deformations on $\text{SO}(n)$ ($n \geq 9$), $\text{Spin}(n)$ ($n \geq 9$), and $\text{SU}(n)$ ($n \geq 6$).

We will now use the ideas from the previous subsection to construct isospectral left invariant metrics on *irreducible* compact Lie groups. More precisely, we embed the above products $\text{SO}(m) \times T^2$ etc. into certain irreducible groups and use Proposition 3.1 to obtain isospectral metrics on these.

Example 3.8.

Let $G := \text{SO}(m + 4)$ and $\mathfrak{g} := \mathfrak{so}(m + 4) = T_e G$; assume $m \geq 5$. Let g_0 be a bi-invariant metric on G , and denote the corresponding Ad_G -invariant scalar product on \mathfrak{g} by g_0 again. Let $K_1 := \text{SO}(m)$ and $K_2 := \text{SO}(4)$. Since $\text{SO}(m) \times \text{SO}(4)$ is canonically embedded in $\text{SO}(m + 4)$, we will from now on consider K_1 and K_2 as commuting subgroups of G which are orthogonal with respect to the Killing metric on G , and consequently with respect to g_0 . Let H be a maximal torus in K_2 , endowed with the invariant metric induced by g_0 . We denote by $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}_1, \mathfrak{k}_2$ the Lie algebras of G, H, K_1 , and K_2 , respectively. Note that

$$(9) \quad [\mathfrak{k}_1, \mathfrak{h}] = 0 \quad \text{and} \quad \mathfrak{k}_1 \perp_{g_0} \mathfrak{h},$$

where \perp_{g_0} denotes orthogonality with respect to g_0 . Since H is two-dimensional, there exist continuous families of linear maps $j(t) : \mathfrak{h} \rightarrow \mathfrak{k}_1$ satisfying the conditions 1.) and 2.) from Example 3.3. (Recall that there even exists a Zariski open subset \mathcal{U} of the space of linear maps from \mathfrak{h} to \mathfrak{k}_1 such that each element of \mathcal{U} is contained in a continuous family satisfying 1.) and 2.)

Let $\{j(t)\}_{t \in (-\varepsilon, \varepsilon)}$ be such a family. As in Example 3.3 we conclude from condition 1.) that for each $Z \in \mathfrak{h}$ we have $j_Z(t) \in \text{Ad}_{K_1}(j_Z(0))$ for all t . We now interpret $j(t) : \mathfrak{h} \rightarrow \mathfrak{k}_1 \subset \mathfrak{g}$ as a linear map from \mathfrak{h} to \mathfrak{g} and define $\lambda(t) := {}^T j(t) : \mathfrak{g} \rightarrow \mathfrak{h}$ as the transpose of $j(t)$ with respect to g_0 .

We claim that the maps $\lambda(t)$ pairwise satisfy the conditions of Proposition 3.1. In fact, by (9) we have that \mathfrak{k}_1 is g_0 -orthogonal both to \mathfrak{h} and \mathfrak{u} , where \mathfrak{u} is the g_0 -orthogonal complement of the centralizer $\mathfrak{z}(\mathfrak{h})$ of \mathfrak{h} in \mathfrak{g} . Thus $\mathfrak{h} \oplus \mathfrak{u} \perp \mathfrak{k}_1 \supseteq \text{Im } j(t)$, which implies that $\mathfrak{h} \oplus \mathfrak{u} \subseteq \ker \lambda(t)$ for all t . Hence the first condition of Proposition 3.1 is satisfied. Moreover, for each $Z \in \mathfrak{h}$ we have $\lambda_Z(t) \in \text{Ad}_{K_1}^*(\lambda_Z(0))$ for all t by the analogous property of the $j_Z(t)$. Since K_1 commutes with H , condition (*3) of Proposition 3.1 is satisfied, too. We thus get isospectral families

$$(\text{SO}(n), g_{\lambda(t)})$$

for each $n = m + 4 \geq 5 + 4 = 9$, where $g_{\lambda(t)}$ is the left invariant metric associated with $\lambda(t)$ and g_0 as in the proposition.

Instead of G, K_1, K_2 we may as well consider their universal coverings $\tilde{G} := \text{Spin}(m + 4)$ and $\tilde{K}_1 \times \tilde{K}_2 := \text{Spin}(m) \times \text{Spin}(4) \subset \tilde{G}$, endowed with a bi-invariant metric \tilde{g}_0 . Note that $\text{Ad}_{\tilde{K}_1}$ -orbits in \mathfrak{k}_1 are the same as Ad_{K_1} -orbits, and that \mathfrak{h} is the Lie algebra of some two-dimensional torus \tilde{H} in \tilde{K}_2 which commutes with \tilde{K}_1 and is \tilde{g}_0 -orthogonal to \tilde{K}_1 . Hence our above arguments go through to show that the family of linear maps $\lambda(t) : \mathfrak{g} \rightarrow \mathfrak{h}$ satisfies the conditions of Proposition 3.1 also with respect to \tilde{G}, \tilde{H} , and \tilde{g}_0 . Thus we also obtain isospectral families

$$(\text{Spin}(n), g_{\lambda(t)})$$

for each $n \geq 9$, where $g_{\lambda(t)}$ is the left invariant metric associated with $\lambda(t)$ and \tilde{g}_0 as in Proposition 3.1.

Concerning local non-isometry, we have by Proposition 3.15 of Section 3.3 below that $\|\text{Ric}_{g_{\lambda(t)}}\|^2$ is nonconstant in t , and by Corollary 3.17 the manifolds are not pairwise isospectral on 1-forms.

Example 3.9.

We replace the groups G, K_1, K_2 appearing in Example 3.8 by $G := \text{SU}(m + 3)$ with $m \geq 3$, $K_1 := \text{SU}(m)$, and $K_2 := \text{SU}(3)$. Again, we consider K_1 and K_2 as commuting subgroups of G which are orthogonal with respect to the Killing metric. We choose a bi-invariant metric g_0 on G (in particular, g_0 is proportional to the Killing metric) and a maximal, hence two-dimensional, torus H in K_2 .

Using the isospectral families $j(t) : \mathfrak{h} \rightarrow \mathfrak{k}_1 \subset \mathfrak{g}$ from Proposition 3.6(i) this time, we obtain continuous families of linear maps $\lambda(t) := {}^T j(t) : \mathfrak{g} \rightarrow \mathfrak{h}$ which pairwise satisfy the conditions of Proposition 3.1; the arguments read exactly as in Example 3.8. We thus obtain isospectral families

$$(\text{SU}(n), g_{\lambda(t)})$$

for each $n = m + 3 \geq 3 + 3 = 6$, where $g_{\lambda(t)}$ is the left invariant metric associated with $\lambda(t)$ and g_0 as in Proposition 3.1.

Concerning non-isometry and non-isospectrality on 1-forms, see again Proposition 3.15 / Corollary 3.17 below.

3.2.3 Isospectral deformations on $\text{SO}(8)$ and $\text{Spin}(8)$.

Recall that in the previous subsection, we modified Example 3.3 (respectively 3.7) to obtain isospectral families of metrics on the irreducible Lie groups of Example 3.8 (respectively 3.9) by considering the canonical embeddings $K_1 \times H = \text{SO}(m) \times T^2 \hookrightarrow \text{SO}(m) \times \text{SO}(4) \hookrightarrow \text{SO}(m + 4)$ and $K_1 \times H = \text{SU}(m) \times T^2 \hookrightarrow \text{SU}(m) \times \text{SU}(3) \hookrightarrow \text{SU}(m + 3)$, respectively. The lowest dimensions of irreducible examples obtained in this way are $35 = \dim(\text{SU}(6))$ and $36 = \dim(\text{SO}(9))$. As a final application of Proposition 3.1, we now construct isospectral families of left invariant metrics on the 28-dimensional irreducible Lie groups $\text{SO}(8)$ and $\text{Spin}(8)$ by arranging K_1 and H in a more economical way.

Example 3.10.

For each $m \in \mathbb{N}$ let $\Psi : \mathbb{C}^m \rightarrow \mathbb{R}^{2m}$ be the isomorphism of real vector spaces which sends the standard basis vector $e_k \in \mathbb{C}^m$ to $e_k \in \mathbb{R}^{2m}$ and $ie_k \in \mathbb{C}^m$ to $e_{m+k} \in \mathbb{R}^{2m}$ for each $k = 1, \dots, m$. Consider the injective Lie algebra homomorphism $\varphi : \mathfrak{su}(m) \ni X \mapsto \Psi X \Psi^{-1} \in \mathfrak{so}(2m)$ and the associated homomorphic embeddings $\Phi : \mathrm{SU}(m) \rightarrow \mathrm{SO}(2m)$ and $\tilde{\Phi} : \mathrm{SU}(m) \rightarrow \mathrm{Spin}(2m)$. Let $K_1 := \mathrm{Im} \Phi \subset \mathrm{SO}(2m) =: G_1$ and $\tilde{K}_1 := \mathrm{Im} \tilde{\Phi} \subset \mathrm{Spin}(2m) =: \tilde{G}_1$. We define $G := \mathrm{SO}(2m+2)$, $\tilde{G} := \mathrm{Spin}(2m+2)$, $G_2 := \mathrm{SO}(2)$, $\tilde{G}_2 := \mathrm{Spin}(2)$, and consider the canonical embeddings $G_1 \times G_2 \hookrightarrow G$ and $\tilde{G}_1 \times \tilde{G}_2 \hookrightarrow \tilde{G}$ which allow us to consider G_1 and G_2 (respectively $\tilde{G}_1 \times \tilde{G}_2$) as commuting subgroups of G (respectively \tilde{G}) which are orthogonal with respect to the Killing metric. Denote by $\mathfrak{k}_1, \mathfrak{g}_1, \mathfrak{g}$ the Lie algebras of K_1, G_1 , and G , respectively.

Let $J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in \mathfrak{so}(2m) = \mathfrak{g}_1$, where I denotes the m -dimensional unit matrix. Note that J commutes with $\mathrm{Im} \varphi = \mathfrak{k}_1$. We define two-dimensional tori H, \tilde{H} in G, \tilde{G} by

$$\begin{aligned} H &:= \exp(\mathbb{R}J) \times G_2 \subset G_1 \times G_2 \subset G, \\ \tilde{H} &:= \widetilde{\exp}(\mathbb{R}J) \times \tilde{G}_2 \subset \tilde{G}_1 \times \tilde{G}_2 \subset \tilde{G}. \end{aligned}$$

For the Lie algebra \mathfrak{h} of H (resp. \tilde{H}) we then have

$$(10) \quad [\mathfrak{k}_1, \mathfrak{h}] = 0 \quad \text{and} \quad \mathfrak{k}_1 \perp_{g_0} \mathfrak{h},$$

where g_0 is any bi-invariant metric on G (resp. \tilde{G}) and \perp_{g_0} denotes orthogonality with respect to g_0 .

Now assume $m \geq 3$, and let $\{\hat{j}(t)\}_{t \in (-\varepsilon, \varepsilon)}$ be a family of linear maps from \mathfrak{h} to $\mathfrak{su}(m)$ satisfying conditions 1.) and 2.) from Proposition 3.6(i). Let

$$j(t) := \varphi \circ \hat{j}(t) : \mathfrak{h} \rightarrow \mathfrak{so}(2m) = \mathfrak{g}_1 \subset \mathfrak{so}(2m+2) = \mathfrak{g}.$$

From condition 1.) on the $\hat{j}(t)$ and the fact that Φ and $\tilde{\Phi}$ are homomorphisms with $\Phi_* = \tilde{\Phi}_* = \varphi$, it follows that for each $Z \in \mathfrak{h}$ we have $j_Z(t) \in \mathrm{Ad}_{K_1}(j_Z(0)) = \mathrm{Ad}_{\tilde{K}_1}(j_Z(0))$ for all t . We interpret the $j(t)$ as linear maps from \mathfrak{h} to \mathfrak{g} and define $\lambda(t) := {}^T j(t) : \mathfrak{g} \rightarrow \mathfrak{h}$ as the transpose of $j(t)$ with respect to g_0 . From (10) we conclude, exactly as in Example 3.7/3.8, that the $\lambda(t)$ pairwise satisfy the conditions of Proposition 3.1 applied to (G, g_0) , resp. to (\tilde{G}, g_0) . We thus obtain isospectral families

$$(\mathrm{SO}(2m+2), g_{\lambda(t)}) \quad \text{and} \quad (\mathrm{Spin}(2m+2), g_{\lambda(t)})$$

for all $2m+2 \geq 2 \cdot 3 + 2 = 8$, where $g_{\lambda(t)}$ is the left invariant metric associated with $\lambda(t)$ and g_0 as in Proposition 3.1. In particular, for $m = 3$ we get continuous families of left invariant isospectral metrics on $\mathrm{SO}(8)$, resp. on $\mathrm{Spin}(8)$.

Finally note that condition 2.) on the $\hat{j}(t)$ implies that also $\mathrm{tr}((j_{Z_1}(t)^2 + j_{Z_2}(t)^2)^2)$ is nonconstant in t . In fact, $\mathrm{tr}((j_{Z_1}(t)^2 + j_{Z_2}(t)^2)^2) = 2 \mathrm{tr}((\hat{j}_{Z_1}(t)^2 + \hat{j}_{Z_2}(t)^2)^2)$, which is nonconstant in t by condition 2.).

Once more, the Ricci tensors of these manifolds have different norms (Proposition 3.15), and the manifolds are not pairwise isospectral for the Laplace operator acting on 1-forms (Corollary 3.17).

norms (Theorem 3.14). In particular, it turns out that in all the isospectral families from Section 3.2 the norm of the Ricci tensor varies during the deformation (Proposition 3.15). This implies not only that the manifolds are not pairwise locally isometric, but also, as can be seen by using heat invariants, that they are not isospectral for the Laplace operator acting on 1-forms (Corollary 3.17).

It should be mentioned that the scalar curvature can of course *not* be used here to distinguish between the metrics: Since the manifolds are homogeneous, the associated scalar curvature is constant on each of them; the fact that volume and total scalar curvature are heat invariants thus implies that this constant is the same for all metrics in the isospectral family.

We fix certain objects and notations which we will use throughout this section.

Notation 3.13.

(i) Let G be a compact Lie group with Lie algebra \mathfrak{g} and a bi-invariant metric g_0 . Let $H \subset G$ be a torus in G with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, and denote by \mathfrak{u} the g_0 -orthogonal complement of the centralizer $\mathfrak{z}(\mathfrak{h})$ of \mathfrak{h} in \mathfrak{g} .

(ii) We consider linear maps $\lambda : \mathfrak{g} \rightarrow \mathfrak{g}$ whose image is contained in \mathfrak{h} and which satisfy $\lambda|_{\mathfrak{h} \oplus \mathfrak{u}} = 0$. For any such λ , we let g_λ be the left invariant metric on G which corresponds to the scalar product $(\text{Id} + \lambda)^* g_0$ on \mathfrak{g} . We denote this scalar product on \mathfrak{g} , and the corresponding ones on tensors, by $\langle \cdot, \cdot \rangle_\lambda$, and we let Ric^λ be the Ricci tensor associated with g_λ . For $Z \in \mathfrak{h}$ we denote by λ_Z the 1-form $\langle \lambda(\cdot), Z \rangle_0 \in \mathfrak{g}^*$. We define $j : \mathfrak{g} \rightarrow \mathfrak{g}$ as the transpose of λ with respect to $\langle \cdot, \cdot \rangle_0$; note that j vanishes on the g_0 -orthogonal complement of \mathfrak{h} , and its image is contained in $\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{h}^\perp$.

(iii) For any $X \in \mathfrak{g}$ we write $\tilde{X} = (\text{Id} - \lambda)(X)$. Note that $\langle \tilde{X}, \tilde{Y} \rangle_\lambda = \langle X, Y \rangle_0$ for all $X, Y \in \mathfrak{g}$. Finally, we choose g_0 -orthonormal bases $\{Z_1, \dots, Z_r\}$ of $\mathfrak{h} \subset \mathfrak{g}$ and $\{V_1, \dots, V_d\}$ of \mathfrak{g} .

Theorem 3.14. *Let $\lambda, \lambda' : \mathfrak{g} \rightarrow \mathfrak{h} \subset \mathfrak{g}$ be two linear maps as above which moreover satisfy condition (*3) of Proposition 3.1; i.e., for every $Z \in \mathfrak{h}$ there exists $a_Z \in G$ such that $\lambda'_Z = \text{Ad}_{a_Z}^* \lambda_Z$ and $\text{Ad}_{a_Z}|_{\mathfrak{h}} = \text{Id}|_{\mathfrak{h}}$. Then we have, using the above notation:*

$$(11) \quad \|\text{Ric}^\lambda\|_\lambda^2 - \|\text{Ric}^{\lambda'}\|_{\lambda'}^2 = \frac{1}{4} \left(\sum_{i,k=1}^r \text{tr}((\text{ad}_{j_{Z_i}})^2 (\text{ad}_{j_{Z_k}})^2) - \sum_{i,k=1}^r \text{tr}((\text{ad}_{j'_{Z_i}})^2 (\text{ad}_{j'_{Z_k}})^2) \right).$$

We postpone the proof of Theorem 3.14 to the end of this section and first deduce from it that the norm of the Ricci tensor varies indeed in all the isospectral families of left invariant metrics given in the examples in Section 3.2.

Proposition 3.15. *Let $g_{\lambda(t)}$ be any of the isospectral families of left invariant metrics from Example 3.3, 3.7, 3.8, 3.9, or 3.10 of Section 3.2. Then $\|\text{Ric}^{\lambda(t)}\|_{\lambda(t)}^2$ is nonconstant in t .*

Proof. First of all, note that in Theorem 3.14 the right hand side of (11) is zero if and only if the two sums running only over pairs i, k with $i \neq k$ are equal. In

fact, $\operatorname{tr}((\operatorname{ad}_{j_{Z_i}})^4) = \operatorname{tr}((\operatorname{ad}_{j'_{Z_i}})^4)$ for all i because j_{Z_i} and j'_{Z_i} are conjugate by an automorphism of \mathfrak{g} by condition (*3). Throughout Section 3.2 we worked with $\dim \mathfrak{h} = 2$; hence we only need to show that

$$(12) \quad \operatorname{tr}((\operatorname{ad}_{j_{Z_1}(t)})^2(\operatorname{ad}_{j_{Z_2}(t)})^2) \neq \text{const in } t$$

for each of the families $j(t) = {}^T \lambda(t) : \mathfrak{h} \rightarrow \mathfrak{g}$ from the examples in Section 3.2.

Recall that in some of those examples, \mathfrak{g} was equal to a matrix algebra $\mathfrak{m} = \mathfrak{so}(n)$ or $\mathfrak{m} = \mathfrak{su}(n)$ (Examples 3.8, 3.9, 3.10); in the remaining examples 3.3 and 3.7, \mathfrak{g} was the direct sum of such an algebra \mathfrak{m} with an abelian Lie algebra. In each case, the images of the maps $j(t) : \mathfrak{h} \rightarrow \mathfrak{g}$ were contained in \mathfrak{m} . In (12) we can therefore interpret “ad” as the adjoint representation of \mathfrak{m} , and “tr” as the trace over \mathfrak{m} .

We want to apply the formulas given in Lemma 3.16 below in order to simplify (12). For this, we first recall that the families $j(t) : \mathfrak{h} \rightarrow \mathfrak{m}$ are isospectral (in the sense of Definition 1.10 or Definition 3.5, respectively), which means that for each $Z \in \mathfrak{h}$ the $j_Z(t)$ are all conjugate to each other by elements of $O(n)$, resp. $SU(n)$; hence $\operatorname{tr}(j_Z(t))^2$ is constant in t . Moreover, $2 \operatorname{tr}(j_{Z_1}(t)j_{Z_2}(t))$ equals the coefficient at su of $\operatorname{tr}(j_{sZ_1+uZ_2}(t)^2)$ and is thus constant in t . Finally, $2 \operatorname{tr}(j_{Z_1}(t)j_{Z_2}(t)j_{Z_1}(t)j_{Z_2}(t)) + 4 \operatorname{tr}(j_{Z_1}(t)^2j_{Z_2}(t)^2)$ is also constant in t since it equals the coefficient at s^2u^2 of $\operatorname{tr}(j_{sZ_1+uZ_2}(t)^4)$. Lemma 3.16 thus implies in our context:

$$\text{If } \mathfrak{m} = \mathfrak{so}(n) \text{ then } \operatorname{tr}((\operatorname{ad}_{j_{Z_1}(t)})^2(\operatorname{ad}_{j_{Z_2}(t)})^2) = \text{const} + (n-2)\operatorname{tr}(j_{Z_1}(t)^2j_{Z_2}(t)^2);$$

$$\text{if } \mathfrak{m} = \mathfrak{su}(n) \text{ then } \operatorname{tr}((\operatorname{ad}_{j_{Z_1}(t)})^2(\operatorname{ad}_{j_{Z_2}(t)})^2) = \text{const} + 2n \operatorname{tr}(j_{Z_1}(t)^2j_{Z_2}(t)^2),$$

where “const” means constant in t . Note that $n-2 \neq 0$ since in all examples we needed $n > 2$. Thus in any of the isospectral families $j(t) : \mathfrak{h} \rightarrow \mathfrak{m}$ from Section 3.2 we have that (12) is equivalent to

$$(13) \quad \operatorname{tr}(j_{Z_1}(t)^2j_{Z_2}(t)^2) \neq \text{const in } t.$$

But this was indeed always the case. In fact, in all our examples we had $\operatorname{tr}((j_{Z_1}(t)^2 + j_{Z_2}(t)^2)^2) \neq \text{const in } t$ by condition 2.) of Example 3.3 (which was assumed in Examples 3.3, 3.8, and shown to hold in Example 3.10), resp. condition 2.) of Proposition 3.6(i) (which was assumed in Examples 3.7, 3.9). This implies (13) since $\operatorname{tr}(j_{Z_1}(t)^4)$ and $\operatorname{tr}(j_{Z_2}(t)^4)$ are constant in t by the isospectrality assumption. \square

In the proof of Proposition 3.15 we have used the following formulas for which we did not find a reference.

Lemma 3.16.

- (i) *Let $X, Y \in \mathfrak{so}(n)$ and ad be the adjoint representation of $\mathfrak{so}(n)$ on itself. Then*

$$\begin{aligned} & \operatorname{tr}((\operatorname{ad}_X)^2(\operatorname{ad}_Y)^2) \\ &= (n-6)\operatorname{tr}(X^2Y^2) - 2 \operatorname{tr}(XYXY) + \operatorname{tr}(X^2)\operatorname{tr}(Y^2) + 2(\operatorname{tr}(XY))^2. \end{aligned}$$

- (ii) *Let $X, Y \in \mathfrak{su}(n)$ and ad be the adjoint representation of $\mathfrak{su}(n)$ on itself. Then*

$$\operatorname{tr}((\operatorname{ad}_X)^2(\operatorname{ad}_Y)^2) = 2n \operatorname{tr}(X^2Y^2) + 2 \operatorname{tr}(X^2)\operatorname{tr}(Y^2) + 4(\operatorname{tr}(XY))^2.$$

Proof. (i) Note that the adjoint representation of $\mathfrak{so}(n)$ on itself is equivalent to the canonical representation ρ of $\mathfrak{so}(n)$ on $\bigwedge^2 \mathbb{R}^n$, given by $\rho_X(y \wedge v) = Xy \wedge v + y \wedge Xv$. Let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbb{R}^n , and define a scalar product on $\bigwedge^2 \mathbb{R}^n$ by $\langle y \wedge v, w \wedge z \rangle = \langle y, w \rangle \langle v, z \rangle - \langle y, z \rangle \langle v, w \rangle$. Then

$$\mathrm{tr}(\rho_X^2 \rho_Y^2) = \frac{1}{2} \sum_{i,k=1}^n \langle \rho_X^2 \rho_Y^2(e_i \wedge e_k), (e_i \wedge e_k) \rangle.$$

That this is indeed equal to the right hand side of the formula in (i) follows by straightforward calculation.

(ii) First consider the adjoint representation of the complex Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ of dimension n^2 on itself, which is equivalent to the canonical representation ρ of $\mathfrak{gl}(n, \mathbb{C})$ on $(\mathbb{C}^n)^* \otimes \mathbb{C}^n$ given by $\rho_X(y^*v) = -({}^t X y)^*v + y^*(Xv)$. Similarly as in (i) we obtain by direct computation that

$$\begin{aligned} \mathrm{tr}(\rho_X^2 \rho_Y^2) &= 2n \mathrm{tr}(X^2 Y^2) + 2 \mathrm{tr}(X^2) \mathrm{tr}(Y^2) + 4(\mathrm{tr}(XY))^2 \\ &\quad - 4 \mathrm{tr}(X) \mathrm{tr}(XY^2) - 4 \mathrm{tr}(Y) \mathrm{tr}(X^2 Y) \end{aligned}$$

for all $X, Y \in \mathfrak{gl}(n, \mathbb{C})$. For $X, Y \in \mathfrak{su}(n)$ the last two terms vanish. Moreover, for $X, Y \in \mathfrak{su}(n)$ the trace of $\rho_X^2 \rho_Y^2$ on $\mathfrak{gl}(n, \mathbb{C})$, interpreted now as the *real* Lie algebra $\mathfrak{u}(n) \oplus i\mathfrak{u}(n)$ of dimension $2n^2$, equals two times the right hand side of the above formula on the one hand, and two times the trace of $\mathrm{ad}_X^2 \mathrm{ad}_Y^2$ on $\mathfrak{u}(n)$ on the other hand. Since $\mathfrak{u}(n)$ is the sum of its center (spanned by $i\mathrm{Id}$) and $\mathfrak{su}(n)$, the assertion of (ii) now follows. \square

Corollary 3.17. *In all examples given in Section 3.2, the isospectral manifolds $(G, g_{\lambda(t)})$ are not pairwise isospectral for the Laplace operator acting on 1-forms.*

More generally, if $(M, g), (M', g')$ is any pair of homogeneous manifolds which are isospectral for the Laplace operator on functions and for which the (constant) functions $\|\mathrm{Ric}^g\|_g^2$ and $\|\mathrm{Ric}^{g'}\|_{g'}^2$ are nonequal, then the associated Laplace operators on 1-forms are not isospectral.

Proof. We only need to prove the second statement since by Proposition 3.15 the norm of the Ricci tensors associated with the metrics $g_{\lambda(t)}$ from Section 3.2 does change nontrivially as t varies.

For any Riemannian metric g on a closed Riemannian manifold M the heat invariants for the associated Laplace operator on p -forms ($0 \leq p \leq \dim M$) are the coefficients $a_i^p(g)$ occurring in the asymptotic expansion

$$\mathrm{tr}(\exp(-s\Delta_g^p)) \sim (4\pi s)^{-\dim M/2} \sum_{i=0}^{\infty} a_i^p(g) s^i \quad \text{for } s \searrow 0.$$

By [18], Theorem 4.8.18 we have

$$\begin{aligned} a_0^0(g) &= \mathrm{vol}(M, g), \quad a_1^0(g) = \frac{1}{6} \int_M \mathrm{scal}^g \, d\mathrm{vol}_g, \\ a_2^0(g) &= \frac{1}{360} \int_M (5(\mathrm{scal}^g)^2 - 2\|\mathrm{Ric}^g\|_g^2 + 2\|R^g\|_g^2) \, d\mathrm{vol}_g, \\ a_{\frac{1}{2}}^1(g) &= a_2^0(g) \cdot \dim M - \frac{1}{12} \int_M (2(\mathrm{scal}^g)^2 - 6\|\mathrm{Ric}^g\|_g^2 + \|R^g\|_g^2) \, d\mathrm{vol}_g, \end{aligned}$$

where scal^g , Ric^g , and R^g denote the scalar curvature, Ricci tensor, and curvature tensor associated with g . The first two of the above heat invariants imply that if (M, g) and (M', g') are homogeneous and isospectral on functions, then their (constant) scalar curvatures are the same; in particular, we then also have $\int_M (\text{scal}^g)^2 d\text{vol}_g = \int_{M'} (\text{scal}^{g'})^2 d\text{vol}_{g'}$. By $a_2^0(g) = a_2^0(g')$, the numbers $x := \int_M \|\text{Ric}^g\|_g^2 d\text{vol}_g - \int_{M'} \|\text{Ric}^{g'}\|_{g'}^2 d\text{vol}_{g'}$ and $y := \int_M \|R^g\|_g^2 d\text{vol}_g - \int_{M'} \|R^{g'}\|_{g'}^2 d\text{vol}_{g'}$ satisfy $-2x + 2y = 0$. If now in addition the two manifolds were isospectral on 1-forms, then $a_2^1(g) = a_2^1(g')$ and thus $-6x + y = 0$. These two equations together imply $x = y = 0$; but $x = 0$ contradicts our assumption. \square

The rest of this section is devoted to the proof of Theorem 3.14. We continue to use Notation 3.13; recall in particular that we consider linear maps $\lambda : \mathfrak{g} \rightarrow \mathfrak{g}$ with image in \mathfrak{h} and $\lambda|_{\mathfrak{h} \oplus \mathfrak{u}} = 0$, that $j = {}^T \lambda$ with respect to g_0 , and that $\tilde{X} = X - \lambda(X)$ for $X \in \mathfrak{g}$. First we need a formula for the Ricci tensor Ric^λ of (G, g_λ) .

Lemma 3.18. *For all $X \in \mathfrak{g}$ we have*

$$\text{Ric}^\lambda(\tilde{X}, \tilde{X}) = \text{Ric}^0((\text{Id} + j)X, (\text{Id} + j)X) - \langle \text{ad}_X, \lambda \circ \text{ad}_X \rangle_0 - \frac{1}{2} \|\lambda \circ \text{ad}_X\|_0^2.$$

Proof. Using the general formula for the Ricci tensor of a homogeneous manifold given in [5], Corollary 7.38, and the fact that $\{\tilde{V}_1, \dots, \tilde{V}_d\}$ is a g_λ -orthonormal basis of \mathfrak{g} , we have

$$(14) \quad \text{Ric}^\lambda(\tilde{X}, \tilde{X}) = -\frac{1}{2} \|\text{ad}_{\tilde{X}}\|_\lambda^2 - \frac{1}{2} \text{tr}((\text{ad}_{\tilde{X}})^2) + \frac{1}{4} \sum_{i,k=1}^d \langle \tilde{X}, [\tilde{V}_i, \tilde{V}_k] \rangle_\lambda^2.$$

We will show that

$$(15) \quad -\frac{1}{2} \|\text{ad}_{\tilde{X}}\|_\lambda^2 - \frac{1}{2} \text{tr}((\text{ad}_{\tilde{X}})^2) = -\langle \text{ad}_X, \lambda \circ \text{ad}_X \rangle_0 - \frac{1}{2} \|\lambda \circ \text{ad}_X\|_0^2 \\ + \langle \text{ad}_X, \text{ad}_X \circ \lambda \rangle_0 - \frac{1}{2} \|\text{ad}_X \circ \lambda\|_0^2$$

and

$$(16) \quad \frac{1}{4} \sum_{i,k=1}^d \langle \tilde{X}, [\tilde{V}_i, \tilde{V}_k] \rangle_\lambda^2 = \text{Ric}^0((\text{Id} + j)X, (\text{Id} + j)X) - \langle \text{ad}_X, \text{ad}_X \circ \lambda \rangle_0 + \frac{1}{2} \|\text{ad}_X \circ \lambda\|_0^2.$$

These two formulas, together with (14), will clearly imply our statement.

Note that $\lambda[X, \lambda(Y)] = 0$ for all $X, Y \in \mathfrak{g}$ since $\text{ad}_{\lambda(Y)}$ annihilates $\mathfrak{z}(\mathfrak{h})$ and thus has image in \mathfrak{u} , on which λ vanishes. Therefore

$$\|\text{ad}_{\tilde{X}}\|_\lambda^2 = \sum_{i=1}^d \|[\tilde{X}, \tilde{V}_i]\|_\lambda^2 = \sum_{i=1}^d \|[X, V_i] - [\lambda(X), V_i] - [X, \lambda(V_i)] + \lambda([X, V_i])\|_0^2 \\ = \|\text{ad}_X - \text{ad}_{\lambda(X)} - \text{ad}_X \circ \lambda + \lambda \circ \text{ad}_X\|_0^2.$$

Moreover,

$$-\mathrm{tr}((\mathrm{ad}_{\tilde{X}})^2) = \sum_{i=1}^d \|\tilde{X}, V_i\|_0^2 = \sum_{i=1}^d \|[X, V_i] - [\lambda(X), V_i]\|_0^2 = \|\mathrm{ad}_X - \mathrm{ad}_{\lambda(X)}\|_0^2.$$

Thus

$$-\|\mathrm{ad}_{\tilde{X}}\|_\lambda^2 - \mathrm{tr}((\mathrm{ad}_{\tilde{X}})^2) = 2\langle \mathrm{ad}_X - \mathrm{ad}_{\lambda(X)}, \mathrm{ad}_X \circ \lambda - \lambda \circ \mathrm{ad}_X \rangle_0 - \|\mathrm{ad}_X \circ \lambda - \lambda \circ \mathrm{ad}_X\|_0^2.$$

Since $\mathrm{ad}_{\lambda(X)}$ annihilates $\mathfrak{z}(\mathfrak{h})$ and has image in \mathfrak{u} , it is g_0 -orthogonal to both of $\mathrm{ad}_X \circ \lambda$ (which annihilates $\mathfrak{u} = \mathfrak{z}(\mathfrak{h})^\perp$) and $\lambda \circ \mathrm{ad}_X$ (whose image is contained in $\mathfrak{h} \subset \mathfrak{u}^\perp$). Moreover, $\mathrm{ad}_X \circ \lambda$ has image in \mathfrak{u} and is therefore g_0 -orthogonal to $\lambda \circ \mathrm{ad}_X$. Formula (15) now follows; it remains to show formula (16). We have

$$\begin{aligned} \sum_{i,k=1}^d \langle \tilde{X}, [\tilde{V}_i, \tilde{V}_k] \rangle_\lambda^2 &= \sum_{i,k=1}^d \langle X, [V_i, V_k] - [V_i, \lambda(V_k)] + [V_k, \lambda(V_i)] + \lambda([V_i, V_k]) \rangle_0^2 \\ &= \sum_{i,k=1}^d (\langle X + j(X), [V_i, V_k] \rangle_0^2 + \langle X, [V_i, \lambda(V_k)] - [V_k, \lambda(V_i)] \rangle_0^2 \\ &\quad - 4\langle X, [V_i, V_k] \rangle_0 \langle X, [V_i, \lambda(V_k)] \rangle_0 - 4\langle X, \lambda([V_i, V_k]) \rangle_0 \langle X, [V_i, \lambda(V_k)] \rangle_0) \\ &= \sum_{k=1}^d (\|[X + j(X), V_k]\|_0^2 + 2\|[X, \lambda(V_k)]\|_0^2 + 2\langle X, [V_k, \lambda([X, \lambda(V_k)])] \rangle_0^2 \\ &\quad - 4\langle [X, V_k], [X, \lambda(V_k)] \rangle_0 + 4\langle X, \lambda([X, \lambda(V_k)], V_k] \rangle_0) \\ &= 4\mathrm{Ric}^0(X + j(X), X + j(X)) + 2\|\mathrm{ad}_X \circ \lambda\|_0^2 + 0 - 4\langle \mathrm{ad}_X, \mathrm{ad}_X \circ \lambda \rangle_0 + 0. \end{aligned}$$

Here, the third term is zero because of $\lambda([X, \lambda(\cdot)]) = 0$ (see above); to see that the fifth term is zero, we assume the g_0 -orthonormal basis $\{V_1, \dots, V_d\}$ to be adapted to the g_0 -orthogonal decomposition $\mathfrak{g} = (\mathfrak{z}(\mathfrak{h}) \cap \mathfrak{h}^\perp) \oplus (\mathfrak{h} \oplus \mathfrak{u})$. The term $\lambda([X, \lambda(V_k)], V_k]$ vanishes for $V_k \in \mathfrak{h} \oplus \mathfrak{u} \subseteq \ker \lambda$; but for $V_k \in \mathfrak{z}(\mathfrak{h}) \cap \mathfrak{h}^\perp$ (which is obviously a Lie subalgebra of \mathfrak{g}), we have $[\mathfrak{u}, V_k] \subseteq \mathfrak{u}$ and thus $\lambda([X, \lambda(V_k)], V_k] = 0$ (recall that $\mathrm{ad}_X \circ \lambda$ has image in $\mathfrak{u} \subseteq \ker \lambda$). Formula (16) now follows. \square

Proof of Theorem 3.14.

For all $X, Y \in \mathfrak{g}$ we have

$$\begin{aligned} \langle \mathrm{ad}_X, \lambda \circ \mathrm{ad}_Y \rangle_0 &= \sum_{k=1}^d \langle [X, V_k], \lambda([Y, V_k]) \rangle_0 = \sum_{k=1}^d \sum_{i=1}^r \langle [X, V_k], Z_i \rangle_0 \langle j_{Z_i}, [Y, V_k] \rangle_0 \\ &= \sum_{i=1}^r \langle [X, Z_i], [Y, j_{Z_i}] \rangle_0 = - \sum_{i=1}^r \langle \mathrm{ad}_{j_{Z_i}} \mathrm{ad}_{Z_i} X, Y \rangle_0 \end{aligned}$$

which is, in particular, symmetric in X and Y since $j_{Z_i} \in \mathfrak{z}(\mathfrak{h})$ commutes with $Z_i \in \mathfrak{h}$. Similarly,

$$\begin{aligned} \langle \lambda \circ \mathrm{ad}_X, \lambda \circ \mathrm{ad}_Y \rangle_0 &= \sum_{k=1}^d \langle \lambda([X, V_k]), \lambda([Y, V_k]) \rangle_0 \\ &= \sum_{k=1}^d \sum_{i=1}^r \langle j_{Z_i}, [X, V_k] \rangle_0 \langle j_{Z_i}, [Y, V_k] \rangle_0 = - \sum_{i=1}^r \langle (\mathrm{ad}_{j_{Z_i}})^2 X, Y \rangle_0. \end{aligned}$$

We can thus reformulate Lemma 3.18 as

$$(17) \quad \begin{aligned} \text{Ric}^\lambda(\tilde{X}, \tilde{Y}) &= \langle \text{Ric}^0(\text{Id} + j)X, (\text{Id} + j)Y \rangle_0 + \sum_{i=1}^r \langle \text{ad}_{Z_i} \text{ad}_{j_{Z_i}} X, Y \rangle_0 \\ &+ \frac{1}{2} \sum_{i=1}^r \langle (\text{ad}_{j_{Z_i}})^2 X, Y \rangle_0. \end{aligned}$$

For any given pair $Z, W \in \mathfrak{h}$ there exist, by our assumption on λ and λ' , elements $a_{s,u} \in G$ such that $j'_{sZ+uW} = \text{Ad}_{a_{s,u}} j_{sZ+uW}$ and $\text{Ad}_{a_{s,u}}|_{\mathfrak{h}} = \text{Id}|_{\mathfrak{h}}$ for all $s, u \in \mathbb{R}$. Note that the endomorphism $\sum_{i=1}^d \text{ad}_{V_i}^2 = -4 \text{Ric}^0$ of \mathfrak{g} restricts to a scalar multiple of the Casimir operator of the adjoint action on each irreducible component of \mathfrak{g} and thus commutes with every inner automorphism. Consequently,

$$\begin{aligned} &\langle \text{Ric}^0(\text{Id} + j')(sZ + uW), (\text{Id} + j')(sZ + uW) \rangle_0 \\ &= \langle \text{Ad}_{a_{s,u}} \text{Ric}^0(\text{Id} + j)(sZ + uW), \text{Ad}_{a_{s,u}}(\text{Id} + j)(sZ + uW) \rangle_0 \\ &= \langle \text{Ric}^0(\text{Id} + j)(sZ + uW), (\text{Id} + j)(sZ + uW) \rangle_0 \end{aligned}$$

for all $s, u \in \mathbb{R}$; comparing the coefficients at su , we obtain that $\langle \text{Ric}^0(\text{Id} + j')Z, (\text{Id} + j')W \rangle_0 = \langle \text{Ric}^0(\text{Id} + j)Z, (\text{Id} + j)W \rangle_0$ for all $Z, W \in \mathfrak{h}$. Since the j_{Z_i} commute with \mathfrak{h} , we have by (17) that these expressions equal $\text{Ric}^\lambda(Z, W)$ and $\text{Ric}^{\lambda'}(Z, W)$, respectively. We conclude

$$(18) \quad \text{Ric}^\lambda|_{\mathfrak{h} \times \mathfrak{h}} = \text{Ric}^{\lambda'}|_{\mathfrak{h} \times \mathfrak{h}}.$$

For $Z \in \mathfrak{h}$ and $X \in \mathfrak{h}^\perp$ we have $j_X = j'_X = 0$ and we obtain, similarly as above, $\langle \text{Ric}^0(\text{Id} + j')Z, X \rangle_0 = \langle \text{Ad}_{a_Z} \text{Ric}^0(\text{Id} + j)Z, X \rangle_0 = \langle \text{Ric}^0(\text{Id} + j)Z, \text{Ad}_{a_Z}^{-1} X \rangle_0$. Since Ad_{a_Z} preserves \mathfrak{h}^\perp , this implies by (17) that

$$(19) \quad \|\text{Ric}^\lambda|_{\mathfrak{h} \times \mathfrak{h}^{\perp \lambda}}\|_\lambda^2 = \|\text{Ric}^{\lambda'}|_{\mathfrak{h} \times \mathfrak{h}^{\perp \lambda'}}\|_{\lambda'}^2,$$

where $\mathfrak{h}^{\perp \lambda}$ denotes the g_λ -orthogonal complement of \mathfrak{h} in \mathfrak{g} . Note also that

$$\begin{aligned} &\sum_{i=1}^r \langle \text{Ric}^0, \text{ad}_{Z_i} \text{ad}_{j'_{Z_i}} + \frac{1}{2}(\text{ad}_{j'_{Z_i}})^2 \rangle_0 \\ &= \sum_{i=1}^r \langle \text{Ric}^0, \text{Ad}_{a_{Z_i}}(\text{ad}_{Z_i} \text{ad}_{j_{Z_i}} + \frac{1}{2}(\text{ad}_{j_{Z_i}})^2) \text{Ad}_{a_{Z_i}}^{-1} \rangle_0 \\ &= \sum_{i=1}^r \langle \text{Ric}^0, \text{ad}_{Z_i} \text{ad}_{j_{Z_i}} + \frac{1}{2}(\text{ad}_{j_{Z_i}})^2 \rangle_0. \end{aligned}$$

Moreover, both of ad_{Z_i} and $\text{ad}_{j_{Z_i}}$ vanish on \mathfrak{h} and have image in \mathfrak{h}^\perp because of $Z_i, j_{Z_i} \in \mathfrak{z}(\mathfrak{h})$. Therefore, again by (17):

$$(20) \quad \begin{aligned} \|\text{Ric}^{g_\lambda}|_{\mathfrak{h}^{\perp \lambda} \times \mathfrak{h}^{\perp \lambda}}\|_\lambda^2 &= \|\text{Ric}^0|_{\mathfrak{h}^{\perp \lambda} \times \mathfrak{h}^{\perp \lambda}}\|_0^2 + 2 \sum_{i=1}^r \langle \text{Ric}^0, \text{ad}_{Z_i} \text{ad}_{j_{Z_i}} + \frac{1}{2}(\text{ad}_{j_{Z_i}})^2 \rangle_0 \\ &+ \left\| \sum_{i=1}^r (\text{ad}_{Z_i} \text{ad}_{j_{Z_i}} + \frac{1}{2}(\text{ad}_{j_{Z_i}})^2) \right\|_0^2. \end{aligned}$$

As we just saw, only the third summand in this expression might differ from the corresponding one for λ' . We have

$$\begin{aligned}
(21) \quad & \left\| \sum_{i=1}^r (\text{ad}_{Z_i} \text{ad}_{j_{Z_i}} + \frac{1}{2} (\text{ad}_{j_{Z_i}})^2) \right\|_0^2 \\
&= \sum_{i,k=1}^r \text{tr}(\text{ad}_{Z_i} \text{ad}_{j_{Z_i}} \text{ad}_{Z_k} \text{ad}_{j_{Z_k}}) + \sum_{i,k=1}^r \text{tr}(\text{ad}_{Z_i} \text{ad}_{j_{Z_i}} (\text{ad}_{j_{Z_k}})^2) \\
&\quad + \frac{1}{4} \sum_{i,k=1}^r \text{tr}((\text{ad}_{j_{Z_i}})^2 (\text{ad}_{j_{Z_k}})^2).
\end{aligned}$$

Noting again that the Z_i commute with the j_{Z_k} , we see that $\text{tr}(\text{ad}_{Z_i} \text{ad}_{Z_k} \text{ad}_{j'_{Z_i}} \text{ad}_{j'_{Z_k}})$ equals half the coefficient at su of

$$\begin{aligned}
\text{tr}(\text{ad}_{Z_i} \text{ad}_{Z_k} (\text{ad}_{j'_{sZ_i+uZ_k}})^2) &= \text{tr}(\text{Ad}_{a_{s,u}} \text{ad}_{Z_i} \text{ad}_{Z_k} (\text{ad}_{j_{sZ_i+uZ_k}})^2 \text{Ad}_{a_{s,u}}^{-1}) \\
&= \text{tr}(\text{ad}_{Z_i} \text{ad}_{Z_k} (\text{ad}_{j_{sZ_i+uZ_k}})^2).
\end{aligned}$$

Similarly, $\text{tr}(\text{ad}_{Z_i} \text{ad}_{j'_{Z_i}} (\text{ad}_{j'_{Z_k}})^2)$ equals one third of the coefficient at su^2 of

$$\begin{aligned}
\text{tr}(\text{ad}_{Z_i} (\text{ad}_{j'_{sZ_i+uZ_k}})^3) &= \text{tr}(\text{Ad}_{a_{s,u}} \text{ad}_{Z_i} (\text{ad}_{j_{sZ_i+uZ_k}})^3 \text{Ad}_{a_{s,u}}^{-1}) \\
&= \text{tr}(\text{ad}_{Z_i} (\text{ad}_{j_{sZ_i+uZ_k}})^3).
\end{aligned}$$

This, together with the formulas (18)–(21), implies the statement of the theorem. \square

3.4 Infinitesimal spectral rigidity of bi-invariant metrics.

In Section 3.2 we used Proposition 3.1 to construct many examples of continuous families $g_{\lambda(t)}$ of left invariant, isospectral, non-isometric metrics on compact Lie groups G . As we saw, these families can occur arbitrarily close to a bi-invariant metric g_0 on G (Remark 3.12(i)). Obviously, however, our construction never yields any isospectral deformations containing g_0 itself. In fact, if $\lambda = 0$ and λ, λ' satisfy condition (*3) of Proposition 3.1, then also $\lambda' = 0$.

A natural question to ask in this context is whether nontrivial, continuous isospectral deformations of bi-invariant metrics within the class of left invariant metrics, even though not available by our construction, might exist nevertheless. By the following theorem the answer to this question is no.

Theorem 3.19. *Let G be a compact Lie group and g_0 be a bi-invariant metric on G . Let $\eta > 0$ and $\{g(t)\}_{t \in (-\eta, \eta)}$ be a continuous family of left invariant metrics on G such that $g(0) = g_0$. If the metrics $g(t)$ are pairwise isospectral, then $g(t) \equiv g_0$ for all t .*

Proof. As in the ‘‘algebraic proof’’ of Proposition 3.1, denote the right-regular unitary representation of G on $L^2(G)$ by ρ . Let $U \subseteq L^2(G)$ be a linear subspace which

is invariant under ρ and irreducible. Since G is compact and the action is unitary, any such U is finite dimensional. If g is any left invariant metric on G then we have by (7) that $\Delta_g = -\sum_{i=1}^d X_i^2 = -\sum_{i=1}^d (\rho_* X_i)^2$, where $\{X_1, \dots, X_d\}$ is a left invariant g -orthonormal frame. Since U is invariant under ρ it is also invariant under Δ_g and thus spanned by eigenfunctions; in particular $U \subset C^\infty(G)$ because U is finite dimensional.

Now consider our family $\Delta_{g(t)}$ restricted to U . For every t , $\text{spec}(\Delta_{g(t)}|_U)$ is contained in the discrete set $\text{spec}(G, g(t))$ which is independent of t by assumption. Since $g(t)$ and therefore $\text{spec}(\Delta_{g(t)}|_U)$ depends continuously on t , $\text{spec}(\Delta_{g(t)}|_U)$ must be independent of t , too.

The metric $g(0) = g_0$ is bi-invariant, hence $\Delta_{g(0)}$ commutes with right translations (which are g_0 -isometries) and therefore with the representation ρ . Since U is irreducible, it follows from Schur's lemma that $\Delta_{g(0)}|_U$ is a multiple of the identity. Now $\text{spec}(\Delta_{g(t)}|_U) = \text{spec}(\Delta_{g(0)}|_U)$ implies that $\Delta_{g(t)}|_U$ too is a multiple of the identity and equals $\Delta_{g(0)}|_U$.

Recall that $L^2(G)$ is a sum of invariant, irreducible subspaces such as the one we just considered. Since $\Delta_{g(t)}$ and $\Delta_{g(0)}$ coincide on each of these, they coincide completely on $C^\infty(G)$. But if two Riemannian metrics on a manifold have equal Laplacians then they are themselves equal. Thus $g(t) \equiv g(0)$ for all t , as claimed. \square

4. Conformally equivalent manifolds which are isospectral and not locally isometric

In the first section of this chapter we present a canonical generalization of Theorem 1.6 / Proposition 1.8 from Chapter 1; see Theorem 4.3 / Proposition 4.5. Here the fibers of the torus bundles under consideration are in general no longer totally geodesic. In particular, we hereby leave the context of Theorem 1.3 which was our general starting point in Chapter 1. However, Theorem 4.3 and Proposition 4.5 can be regarded as special versions of another theorem which was established recently by C. Gordon and Z. Szabó in [23]; see Remark 4.4(ii) below.

In Section 4.2 we use Proposition 4.5 to construct the first pairs of isospectral manifolds which are conformally equivalent and not locally isometric. Note that in exactly one instance there have previously been examples (even continuous families) of isospectral, conformally equivalent manifolds; namely, those constructed in 1990 by R. Brooks and C. Gordon [6]. However, the manifolds in these isospectral families had (as all examples of isospectral manifolds known at that time) a common Riemannian covering and thus were locally isometric.

For proving that our new examples are not locally isometric we use results from Chapter 3 to show that the preimages of the maximal scalar curvature on the two manifolds are not locally isometric because their Ricci tensors (associated with the induced metrics) have different norms; see Proposition 4.7.

4.1 Isospectral torus bundles whose fibers are not totally geodesic.

Notation 4.1. Let H be a torus with Lie algebra $\mathfrak{h} = T_e H$, and let H be equipped

with a fixed invariant metric. Let M be a principal H -bundle over a closed Riemannian manifold (N, h) , and let $\varphi, \psi \in C^\infty(N, \mathbb{R}_+)$.

- (i) Given a connection form ω on M we denote by $g_{\omega, \varphi, \psi}$ the unique H -invariant Riemannian metric on M which satisfies:
 - 1.) For each $p \in N$, the induced metric $g_{\omega, \varphi, \psi}|_{\pi_H^{-1}(p)}$ on the fiber over p equals $\varphi(p)$ times the given invariant metric (induced from the metric on H).
 - 2.) The projection $\pi_H : M \rightarrow N$ is a Riemannian submersion with respect to $g_{\omega, \varphi, \psi}$ on M and ψh on N .
 - 3.) The ω -horizontal distribution $\ker \omega$ is $g_{\omega, \varphi, \psi}$ -orthogonal to the fibers. In particular, note that $g_{\omega, \varphi, \psi}(X, Z) = \tilde{\varphi}(x)\omega_Z(X)$ for all $X \in T_x M$ and $Z \in \mathfrak{h}$, where $\tilde{\varphi}$ denotes the lift of φ to M .
- (ii) If M is the trivial bundle $N \times H$ and λ is an \mathfrak{h} -valued 1-form on N we write $g_{\lambda, \varphi, \psi} := g_{\omega, \varphi, \psi}$, where ω is the connection form on $N \times H$ defined by $\omega(X, Z) = \lambda(X) + Z$ for all $(X, Z) \in T(N \times H) \cong TN \times \mathfrak{h}$.

Remark 4.2. The metric g_ω which was defined in Notation 1.5(iii) is just the same as $g_{\omega, 1, 1}$. In other words, introducing the metric $g_{\omega, \varphi, \psi}$ on M can be described as first introducing g_ω and then stretching vertical vectors by $\tilde{\varphi}^{1/2}$ and horizontal vectors by $\tilde{\psi}^{1/2}$, where $\tilde{\varphi}$ and $\tilde{\psi}$ are the lifts of φ and ψ to M . In the context of 4.1(ii), introducing $g_{\lambda, \varphi, \psi}$ can be described analogously, this time by first introducing the metric g_λ on $N \times H$ which was defined in Notation 1.5(iv).

Note also that for $\psi := \varphi$, the metric $g_{\omega, \varphi, \varphi}$ equals $\tilde{\varphi}g_\omega$ which is conformally equivalent to g_ω . Similarly we have $g_{\lambda, \varphi, \varphi} = \tilde{\varphi}g_\lambda$ in the context of 4.1(ii).

Theorem 4.3. *Let (N, h) be a closed Riemannian manifold and H be a torus equipped with an invariant metric. Let M be a principal H -bundle over (N, h) , let ω, ω' be two connection forms on M , and let $\varphi, \varphi', \psi, \psi' \in C^\infty(N, \mathbb{R}_+)$. Assume:*

- (*5) *For every $Z \in \mathfrak{h}$ there exists a bundle automorphism $F_Z : M \rightarrow M$ which induces an isometry \bar{F}_Z on the base manifold (N, h) and satisfies*

$$\omega'_Z = F_Z^* \omega_Z, \quad \varphi' = \bar{F}_Z^* \varphi, \quad \text{and} \quad \psi' = \bar{F}_Z^* \psi.$$

Then $(M, g_{\omega, \varphi, \psi})$ and $(M, g_{\omega', \varphi', \psi'})$ are isospectral.

Proof. In the following we write $g = g_{\omega, \varphi, \psi}$ and $g' = g_{\omega', \varphi', \psi'}$. Let $\mathcal{H} = L^2(M, g) = L^2(M, g')$. For any closed connected subgroup $W \subset H$ of codimension 1 we denote, as in the proof of Theorem 1.3, by \mathcal{H}_W the space of W -invariant functions in \mathcal{H} . Let $\mathcal{C}_W := C^\infty(M) \cap \mathcal{H}_W$. Finally we denote by \mathcal{H}_0 the space of H -invariant functions in \mathcal{H} and let $\mathcal{C}_0 := C^\infty(M) \cap \mathcal{H}_0$. We claim that

$$(22) \quad \Delta_{g'}|_{\mathcal{C}_W} = (F_Z^* \circ \Delta_g \circ F_Z^{*-1})|_{\mathcal{C}_W},$$

where $Z \in \mathfrak{h} \setminus \{0\}$ is chosen orthogonal to $T_e W$, and F_Z is as in (*5). Note that F_Z , being a bundle automorphism, leaves the spaces \mathcal{C}_W and $\mathcal{C}_0 \subset \mathcal{C}_W$ invariant. Therefore equation (22) implies $\text{spec}(\Delta_g|_{\mathcal{H}_0}) = \text{spec}(\Delta_{g'}|_{\mathcal{H}_0})$ and $\text{spec}(\Delta_g|_{\mathcal{H}_W}) = \text{spec}(\Delta_{g'}|_{\mathcal{H}_W})$. Since W was arbitrary, we will then be done by the decomposition (1) from the proof of Theorem 1.3.

It remains to prove (22). In contrast to the situation of Theorem 1.6, the fibers of the Riemannian submersions $\pi_H : (M, g) \rightarrow (N, \psi h)$ and $\pi_H : (M, g') \rightarrow (N, \psi' h)$

are in general not totally geodesic now (unless φ is constant). The proof of equation (22) has to take into account the mean curvature vector fields V (on (M, g)) and V' (on (M, g')) of the H -orbits in M . We first compute V and V' . Let ∇ be the Levi-Civita connection of g , and denote by $\tilde{\varphi}, \tilde{\psi}$ the lifts of φ, ψ to M . For any $Z \in \mathfrak{h}$ the vector field $\nabla_Z Z$ is obviously g -orthogonal to the H -orbits and thus ω -horizontal. For any ω -horizontal, H -invariant vector field X on M we have, noting that X commutes with Z :

$$\begin{aligned} \psi h(\pi_{H^*}(\nabla_Z Z), \pi_{H^*} X) &= g(\nabla_Z Z, X) = Z(g(Z, X)) - g(Z, \nabla_Z X) \\ &= Z(\tilde{\varphi} g_\omega(Z, X)) - g(Z, \nabla_X Z) \\ &= 0 - \frac{1}{2} X(\tilde{\varphi} g_\omega(Z, Z)) = -\frac{1}{2} |Z|^2 X(\tilde{\varphi}) \\ &= -\frac{1}{2} |Z|^2 \psi h\left(\frac{1}{\tilde{\psi}} \text{grad}_h \varphi, \pi_{H^*} X\right). \end{aligned}$$

Letting Z run through g -orthonormal bases of the tangent spaces to the H -orbits and summing up, we get

$$V = \omega\text{-horizontal lift of } \frac{-\dim \mathfrak{h}}{2\varphi\psi} \text{grad}_h \varphi.$$

Analogously, V' is the ω' -horizontal lift of $\frac{-\dim \mathfrak{h}}{2\varphi'\psi'} \text{grad}_h \varphi'$.

Now let $x \in M$ and $p = \pi_H(x) \in N$. Choose a local frame $\{E_1, \dots, E_n\}$ on a neighbourhood U of p such that $\{E_1(p), \dots, E_n(p)\}$ is a $\psi' h$ -orthonormal basis of $T_p N$, and such that the integral curves of the E_i through p are geodesics in $(N, \psi' h)$. Denote the ω' -horizontal lift of E_i to $\pi_H^{-1}(U) \subseteq M$ by X_i . Since $\pi_H : (M, g') \rightarrow (N, \psi' h)$ is a Riemannian submersion, the integral curves of X_i through x are geodesics in (M, g') . Thus

$$\Delta_{g'}|_x = -\sum_{i=1}^n X_i|_x X_i + \tilde{\varphi}'(x)^{-1} \Delta_{\mathfrak{h}}|_x + V'|_x,$$

where $\Delta_{\mathfrak{h}} := -\sum_{k=1}^r Z_k^2$ and $\{Z_1, \dots, Z_r\}$ is an orthonormal basis of \mathfrak{h} . Now let $y := F_Z(x)$ and $Y_i := F_{Z^*}(X_i)$. Then the Y_i are H -invariant vector fields defined in an H -invariant neighbourhood of y . Since $\omega'_Z = F_Z^* \omega_Z$, $\varphi' = \bar{F}_Z^* \varphi$, $\psi' = \bar{F}_Z^* \psi$, and \bar{F}_Z is an isometry of (N, h) , the vector field $F_{Z^*} V'$ equals V up to errors tangent to the W -orbits. Moreover, each Y_i is ω -horizontal up to errors tangent to the W -orbits. We write $Y_i = A_i + U_i$, where A_i is ω -horizontal and U_i is tangent to the W -orbits. Note that A_i and U_i are again H -invariant, and $[A_i, U_i]$ is tangent to the W -orbits. Hence for $f \in \mathcal{C}_W$ we have

$$V|_x(f) = V'|_x(F_Z^* f) \quad \text{and} \quad A_i|_y A_i(f) = Y_i|_y Y_i(f) = X_i|_x X_i(F_Z^* f).$$

Since F_Z induces an isometry from $(N, \psi' h)$ to $(N, \psi h)$ the $A_i|_y$ are g -orthonormal, and thus

$$\Delta_g|_y = -\sum_{i=1}^n A_i|_y A_i + \tilde{\varphi}(y)^{-1} \Delta_{\mathfrak{h}}|_y + V|_y.$$

Therefore we have indeed

$$\begin{aligned} (\Delta_g f)(y) &= \left(- \sum_{i=1}^n X_i^2(F_Z^* f) + \tilde{\varphi}'^{-1} \Delta_{\mathfrak{h}}(F_Z^* f) + V'(F_Z^* f) \right)(x) \\ &= (\Delta_{g'}(F_Z^* f))(x) = ((F_Z^{*-1} \circ \Delta_{g'} \circ F_Z^*)f)(y) \end{aligned}$$

for each $f \in \mathcal{C}_W$. \square

Remarks 4.4.

(i) In the special case $\varphi = \psi = 1$, the accordingly simplified proof constitutes an alternative proof for Theorem 1.6 from Chapter 1 (recall Remark 1.7).

(ii) In turn, there is also an alternative proof of the above Theorem 4.3 along the lines of the proof of Theorem 1.6 given in Chapter 1. It involves showing that F_Z (for $Z \neq 0$ in the orthogonal complement of $T_e W$) induces an isometry from $(M/W, g_{\omega', \varphi', \psi'}^W)$ to $(M/W, g_{\omega, \varphi, \psi}^W)$ which, moreover, carries the projected $g_{\omega', \varphi', \psi'}$ -mean curvature vector field of the W -orbits to the projected $g_{\omega, \varphi, \psi}$ -mean curvature vector field of the W -orbits.

From this version of the proof one sees immediately that Theorem 4.3 is actually a special case of a theorem which was proven recently by C. Gordon and Z. Szabó (Theorem 1.2 in [23]). Their theorem reads like Theorem 1.3 by C. Gordon in Chapter 1, with the following changes: Condition (i) is dropped; in condition (ii), “isospectral” is replaced by “isometric”, and moreover it is required that there exists an isometry between the quotient manifolds which intertwines the projected mean curvature vector fields of the W -orbits.

(iii) In [23], C. Gordon and Z. Szabó apply the theorem mentioned in (ii), and a version of it for the case of manifolds with boundary, to construct a specific class of interesting new examples of isospectral, locally non-isometric manifolds which arise as torus bundle whose fibers are not totally geodesic. The manifolds are diffeomorphic to products of spheres with tori, resp. balls (or bounded cylinders) with tori. In the case of manifolds with boundary, they obtain continuous isospectral families of negatively curved manifolds, which contrasts with the spectral rigidity result by C. Croke and V. Sharafutdinov for closed negatively curved manifolds [14].

Without going into detail, we mention here that those examples which Gordon and Szabó construct in the case of closed manifolds can also be viewed as arising from our (more special) Theorem 4.3; more precisely, from Proposition 4.5 below.

(iv) As we mentioned at the beginning of this chapter, the only examples of conformally equivalent, isospectral manifolds which were previously known had been given by R. Brooks and C. Gordon in [6]. We note here, again without giving details, that those examples too can be interpreted as an application of Theorem 4.3.

We finish this preparatory section by specializing Theorem 4.3 to the case of products; the following proposition is related to Proposition 1.8 in the same way as Theorem 4.3 is to Theorem 1.6. We use Notation 4.1(ii).

Proposition 4.5. *Let (N, h) be a closed Riemannian manifold and H be a torus equipped with an invariant metric. Let $\mathfrak{h} = T_e H$ and λ, λ' be two \mathfrak{h} -valued 1-forms on N . Let $\varphi, \psi, \varphi', \psi' \in C^\infty(N, \mathbb{R}_+)$. Assume:*

- (*6) *For every $Z \in \mathfrak{h}$ there exists an isometry f_Z of (N, h) which satisfies*
 $\lambda'_Z = f_Z^* \lambda_Z$, $\varphi' = f_Z^* \varphi$, and $\psi' = f_Z^* \psi$.

Then $(N \times H, g_{\lambda, \varphi, \psi})$ and $(N \times H, g_{\lambda', \varphi', \psi'})$ are isospectral.

Proof. The connection forms ω, ω' associated with λ, λ' satisfy condition (*5) from Theorem 4.3. In fact, $F_Z := (f_Z, \text{Id}) : N \times H \rightarrow N \times H$ has all the properties required there. \square

4.2 Conformally equivalent examples.

We will now apply Proposition 4.5 to construct the first examples of isospectral manifolds which are conformally equivalent and not locally isometric. The idea is to find, in the context of Proposition 4.5, a situation where $\lambda = \lambda'$ and nevertheless there exists a pair of functions $\varphi = \psi$ and $\varphi' = \psi'$ such that condition (*6) is satisfied *nontrivially*; that is, the isometries f_Z cannot be chosen independently of Z .

Example 4.6. Let K be a compact Lie group with Lie algebra $\mathfrak{k} = T_e K$, and let h be a bi-invariant metric on K . Let H be a torus with Lie algebra $\mathfrak{h} = T_e H$, equipped with an invariant metric. Let $\lambda, \lambda' : \mathfrak{k} \rightarrow \mathfrak{h}$ be two linear maps which satisfy condition (*4) from Corollary 3.2; i.e., for each $Z \in \mathfrak{h}$ there exists $a_Z \in K$ such that $\lambda'_Z = \text{Ad}_{a_Z}^* \lambda_Z$. We endow $K \times K$ with the product metric \bar{h} , and define

$$\bar{\lambda} : \mathfrak{k} \oplus \mathfrak{k} \ni (X, Y) \mapsto \lambda(X) + \lambda'(Y) \in \mathfrak{h}.$$

Choose a class function $\varphi \in C^\infty(K, \mathbb{R}_+)$ (i.e., one that is invariant under inner automorphisms of K), and define $\varphi_1, \varphi_2 : K \times K \rightarrow \mathbb{R}_+$ by

$$\varphi_1(x, y) := \varphi(x), \quad \varphi_2(x, y) := \varphi(y)$$

for all $x, y \in K$. Denote the lifts of φ_1, φ_2 to $K \times K \times H$ by $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, respectively. We claim that the conformally equivalent manifolds

$$(K \times K \times H, \tilde{\varphi}_1 g_{\bar{\lambda}}) \quad \text{and} \quad (K \times K \times H, \tilde{\varphi}_2 g_{\bar{\lambda}})$$

are isospectral by Proposition 4.5, where $g_{\bar{\lambda}}$ is the left invariant metric on $K \times K \times H$ associated with \bar{h} and $\bar{\lambda}$ as in Corollary 3.2.

In fact, for any given $Z \in \mathfrak{h}$ choose $a_Z \in K$ such that $\lambda'_Z = \text{Ad}_{a_Z}^* \lambda_Z$, and define

$$f_Z : K \times K \ni (x, y) \mapsto (I_{a_Z}(y), I_{a_Z}^{-1}(x)) \in K \times K,$$

where I_{a_Z} denotes conjugation by a_Z . Then f_Z is an isometry by the bi-invariance of \bar{h} ; moreover,

$$\begin{aligned} (f_Z^* \bar{\lambda}_Z)(X, Y) &= \lambda_Z(\text{Ad}_{a_Z}(Y)) + \lambda'_Z(\text{Ad}_{a_Z}^{-1}(X)) = \lambda'_Z(Y) + \lambda_Z(X) \\ &= \bar{\lambda}_Z(X, Y) \end{aligned}$$

for all $(X, Y) \in \mathfrak{k} \oplus \mathfrak{k}$. Finally, $(f_Z^* \varphi_2)(x, y) = \varphi_2(I_{a_Z}(y), I_{a_Z}^{-1}(x)) = \varphi(x) = \varphi_1(x, y)$ for all $x, y \in K$ since φ is a class function. So f_Z satisfies all the conditions from Proposition 4.5, with $\bar{\lambda}$ playing the role of both λ and λ' from the proposition, φ_1 playing the role of $\varphi = \psi$, and φ_2 the role of $\varphi' = \psi'$. Recall from Remark 4.2 that $g_{\bar{\lambda}, \varphi_1, \varphi_1} = \tilde{\varphi}_1 g_{\bar{\lambda}}$ and $g_{\bar{\lambda}, \varphi_2, \varphi_2} = \tilde{\varphi}_2 g_{\bar{\lambda}}$.

The following proposition shows that in many cases the two resulting isospectral, conformally equivalent manifolds are not locally isometric.

Proposition 4.7. *In the context of Example 4.6 assume that*

- (i) $K = \mathrm{SO}(m)$, $m \geq 5$, or
- (ii) $K = \mathrm{SU}(m)$, $m \geq 3$,

and that the torus H is two-dimensional. For the isospectral pair $\lambda, \lambda' : \mathfrak{k} \rightarrow \mathfrak{h}$ (as in the example) assume that $\|\mathrm{Ric}^{g_\lambda}\|^2 \neq \|\mathrm{Ric}^{g_{\lambda'}}\|^2$, where g_λ and $g_{\lambda'}$ are the associated left invariant metrics on $K \times H$. Finally, define a class function φ on K by

$$\varphi(x) := e^{2\varepsilon \mathrm{tr}(x)/m},$$

where tr denotes the trace on $(m \times m)$ -matrices, and $0 < \varepsilon < 1/8$. Then the conformal metrics $\tilde{\varphi}_1 g_{\bar{\lambda}}$ and $\tilde{\varphi}_2 g_{\bar{\lambda}}$ on $K \times K \times H$, defined as in Example 4.6, are not locally isometric.

More precisely, the preimage in $K \times K \times H$ of the maximal scalar curvature of $\tilde{\varphi}_1 g_{\bar{\lambda}}$, resp. of $\tilde{\varphi}_2 g_{\bar{\lambda}}$, is a submanifold which, when endowed with the induced metric, is isometric to $(K \times H, e^{2\varepsilon} g_\lambda)$, respectively $(K \times H, e^{2\varepsilon} g_{\lambda'})$, whose Ricci tensors have different norms by the choice of λ and λ' .

Recall that if K and H are of the above type then isospectral pairs $\lambda, \lambda' : \mathfrak{k} \rightarrow \mathfrak{h}$ with $\|\mathrm{Ric}^{g_\lambda}\|^2 \neq \|\mathrm{Ric}^{g_{\lambda'}}\|^2$ exist indeed, even many of them. See the examples 3.3/3.7 in connection with Proposition 3.15.

Proof. Let Id denote the neutral element of $K = \mathrm{SO}(m)$, resp. $K = \mathrm{SU}(m)$, where $m \geq 5$, resp. $m \geq 3$. We claim that for our choice of φ the preimage of the maximal scalar curvature of $(K \times K \times H, \tilde{\varphi}_1 g_{\bar{\lambda}})$, resp. $(K \times K \times H, \tilde{\varphi}_2 g_{\bar{\lambda}})$, is precisely

$$\{\mathrm{Id}\} \times K \times H, \quad \text{resp.} \quad K \times \{\mathrm{Id}\} \times H.$$

These submanifolds, endowed with the metric induced by $\tilde{\varphi}_1 g_{\bar{\lambda}}$, resp. $\tilde{\varphi}_2 g_{\bar{\lambda}}$, are isometric to $(K \times H, e^{2\varepsilon} g_{\lambda'})$, resp. $(K \times H, e^{2\varepsilon} g_\lambda)$, by the definition of $\bar{\lambda}$ and the fact that $\varphi_1(\mathrm{Id}, x) = \varphi_2(x, \mathrm{Id}) = \varphi(\mathrm{Id}) = e^{2\varepsilon}$ for all $x \in K$. The statement of the proposition will thus follow.

We now prove our above claim. Since K is irreducible by assumption, the bi-invariant metric h is a scalar multiple of the Killing metric $-B : (X, Y) \mapsto -\mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y)$; let $c \in \mathbb{R}$ be such that $h = c^2 \cdot (-B)$. Define $\tau(x) := \varepsilon \mathrm{tr}(x)/m$, $\tau_1(x, y) := \tau(x)$ for $(x, y) \in K \times K$, and let $\tilde{\tau}_1$ be the lift of τ_1 to $K \times K \times H$; we thus have $\tilde{\varphi}_1 = \exp(2\tilde{\tau}_1)$. Note that $(\Delta_{g_{\bar{\lambda}}} \tilde{\tau}_1)(x, y, z) = (\Delta_h \tau)(x) = \mu \tau(x) = \mu \tilde{\tau}_1(x, y, z)$ for all $(x, y, z) \in K \times K \times H$, where

$$\mu = \frac{1}{c^2} \cdot \frac{1}{m} \cdot \frac{1}{k(m)} \cdot \dim K$$

with $k(m) = m - 2$ in case (i) and $k(m) = 2m$ in case (ii). By [5], Theorem 1.159 we have

$$\mathrm{scal}^{\tilde{\varphi}_1 g_{\bar{\lambda}}} = \frac{\alpha + \beta \tilde{\tau}_1 - (N - 1)(N - 2) \|\tilde{\tau}_1\|_{g_{\bar{\lambda}}}^2}{\exp(2\tilde{\tau}_1)},$$

where $N := \dim(K \times K \times H) = 2 \dim K + 2$, $\alpha := \mathrm{scal}^{g_{\bar{\lambda}}}$, and $\beta := 2(N - 1)\mu$. Note that $\alpha \leq \mathrm{scal}^{\tilde{h}} = 2 \mathrm{scal}^h = \frac{2}{c^2} \cdot \frac{\dim K}{4}$, where the inequality follows from Proposition 2.8.

We observe that the function $\mathbb{R} \ni s \mapsto \frac{\alpha + \beta s}{\exp(2s)} \in \mathbb{R}$ is strictly monotonously increasing in $s \in (-\infty, \frac{1}{2} - \frac{\alpha}{\beta}]$. The image of our $\tilde{\tau}_1$ is contained in this interval: In fact, we obviously have $\max \tilde{\tau}_1 = \max \tau = \varepsilon$ and

$$\frac{\alpha}{\beta} \leq \frac{\dim K / 2c^2}{2(2 \dim K + 1) \dim K / c^2 m k(m)} \leq \frac{m k(m)}{8 \dim K},$$

which in case (i) equals $\frac{2m(m-2)}{8m(m-1)} \leq \frac{1}{4} < \frac{1}{2} - \varepsilon$, and in case (ii) equals $\frac{2m^2}{8(m^2-1)}$ which for $m \geq 3$ is also smaller than $\frac{1}{2} - \varepsilon$ by the choice of ε .

Since additionally we have $d\tilde{\tau}_1|_p = 0$ if $\tilde{\tau}_1(p)$ is maximal, we conclude that $\text{scal}^{\tilde{\varphi}_1 g_\lambda}$ attains its maximum precisely in those points where $\tilde{\tau}_1$ does so; namely, in $\{\text{Id}\} \times K \times H$, as claimed. In the same way we show that $\text{scal}^{\tilde{\varphi}_2 g_\lambda}$ attains its maximum precisely in $K \times \{\text{Id}\} \times H$. \square

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