ISOSPECTRAL ORBIFOLDS WITH DIFFERENT MAXIMAL ISOTROPY ORDERS

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Abstract. We construct pairs of compact Riemannian orbifolds which are isospectral for the Laplace operator on functions such that the maximal isotropy order of singular points in one of the orbifolds is higher than in the other. In one type of examples, isospectrality arises from a version of the famous Sunada theorem which also implies isospectrality on p-forms; here the orbifolds are quotients of certain compact normal homogeneous spaces. In another type of examples, the orbifolds are quotients of Euclidean $\mathbb{R}^3$ and are shown to be isospectral on functions using dimension formulas for the eigenspaces developed in [12]. In the latter type of examples the orbifolds are not isospectral on 1-forms. Along the way we also give several additional examples of isospectral orbifolds which do not have maximal isotropy groups of different size but other interesting properties.

1. Introduction

This paper is concerned with the spectral geometry of compact Riemannian orbifolds. The notion of Riemannian orbifolds is a generalization of the notion of Riemannian manifolds. In a Riemannian orbifold each point has a neighborhood which can be identified with the quotient of an open subset of a Riemannian manifold by some finite group of isometries acting on this subset.

We omit the exact definitions for general Riemannian orbifolds, which can be found, e.g., in [16], [20], [2], [21], because actually we will be dealing in this article only with the special case of so-called "good" Riemannian orbifolds. A good Riemannian orbifold $\mathcal{O}$ is the quotient of a Riemannian manifold (M, g) by some group of isometries Γ which acts effectively and properly discontinuously on M; that is, for each compact subset $K \subset M$, the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is finite. Let $p : M \to \Gamma\backslash M = \mathcal{O}$ be the canonical projection. For $x \in \mathcal{O}$, the isotropy group $\mathrm{Iso}(x)$ of x is defined as the isomorphism class of the stabilizer $\Gamma_{\tilde{x}} := \{\gamma \in \Gamma \mid \gamma\tilde{x} = \tilde{x}\}$ of $\tilde{x}$ in Γ, where $\tilde{x}$ is any point in the preimage $p^{-1}(x) \subset M$ of x. Note that $\mathrm{Iso}(x)$ is well-defined because for any $\tilde{x}' \in p^{-1}(x)$ the groups $\Gamma_{\tilde{x}}$ and $\Gamma_{\tilde{x}'}$ are conjugate in Γ. By abuse of notation we will sometimes call $\Gamma_{\tilde{x}}$ (instead of its isomorphism class) the isotropy group of $x = p(\tilde{x})$. If $\mathrm{Iso}(x)$ is nontrivial then x is called a singular point of $\mathcal{O}$, and the (finite) number $\#\mathrm{Iso}(x)$ is called its isotropy order.

The space $C^\infty(\mathcal{O})$ of smooth functions on a good Riemannian orbifold $\mathcal{O} = \Gamma\backslash M$ may be defined as the space $C^\infty(M)^\Gamma$ of Γ-invariant smooth functions on M. Similarly,

Key words and phrases. Laplace operator, isospectral orbifolds, isotropy orders
2000 *Mathematics Subject Classification.* 58J53, 58J50, 53C20.
All three authors were partially supported by DFG Sonderforschungsbereich 647.

smooth k-forms on $\mathcal{O}$ are defined as Γ-invariant smooth k-forms on M. Since the Laplace operator Δ_g on (M, g) commutes with isometries and thus preserves Γ-invariance, it preserves the space $C^\infty(\mathcal{O})$, and its restriction to this space is called the Laplace operator on functions on $\mathcal{O}$. Similarly, the Laplace operator on k-forms on $\mathcal{O}$ is the restriction of $dd^* + d^*d : \Omega_k(M) \to \Omega_k(M)$ to the space of Γ-invariant k-forms. Again, these notions can be suitably defined also on general Riemannian orbifolds and coincide with the given ones on good Riemannian orbifolds. On every compact connected Riemannian orbifold the Laplace operator on functions has a discrete spectrum of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$ with finite multiplicities; see [3]. For a good orbifold $\mathcal{O}$ as above, the eigenspace $E_\lambda(\mathcal{O})$ associated with the eigenvalue λ of $\mathcal{O}$ is canonically identified with the subspace of Γ-invariant elements of the space $E_\lambda(M, g)$ of eigenfunctions associated with this eigenvalue on (M, g). Two compact Riemannian orbifolds are called isospectral if they have the same spectrum.

To which extent does the Laplace spectrum determine the geometry of a compact Riemannian orbifold, and, in particular, the structure of its singularities? There exist some positive results in this direction. An important general observation is that a compact Riemannian orbifold which is not a manifold (i.e., has singular points) can never be isospectral to a Riemannian manifold with which it shares a common Riemannian covering. This is shown in [10] using an asymptotic expansion by H. Donnelly of the heat trace for good compact Riemannian orbifolds; his result was made more explicit and generalized to non-good orbifolds in [8]. It is not known whether the statement concerning nonisospectrality of manifolds and orbifolds remains true without the condition of a common Riemannian covering. E. Dryden and A. Strohmaier showed that on oriented compact hyperbolic orbifolds in dimension two, the spectrum completely determines the types and numbers of singular points [9]. Independently, this had also been shown by the first author together with P.G. Doyle (unpublished). By a result of E. Stanhope, only finitely many isotropy groups can occur in a family of isospectral orbifolds satisfying a uniform lower bound on the Ricci curvature [18]. On the other hand, N. Shams, E. Stanhope, and D. Webb have constructed arbitrarily large (finite) families of mutually isospectral Riemannian orbifolds such that each of these contains an isotropy group which does not occur in any of the other orbifolds of the family [17]. More precisely, for the maximal isotropy orders occurring in the orbifolds of such a family, the corresponding isotropy groups all have the same order, but are mutually nonisomorphic. A natural question arising in this context is whether it might be possible that two isospectral orbifolds have maximal isotropy groups which are not only nonisomorphic but even of different size. The only previously known examples of this kind concerned pairs of orbifolds with disconnected topology [7]. The present paper, however, exhibits several kinds of examples of isospectral connected orbifolds with different maximal isotropy orders; thus, using a popular formulation: You cannot hear the maximal isotropy order of an orbifold.

The paper is organized as follows:

In Section 2 we recall Bérard's, Ikeda's and Pesce's versions of the Sunada theorem and apply it to obtain a general construction of pairs of isospectral orbifolds with different maximal isotropy orders (Theorem 2.5, Corollary 2.6), as well as some explicit examples.

In this approach, the orbifolds arise as quotients of Riemannian manifolds which are locally isometric to a compact Lie group with a biinvariant metric, or, more generally, to a homogeneous space.

In Section 3 we recall some formulas developed by R. Miatello and the first author concerning the spectrum of flat manifolds and orbifolds. We use these to obtain several isospectral pairs of compact flat 3-dimensional orbifolds, among these also a pair with different maximal isotropy orders (Example 3.3). In another example (Example 3.5), the maximal isotropy groups are of the same size but not isomorphic, as in the examples by Shams, Stanhope, and Webb [17]. Moreover, the sets of singular points of maximal isotropy order have different dimension in the two orbifolds. Example 3.10 is another example of this kind. In Examples 3.7 and 3.9, all nontrivial isotropy groups are isomorphic, but again the topology of the singular sets is different. These two examples are obtained by the classical Sunada construction. Their existence within the context of flat 3-dimensional orbifolds is interesting because it is known [15], [6] that there do not exist nontrivial pairs of Sunada isospectral flat manifolds in dimension three. See [21] for a more detailed treatment of some of the examples in this section.

The first author would like to thank Humboldt-Universität zu Berlin, and especially Dorothee Schueth, for the great hospitality during his one year stay there.

2. Sunada isospectral orbifolds

The famous Sunada theorem [19] gives a general method for constructing isospectral manifolds and orbifolds. In order to formulate it and the versions given by P. Bérard and A. Ikeda which we will use here, one needs the notion of almost conjugate subgroups.

Definition 2.1. Let G be a group. Two finite subgroups Γ_1, Γ_2 of G are called *almost conjugate in* G if each conjugacy class $[b]_G$ in G intersects Γ_1 and Γ_2 in the same number of elements: $\#([b]_G \cap \Gamma_1) = \#([b]_G \cap \Gamma_2)$.

The classical version of the Sunada theorem says that if G is a finite group acting by isometries on a compact Riemannian manifold (M, g), and if Γ_1 and Γ_2 are almost conjugate subgroups of G acting without fixed points on M, then the quotient manifolds $\Gamma_1\backslash M$, $\Gamma_2\backslash M$, each endowed with the metric induced by g, are isospectral. If one drops the condition that Γ_1 and Γ_2 act without fixed points then the statement remains true in the context of Riemannian orbifolds, as shown by P. Bérard [1]. Finally, A. Ikeda [11] showed that the Sunada theorem still holds in the case that G is the whole (necessarily compact) group of isometries of (M, g), or any subgroup of the latter (as his proof allows). Although he did not formulate this result for orbifolds, the proof he gives in the manifold context carries over verbatim to the orbifold case. Independently, H. Pesce [14] had already given a version of the Sunada theorem for compact, not necessarily finite G, with a slightly different (but equivalent) formulation of the almost conjugacy condition in representation theoretic terms. Thus, one has the following theorem (which can also be interpreted as a special case of a much more general result by D. DeTurck and C. Gordon [5]):

Theorem 2.2 ([19], [1], [11], [14])**.** *Let (M, g) be a compact Riemannian manifold, and let G be a group which acts by isometries on (M, g). If Γ_1 and Γ_2 are two finite subgroups* *which are almost conjugate in G, then the compact Riemannian orbifolds $\mathcal{O}_1 := \varGamma_1\backslash M$ and $\mathcal{O}_2 := \varGamma_2\backslash M$ are isospectral.*

Note that we have not assumed effectiveness of the action of the $\varGamma_i$ on M. However, by identifying $\varGamma_i\backslash M$ with $\tilde{\varGamma}_i\backslash M$, where $\tilde{\varGamma}_i$ is the quotient of $\varGamma_i$ by the kernel of its action, this orbifold is again seen to be a good Riemannian orbifold in the sense of the introduction.

We briefly sketch Ikeda's particularly simple proof of Theorem 2.2: Since G acts by isometries, its canonical action on $C^\infty(M)$ commutes with the Laplace operator Δ_g; in particular, it preserves the corresponding eigenspaces $E_\lambda(M,g)$. Fix λ, let $V := E_\lambda(M,g)$, and denote the action of G on V by ρ. Note that V is finite dimensional since M is compact. We have to show that for $i = 1,2$, the $\varGamma_i$-invariant subspaces $V^{\varGamma_i} \cong E_\lambda(\mathcal{O}_i)$ of V have the same dimension. But this dimension is the trace of the projection operator $(\#\varGamma_i)^{-1}\sum_{\gamma\in\varGamma_i}\rho_\gamma$; it is thus equal to $(\#\varGamma_i)^{-1}\sum_{\gamma\in\varGamma_i}\mathrm{tr}(\rho_\gamma)$. Since there exists a bijection from $\varGamma_1$ to $\varGamma_2$ which preserves conjugacy classes in G, and thus traces, the two numbers are indeed the same for $i = 1,2$.

Remark 2.3. Sunada-isospectral orbifolds (i.e., isospectral orbifolds arising from Theorem 2.2) are actually isospectral on k-forms for all k; see the articles cited above. In fact, the above proof goes through without change if one replaces smooth functions by smooth k-forms.

If $\varGamma_1$ and $\varGamma_2$ are not only almost conjugate, but conjugate in G then the situation becomes trivial; in fact, if $\varGamma_2 = a\varGamma_1 a^{-1}$ for some $a \in G$ then $a : M \to M$ induces an isometry between the Riemannian orbifolds $\varGamma_1\backslash M$ and $\varGamma_2\backslash M$. Fortunately there exist many triples $(G, \varGamma_1, \varGamma_2)$ where the $\varGamma_i$ are almost conjugate, but not conjugate in G. One example which we are going to use is the following:

Example 2.4. Let $G := SO(6)$. Writing diagonal matrices in G as the vectors of their entries on the diagonal, define

$$
\begin{aligned}
\varGamma_1 := \{&(1,1,1,1,1,1), (-1,-1,-1,-1,-1,-1),\\
&(-1,-1,1,1,1,1), (-1,1,-1,1,1,1), (1,-1,-1,1,1,1),\\
&(-1,1,1,-1,-1,-1), (1,-1,1,-1,-1,-1), (1,1,-1,-1,-1,-1)\},\\
\varGamma_2 := \{&(1,1,1,1,1,1), (-1,-1,-1,-1,-1,-1),\\
&(-1,-1,1,1,1,1), (1,1,-1,-1,1,1), (1,1,1,1,-1,-1),\\
&(-1,-1,-1,-1,1,1), (-1,-1,1,1,-1,-1), (1,1,-1,-1,-1,-1)\}.
\end{aligned}
$$

Obviously there is a bijection from $\varGamma_1$ to $\varGamma_2$ preserving conjugacy classes in G; thus the two subgroups are almost conjugate in G. (Actually, the two groups can be seen to be almost conjugate by elements of the group A_6 of even permutation matrices in G, and thus almost conjugate *in* the finite subgroup of G generated by $\varGamma_1 \cup \varGamma_2 \cup A_6$.) This example corresponds to a certain pair of linear codes in $\mathbb{Z}_2^6$ with the same weight enumerator, mentioned in [4]. The groups $\varGamma_1$ and $\varGamma_2$ are not conjugate in $G = SO(6)$ because $\varGamma_1$ has a four-element subgroup acting as the identity on some three-dimensional subspace of $\mathbb{R}^3$ (namely, on $\mathrm{span}\{e_4, e_5, e_6\}$), while no four-element subgroup of $\varGamma_2$ acts as the identity on any three-dimensional subspace of $\mathbb{R}^3$.

The following observation is the main point of this section:

Theorem 2.5. *Let G be a compact Lie group and H be a closed Lie subgroup of G. Choose a left invariant Riemannian metric on G which is also right invariant under H. Let g denote the corresponding Riemannian metric on the quotient manifold $M := G/H$ such that the canonical projection $G \to G/H$ becomes a Riemannian submersion. Let $\mathit{\Gamma}_1$ and $\mathit{\Gamma}_2$ be two finite subgroups of G which are almost conjugate in G.*

(i) *The compact Riemannian orbifold quotients $\mathcal{O}_1 := \mathit{\Gamma}_1\backslash M$ and $\mathcal{O}_2 := \mathit{\Gamma}_2\backslash M$ of (M, g) are isospectral.*

(ii) *Let $m(\mathit{\Gamma}_i, H) := \max_{a\in G} \#(\mathit{\Gamma}_i \cap aHa^{-1})$ and $n(\mathit{\Gamma}_i, H) := \#(\mathit{\Gamma}_i \cap \bigcap_{a\in G} aHa^{-1})$ for $i = 1, 2$. Then $m(\mathit{\Gamma}_i, H) : n(\mathit{\Gamma}_i, H)$ is the maximal isotropy order of singular points in $\mathcal{O}_i$. Moreover, $n(\mathit{\Gamma}_1, H) = n(\mathit{\Gamma}_2, H)$. In particular, if $m(\mathit{\Gamma}_1, H) \neq m(\mathit{\Gamma}_2, H)$ then $\mathcal{O}_1$ and $\mathcal{O}_2$ have different maximal isotropy orders.*

Proof. (i) This follows from Theorem 2.2 because G acts by isometries on the homogeneous space $(M, g) = (G/H, g)$.

(ii) Let $a \in G$. Then the stabilizer in $\mathit{\Gamma}_i$ of the point $aH \in M$ is the group $\{\gamma \in \mathit{\Gamma}_i \mid \gamma aH = aH\} = \{\gamma \in \mathit{\Gamma}_i \mid \gamma \in aHa^{-1}\}$; that is,

$$(\mathit{\Gamma}_i)_{aH} = \mathit{\Gamma}_i \cap aHa^{-1}. \tag{1}$$

Moreover, the kernel of the action of $\mathit{\Gamma}_i$ on G/H is $\mathit{\Gamma}_i \cap \bigcap_{a\in G} aHa^{-1}$. This implies the formula for the maximal isotropy orders. For the statement about the numbers $n(\mathit{\Gamma}_i, H)$ let $\Phi : \mathit{\Gamma}_1 \to \mathit{\Gamma}_2$ be a bijection which preserves G-conjugacy classes. Note that $N := \bigcap_{a\in G} aHa^{-1}$ is a normal subgroup of G. Hence Φ restricts to a bijection from $\mathit{\Gamma}_1 \cap N$ to $\mathit{\Gamma}_2 \cap N$. □

Corollary 2.6. *Let G be a compact Lie group and $\mathit{\Gamma}_1$, $\mathit{\Gamma}_2$ be two almost conjugate, non-conjugate finite subgroups of G. Choose a biinvariant metric on G, and denote the induced metric on the quotient manifold $M := G/\mathit{\Gamma}_1$ by g. Then the compact Riemannian orbifold quotients $\mathcal{O}_1 := \mathit{\Gamma}_1\backslash M$ and $\mathcal{O}_2 := \mathit{\Gamma}_2\backslash M$ of (M, g) are isospectral and have different maximal isotropy orders.*

Proof. This follows immediately from Theorem 2.5 with $H := \mathit{\Gamma}_1$. In fact, we have $m(\mathit{\Gamma}_1, \mathit{\Gamma}_1) = \#\mathit{\Gamma}_1 = \#\mathit{\Gamma}_2$; if this were equal to $m(\mathit{\Gamma}_2, \mathit{\Gamma}_1)$ then $\mathit{\Gamma}_1$ and $\mathit{\Gamma}_2$ would be conjugate by some $a \in G$, contradicting the hypothesis. □

Example 2.7. The following is an example for Theorem 2.5 not arising from the corollary. Let $G, \mathit{\Gamma}_1, \mathit{\Gamma}_2$ be the groups from Example 2.4. Let $H \cong SO(3)$ be the subgroup of G consisting of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & I_3 \end{pmatrix},$$

where I_3 denotes the unit element in $SO(3)$. Then $M := G/H = SO(6)/SO(3)$ is the Stiefel manifold $V_{6,3}$ of orthonormal 3-frames in euclidean $\mathbb{R}^6$; the point $aH \in M$ corresponds to the 3-frame formed by the three last column vectors of the matrix $a \in SO(6)$. Note that G acts effectively on M. Choose a biinvariant metric on $SO(6)$ (or any left invariant metric which is also right invariant under H) and endow M with the corresponding homogeneous metric. By Theorem 2.5, the compact Riemannian orbifolds $\mathcal{O}_1 := \Gamma_1\backslash M$ and $\mathcal{O}_2 := \Gamma_2\backslash M$ are isospectral. Moreover, the point $eH \in M$ (corresponding to the orthonormal 3-frame (e_4, e_5, e_6) in $V_{6,3}$, where e_i denotes the i-th standard unit vector) is stabilized by four elements in Γ_1, namely, the elements of $\Gamma_1 \cap H$ (recall (1)). The same point is also stabilized by some two-element subgroup of Γ_2. On the other hand, no four-element subgroup of Γ_2 stabilizes any point in M: Such a point would have to correspond to an orthonormal 3-frame each of whose vectors is contained in the intersection of the 1-eigenspaces of the group elements; but for each four-element subgroup of Γ_1 this intersection is at most two-dimensional. Since obviously no point in M (not even any single unit vector in $\mathbb{R}^6$) is stabilized by the whole group Γ_1, we see that $\mathcal{O}_1$ has maximal isotropy order four, while $\mathcal{O}_2$ has maximal isotropy order two. In the notation of Theorem 2.5, $m(\Gamma_1, H) = 4$, $m(\Gamma_2, H) = 2$, and $n(\Gamma_1, H) = n(\Gamma_2, H) = 1$.

Example 2.8. Let G, Γ_1, Γ_2 again be as in the previous example, and let g be the Riemannian metric on $M := G/\Gamma_1$ induced by a biinvariant metric on $G = SO(6)$. Then the Riemannian orbifold quotients $\mathcal{O}_1 := \Gamma_1\backslash M$ and $\mathcal{O}_2 := \Gamma_2\backslash M$ of (M, g) are isospectral and have different maximal isotropy orders by Corollary 2.6.

More precisely, the maximal isotropy order of singular points in $\mathcal{O}_1$ is $m(\Gamma_1, \Gamma_1) : 2 = 4$, while in $\mathcal{O}_2$ it is $m(\Gamma_2, \Gamma_1) : 2 = 2$. In fact, $N := \bigcap_{a\in G} a\Gamma_1 a^{-1} \subset \Gamma_1 \cap \Gamma_2$ is the subgroup $\{\pm I_6\}$ of order 2, and we have $m(\Gamma_2, \Gamma_1) = 4$ because a four-element subgroup of Γ_1 which contains $-I_6$ is conjugate by some $a \in G$ (for example, a permutation matrix) to a subgroup of Γ_2.

Example 2.9. Another variation of the above examples, but *not* leading to different maximal isotropy orders, is obtained by letting G act canonically on the standard unit sphere $(M, g) := S^5$; in our above approach, this corresponds to letting $H := SO(5)$. As one immediately sees, the isotropy group of maximal order in $\mathcal{O}_i := \Gamma_i\backslash S^5$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ for both $i = 1, 2$. Nevertheless it is possible to distinguish between $\mathcal{O}_1$ and $\mathcal{O}_2$ by using the *topology* of the set $\mathcal{S}_i \subset \mathcal{O}_i$ of singularities with maximal isotropy orders, that is, the image in $\mathcal{O}_i$ of the set of points in S^5 whose stabilizer in Γ_i consists of four elements: The set $\mathcal{S}_1$ is the disjoint union of one copy of $\mathbb{R}P^2$ (the image of the unit sphere in $\mathrm{span}\{e_4, e_5, e_6\}$) and of three points (the images of $\pm e_1$, $\pm e_2$, and $\pm e_3$). The set $\mathcal{S}_2$, in contrast, is the disjoint union of three copies of S^1 (the images of the unit spheres in $\mathrm{span}\{e_1, e_2\}$, $\mathrm{span}\{e_3, e_4\}$, and $\mathrm{span}\{e_5, e_6\}$).

Remark 2.10. (i) The fact that the topological structure of certain singular strata can be different in isospectral orbifolds has also been shown in [17]; a new feature in Example 2.9 is that this concerns the set of points of *maximal* isotropy order. We will reencounter the analogous situation in certain isospectral pairs of flat 3-dimensional orbifolds; see Examples 3.5, 3.7, 3.9, and 3.10.

(ii) It is easy to see that for almost conjugate pairs Γ_1, Γ_2 of *diagonal* subgroups of $SO(n)$, necessarily containing only ± 1 as entries (as the pair used in the above examples), the corresponding actions on $S^{n-1} \cong SO(n)/SO(n-1)$ will always have the same maximal isotropy order (and isomorphic maximal isotropy groups $\mathbb{Z}_2^k$ for some k). We do not know whether there exist pairs of almost conjugate finite subgroups Γ_1 and Γ_2 of $SO(n)$ which satisfy $m(\Gamma_1, SO(n-1)) \neq m(\Gamma_2, SO(n-1))$ and would thus yield, by Theorem 2.5, isospectral *spherical* orbifolds with different maximal isotropy orders.

Remark 2.11. Once one has a pair of isospectral compact Riemannian orbifolds $\mathcal{O}_1$, $\mathcal{O}_2$ with different maximal isotropy orders, then one immediately obtains for each $m \in \mathbb{N}$ a family of $m+1$ mutually isospectral Riemannian orbifolds $\mathcal{V}_0, \dots, \mathcal{V}_m$ with pairwise different maximal isotropy orders; one just defines $\mathcal{V}_i$ as the Riemannian product of i times $\mathcal{O}_1$ and $m-i$ times $\mathcal{O}_2$. The Riemannian product of two *good* Riemannian orbifolds (as are all orbifolds in our examples) $\mathcal{O} = \Gamma\backslash M$ and $\mathcal{O}' = \Gamma'\backslash M'$ of (M, g), resp. (M', g'), is defined as $(\Gamma \times \Gamma')\backslash(M \times M')$, where $M \times M'$ is endowed with the Riemannian product metric associated with g and g'.

3. Isospectral flat orbifolds in dimension three

A Riemannian orbifold $\mathcal{O}$ is called flat if each point in $\mathcal{O}$ has a neighborhood which is the quotient of an open subset of $\mathbb{R}^n$, endowed with the euclidean metric, by a finite group of Riemannian isometries. It can be shown that every flat orbifold is good [20]; hence, it is the quotient of a flat Riemannian manifold by some group of isometries acting properly discontinuously.

Let us recall some facts from the theory of quotients of standard euclidean space $(\mathbb{R}^n, g)$ by groups of isometries; see [22]. The isometry group $I(\mathbb{R}^n, g)$ is the semidirect product $O(n) \ltimes \mathbb{R}^n$ consisting of all transformations BL_b with $B \in O(n)$ and $b \in \mathbb{R}^n$, where L_b is the translation $x \mapsto x + b$ of $\mathbb{R}^n$. Note that

$$L_bB = BL_{B^{-1}b},\ BL_bB^{-1} = L_{Bb},\ \text{and } (BL_b)^{-1} = B^{-1}L_{-Bb}. \tag{2}$$

The compact-open topology on $I(\mathbb{R}^n, g)$ coincides with the canonical product topology on $O(n) \times \mathbb{R}^n$. A subgroup Γ of $I(\mathbb{R}^n, g)$ acts properly discontinuously with compact quotient on $\mathbb{R}^n$ if and only if it is discrete and cocompact in $I(\mathbb{R}^n, g)$. Such a group is called a crystallographic group. If, in addition, Γ is torsion-free, then it acts without fixed points on $\mathbb{R}^n$, and $\Gamma\backslash\mathbb{R}^n$ is a flat Riemannian manifold. Conversely, every compact flat Riemannian manifold is isometric to such a quotient. If the condition that Γ be torsion-free is dropped then $\Gamma\backslash\mathbb{R}^n$ is a compact good Riemannian orbifold which is flat. Conversely, if $\mathcal{O}$ is any compact flat Riemannian orbifold (and is thus, as mentioned above, a good orbifold), then there exists a crystallographic group $\Gamma \subset I(\mathbb{R}^n, g)$ such that $\mathcal{O}$ is isometric to $\Gamma\backslash\mathbb{R}^n$.

If Γ is a crystallographic group acting on $\mathbb{R}^n$ then the translations in Γ form a normal, maximal abelian subgroup L_Λ where Λ is a cocompact lattice in $\mathbb{R}^n$; the quotient group $\bar{\Gamma} := \Gamma/L_\Lambda$ is finite. The flat torus $T_\Lambda := L_\Lambda\backslash\mathbb{R}^n$ covers $\mathcal{O} := \Gamma\backslash\mathbb{R}^n$ because L_Λ is normal in Γ. More precisely, we have $\mathcal{O} \cong \bar{\Gamma}\backslash T_\Lambda$, where $\gamma L_\Lambda \in \bar{\Gamma}$ acts on T_Λ as the map $\bar{\gamma} : T_\Lambda \to T_\Lambda$ induced by $\gamma : \mathbb{R}^n \to \mathbb{R}^n$. Let $F \subset O(n)$ be the image of the canonical projection from $\Gamma \subset O(n) \ltimes \mathbb{R}^n$ to $O(n)$. This projection has kernel L_Λ; thus we have $F \cong \bar{\Gamma}$.

Let $k \in \{0, \dots, n\}$. For $\mu \geq 0$ let $H_{k,\mu}(T_\Lambda)$ denote the space of smooth k-forms on T_Λ which are eigenforms associated with the eigenvalue $4\pi^2\mu$. Then the multiplicity of $4\pi^2\mu$ as an eigenvalue for the Laplace operator on k-forms on the Riemannian orbifold

$\mathcal{O} = \Gamma\backslash\mathbb{R}^n = \bar{\Gamma}\backslash T_\Lambda$ equals the dimension of the subspace

$$H_{k,\mu}(T_\Lambda)^{\bar{\Gamma}} = \{\omega \in H_{k,\mu}(T_\Lambda) \mid \bar{\gamma}^*\omega = \omega \ \forall \bar{\gamma} \in \bar{\Gamma}\}$$

(which might be zero). This dimension can be computed using the formula from the following theorem.

Theorem 3.1 ([12], [13]). *Let $d_{k,\mu}(\Gamma) := \dim H_{k,\mu}(T_\Lambda)^{\bar{\Gamma}}$. Then*

$$d_{k,\mu}(\Gamma) = (\#F)^{-1} \sum_{B \in F} \mathrm{tr}_k(B) e_{\mu,B}(\Gamma), \quad \textit{where} \;\; e_{\mu,B}(\Gamma) := \sum_{\substack{v \in \Lambda^*, \|v\|^2 = \mu \\ Bv = v}} e^{2\pi i \langle v, b\rangle}$$

with b chosen such that $BL_b \in \Gamma$, the trace of B acting on the $\binom{n}{k}$-dimensional space of alternating k-linear forms on $\mathbb{R}^n$ as pullback by B^{-1} is denoted by $\mathrm{tr}_k(B)$, and where $\Lambda^* := \{v \in \mathbb{R}^n \mid \langle v, \lambda\rangle \in \mathbb{Z} \ \forall \lambda \in \Lambda\}$ is the dual lattice associated with Λ.

Notation and Remarks 3.2.
(i) Note that $\mathrm{tr}_0(B) = 1$ and $\mathrm{tr}_1(B) = \mathrm{tr}(B^{-1}) = \mathrm{tr}({}^tB) = \mathrm{tr}(B)$ for all $B \in O(n)$.
(ii) For $k = 0$ we write $d_\mu := d_{0,\mu}$. Thus $d_\mu(\Gamma)$ will be the multiplicity of $4\pi^2\mu$ as an eigenvalue for the Laplace operator on functions on $\mathcal{O}$.

The following is an example of two isospectral flat three-dimensional orbifolds with different maximal isotropy orders.

Example 3.3. Let Λ be the lattice $2\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}$ in $\mathbb{R}^3$. Define

$$\tau := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \chi_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ \chi_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ \chi_3 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$b_1 := e_1, \ b_2 := 0, \ b_3 := -e_1 \in \mathbb{R}^3.$$

Let Γ_1 be the subgroup of $I(\mathbb{R}^3)$ generated by L_Λ and τ, and let Γ_2 be generated by L_Λ and the maps $\rho_j := \chi_j \circ L_{b_j}$ $(j = 1, 2, 3)$. Using (2) one easily checks that

$$\Gamma_1 = \{\tau^j L_\lambda \mid j \in \{0,1,2,3\}, \lambda \in \Lambda\} \text{ and } \Gamma_2 = \{\rho_j L_\lambda \mid j \in \{0,1,2,3\}, \lambda \in \Lambda\},$$

where $\rho_0 := \mathrm{Id}$. Since these are discrete and cocompact subgroups of $O(3) \ltimes \mathbb{R}^3$, we obtain two compact flat orbifolds

$$\mathcal{O}_1 := \Gamma_1\backslash\mathbb{R}^3, \text{ and } \mathcal{O}_2 := \Gamma_2\backslash\mathbb{R}^3.$$

It is not difficult to see that the unit cube $[0,1]^3$ is a fundamental domain for the action of Γ_1, resp. Γ_2, on $\mathbb{R}^3$, and that the identifications on the sides are as given in the following two figures, where the top and bottom sides are identified by the vertical translation L_{e_3}.

In Figure 1, describing $\mathcal{O}_1$, the element $\tau \in \Gamma_1$ accounts for the side identification denoted by P, and $\tau L_{-2e_2} \in \Gamma_1$ for the one denoted by F. Note that $\mathcal{O}_1$ is actually the Riemannian product of a two-dimensional so-called 442-orbifold and a circle of length one. (A 442-orbifold has two cone points of order 4 and one cone point of order 2.) In Figure 2 which describes $\mathcal{O}_2$, the elements of Γ_2 which account for the side identifications denoted by F, L, Δ, P are ρ_3, $\rho_3 L_{-2e_2}$, $\rho_2 L_{-e_3}$, and $\rho_2 L_{-2e_1-e_3}$, respectively.

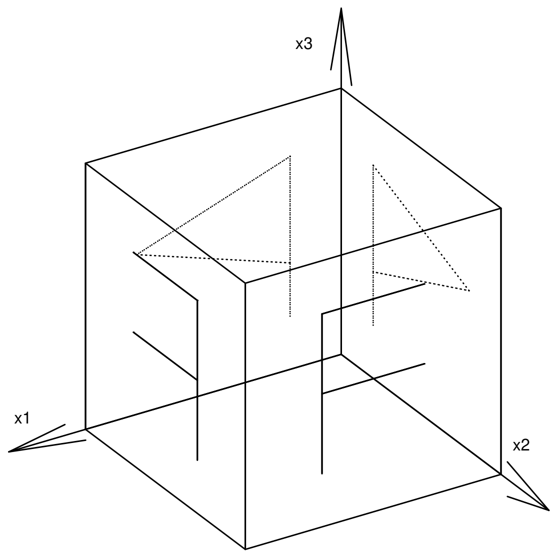

FIGURE 1. The underlying space of $\mathcal{O}_1$ as a quotient of the unit cube

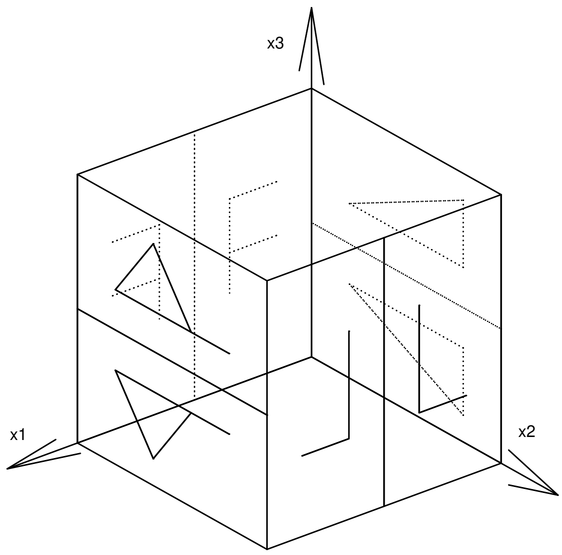

FIGURE 2. The underlying space of $\mathcal{O}_2$ as a quotient of the unit cube

Isotropy groups: It is clear that the isotropy groups both in $\mathcal{O}_1$ and $\mathcal{O}_2$ can have *at most* order four because L_Λ has index four in Γ_i and a point in $\mathbb{R}^3$ cannot be fixed simultaneously by two isometries that differ by a nontrivial translation.

Since τ is a quarter rotation around the axis spanned by e_3, the four-element subgroup $\{\mathrm{Id}, \tau, \tau^2, \tau^3\} \cong \mathbb{Z}_4$ of Γ_1 pointwise fixes the edge $\{(0,0,x_3) \mid 0 \leq x_3 \leq 1\}$ of the fundamental cube; thus $\mathcal{O}_1$ has maximal isotropy order four. The other points in the fundamental domain with nontrivial stabilizer in Γ_1 are $\{(1,1,x_3) \mid 0 \le x_3 \le 1\}$, pointwise fixed by the four-element group generated by τL_{-2e_2}, and $\{(0,1,x_3) \mid 0 \le x_3 \le 1\}$ (identified with $\{(1,0,x_3) \mid 0 \le x_3 \le 1\}$ via the identifications marked F or P in Figure 1), pointwise fixed by $\{\mathrm{Id}, \tau^2 L_{-2e_2}\} \cong \mathbb{Z}_2$. So the singular set in $\mathcal{O}_1$ consists of three copies of S^1, each of length one, two of them with isotropy group $\mathbb{Z}_4$ and one with $\mathbb{Z}_2$. (Of course, these three components correspond to the three cone points of the 442-orbifold mentioned above.)

In $\mathcal{O}_2$ there are no points with isotropy order four. Otherwise, there would have to exist a point in $\mathbb{R}^3$ fixed by three elements of the form $\rho_1 L_\lambda$, $\rho_2 L_\mu$, $\rho_3 L_\nu$ with $\lambda, \mu, \nu \in \Lambda$. But $(\rho_1 L_\lambda)^2 = (\chi_1 L_{e_1+\lambda})^2 = L_{(\chi_1^{-1}+\mathrm{Id})(e_1+\lambda)}$. In order to fix a point, this translation would have to be trivial; in particular, the first coordinate of $e_1 + \lambda$ would have to vanish. This contradicts $\lambda \in \Lambda$. Thus, the points in $\mathcal{O}_2$ which do have nontrivial isotropy all have isotropy group $\mathbb{Z}_2$. The singular set in $\mathcal{O}_2$ consists of four copies of S^1: Two of length two, corresponding to the horizontal edges and middle segments in the faces of the fundamental cube marked by P and Δ in Figure 2, and two of length one, corresponding to the middle vertical segments on the faces marked by L and F.

Isospectrality: Let $\mu \ge 0$. The space of eigenfunctions associated with the eigenvalue $4\pi^2\mu$ on $\mathcal{O}_i$ has dimension $d_\mu(\Gamma_i)$ $(i = 1,2)$ which we compute using Theorem 3.1 with $k = 0$. We have $F_1 = \{\mathrm{Id}, \tau, \tau^2, \tau^3\}$ and $F_2 = \{\mathrm{Id}, \chi_1, \chi_2, \chi_3\}$. Obviously, $d_0(\Gamma_i) = 1$ for both $i = 1,2$. Let $\mu > 0$. For $B = \mathrm{Id}$, we get $e_{\mu,\mathrm{Id}}(\Gamma_i) = \#\{v \in \Lambda^* \mid \|v\|^2 = \mu\} =: e_{\mu,\mathrm{Id}}$ for both $i = 1,2$. Note that $\Lambda^* = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \mathbb{Z}$. The only vectors of length $\sqrt{\mu}$ in $\mathbb{R}^3$ which are fixed by some nontrivial element of F_i are $\pm\sqrt{\mu}e_3$ for $i = 1$ and $\pm\sqrt{\mu}e_j$ $(j = 1,2,3)$ for $i = 2$. Therefore, if $\sqrt{\mu} \notin \frac{1}{2}\mathbb{N}$ then no $v \in \Lambda^*$ of length $\sqrt{\mu}$ is fixed by any nontrivial element of the F_i, and thus $d_\mu(\Gamma_1) = \frac{1}{4}e_{\mu,\mathrm{Id}} = d_\mu(\Gamma_2)$. If $\sqrt{\mu} \in \mathbb{N}$ then

$$e_{\mu,\tau^j}(\Gamma_1) = e^{2\pi i\langle\sqrt{\mu}e_3, 0\rangle} + e^{2\pi i\langle-\sqrt{\mu}e_3, 0\rangle} = 2$$

for $j = 1,2,3$, and

$$\begin{aligned}
e_{\mu,\chi_1}(\Gamma_2) &= e^{2\pi i\langle\sqrt{\mu}e_1, e_1\rangle} + e^{2\pi i\langle-\sqrt{\mu}e_1, e_1\rangle} = 2,\\
e_{\mu,\chi_2}(\Gamma_2) &= e^{2\pi i\langle\sqrt{\mu}e_2, 0\rangle} + e^{2\pi i\langle-\sqrt{\mu}e_2, 0\rangle} = 2,\\
e_{\mu,\chi_3}(\Gamma_2) &= e^{2\pi i\langle\sqrt{\mu}e_3, -e_1\rangle} + e^{2\pi i\langle-\sqrt{\mu}e_3, -e_1\rangle} = 2,
\end{aligned}$$

hence $d_\mu(\Gamma_1) = \frac{1}{4}(e_{\mu,\mathrm{Id}} + 6) = d_\mu(\Gamma_2)$. Finally, if $\sqrt{\mu} \in \mathbb{N}_0 + \frac{1}{2}$ then $\pm\sqrt{\mu}e_3 \notin \Lambda^*$ and thus $e_{\mu,\tau^j}(\Gamma_1) = 0$ for $j = 1,2,3$ and $e_{\mu,\chi_3}(\Gamma_2) = 0$; moreover,

$$\begin{aligned}
e_{\mu,\chi_1}(\Gamma_2) &= e^{2\pi i\langle\sqrt{\mu}e_1, e_1\rangle} + e^{2\pi i\langle-\sqrt{\mu}e_1, e_1\rangle} = -2,\\
e_{\mu,\chi_2}(\Gamma_2) &= e^{2\pi i\langle\sqrt{\mu}e_2, 0\rangle} + e^{2\pi i\langle-\sqrt{\mu}e_2, 0\rangle} = 2,
\end{aligned}$$

hence $d_\mu(\Gamma_1) = \frac{1}{4}e_{\mu,\mathrm{Id}} = d_\mu(\Gamma_2)$. We have now shown that $d_\mu(\Gamma_1) = d_\mu(\Gamma_2)$ for every $\mu \ge 0$; that is, $\mathcal{O}_1$ and $\mathcal{O}_2$ are isospectral on functions.

Remark 3.4. The orbifolds $\mathcal{O}_1$ and $\mathcal{O}_2$ from the previous example are *not* isospectral on 1-forms, as we can compute by using Theorem 3.1 with $k = 1$. Note that $\mathrm{tr}(\mathrm{Id}) = 3$, $\mathrm{tr}(\tau) = \mathrm{tr}(\tau^3) = 1$ and $\mathrm{tr}(\tau^2) = \mathrm{tr}(\chi_j) = -1$ for $j = 1,2,3$. Now consider $\mu > 0$ with

$\sqrt{\mu} \in \mathbb{N}$. Adjusting the trace coefficients in the corresponding computation above, we get

$$d_{1,\mu}(\Gamma_1) = \frac{1}{4}(3e_{\mu,\mathrm{Id}} + 2 - 2 + 2) \neq \frac{1}{4}(3e_{\mu,\mathrm{Id}} - 2 - 2 - 2) = d_{1,\mu}(\Gamma_2).$$

In the following pair of isospectral flat orbifolds, the maximal isotropy orders coincide, but the maximal isotropy groups are not isomorphic, similarly as in the spherical examples from [17]. In contrast to those examples from [17], the sets of singularities with maximal isotropy order will have different dimensions in the two orbifolds.

Example 3.5. Let $\Lambda := 2\mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{R}^3$. Define $\tau, \chi_1, \chi_2, \chi_3$ as in Example 3.3, let Γ_1 be generated by L_Λ and τ, and let Γ_2 be generated by L_Λ and the $\rho_j := \chi_j$ ($j = 1, 2, 3$); note that the ρ_j have no translational parts this time. Again we confirm, using (2), that

$$\Gamma_1 = \{\tau^j L_\lambda \mid j \in \{0, 1, 2, 3\}, \lambda \in \Lambda\} \text{ and } \Gamma_2 = \{\rho_j L_\lambda \mid j \in \{0, 1, 2, 3\}, \lambda \in \Lambda\}$$

(where $\rho_0 := \mathrm{Id}$), and we obtain two compact flat orbifolds $\mathcal{O}_1 := \Gamma_1 \backslash \mathbb{R}^3$ and $\mathcal{O}_2 := \Gamma_2 \backslash \mathbb{R}^3$. This time, $[0, 1] \times [0, 1] \times [0, 2]$ is a fundamental domain for the action of Γ_1, resp. Γ_2, on $\mathbb{R}^3$. The side identifications are given in Figure 3; the top and bottom sides are again identified via the corresponding translation L_{2e_3}.

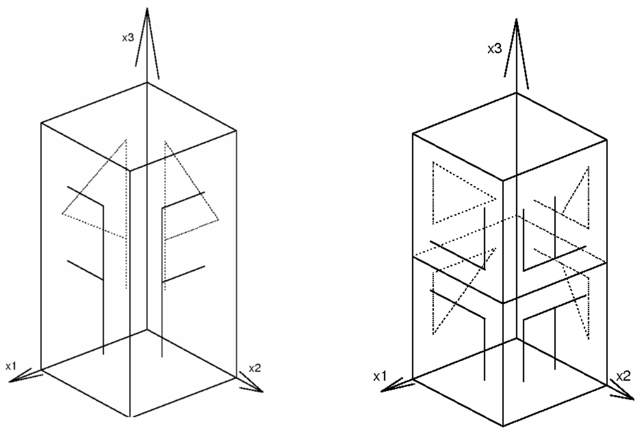

FIGURE 3. The underlying spaces of $\mathcal{O}_1$ and $\mathcal{O}_2$ from Example 3.5

The orbifold $\mathcal{O}_1$ which is pictured on the left hand side of Figure 3 is just a double covering of the first orbifold from the previous example in Figure 1; the explanations concerning the side identifications and the isotropy groups are the same as before, except that now all the vertical circles have length 2. As for the right hand side of Figure 3, showing $\mathcal{O}_2$, the elements of Γ_2 which account for the side identifications denoted by F, Δ, L, P are $\rho_1 L_{-2e_2-2e_3}$, $\rho_1 L_{-2e_3}$, $\rho_2 L_{-2e_1-2e_3}$, and $\rho_2 L_{-2e_3}$, respectively.

Isotropy groups: One easily verifies that for $j = 1, 2, 3$, an element $\rho_j L_\lambda \in \Gamma_2$ has fixed points if and only if $\lambda_j = 0$ (that is, $(\rho_j L_\lambda)^2 = \mathrm{Id}$), and in this case the fixed point set is the line $\{-\frac{1}{2}\lambda + re_j \mid r \in \mathbb{R}\}$. Since $\Lambda = 2\mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z}$, the points in $\mathbb{R}^3$ with exactly two coordinates in $\mathbb{Z}$ have isotropy group $\mathbb{Z}_2$, while those in $\mathbb{Z}^3$ have isotropy group isomorphic to $\{\mathrm{Id}, \rho_1, \rho_2, \rho_3\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus (taking identifications into account), the singular set in $\mathcal{O}_2$ consists of eight points with isotropy group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and of twelve open segments of length one with isotropy group $\mathbb{Z}_2$.

Since the maximal isotropy group occurring in $\mathcal{O}_1$ was $\mathbb{Z}_4$, the maximal isotropy orders coincide here, but the maximal isotropy groups are nonisomorphic. Moreover, the set of singular points with maximal isotropy has dimension one in $\mathcal{O}_1$ and dimension zero in $\mathcal{O}_2$.

Isospectrality: We continue to use the notation from the isospectrality discussion in Example 3.3 and note that now $\Lambda^* = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$. We have $d_\mu(\Gamma_1) = \frac{1}{4}e_{\mu,\mathrm{Id}} = d_\mu(\Gamma_2)$ if $\sqrt{\mu} \notin \frac{1}{2}\mathbb{N}$; if $\sqrt{\mu} \in \frac{1}{2}\mathbb{N}$ then $e_{\mu,\tau^j}(\Gamma_1) = 2$, $e_{\mu,\rho_j}(\Gamma_2) = 2$ for $j = 1, 2, 3$, hence $d_\mu(\Gamma_1) = \frac{1}{4}(e_{\mu,\mathrm{Id}} + 6) = d_\mu(\Gamma_2)$. Thus $\mathcal{O}_1$ and $\mathcal{O}_2$ are isospectral on functions.

Remark 3.6. Similarly as in Remark 3.4, one shows that here $d_{1,\mu}(\Gamma_1) \neq d_{1,\mu}(\Gamma_2)$ for $\sqrt{\mu} \in \frac{1}{2}\mathbb{N}$. Thus, $\mathcal{O}_1$ and $\mathcal{O}_2$ from Example 3.5 are again *not* isospectral on 1-forms, and, in particular, not Sunada-isospectral.

The following two examples are pairs of compact flat three-dimensional orbifolds which are Sunada-isospectral; recall that we mean by this: which arise from Theorem 2.2. Actually, the group G from the theorem will even be finite here. The existence of such pairs in the category of three-dimensional flat orbifolds is noteworthy because there are no such pairs in the category of flat three-dimensional manifolds. In fact, as shown by J.H. Conway and the first author in [15], there is exactly one pair, up to scaling, of isospectral flat manifolds in dimension three. But the manifolds in that pair are not isospectral on 1-forms [6], and thus not Sunada-isospectral.

Example 3.7. Let $\Lambda := \mathbb{Z} \times \mathbb{Z} \times \frac{1}{\sqrt{2}}\mathbb{Z} \subset \mathbb{R}^3$. Define

$$\tau := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \rho := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let Γ_1 be generated by L_Λ and τ, and let Γ_2 be generated by L_Λ and ρ. Then

$$\Gamma_1 = \{\tau^j L_\lambda \mid j \in \{0,1\}, \lambda \in \Lambda\} \text{ and } \Gamma_2 = \{\rho^j L_\lambda \mid j \in \{0,1\}, \lambda \in \Lambda\}.$$

Let $\mathcal{O}_1 := \Gamma_1\backslash\mathbb{R}^3$ and $\mathcal{O}_2 := \Gamma_2\backslash\mathbb{R}^3$. A fundamental domain for the action of Γ_1, resp. Γ_2, on $\mathbb{R}^3$, is given by the prism of height $1/\sqrt{2}$ over the triangle with vertices 0, e_1, e_2. The side identifications are given in Figure 4 (where once more the top and bottom sides are identified via a vertical translation).

No isotropy groups of order greater than two can occur now, since $\tau^2 = \rho^2 = \mathrm{Id}$, thus Λ is of index two in Γ_1 and Γ_2. Therefore, all singular points in $\mathcal{O}_1$ and $\mathcal{O}_2$ have isotropy group $\mathbb{Z}_2$.

The points $x \in \mathbb{R}^3$ which are fixed by an element of the form $\tau L_\lambda \in \Gamma_1$ must satisfy $(x_1 + x_2, x_1 + x_2, 2x_3) = -(\lambda_2, \lambda_1, \lambda_3)$. These are exactly those x with $x_3 \in \frac{1}{2\sqrt{2}}\mathbb{Z}$ and $x_1 + x_2 \in \mathbb{Z}$. Thus (taking identifications into account), the singular set in $\mathcal{O}_1$ consists of two copies of S^1 of length $\sqrt{2}$, corresponding to the horizontal segments in the face of the fundamental domain denoted by F on the left hand side of Figure 4.

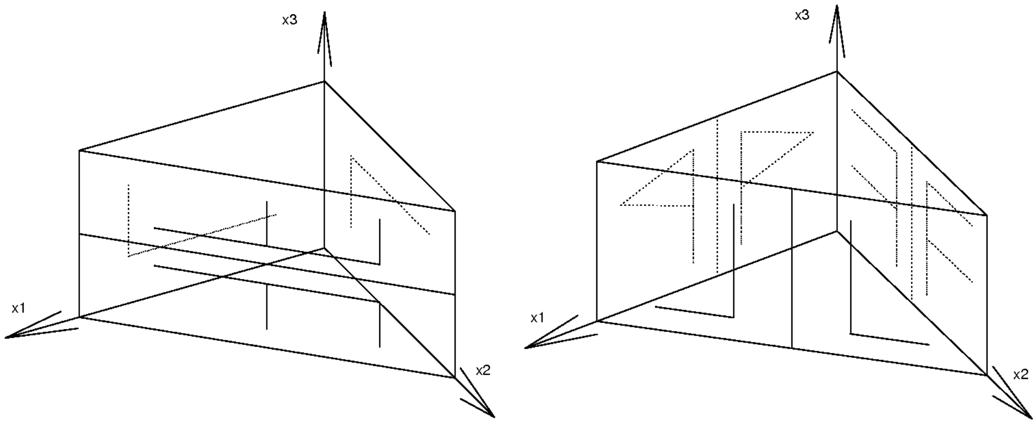

FIGURE 4. The underlying spaces of $\mathcal{O}_1$ and $\mathcal{O}_2$ from Example 3.7

The orbifold $\mathcal{O}_2$ is the Riemannian product of a two-dimensional orbifold called a 4-pillow or 2222-orbifold (here in the form of a square of side length $\frac{1}{2}$) and a circle of length $\frac{1}{\sqrt{2}}$. Accordingly, its singular set consists of four copies of S^1 of length $\frac{1}{\sqrt{2}}$ (corresponding to the vertical segments over the points $0, \frac{1}{2}e_1, \frac{1}{2}e_2, \frac{1}{2}(e_1+e_2)$; note that the points e_1 and e_2 are identified with 0). So, also in this pair of Sunada-isospectral (see below) flat orbifolds, the singular sets have different topology.

Sunada isospectrality: Define the sublattice $\Lambda' := \mathrm{span}_{\mathbb{Z}}\{(1,1,0),(1,-1,0),(0,0,\sqrt{2})\}$ of Λ, and let $T_{\Lambda'} := L_{\Lambda'}\backslash\mathbb{R}^3$. We will see that $\mathcal{O}_1 \cong G_1\backslash T_{\Lambda'}$ and $\mathcal{O}_2 \cong G_2\backslash T_{\Lambda'}$ for two eight-element groups G_1, G_2 of isometries of $T_{\Lambda'}$ which are almost conjugate in a certain finite subgroup of the isometry group of $T_{\Lambda'}$. Here, we use the symbol $\cong$ to denote that two orbifolds are isometric.

One easily sees that Λ' has index four in Λ, and that a full set of representatives of $\Lambda/\Lambda' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is given by $\{0, e_2, \frac{1}{\sqrt{2}}e_3, e_2 + \frac{1}{\sqrt{2}}e_3\}$. Since Λ' is invariant under τ and ρ, these isometries of $\mathbb{R}^3$ descend to isometries $\overline{\tau}$ and $\overline{\rho}$ of $T_{\Lambda'} := L_{\Lambda'}\backslash\mathbb{R}^3$; trivially, also translations L_λ descend to isometries $\overline{L_\lambda}$ of $T_{\Lambda'}$. Define the groups

$$G_1 := \{\overline{\tau^j L_\lambda} \mid j \in \{0,1\}, \lambda \in \Lambda/\Lambda'\} \text{ and } G_2 := \{\overline{\rho^j L_\lambda} \mid j \in \{0,1\}, \lambda \in \Lambda/\Lambda'\}.$$

It is not hard to verify that

$$G_1\backslash T_{\Lambda'} \cong \mathcal{O}_1 \text{ and } G_2\backslash T_{\Lambda'} \cong \mathcal{O}_2.$$

We are looking for a bijection from G_1 to G_2 preserving conjugacy classes in the isometry group of $T_{\Lambda'}$. Let

$$A := \begin{pmatrix} -1/2 & -1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}.$$

Note that $A\tau A^{-1} = \rho$ and $A(\Lambda') = \Lambda'$. Let $H \subset O(3)$ be the subgroup generated by τ, ρ, and A. Note that H is finite since it preserves the lattice Λ'. Define $\Phi : G_1 \to G_2$ by $\Phi(\overline{L_\lambda}) := \overline{L_\lambda}$ for $\lambda \in \Lambda/\Lambda'$ and

$$\Phi(\overline{\tau}) := \overline{\rho},\ \ \Phi(\overline{\tau L_{e_2}}) := \overline{\rho L_{\frac{1}{\sqrt{2}}e_3}},\ \ \Phi(\overline{\tau L_{\frac{1}{\sqrt{2}}e_3}}) := \overline{\rho L_{e_2}},\ \ \Phi(\overline{\tau L_{e_2+\frac{1}{\sqrt{2}}e_3}}) := \overline{\rho L_{e_2+\frac{1}{\sqrt{2}}e_3}}.$$

We claim that Φ preserves conjugacy classes in the finite subgroup

$$G := \{\overline{BL_b} \mid B \in H,\ b \in (\Lambda'/4)/\Lambda'\}$$

of the isometry group of $T_{\Lambda'}$. This follows from the relation

$$A\tau L_\lambda A^{-1} = \rho L_{A\lambda}$$

in connection with the following formulas, where $b := \frac{1}{4}e_1 - \frac{1}{4}e_2 \in \Lambda'/4$:

$$\begin{aligned} L_b^{-1}(\rho L_{Ae_2})L_b &= \rho L_{-\frac{1}{\sqrt{2}}e_3} \sim \rho L_{\frac{1}{\sqrt{2}}e_3} \\ L_b(\rho L_{A(\frac{1}{\sqrt{2}}e_3)})L_b^{-1} &= \rho L_{e_2} \\ \rho L_{A(e_2+\frac{1}{\sqrt{2}}e_3)} &= \rho L_{e_2-\frac{1}{\sqrt{2}}e_3} \sim \rho L_{e_2+\frac{1}{\sqrt{2}}e_3} \end{aligned}$$

Here, the sign $\sim$ between two isometries of $\mathbb{R}^3$ means that they differ by a translation in $L_{\Lambda'}$ and thus induce the same isometry of $T_{\Lambda'}$. So $\mathcal{O}_1$ and $\mathcal{O}_2$ are indeed Sunada-isospectral; in particular, they are isospectral on k-forms for all k.

Remark 3.8. It is an interesting open question whether there exists a pair of compact flat orbifolds which are k-isospectral for all k and have different maximal isotropy orders. Another open question is whether a pair of compact flat orbifolds which are k-isospectral for all k must necessarily be Sunada-isospectral.

Example 3.9. Another pair of Sunada-isospectral orbifolds is given as follows. Let $\Lambda := 2\mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z}$,

$$\chi_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \chi_2 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \chi_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$b_1 := e_1 + e_2,\ b_2 := 0,\ b_3 := e_1 + e_2,\ b_1' := e_3,\ b_2' := 0,\ b_3' := e_3 \in \mathbb{R}^3.$$

Set $\rho_j := \chi_j \circ L_{b_j}$, $\rho_j' := \chi_j \circ L_{b_j'}$, $\rho_0 = \rho_0' = \mathrm{Id}$ and observe that

$$\Gamma_1 := \{\rho_j L_\lambda \mid j \in \{0,1,2,3\}, \lambda \in \Lambda\} \text{ and } \Gamma_2 := \{\rho_j' L_\lambda \mid j \in \{0,1,2,3\}, \lambda \in \Lambda\}$$

are discrete and cocompact subgroups of $O(3) \ltimes \mathbb{R}^3$. Note that the orbifolds $\mathcal{O}_1 := \Gamma_1\backslash\mathbb{R}^3$ and $\mathcal{O}_2 := \Gamma_2\backslash\mathbb{R}^3$ are not orientable. For both, a fundamental domain is given by $[0,1] \times [0,1] \times [0,2]$. The boundary identifications are shown in Figure 5, where we omit the identifications by L_{2e_3} as usual. Note that the underlying topological space of $\mathcal{O}_1$ is the product of a projective plane and a circle.

Using the notation introduced at the beginning of this section, we note that $\mathcal{O}_i \cong \bar{\Gamma}_i\backslash T_\Lambda$ for $i = 1, 2$, where $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ are the following subgroups of the isometry group of T_Λ:

$$\bar{\Gamma}_1 = \{\overline{\rho_j} \mid j \in \{0,1,2,3\}\},\ \bar{\Gamma}_2 = \{\overline{\rho_j'} \mid j \in \{0,1,2,3\}\}.$$

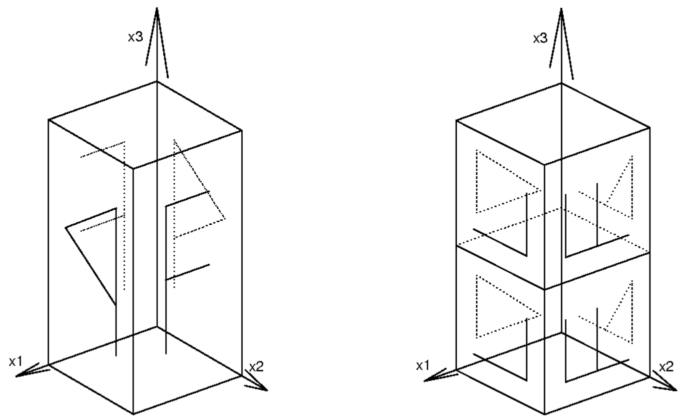

FIGURE 5. The underlying spaces of $\mathcal{O}_1$ and $\mathcal{O}_2$ from Example 3.9

It is not difficult to see that the groups $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ are almost conjugate in the finite group G generated by $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\{\overline{BL_b} \mid B \in P(3), b \in (\Lambda/4)/\Lambda\}$, where $P(3) \subset O(3)$ denotes the group of permutation matrices. Hence, $\mathcal{O}_1$ and $\mathcal{O}_2$ are Sunada-isospectral. Alternatively, one can apply the methods developed in [13], Section 3, to verify that the two orbifolds are Sunada-isospectral.

However, $\mathcal{O}_1$ and $\mathcal{O}_2$ are not isometric; in fact, their respective singular sets have different numbers of components. For each $i = 1, 2$ the points in $\mathbb{R}^3$ which are fixed by nontrivial elements of Γ_i are given by the set $\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}$. Each of these points is fixed by exactly one nontrivial group element and thus has isotropy $\mathbb{Z}_2$. Taking identifications into account (recall Figure 5), we observe that in $\mathcal{O}_1$ the singular set consists of two copies of S^1 of length two, whereas in $\mathcal{O}_2$ it consists of four copies of S^1 of length one.

Finally, we present another pair of (non-Sunada) isospectral orbifolds with properties similar to the pair from Example 3.5, this time with nonisomorphic maximal isotropy groups of order six.

Example 3.10. Let $\Lambda := \operatorname{span}_{\mathbb{Z}}\{(2,0,0), (1,\sqrt{3},0), (0,0,1)\}$ and

$$H := \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Note that H is just the rotation by $\pi/3$ around the x_3-axis. Now

$$\Gamma_1 := \{H^j L_\lambda \mid j \in \{0, \dots, 5\}, \lambda \in \Lambda\}, \ \Gamma_2 := \{H^{2j} R^k L_\lambda \mid j \in \{0,1,2\}, k \in \{0,1\}, \lambda \in \Lambda\}$$

are crystallographic groups acting on $\mathbb{R}^3$. For both $i = 1, 2$, a fundamental domain of the action of Γ_i on $\mathbb{R}^3$ is given by the prism of height one over the triangle with vertices $(0,0,0)$, $(2,0,0)$, $(1, 1/\sqrt{3}, 0)$ (compare Figure 6 where we again omit the identifications by L_{e_3}).

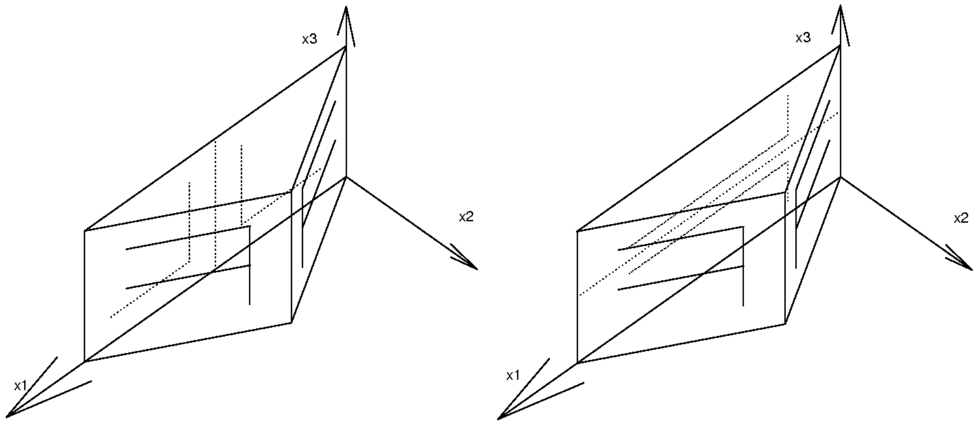

FIGURE 6. The underlying spaces of $\mathcal{O}_1$ and $\mathcal{O}_2$ from Example 3.10

Using Theorem 3.1 one shows that the two orbifolds $\mathcal{O}_1 := \Gamma_1\backslash\mathbb{R}^3$ and $\mathcal{O}_2 := \Gamma_2\backslash\mathbb{R}^3$ are isospectral on functions but not on 1-forms. It is not hard to verify that the maximal isotropy group is $\mathbb{Z}_6$ in the case of $\mathcal{O}_1$ and D_6 (the dihedral group with six elements) in the case of $\mathcal{O}_2$. Just as in Example 3.5, the sets of points with maximal isotropy have different dimensions: In $\mathcal{O}_1$, it is a circle of length one (the image of the x_3-axis), while in $\mathcal{O}_2$ it consists of only two points (the images of $(0,0,0)$ and $(0,0,1/2)$). Note that $\mathcal{O}_1$ is the product of a 236-orbifold with a circle of length one. So its other nontrivial isotropy groups are $\mathbb{Z}_2$ and $\mathbb{Z}_3$, and the corresponding singular points each time form another circle of length one. In $\mathcal{O}_2$ there are two open segments of length two consisting of points with isotropy group $\mathbb{Z}_2$ (corresponding to the horizontal segments in Figure 6). The set of points with isotropy $\mathbb{Z}_3$ consists of the open segment of length 1/2 which joins the two points with maximal isotropy and of the circle of length one corresponding to the vertical edge through the point $(1, 1/\sqrt{3}, 0)$.

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