Constructing isospectral metrics via principal connections

Dorothee Schueth

Mathematisches Institut, Universität Bonn, Beringstr. 6, D-53115 Bonn, Germany
schueth@math.uni-bonn.de

1 Recent developments concerning isospectral manifolds

The spectrum of a closed Riemannian manifold is the eigenvalue spectrum of the associated Laplace operator acting on functions, counted with multiplicities; two manifolds are said to be isospectral if their spectra coincide. Spectral geometry deals with the mutual influences between the spectrum of a Riemannian manifold and its geometry. To which extent does the spectrum determine the geometry?

For example, the spectrum determines a sequence of so-called heat invariants, the first few of which are the dimension, the volume, and the total scalar curvature (see e.g. [4]). Some Riemannian manifolds are known to be completely characterized by their spectra; e.g., flat tori in dimensions two and three [13], or round spheres in dimensions up to six [20].

In order to find geometric properties which are not determined by the spectrum, one needs examples of isospectral manifolds. One method of constructing such manifolds which was very productive in the 1980's is the so-called Sunada method and its generalizations (see [17], [5], [2], [3], [12]). This construction yielded pairs (or families) of isospectral manifolds arising as quotients of a common Riemannian covering manifold by different discrete subgroups of isometries. In particular, these pairs were always locally isometric, and their geometries could be distinguished only by global properties.

The first examples of locally nonisometric manifolds were found in the first half of the 1990's by Zoltan I. Szabó [18] (in the case with boundary) and Carolyn Gordon [6]. In both cases, the manifolds arose from very special constructions; isospectrality was proven, at first, more or less by explicit computation of the spectra, which happened to be possible here. However, in a second paper on her examples [7], Gordon interpreted them as instances of the following general principle:

Theorem 1. [7] *If a torus acts on two Riemannian manifolds freely and isometrically with totally geodesic orbits, and if the quotients of the manifolds by any subtorus of codimension at most one are isospectral when endowed with the submersion metric, then the original two manifolds are isospectral.*

Several other examples of locally nonisometric isospectral manifolds, all in some sense related to Szabó's and Gordon's original examples, were subsequently found, and proven to be isospectral by the above theorem ([11], [9], [14]). Some of these examples show that neither the range of the scalar curvature function [9] nor certain curvature integrals like $\int_M \mathrm{scal}^2 \, dvol_g$, $\int_M \|\mathrm{Ric}\|^2 \, dvol_g$, $\int_M \|R\|^2 \, dvol_g$ are determined by the spectrum [14]. The latter is particularly remarkable because a certain linear combination of these three terms is a heat invariant, thus spectrally determined.

It is not very obvious from the theorem how to find new applications. A main motive of the author's habilitation thesis [15] which was (just as [9], [14], and [16]) written under partial support of the SFB 256 at Bonn, was to give a specialized version which would be useful for finding new situations in which Gordon's theorem applied. Such a specialized version is the following:

Let T be a torus, endowed with a fixed left invariant metric, and let $P \to B$ be a principal T-bundle, where B is a closed and connected Riemannian manifold. For any principal connection ω on P there is a unique Riemannian metric g_ω on P such that the projection $P \to B$ is a Riemannian submersion, the horizontal distribution is equal to $\ker \omega$, and the induced metric on each fiber is the given metric on T. With respect to this metric g_ω, the fibers are totally geodesic.

Theorem 2. [15] ("Connection Technique") *Let ω, ω' be principal connections on P. Suppose that for each μ in the dual of the Lie algebra $\mathfrak{z}$ of T there is a bundle automorphism $F_\mu : P \to P$ which factors over an isometry of B and satisfies $\mu \circ \omega = F_\mu^*(\mu \circ \omega')$. Then (P, g_ω) and $(P, g_{\omega'})$ are isospectral.*

The idea of the proof is to show that if $W < T$ is a subtorus of codimension one and $\mu \in \mathfrak{z}^*$ is such that $\ker \mu = T_e W$, then F_μ induces an isometry from $(P/W, g_\omega^W)$ to $(P/W, g_{\omega'}^W)$, where g_ω^W and $g_{\omega'}^W$ denote the submersion metrics.

Ballmann [1] gave an interesting extension of Theorem 2 to other fiber bundles associated with P.

One application of Theorem 2 was the construction of isospectral, locally nonisometric metrics on $S^2 \times T^2$ [15]; thereby, dimension four became the lowest dimension of manifolds on which such metrics are so far known to exist. Geometrically, these isospectral metrics on $S^2 \times T^2$ can be distinguished by the dimension of the locus of the maximal scalar curvature. Another application of Theorem 2 was the construction of left invariant isospectral metrics on compact Lie groups [15]; see more about this in Section 2.

Soon afterwards, Carolyn Gordon [8] discovered a new version of her original theorem in which the torus actions were no longer required to be free. (Earlier there had already been versions, first by Gordon and Szabó [10], then by the author [15] in which the T-orbits were not anymore required to be totally geodesic.) As an application, Gordon obtained isospectral metrics on spheres, in dimension at least eight; together with certain examples of very different type found independently by Z.I. Szabó [19], these were the first examples of isospectral spheres. The author established a specialized formulation of Gordon's new theorem, in the spirit of the above connection technique:

Let (M, g_0) be a closed and connected Riemannian manifold on which a torus T with Lie algebra $\mathfrak{z}$ acts effectively by isometries. For $Z \in \mathfrak{z}$ we denote by Z^* the induced vectorfield on M. We call a 1-form λ on M *horizontal* if it vanishes on all Z^*. With every $\mathfrak{z}$-valued, T-invariant, horizontal 1-form on M we associate a Riemannian metric $g_\lambda : (X, Y) \mapsto g_0(X + \lambda(X)^*, Y + \lambda(Y)^*)$ on M.

Theorem 3. **[16]** *Let λ, λ' be $\mathfrak{z}$-valued, T-invariant, horizontal 1-forms on M. If for every $\mu \in \mathfrak{z}^*$ there is a T-equivariant isometry F_μ of (M, g_0) such that $\mu \circ \lambda = F_\mu^*(\mu \circ \lambda')$, then (M, g_λ) and $(M, g_{\lambda'})$ are isospectral.*

Remark 1. Note that the union $\hat{M}$ of principal orbits in M is a principal T-bundle again. If we let ω_λ denote the principal connection on $\hat{M}$ whose kernel is the horizontal distribution with respect to g_λ , then $\omega_\lambda = \omega_0 + \lambda$. In particular, the condition $\mu \circ \lambda = F_\mu^*(\mu \circ \lambda')$ from Theorem 3 is equivalent to $\mu \circ \omega_\lambda = F_\mu^*(\mu \circ \omega_{\lambda'})$, which is a condition on connection forms just as in Theorem 2.

Theorem 3 was used in [16] to construct examples of isospectral metrics on spheres in dimension five. A sketch of proof for Theorem 3 is as follows: Consider the Sobolev spaces $\mathcal{H} := H^{1,2}(M, g_\lambda)$ and $\mathcal{H}' := H^{1,2}(M, g_{\lambda'})$, and decompose each as an orthogonal sum of subspaces $\mathcal{H}_\mu , \mathcal{H}'_\mu$ using Fourier decomposition with respect to the unitary T-action on $\mathcal{H}$ and $\mathcal{H}'$. Here, μ runs through the dual lattice $\mathcal{L}^* \subset \mathfrak{z}^*$ of $T = \mathfrak{z}/\mathcal{L}$. We claim that $F_\mu^* : \mathcal{H}'_\mu \to \mathcal{H}_\mu$ preserves both the L^2- and the $H^{1,2}$-norms, whence the theorem will follow by the variational characterization of eigenvalues via the Rayleigh quotient. Preservation of L^2-norms is trivial since F_μ is a g_0-isometry and $g_0 , g_\lambda , g_{\lambda'}$ all have the same volume element. That $F_\mu^* : \mathcal{H}'_\mu \to \mathcal{H}_\mu$ also preserves the norm of the gradient (hence the $H^{1,2}$-norm) can now be derived using the condition $\mu \circ \lambda = F_\mu^*(\mu \circ \lambda')$.

Actually Theorem 3 applies to the case of manifolds with boundary as well. In that case, both Dirichlet and Neumann isospectrality is implied; see [16], [8].

Note that it is crucial that F_μ may depend on μ. If all F_μ are equal, then $\lambda = F_\mu^*\lambda'$ (for any μ), which is easily seen to imply that g_λ and $g_{\lambda'}$ are isometric. In the opposite case we say that λ, λ' satisfy the conditions of Theorem 3 *nontrivially.*

At this point, the reader might be wondering how one is able to produce examples of $\mathfrak{z}$-valued 1-forms λ, λ' which do satisfy the conditions of Theorem 3 nontrivially. The next section describes one fruitful way of doing this.

2 Almost conjugate subspaces of compact Lie algebras and associated principal connections

We start this section by describing families of almost conjugate subspaces in certain classical compact Lie algebras. Such families have repeatedly been used (at first in [7] and [11]) as ingredients in earlier constructions of isospectral manifolds which either arose from, or can be viewed as arising from, Theorem 3. We will try to explain as generally as possible how these almost conjugate subspaces can be used to produce nontrivial pairs of 1-forms satisfying the conditions of Theorem 3. Several known examples will serve as illustration. In Section 3, we will then give some new applications.

Note that the isospectral, locally nonisometric metrics on the low-dimensional manifolds $S^2 \times T^2$ [15] and S^5 or $S^3 \times S^1$ [16] do, however, *not* arise from the approach discussed here.

In the following, let H be a compact connected semisimple Lie group with Lie algebra $\mathfrak{h}$, and let $\mathfrak{z}$ be a euclidean vector space.

Definition 1. (i) *Two linear maps* $j, j' : \mathfrak{z} \to \mathfrak{h}$ *are called* isospectral *if for each* $Z \in \mathfrak{z}$ *there is* $a_Z \in H$ *such that* $j'_Z = \mathrm{Ad}_{a_Z}(j_Z)$.
(ii) j *and* j' *are called* equivalent *if there is* $\Phi \in \mathrm{Aut}(\mathfrak{h})$ *and* $C \in \mathrm{O}(\mathfrak{z})$ *such that* $j'_Z = \Phi(j_{C(Z)})$ *for all* $Z \in \mathfrak{z}$.

Note that if j, j' are isospectral, then the linear subspaces $j(\mathfrak{z})$ and $j'(\mathfrak{z})$ of $\mathfrak{h}$ are "almost conjugate" in the sense that there is a linear bijection (given by $j_Z \leftrightarrow j'_Z$) between them which preserves adjoint orbits. The following facts turn out to be very pleasant with respect to isospectral purposes:

Lemma 1. *Let* $\mathfrak{z} = \mathbb{R}^2$*, equipped with the standard metric, and denote by* $\mathcal{J}$ *the vector space of all linear maps from* $\mathfrak{z}$ *to* $\mathfrak{h}$.

(i) [11] *If* $\mathfrak{h} = \mathfrak{so}(m)$*, where* m *is any positive integer other than* $1, 2, 3, 4,$ *or* 6*, then there is a Zariski open subset* $\mathcal{O}$ *of* $\mathcal{J}$ *such that each* $j \in \mathcal{O}$ *belongs to a d-parameter family of isospectral, inequivalent elements of* $\mathcal{J}$*. Here* $d \geq m(m-1)/2 - [m/2]([m/2]+2) > 1$*. For* $m = 6$*, there exist at least* 1*-parameter families in* $\mathcal{J}$ *with these properties.*
(ii) [15] *If* $\mathfrak{h} = \mathfrak{su}(m)$*, where* $m \geq 3$*, then there is a Zariski open subset* $\mathcal{O}$ *of* $\mathcal{J}$ *such that each* $j \in \mathcal{O}$ *belongs to a continuous family of isospectral, inequivalent elements of* $\mathcal{J}$.

The definitions of isospectrality and equivalence which were used in [11] and [15] were different in minor ways from our above definition; however, it is not hard to see that this does not affect the statements of Lemma 1.

Now suppose that our compact Lie group H acts on a compact connected Riemannian manifolds N by isometries. Then each $j_Z \in \mathfrak{h}$ induces a vectorfield j_Z^* on N; by taking the dual with respect to the Riemannian metric on N, we obtain a 1-form λ_Z on N.

Key observation: *If $j_Z\,, j'_Z$ belong to the same adjoint orbit then the associated* 1*-forms $\lambda_Z\,, \lambda'_Z$ on N belong to the same $H \subset \mathrm{Isom}(N)$-orbit.*

This observation, and how one may use it systematically, was already mentioned, but not further elaborated on, in the author's Remark 3.4 in [15].

Our first application is the construction of isospectral metrics on $M := N \times T$, where T is a torus with Lie algebra $\mathfrak{z}$. Let T be equipped with a fixed left invariant metric, and denote by g_0 the Riemannian product metric on M. Given any linear map $j : \mathfrak{z} \to \mathfrak{h}$, we consider the associated 1-forms λ_Z $(Z \in \mathfrak{z})$ on N as above. We define a $\mathfrak{z}$-valued 1-form λ on N by $\langle \lambda(X), Z\rangle = \lambda_Z(X)$ for all $Z \in \mathfrak{z}$ and $X \in TN$, where $\langle\ ,\ \rangle$ is the given euclidean inner product on $\mathfrak{z}$. Let the pullback of λ to $M = N \times T$ be denoted λ again. Note that λ is T-invariant and horizontal by construction.

Proposition 1. *Let H act on N by isometries. If $j, j' : \mathfrak{z} \to \mathfrak{h}$ are isospectral, then the associated $\mathfrak{z}$-valued* 1*-forms λ, λ' on $M = N \times T$ satisfy the conditions of Theorem 3 on (M, g_0); in particular, (M, g_λ) and $(M, g_{\lambda'})$ are isospectral.*

Proof. Let $\mu \in \mathfrak{z}^*$, and let $Z \in \mathfrak{z}$ be the dual vector with respect to the inner product on $\mathfrak{z}$. Choose $a_Z \in H$ such that $j'_Z = \mathrm{Ad}_{a_Z}(j_Z)$. Then straightforward calculation shows that the T-equivariant isometry $F_\mu := (a_Z\,, \mathrm{Id})$ of $M = N \times T$ satisfies $\mu \circ \lambda = F^*_\mu(\mu \circ \lambda')$. □

Example 1. The following known examples of isospectral manifolds can be viewed as applications of Proposition 1:

(i) The isospectral families of left invariant metrics on $H \times T^2$ from [15], where H is one of $\mathrm{SO}(m \geq 5)$, $\mathrm{Spin}(m \geq 5)$, or $\mathrm{SU}(m \geq 3)$. Here we have $\mathfrak{z} = \mathbb{R}^2$, $\mathfrak{h} = \mathfrak{so}(m)$ (resp. $\mathfrak{su}(m)$), and the Riemannian manifold N is H, endowed with a biinvariant metric. We consider the action of H on itself from the left and use the isospectral families of maps from Lemma 1. Actually, our present construction leads to *right* invariant vectorfields j_Z and 1-forms λ_Z, hence to right invariant isospectral metrics on $H \times T^2$. However, under the canonical identification of right invariant metrics with left invariant ones via the map $a \mapsto a^{-1}$, these classes of isospectral families of metrics on $H \times T^2$ are in fact the same. The author showed in [15] that, generically, these isospectral homogeneous metrics differ by the norm of the associated Ricci tensors.

(ii) The isospectral families of metrics on $S^{m-1\geq 4} \times T^2$ from [9]. Here $\mathfrak{z} = \mathbb{R}^2$, $\mathfrak{h} = \mathfrak{so}(m)$, $H = \mathrm{SO}(m)$, and $N = S^{m-1}$ endowed with a round metric and with the canonical action of H from the left. Again one uses the isospectral maps from Lemma 1(i). The original construction of these isospectral families of metrics on $S^{2m-1} \times T^2$ was done in a very different context; the manifolds were viewed as submanifolds of certain Riemannian nilmanifolds. Moreover, it was shown in [9] that the maximum of the scalar curvature is in general nonconstant during these isospectral deformations.

We are now going to give an extension of Proposition 1 which produces isospectral metrics on irreducible manifolds as well:

Let T again be a torus with Lie algebra $\mathfrak{z}$, and suppose that $H \times T$ acts on a compact connected Riemannian manifold (M, g_0) by isometries. With any linear map $j : \mathfrak{z} \to \mathfrak{h}$ we associate vectorfields j^*_Z , 1-forms λ_Z and a $\mathfrak{z}$-valued 1-form λ on M exactly as we did above on N. In short,

$$\langle \lambda(X), Z \rangle = g_0(j^*_Z(p), X)$$

for all $Z \in \mathfrak{z}$ and $X \in T_pM$ $(p \in M)$, where $\langle\ ,\ \rangle$ is the given inner product on $\mathfrak{z}$ (not to be confused with the metrics induced by g_0 on T-orbits in M).

Note that λ is T-invariant because the actions of H and T on M commute. However, λ will in general not be horizontal; this is the reason for the additional orthogonality assumption in the following result.

Proposition 2. *Let $H \times T$ act on (M, g_0) by isometries. If $j, j' : \mathfrak{z} \to \mathfrak{h}$ are isospectral, and if $j^*_Z(p), j'^*_Z(p) \perp W^*(p)$ for all $Z, W \in \mathfrak{z}$ and $p \in M$, then the associated $\mathfrak{z}$-valued 1-forms λ, λ' on M satisfy the conditions of Theorem 3; in particular, (M, g_λ) and $(M, g_{\lambda'})$ are isospectral.*

Proof. By the orthogonality assumption, λ and λ' are now indeed horizontal. We proceed as in the proof of Proposition 1, this time letting $F_\mu := a_Z \in H$; these maps are T-equivariant because H commutes with T. □

Example 2. The following known examples of isospectral manifolds can be viewed as applications of Proposition 2 (each time, the reservoir of isospectral j-maps from Lemma 1 is used):

(i) Isospectral families of right invariant isospectral metrics on G, corresponding to the left invariant ones from [15], where G is one of $\mathrm{SO}(n \geq 9)$, $\mathrm{Spin}(n \geq 9)$, or $\mathrm{SU}(n \geq 6)$. Here $\mathfrak{z} = \mathbb{R}^2$, $\mathfrak{h}$ is $\mathfrak{so}(m \geq 5)$ or $\mathfrak{su}(m \geq 3)$, H is $\mathrm{SO}(m)$ or $\mathrm{Spin}(m)$ or $\mathrm{SU}(m)$, respectively, and T is a maximal torus in $\mathrm{SO}(4)$ or $\mathrm{Spin}(4)$ or $\mathrm{SU}(3)$, respectively. We consider the left action of $H \times T$ on $M := G$ given by the inclusion $H \times T \subset \mathrm{SO}(m) \times \mathrm{SO}(4) \subset \mathrm{SO}(m+4) = G$ (and similarly for Spin and SU). Let g_0 be a biinvariant metric on G. Then λ, and hence g_λ , is right invariant on G (and left invariant under T). The orthogonality assumption of Proposition 2 is satisfied because H- and T-orbits meet perpendicularly with respect to g_0 .

(ii) Gordon's families of isospectral metrics on $S^{n-1 \geq 8}$ from [8]. Here $\mathfrak{z} = \mathbb{R}^2$, $\mathfrak{h} = \mathfrak{so}(m \geq 5)$, $H = \mathrm{SO}(m)$; T is again a maximal torus in $\mathrm{SO}(4)$, and $M = S^{m+3} \subset \mathbb{R}^m \oplus \mathbb{R}^4$, endowed with the standard metric g_0 . The action of $H \times T \subset \mathrm{SO}(m) \times \mathrm{SO}(4) \subset \mathrm{SO}(m+4) =: G$ on M is the restriction of the canonical action of G on the ambient space.

Remark 2. Further instances of Proposition 2 are the isospectral manifolds constructed in [14] and [10], and the conformally equivalent isospectral metrics on certain products of Lie groups from the last chapter of [15].

How can we get around the orthogonality assumption in Proposition 2? Our last observation in this section is that we can make λ horizontal by "brute force" if necessary: Suppose $H \times T$ acts isometrically on (M, g_0), and let λ be the associated $\mathfrak{z}$-valued, T-invariant 1-form on M as above. Then we define

$$\begin{aligned}\lambda_h(X) &:= \|Z_1^* \wedge \ldots \wedge Z_r^*\|^2 \lambda(X)\\ &\quad - \sum_{k=1}^{r} \langle Z_1^* \wedge \ldots \wedge Z_{k-1}^* \wedge X \wedge Z_{k+1}^* \wedge \ldots \wedge Z_r^*,\, Z_1^* \wedge \ldots \wedge Z_r^* \rangle \lambda(Z_k^*)\end{aligned}$$

for all $X \in TM$, where $\{Z_1, \ldots, Z_r\}$ is a basis of $\mathfrak{z}$, the Z_k^* are the induced vectorfields on M, and on each $\bigwedge^r T_pM$ we use the inner product induced by g_0. Obviously, λ_h is indeed horizontal now, and is again T-invariant.

Proposition 3. *Let $H \times T$ act on (M, g_0) by isometries. If $j, j' : \mathfrak{z} \to \mathfrak{h}$ are isospectral, then the associated 1-forms λ_h, λ'_h on M satisfy the conditions of Theorem 3; in particular, (M, g_{λ_h}) and $(M, g_{\lambda'_h})$ are isospectral.*

Proof. The proof is the same as for Proposition 2; in order to show that $F_\mu := a_Z \in H$ satisfies $\mu \circ \lambda_h = F_\mu^*(\mu \circ \lambda'_h)$, one must now also use the invariance of the vecorfields Z_k^* under the g_0-isometry $a_Z \in H$, which is another consequence of the fact that H and T commute. □

Example 3. There is just one instance of previously known examples which can in hindsight be viewed as an application of Proposition 3 (but not of Proposition 2): Certain continuous isospectral families of metrics on S^7 constructed by the author in [16]. In our present notation, we there had $\mathfrak{z} = \mathbb{R}^2$, $\mathfrak{h} = \mathfrak{su}(3)$, $H = \mathrm{SU}(3)$, $T = \{e^{it}\mathrm{Id} \mid t \in \mathbb{R}\} \times \mathrm{U}(1) \subset \mathrm{U}(3) \times \mathrm{U}(1)$, and $M = S^7 \subset \mathbb{C}^3 \oplus \mathbb{C}$, endowed with the standard metric g_0 and the action of $H \times T \subset U(3) \times U(1) \subset \mathrm{U}(4) =: G$ which is the restriction of the canonical action of G on S^7. (To be precise, our present λ_h differ from those used in [16] by multiplication with the $H \times T$-invariant function $\|Z_2^*\|^2$.)

3 Some new applications

In the earlier examples of isospectral manifolds which served as illustrations for Propositions 1–3 in the previous section, the only occurring actions of the compact simple groups H, or the compact groups $G \supset H \times T$, were the actions of these groups either on themselves, or on some round sphere. Between these two "extremes", one may as well consider their actions on any associated homogeneous space. Continuing to use the notation from Section 2, we obtain:

Corollary 1. *Let $j, j' : \mathfrak{z} \to \mathfrak{h}$ be isospectral.*

(1) *Let $K \subset H$ be a Lie subgroup. For any left invariant metric on H which is right invariant under K, consider the associated H-invariant homogeneous metric on H/K. Applying Proposition 1 to the Riemannian manifold $N := H/K$, we obtain isospectral metrics g_λ, $g_{\lambda'}$ on $(H/K) \times T$.*

(2) *Let* G *be a compact Lie group containing* $H \times T$ *as a Lie subgroup, and let* K *be any Lie subgroup of* G*. Given a left invariant metric on* G *which is right invariant under* K*, let* g_0 *denote the associated* G*-invariant homogeneous metric on* $M := G/K$*. If the orthogonality condition* $j_Z^*(p), j_Z'^*(p) \perp W^*(p)$ *holds with respect to* g_0 *for all* $Z, W \in \mathfrak{z}$ *and* $p \in M$*, then Proposition 2 yields isospectral metrics* $g_\lambda, g_{\lambda'}$ *on* $M = G/K$*.*

(3) *In the context of* (2)*, applying Proposition 3 to* $M = G/K$ *yields isospectral metrics* $g_{\lambda_h}, g_{\lambda'_h}$ *on* G/K *even if the orthogonality condition is not satisfied.*

Remark 3. (i) In Examples 1–3 of Section 2, we have already seen examples of the above corollary. We had $K = \{e\}$ in Examples 1(i) and 2(i), $K = \mathrm{SO}(m-1) \subset \mathrm{SO}(m) = H$ in Example 1(ii), $K = \mathrm{SO}(m+3) \subset \mathrm{SO}(m+4) = G$ in Example 2(ii), and $K = \mathrm{U}(3) \subset \mathrm{U}(4) = G$ in Example 3. In each case, the metrics g_0 were associated to biinvariant metrics on the groups.

(ii) In case the reader has become worried by our silence on general nontriviality statements for the above constructions, he should at least find it reassuring that in Examples 1–3 above, the isospectral manifolds have indeed been proven to be nonisometric under generic conditions, provided that the isospectral maps $j, j' : \mathfrak{z} \to \mathfrak{h}$ are not equivalent ([9], [15], [8], [16]). So the natural expectation that the isospectral manifolds will generically be nonisometric if there is no apparent reason for them to be otherwise was indeed justified in all earlier examples. In order not to make the present contribution too technical, we restrict ourselves to giving a nonisometry proof only for the first (which is the least complicated one) of the three new families of examples below.

Example 4. Isospectral metrics on $N \times T^2$*, where* N *is one of the real Grassmann manifolds* $\mathrm{Gr}_{k,m}$ *of* k*-planes in* $\mathbb{R}^{m\geq 5}$*,* $1 \leq k \leq m-1$*.*

This is a new application of Proposition 1 as specialized in part (1) of Corollary 1. In case $k = 1$ (or $k = m-1$) we get isospectral products of a projective space and a torus; these are Riemannian subcoverings of the manifolds from Example 1(ii), obtained by dividing by the canonical action of $\mathbb{Z}_2$ on the sphere factor.

In the context of part (1) of Corollay 1, we let $\mathfrak{z} = \mathbb{R}^2$, equipped with the standard inner product, $\mathfrak{h} = \mathfrak{so}(m \geq 5)$, $H = \mathrm{SO}(m)$, $K = \mathrm{S}(\mathrm{O}(k) \times \mathrm{O}(m-k)) \subset H$. Consider the biinvariant metric on H given by $\langle X, Y\rangle = \frac{1}{2}\mathrm{tr}({}^tXY)$ and the induced normal homogeneous metric on $N := H/K = \mathrm{Gr}_{k,m}$. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the standard twodimensional torus with the canonical metric, and let g_0 be the Riemannian product metric on $(H/K) \times T$. Then for each pair of isospectral linear maps $j, j' : \mathfrak{z} \to \mathfrak{h}$ the associated metrics $g_\lambda, g_{\lambda'}$ on $(H/K) \times T$ are isospectral. Below we will show that if j and j' are not equivalent (in the sense of Definition 1), and if j' is *generic* in the sense that $j'(\mathfrak{z})$ has trivial centralizer in $\mathfrak{h}$, then these two metrics are not isometric. In particular, there exist d-parameter families of pairwise nonisometric isospectral metrics on $(H/K) \times T$, where d is as in Lemma 1(i).

Example 5. Isospectral metrics on the real Stiefel manifolds $W_{k,n}$ *of orthonormal* k*-frames in* $\mathbb{R}^{n\geq 9}$*,* $1 \leq k \leq n-1$.

This is a new application of Proposition 2 as specialized in part (2) of Corollary 1; in case $k = 1$ we obtain Gordon's isospectral spheres from Example 2(ii) again.

Let $\mathfrak{z} = \mathbb{R}^2$, $\mathfrak{h} = \mathfrak{so}(m \geq 5)$, $H = \mathrm{SO}(m)$, and T be a maximal torus in $\mathrm{SO}(4)$; thus $H \times T \subset \mathrm{SO}(m) \times \mathrm{SO}(4) \subset \mathrm{SO}(m+4) =: G$. Let M be the set of orthonormal k-tuples $(p_1, \ldots, p_k)$ with $p_\ell \in \mathbb{R}^{m+4}$ $(1 \leq \ell \leq k)$, viewed as a submanifold of $(\mathbb{R}^{m+4})^k = \mathbb{R}^{(m+4)k}$ and endowed with the metric g_0 which is induced on M by the standard metric of $\mathbb{R}^{(m+4)k}$. Then G, hence $H \times T$, acts by isometries on M. The orthogonality condition from Proposition 2 is satisfied. In fact, for any $j_Z \in j(\mathfrak{z}) \subset \mathfrak{h} = \mathfrak{so}(m) \subset \mathfrak{so}(m) \oplus \mathfrak{so}(4) \subset \mathfrak{g}$ and $W \in \mathfrak{z} = \mathfrak{so}(4) \subset \mathfrak{so}(m) \oplus \mathfrak{so}(4) \subset \mathfrak{g}$ we have

$$\begin{aligned}\langle j_Z^*(p), W^*(p)\rangle &= \langle (j_Z p_1, \ldots, j_Z p_k), (W p_1, \ldots, W p_k)\rangle \\ &= \sum_{\ell=1}^{k} \langle j_Z p_\ell \,, W p_\ell\rangle = -\sum_{\ell=1}^{k} \langle W j_Z p_\ell \,, p_\ell\rangle = 0\end{aligned}$$

because $W j_Z = 0$ in $\mathrm{End}(\mathbb{R}^{m+4})$. Thus for each pair of isospectral maps $j, j' : \mathfrak{z} \to \mathfrak{h}$ we obtain a pair of isospectral metrics $g_\lambda \,, g_{\lambda'}$ on M.

The manifold M can be identified with $G/K = W_{k,n}$, where $n = m+4$ and $K = \{\mathrm{Id}\} \times \mathrm{SO}(m+4-k) \subset \mathrm{SO}(k) \times \mathrm{SO}(m+4-k) \subset \mathrm{SO}(m+4) = G$. Note, however, that the homogeneous metric on M which we used above is *not* induced by a biinvariant metric on G, except if $k = 1$. Instead, as one can check, it is induced by the left invariant and K-right invariant metric g on G given by

$$g(X, Y) = \langle X, Y\rangle + \langle \mathrm{pr}_k \circ X \circ \mathrm{pr}_k \,, \, \mathrm{pr}_k \circ Y \circ \mathrm{pr}_k\rangle$$

for all $X, Y \in \mathfrak{g} = \mathfrak{so}(m+4) \subset \mathrm{End}(\mathbb{R}^{m+4})$, where $\langle X, Y\rangle = \frac{1}{2}\mathrm{tr}({}^t X Y)$, and pr_k denotes the projection of $\mathbb{R}^{m+4}$ to the first k coordinates. The normal homogeneous metric would *not* have satisfied the orthogonality condition of Proposition 2 except if $k = 1$.

Example 6. Isospectral metrics on the real Grassmann manifolds $\mathrm{Gr}_{k,n}$ *of* k*-planes in* $\mathbb{R}^{n\geq 9}$*,* $1 \leq k \leq n-1$.

This is a new application of Proposition 3 as specialized in part (3) of Corollary 1. In case $k = 1$ (or $k = n-1$) we obtain isospectral projective spaces which are just $\mathbb{Z}_2$-subcoverings of Gordon's isospectral spheres from Example 2(ii).

We let $\mathfrak{z}$, $\mathfrak{h}$, H, T, G, and n be as in the previous example, but now choose $K := \mathrm{S}(\mathrm{O}(k) \times \mathrm{O}(m+4-k))$. Let $j, j' : \mathfrak{z} \to \mathfrak{h}$ be isospectral linear maps.

The normal homogeneous metric g_0 on $M := G/K = \mathrm{Gr}_{k,n}$, induced from the biinvariant metric on G given by $\langle X, Y\rangle = \frac{1}{2}\mathrm{tr}({}^tXY)$, will in general not satisfy the orthogonality condition from Proposition 2; in contrast to the case of Stiefel manifolds in the previous example, there is no remedy for this in choosing a different G-invariant metric on M. Thus the construction from Proposition 3 is our last recourse here. We use the normal homogeneous metric g_0 on M (or any other G-invariant metric) and obtain associated isospectral metrics g_{λ_h} , $g_{\lambda'_h}$ on M, where λ_h , λ'_h are constructed as in Proposition 3 from the $\mathfrak{z}$-valued 1-forms λ, λ' on M associated with j and j'.

We conclude this contribution by giving a nonisometry proof for the isospectral manifolds of Example 4. We use the same notation as there.

In order to simplify our subsequent calculations, it is convenient to view the Grassmann manifold N as a submanifold of the space of symmetric matrices $\mathrm{Sym}(\mathbb{R}^m) \subset \mathrm{End}(\mathbb{R}^m)$ via the identification of each $p = aK \in H/K = N$ with the projection ap_0a^{-1}, where p_0 denotes the projection of $\mathbb{R}^m$ to the first k coordinates. This is just the canonical embedding of $\mathrm{Gr}_{k,m}$ given by identifying a k-plane in $\mathbb{R}^m$ with the orthogonal projection onto this k-plane. Using the inner product $\langle X, Y\rangle = \frac{1}{2}\mathrm{tr}({}^tXY)$ on $\mathrm{End}(\mathbb{R}^m) \supset \mathrm{Sym}(\mathbb{R}^m)$, this identification is actually an isometry with respect to our normal homogeneous metric on N and the one induced from the ambient space $\mathrm{Sym}(\mathbb{R}^m)$. On this new copy of N, we have

$$\lambda_Z(X) = \langle [j_Z , p], X\rangle$$

for all $Z \in \mathfrak{z}$, $X \in T_pN \subset \mathrm{Sym}(\mathbb{R}^m)$, and hence

$$d\lambda_Z(X, Y) = 2\langle [j_Z , X], Y\rangle = -2\langle j_Z , [X, Y]\rangle.$$

Our proof follows the strategy used in [16].

We first show that if j is generic (i.e., if $j(\mathfrak{z})$ has trivial centralizer in $\mathfrak{h}$), then T is a maximal torus in $\mathrm{Isom}(N \times T, g_\lambda)$. Let F_t be a 1-parameter family of isometries of $(N \times T, g_\lambda)$ commuting with T. Then the F_t are T-equivariant and preserve the g_λ-horizontal distribution, hence the associated principal connection $\omega_\lambda = \omega_0 + \lambda$, hence they preserve also $d\omega_\lambda = d\omega_0 + d\lambda = d\lambda$; note that $d\omega_0 = 0$ here. In particular, the F_t induce a 1-parameter family of isometries $\bar{F}_t$ of the normal homogeneous metric on N which preserve $d\lambda$, hence $d\lambda_Z$ for all $Z \in \mathfrak{z}$. But each 1-parameter family of isometries of the Grassmann manifold $N = H/K$ belongs to $H = \mathrm{SO}(m)$. Hence we have $\bar{F}_t : p \mapsto a_tpa_t^{-1}$, where a_t is a 1-parameter family in H. Preservation of $d\lambda$ implies, by the above formula, that

$$\langle j_Z , [X, Y]\rangle \equiv \langle j_Z , a_t[X, Y]a_t^{-1}\rangle = \langle a_t^{-1}j_Za_t , [X, Y]\rangle$$

for all $Z \in \mathfrak{z}$, $t \in \mathbb{R}$, and $X, Y \in T_pN \subset \mathrm{Sym}(\mathbb{R}^m)$. One easily checks that these $[X, Y]$, which are skew-symmetric, span the whole space $\mathfrak{h}$. Thus

$j_Z \equiv a_t^{-1} j_Z a_t$ for all $Z \in \mathfrak{z}$, contradicting the genericity of j; this shows that T is indeed a maximal torus in $\mathrm{Isom}(N \times T, g_\lambda)$ if j is generic.

Now suppose that there were an isometry $F : (N \times T, g_\lambda) \to (N \times T, g_{\lambda'})$. By assumption, j' is generic; hence T is a maximal torus in $\mathrm{Isom}(N \times T, g_{\lambda'})$. Since all maximal tori are conjugate, we can assume (after possibly composing F with an isometry of $g_{\lambda'}$) that conjugation by F maps $T \subset \mathrm{Isom}(N \times T, g_\lambda)$ to $T \subset \mathrm{Isom}(N \times T, g_{\lambda'})$. Let Ψ denote the automorphism of $\mathfrak{z} = T_e T$ induced by conjugation by F. Then $F_*(Z^*) = \Psi(Z)^*$ for all $Z \in \mathfrak{z}$. In particular, $\Psi \in \mathrm{O}(\mathfrak{z})$, and F maps T-orbits to T-orbits. By similar arguments as above, F induces an isometry $\bar{F}$ of the normal homogeneous metric on N satisfying $d\lambda = \Psi \circ (\bar{F}^* d\lambda')$. But every isometry $\bar{F}$ of N is induced by conjugation by some $a \in \mathrm{O}(m)$, possibly combined with the orthocomplementation map $\beta : p \mapsto \mathrm{Id} - p$ in case $m = 2k$. From our above formula for $d\lambda, d\lambda'$ we see that $d\lambda'$ is invariant under β^* and therefore, again by similar arguments as before, that

$$j_Z = a^{-1} j'_{\Psi^{-1}(Z)} a$$

for all $Z \in \mathfrak{z}$ and some $a \in \mathrm{O}(m)$, contradicting the asssumed inequivalence of j and j'. $\square$

References

1. Ballmann, W. (2001): On the construction of isospectral manifolds (preprint)
2. Bérard, P. (1992): Transplantation et isospectralité I. Math. Ann., **292**, 547–559
3. Bérard, P. (1993): Transplantation et isospectralité II. J. London Math. Soc., **48**, 565–576
4. Berger, M., Gauduchon, P., Mazet, E. (1971): Le Spectre d'une Variété Riemannienne. Springer Lect. Notes Math., **194**
5. DeTurck, D., Gordon, C.S. (1987): Isospectral Deformations II: Trace formulas, metrics, and potentials. Comm. Pure Appl. Math., **XL**, 1067–1095
6. Gordon, C.S. (1993): Isospectral closed Riemannian manifolds which are not locally isometric. J. Diff. Geom., **37**, 639–649
7. Gordon, C.S. (1994): Isospectral closed Riemannian manifolds which are not locally isometric: II. In: Brooks, R., Gordon, C., Perry, P. (eds) Geometry of the Spectrum. Contemp. Math., **173**, 121–131
8. Gordon, C.S. (2001): Isospectral deformations of metrics on spheres. Invent. Math., **145**, 317–331
9. Gordon, C.S., Gornet, R., Schueth, D., Webb, D., Wilson, E. (1998): Isospectral deformations of closed Riemannian manifolds with different scalar curvature. Ann. Inst. Fourier, **48**, 593–607
10. Gordon, C.S., Szabó, Z.I. (2001): Isospectral deformations of negatively curved Riemannian manifolds with boundary which are not locally isometric. Duke Math. J. (to appear)
11. Gordon, C.S., Wilson, E. (1997): Continuous families of isospectral Riemannian metrics which are not locally isometric. J. Diff. Geom., **47**, 504–529

12. Pesce, H. (1996): Représentations relativement équivalentes et variétés riemanniennes isospectrales. Comment. Math. Helvetici, **71**, 243–268
13. Schiemann, A. (1997): Ternary positive definite quadratic forms are determined by their theta series. Math. Ann., **308**, 507–517
14. Schueth, D. (1999): Continuous families of isospectral metrics on simply connected manifolds. Ann. of Math., **149**, 287–308
15. Schueth, D. (2001): Isospectral manifolds with different local geometries. J. reine angew. Math., **534**, 41–94
16. Schueth, D. (2001): Isospectral metrics on five-dimensional spheres. J. Diff. Geom. (to appear)
17. Sunada, T. (1985): Riemannian coverings and isospectral manifolds. Ann. of Math., **121**, 169–186
18. Szabó, Z.I. (1997): Locally non-isometric yet super isospectral spaces. Geom. Funct. Anal., **9**, 185–214
19. Szabó, Z.I. (2001): Isospectral pairs of metrics on balls, spheres, and other manifolds with different local geometries. Ann. of Math., **154**, 437–475
20. Tanno, S. (1973): Eigenvalues of the Laplacian of Riemannian manifolds. Tôhoku Math. J., **25**, 391–403